RESULTS OF THE ENUMERATION OF COSTAS ARRAYS OF ORDER 29

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Abstract. The results of the enumeration of Costas arrays of order 29 are presented: except for 16 arrays out of a total of 164, all other arrays found are accounted for by the Golomb and Welch construction methods. These 16 arrays, however, cannot be considered to be new, as they were discovered in the past through a semi-empirical technique. The enumeration was performed on several computer clusters and required the equivalent of 366.55 years of single CPU time.

1. Introduction

Costas arrays [3] are square arrays of dots/1s and blanks/0s, such that there exists exactly one dot per row and column (that is, they are permutation arrays), and such that a) no four dots not lying on a straight line form a parallelogram, b) no four dots lying on a straight line form two equidistant pairs, and c) no three dots lying on a straight line are equidistant. They were first proposed by J.P. Costas in 1965 in the context of SONAR detection in a GE technical report [1], and later in 1984 in a journal publication [2].

There are two known algebraic construction techniques for the production of Costas arrays: the Golomb and Welch methods. Both are based on the theory of finite fields and were introduced and proven in 1984 by S. Golomb and H. Taylor [6, 8]. These methods successfully construct $n \times n$ Costas arrays for infinitely many, but not all, orders n, and they remain the only generally applicable construction techniques for Costas arrays available today.

In this work we present the results of the enumeration of Costas arrays of order 29. These results come approximately eight months and 2.5 years after the results of the enumeration of Costas arrays of order 27 [5] and 28 [4], respectively. *This presentation is meant to be read as a supplement to [4]: the reader is referred therein, in particular, for all relevant definitions not included here.*

2. Definition and basics

Let us begin with the definition of a Costas function/permutation [2, 3]:

Definition 1. Let $[n] := \{1, \ldots, n\}$, $n \in \mathbb{N}$ and consider a bijection $f : [n] \to [n]$; f is a *Costas permutation* iff

$\forall i, j, k$ such that $i, j, i + k, j + k \in [n]$:
$$f(i + k) - f(i) = f(j + k) - f(j) \Rightarrow i = j \text{ or } k = 0.$$

This condition is known as the *Costas property.*

Permutations correspond to permutation arrays by setting the elements of the permutation to denote the positions of the (unique) 1 in the corresponding column of the array, counting from top to bottom: $f(i) = j \Leftrightarrow a_{j,i} = 1$. It is customary to represent the 1's of a permutation array as "dots" and the 0's as "blanks". The terms "array" and "permutation" will henceforth be used interchangeably.

The Costas property is invariant under rotations of the array by 90°, horizontal and vertical flips, and flips around the diagonals, hence a Costas array gives birth to an equivalence class (EC) that contains either eight Costas arrays, or four if the array happens to be symmetric; in the latter case, we say the EC is symmetric. When presenting the results of the enumeration, we will give the lexicographically minimal representative from each EC for brevity; the members of the EC of a Costas array are often referred to as its *polymorphs.*

There exist two algebraic methods for the construction of Costas arrays, known as the Golomb and Welch methods [3, 6, 8, 7], each actually comprising several submethods: relevant to this work are Golomb methods G_0, G_2, and G_3, and Welch methods W_0 and W_2. A further semi-empirical method relying on these two, known as the Rickard method and consisting of two submethods, is currently the only additional method that has successfully led to the discovery of four previously unknown Costas arrays, namely two of order 29 (through the Rickard Welch RW_0 submethod), one of order 36, and one of order 42 [9]. A summary of all these techniques can be found in [4]. Any additional Costas array not resulting from these techniques will be characterized as *sporadic.*

As the Rickard method is not very well known, it is perhaps worth offering a brief description here. Appending a blank row to an exponential W_1 array and then cyclically shifting its rows any number of times results in a non-square array with the Costas property. If a column is then appended on either side with a single dot corresponding to location of the blank row, a new permutation array is obtained, which may have the Costas property (the same effect can be achieved by appending a blank column to a logarithmic W_1 array and then cyclically shifting its columns). Similarly, appending both a blank row and a blank column to a G_2 array and then cyclically shifting both its rows and its columns independently any number of times results in a new square array with the Costas property, which can furthermore be turned into a permutation array through the placement of a new dot at the intersection of the blank row and column, and which may have the Costas property.

