Relative Efficiency of RAS Versus Least Squares Methods of Updating Input-Output Structures, as Adjudged by Application to Irish Data

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The following paper is both theoretical and empirical, being concerned with RAS and Least Squares (LS) methods of updating Irish inter-industry structures from 1964 to 1968. Part 1 has a résumé of the published papers [1] and [2] and some developments, including a description of the LS form used in Part 2, which deals with numerical exercises.

The topic considered in Part 1 is how to distribute residual changes in row and column totals of inter-industry transactions, when all known effects have been allowed for. Thus the subject matter of Part 1 is quite limited by comparison with studies such as those of Seveldson or Middelhoek [3] or of Fonetela *et. al.* [4]. Methods of considerable generality and scope can also be found in papers such as [5], [6] and [7], presented at the September 1972 European Conference of the Econometric Society.

R. C. Geary in [1] surmised* that the LS method of distributing aggregate row and column transaction changes over non-zero individual elements of the transactions matrix is not new. He raised the query as to whether the convergence of the RAS method of distributing such changes has been proved mathematically.

Regarding the actual and implied questions of the previous paragraph, what is undoubtedly the most comprehensive treatment of the whole topic of updating inter-industry structures is the monograph by Bacharach [12] published in 1970. This includes discussion of the Deming and Stephan paper [11] and that of Friedlander [8], each of which is considered separately in this essay. Deming and Stephan [11] in 1940 had the first LS fully worked numerical solution to the

*This author acknowledges valuable comments by Mr. D. Conniffe of Foras Talúntais.

problem of distributing aggregate change over the non-zero elements of the matrix of transactions. In chapter 4 of [12] Bacharach provides a mathematical proof of the convergence of the RAS solution for *non-negative* matrices and also of the uniqueness of such a solution.

It is of interest that Bacharach deals almost exclusively with non-negative matrices, whereas Geary in [1] successfully applies the LS method to a matrix having negative as well as positive entries. Indeed a normal feature of more recently published input-output tables are the artificial transfers of byproducts from one sector to another by means of negative transactions. Geary furthermore appears to be on solid ground in setting his original or basic matrix on the same scale as the revised or estimated matrix, either by a global scaling to give identical grand totals for both matrices or by a columnwise scaling which will not give identical grand totals. Apart from Deming and Stephan [11] and the review of their methods by Friedlander in [8], the other authors including Bacharach do not appear to have adverted to the sound logic behind this initial step—the final estimates must be on the scale of production of the year in question and it makes good sense to scale up before using LS or RAS methods of estimation.

Geary has demonstrated in [1] that satisfactory numerical results emerge over a moderate range of variation in transaction size (i.e. over a scale of 1 to 4 units) for the LS method of uniform distribution of change used in [1]. It is only for extreme variations in transaction size, as will be made clear in Part 2 of this essay, that distributing change in proportion to the original known non-zero transaction values becomes necessary, in order to reduce probability of negative current estimates emerging from transactions which were originally positive. Whether the attempt to find simultaneous positive estimates over an extreme range of transaction size is realistic, or has entered the realm of playing with figures, is an open question. In any event this author has demonstrated in Part 2 of this essay that the proportionate form of the LS solution appears to work satisfactorily in getting positive original entries to yield positive final estimates. Thus either the uniform or the proportionate LS distribution of change may be considered operable—the feasibility of either depends on the base-year data in question.

Apart from the 1940 paper of Deming and Stephan [11], this author has not seen any explicit direct solution of the LS problem until Geary's paper [1] appeared. The LS algebraic solution appears to have been abandoned in favour of two alternative biproportional iterative forms of solution, one form being in more recent years synonymous with the RAS method, and the other form (referred to by Friedlander in [8] as that used by the Registrar General) in less wide usage. The latter form distributes the most recent change pro rata the *first* approximation whereas the RAS distributes change pro rata the *most recent* approximation. Abandonment of the direct LS solution could be due to two causes. (*a*) Until quite recently the solution of fairly large-scale systems of simultaneous equations was considered hazardous, even on computers, for a variety of reasons whereas the iterative process did not require elaborate calculating equipment. (*b*) Deming and Stephan in 1940 and again Friedlander in 1961 believed that one or other of the two iterative methods yielded results identical with those of the LS solution. Friedlander, for instance, in Appendix II of [8] sets out to prove that both iterative methods give identical answers, which coincide with that of the LS proportionate form, unless this author is mistaken in his reading of that appendix. The correctness of the assumptions concerning the identity of forms, either theoretically or in numerical applications, will be questioned below on several counts.

It may be that the assumptions about identity of form have precluded comparison of RAS and LS estimates. Such comparisons were made by Geary in [1] and are made below in Part 2 of this essay and there is a distinct possibility that this approach is fresh.

Neither in Bacharach nor elsewhere in the quoted references has the author succeeded in finding any formal comparison between the logarithmic transform of the RAS and the LS uniform distribution for logarithmic data. The significance of this comparison, as already set out in the author's paper [2] is twofold. (*a*) It explains the difficulties encountered empirically in applying RAS to matrices having negative entries, the difficulty being of course that a negative number does not have its logarithm in the field of real numbers. (*b*) The unique simultaneous-equation solution to the LS distribution of the logarithms of the r_i and s_j multipliers means that the r_i s_j joint multiplier exists and is unique—this approach is an alternative to that of proving that the iterative process converges and gives a unique distribution. Thus the comparison of forms of the transformed RAS and the LS may be a fresh approach to the convergence problem.