3. Enumeration results for order 29

The enumeration found in total 164 Costas arrays, divided into 23 ECs, 5 of which are symmetric. More specifically, breaking down this set per construction method (descriptions of which can be found in [4]), there are:

- 4 G_0-arrays in 1 symmetric EC, constructed in $\mathbb{F}(29)$;
- $64 = 8^2 = \phi^2(30)$ G_2-arrays in 10 ECs (4 symmetric), constructed in $\mathbb{F}(31)$;
- 24 G_3-arrays in 3 ECs, constructed in $\mathbb{F}(32)$;
- 24 W_0-arrays in 3 ECs, constructed in $\mathbb{F}(29)$;
- 32 W_2-arrays in 4 ECs, constructed in $\mathbb{F}(31)$;
- 16 RW_0-arrays in 2 ECs, constructed in $\mathbb{F}(29)$.

The lexicographically minimal polymorphs are shown in Table 2, while the two RW_0-arrays are plotted in Figure 1. As was the case for order $n = 28$, none of the arrays found is sporadic. The total runtime of the project on a single CPU was recorded to be approximately 366.55 years (about 5.2 times as long as the enumeration of Costas arrays of order 28 recorded in [4]), but, due to high parallelization of the tasks, the real time required was approximately 230 days, between 2010/06/08 and 2011/01/24. This shows that, on average, approximately 568 processors were used throughout the run (though, briefly, at peak times, simultaneous use of up to about 2,000 processors was recorded). The memory and storage requirements of the code used were minimal.

Forbidden positions for Costas arrays of order 29 are shown in Figure 2: they correspond to positions in which none of the arrays carries a dot. The resulting cross-shaped region is similar to the cross-shaped regions appearing in the forbidden positions of orders 25 and 27 [5]. There were no forbidden positions in order 28: the cyclic shift property of W_1-arrays, present in all orders $n = p - 1$, p prime (hence 28 as well), implies that the dots of a single such array along with all of its cyclic shifts will span all elements of the array, leaving no gaps.

4. A brief description of the the procedure followed

Our code was essentially the same as the one used by our group for the enumeration of Costas arrays of order 28 [4]. The project was run on several clusters in Ireland:

- UCD's Phaeton cluster, managed by UCD IT Services, and featuring Intel Xeon E5450 and E5540 Quad Core CPUs, at 3.00GHz and 2.53GHz, respectively. We were allocated a maximum of 200 CPUs, while jobs were submitted through a custom-made task-farming Python PBS script.
- UCD's SenseTile cluster, managed by members of UCD CASL, and featuring AMD Opteron 8360SE Quad Core CPUs at 2.51GHz We were allocated 78 reserved CPUs, and jobs were submitted through a custom-made task-farming Python SGE script.
- The Stokes supercomputer of the Irish Centre for High End Computing (ICHEC), featuring Intel Westmere Xeon E5650 Hex Core CPUs at 2.66GHz. We used a recorded total of 1.6 million CPU hours allocated through a Class A ICHEC grant. Bundles of jobs were submitted to be run using Stokes's built-in task-farming facilities, each bundle allocated 288 CPUs and a walltime of half a week (84 hours).
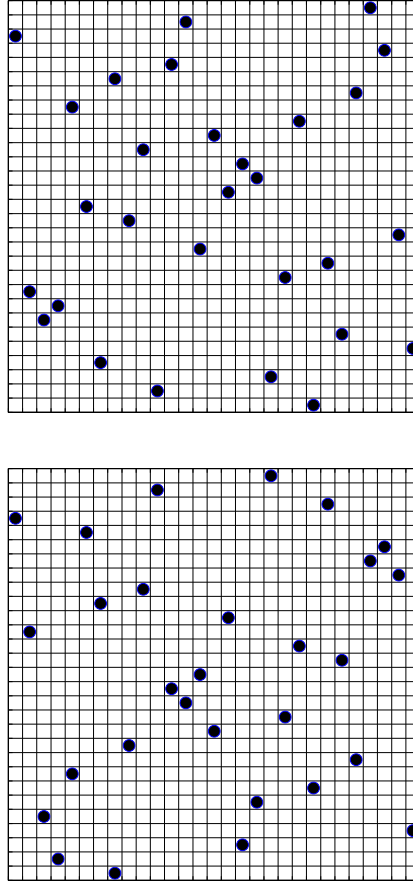
Figure 1. The two lexicographically minimal Welch Rickard Costas arrays of order 29, corresponding to permutations 3 21 23 22 8 15 26 6 16 11 28 5 2 18 10 14 12 13 27 20 9 29 19 24 7 1 4 17 25 (top) and 4 12 25 28 22 5 10 29 20 9 2 16 17 15 19 11 27 24 1 18 13 23 3 14 21 7 6 8 26 (bottom)

- Grid-Ireland, administered by members of Trinity College Dublin's (TCD) School of Computer Science and Statistics, and featuring a variety of C-PUs located all across Ireland. Jobs were initially submitted through the DIANE/GANGA queue submission system, but subsequently a new grid job framework and workflow was developed to comprehensively handle bounded job submission, job status monitoring with error handling, and output retrieval. A walltime of three days was individually applied to each job.
- Lonsdale, administered by the Trinity Centre for High Performance Computing (TCHPC), and featuring AMD Opteron 2350 Quad Core CPUs running at 2.00GHz. Jobs were submitted through custom scripts handling SLURM, the built-in job submission system.
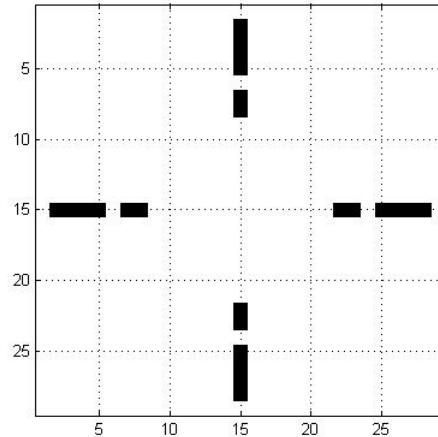
FIGURE 2.   Forbidden positions for Costas arrays of order 29

Cluster	Jobs run	%	CPU years run	%
Stokes (ICHEC)	106,724	54.22	217.37	59.30
Phaeton (UCD)	63,473	32.25	79.72	21.75
SenseTile (CASL)	13,744	6.98	27.12	7.40
Grid-Ireland	10,358	5.26	36.88	10.06
TCHPC (TCD)	2,540	1.29	5.46	1.49
Total	196,839	100	366.55	100

TABLE 1.   Statistics (number of jobs run and CPU time used) per cluster

Specific statistics per cluster are shown in Table 1. Runtimes of individual jobs, when the latter are lexicographically sorted, are shown in 3 (top), while the corresponding histogram is shown in Figure 3 (bottom). In the top figure, 13 individual ramp-shaped parts are clearly discernible, corresponding to the starting integer (2 to 14) of the permutations tested. The high spikes in the second ramp (corresponding to starting integer 3) are due to the slowest processors of Grid-Ireland.

5. Conclusion

Costas arrays of order 29 have been enumerated: there are 164 Costas arrays divided into 23 equivalence classes, five of which are symmetric. Had it not been for the publication of [9] in 2004, two equivalence classes would have been new/sporadic, but, as it stands, no new/sporadic Costas arrays were found, as was the case for order 28.
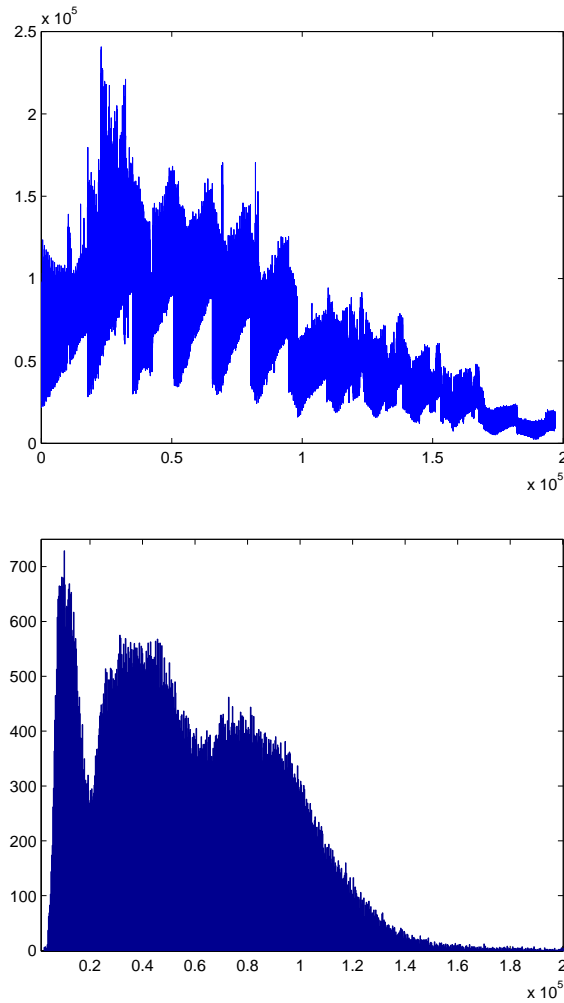
Figure 3.   Top: runtimes of jobs arranged in lexicographical order. Bottom: the histogram of runtimes of the various instances of the project (1000 bins); outlier runtimes of more than 200,000s are not shown.

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1 6 3 28 13 9 7 14 26 25 27 10 19 5 15 18 29 4 22 12 20 2 24 17 21 16 8 23 11	G_3
1 10 3 22 21 23 7 28 14 17 12 9 16 24 15 20 26 11 29 8 4 27 25 5 19 13 2 6 18	G_3
1 11 3 25 23 4 7 21 20 22 16 29 9 6 15 5 12 17 10 26 14 18 2 8 28 24 19 27 13	G_3
1 12 29 3 25 14 26 19 28 27 18 24 7 4 6 11 21 15 16 2 10 8 22 17 5 20 23 13 9	G_2
1 17 22 13 19 23 24 27 11 8 25 15 28 7 18 6 26 12 21 3 5 20 16 14 9 2 10 4 29	G_0/s
1 20 14 16 25 22 23 13 26 12 7 28 21 4 29 11 17 15 6 9 8 18 5 19 24 3 10 27 2	W_0
1 22 21 17 20 28 16 10 19 12 27 25 15 4 9 23 18 29 11 13 26 5 24 2 14 6 3 7 8	W_0
1 25 22 23 13 26 12 7 28 21 4 29 11 17 15 6 9 8 18 5 19 24 3 10 27 2 20 14 16	W_0
2 1 8 12 25 15 18 3 13 22 17 4 9 28 6 29 11 7 21 27 19 10 26 24 5 23 20 14 16	G_2/s
2 8 26 18 25 15 16 19 28 24 12 7 23 9 29 27 21 3 11 4 14 13 10 1 5 17 22 6 20	W_2
2 23 1 7 5 19 29 24 20 9 3 15 22 10 12 21 14 18 17 4 26 11 27 28 25 8 13 16 6	G_2
2 28 10 1 25 18 22 20 23 9 27 29 7 13 12 26 3 8 19 14 11 21 6 15 5 24 16 17 4	G_2
3 9 20 17 25 12 27 13 7 16 28 1 24 29 18 6 8 22 26 21 19 2 23 15 5 4 14 10 11	G_2
3 11 1 10 25 19 14 16 20 4 2 15 27 7 12 8 29 18 6 9 26 23 22 28 5 21 13 24 17	G_2/s
3 21 23 22 8 15 26 6 16 11 28 5 2 18 10 14 12 13 27 20 9 29 19 24 7 1 4 17 25	RW_0
4 12 25 28 22 5 10 29 20 9 2 16 17 15 19 11 27 24 1 18 13 23 3 14 21 7 6 8 26	RW_0
4 26 20 17 5 6 14 10 1 3 9 13 29 11 24 22 21 16 23 28 7 27 12 15 25 18 2 19 8	G_2
6 19 12 20 25 22 18 8 26 17 1 29 28 9 24 13 14 16 2 11 23 3 7 10 5 27 4 21 15	W_2
6 26 23 28 5 8 27 1 16 14 7 18 4 21 12 20 2 15 3 9 10 24 19 29 25 17 11 13 22	G_2/s
7 13 2 17 5 27 28 21 8 6 20 15 19 22 1 24 18 29 14 26 4 3 10 23 25 11 16 12 9	W_2
7 20 28 8 25 15 1 4 23 27 22 14 16 12 6 13 18 17 26 2 29 11 9 24 5 19 10 3 21	G_2/s
8 22 10 19 25 12 28 20 17 21 3 11 13 7 18 14 27 2 1 26 29 9 24 15 5 6 4 23 16	G_2
10 27 28 8 5 3 12 18 22 4 23 15 20 13 29 19 2 1 21 24 26 17 11 7 25 6 14 9 16	W_2

TABLE 2.   The lexicographically minimal polymorphs of Costas permutations of order 29, shown along with the method producing them. A "/s" indicates the EC is symmetric.

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