The papers [1] and [2] and the present paper represent the Irish experience to date. No claim is made that we have anything genuinely original to offer. The simultaneous equations required for solving by the LS method are quite manageable. In fact the author has written a computer programme (in Fortran D) which he will supply to enquirers on demand. It should be noted that the number of simultaneous linear equations is not the whole set of non-zero cell entries (which might be very large indeed) but merely $(m + n - 1)$ where m and n are the numbers of rows and columns respectively. Hence the writer opts firmly for LS treatment in preference to RAS.

The capability of the LS method to include negative transactions gives it one property which the RAS does not possess. The possible weakness of the LS method, however, is the lack of a guarantee that positive base entries will not become negative current estimates. The numerical results in Part 2 of this paper would seem to provide a reasonably satisfactory answer to this possibility. For the data showing such an extreme range of magnitude there is only one possible negative entry, and this is matched by a zero original entry which means the potential negative estimate does not materialise. Bacharach in chapter 11 of [12] comments on his own "Friedlander" numerical results as follows: "Table 6 illustrates the property of the Friedlander estimates which was our main reason for rejecting them in favour of biproportional (RAS) ones—namely, that their non-negativity is not assured. On the other hand, the negative elements of Table 6 are few and are all very small". Thus it appears that the small but definite probability of obtaining negative estimates via the LS approach, no matter how one weights the squared deviations, is an intrinsic property of the LS method, as at present formulated.

1. LEAST SQUARES AND RAS DISTRIBUTION OF ROW AND COLUMN AGGREGATE CHANGES

The problem is to estimate, at base year prices, the missing inter-industry matrix. One may have removed very large entries and those of known or precisely estimated magnitude from the matrix and from the appropriate row and column aggregates, in effect replacing them by zeros. The problem of consistently allowing for price changes is bypassed, under the assumption of constant prices for all values being considered.

A Least Squares Method of Estimation, by R.C. Geary

The LS constrained minimisation method of estimating the non-zero entries, published in [1], of course is not new, having the same approach as that of Friedlander, published in [8], but with three differences:

(1) A solution is directly obtainable without iteration, via a set of simultaneous linear equations. It appears that Friedlander overestimated the number of independent constraints and was consequently unable to solve the system of linear equations resulting from the derivatives.

(2) It is not the same solution as that given by the RAS, as will be shown below. Friedlander appears to have concluded that the LS and iterative (RAS) solution yield identical results. Earlier work indeed seems to make heavy weather of the problem of finding a solution.

(3) It appears to be applicable even when some transactions in the basic matrix are negative, whereas the RAS is known to be unworkable (i.e. fails to converge via iteration), if any of the inter-industry transactions is negative.

What might be the nearest approach to the Geary method in more recent literature appears in Clopper Almon's 1968 paper [10] within the section headed "*Provisional Matrices by Least Squares Balancing*". There is reference to "a linearisation of the RAS technique" whereby "the RAS method and its linearisation should therefore give *essentially* the same results". After some substitution among equations there results "a system of linear equations . . . with as many equations as unknowns. The system, however, is singular, for the row sums of the matrix on the left are all zero. . . . Consequently we may arbitrarily set one of the λ, say the last, equal to zero and drop the last equation from the system. . . . We find that about a dozen iterations produces very exact results."

The three differences between the Geary and Friedlander methods of solution would still seem to apply to Geary versus the linearised RAS technique, as quoted above by Almon. The Geary method would therefore seem to have certain advantages, in simplicity of solution, in adaptability to negative entries, and in being a genuine least squares solution and not some proxy for one.

Notation

m the number of rows of the inter-industry matrix.

n the number of columns of the inter-industry matrix.

N the number of non-zero elements to be estimated, with $N > m+n-1$.

X^o_{j} total input of column j, base year.

X'_{j} total input of column j, current year, assumed known.

ξ_{ij} the value of the transaction in row i of column j of the base year inter-industry matrix, after scaling up by the ratio X'_j/X^o_j i.e. the expected value of element (i, j), via the base structure, prior to distribution of the row and column aggregate changes. There are N non-zero values of ξ_{ij}.

$\xi_{i.}$ $\sum_j \xi_{ij}$ for row i.

$\xi_{.j}$ $\sum_i \xi_{ij}$ for column j.

x'_{ij} the current value of transaction (i, j), to be estimated, with non-zero elements corresponding to those of ξ_{ij}.

$x'_{i.}$ $\sum_j x'_{ij}$ for row i, a specified constant.

$x'_{.j}$ $\sum_i x'_{ij}$ for column j, a specified constant.

x_{ij} given by $x'_{ij}-\xi_{ij}$.

$x_{i.}$ ξx_{ij} for row i, also given by $x'_{i.}-\xi_{i.}$.

$x_{.j}$ $\sum_j x_{ij}$ for column j, also given by $x'_{.j}-\xi_{.j}$.

z the objective function, to be minimised.

$\lambda_{i.}$ Lagrange multiplier for row i constraint.

$\lambda_{.j}$ Lagrange multiplier for column j constraint.

λ Lagrange multiplier for grand total.

r_i RAS multiplier for row i.

s_j RAS multiplier for column j.

Geary proposes to minimise

(0) (i) $$2z = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij})^2 - 2\sum_{i=1}^{m-1} \lambda_{i.} \sum_{j=1}^{n} x_{ij} - 2\sum_{j=1}^{n-1} \lambda_{.j} \sum_{i=1}^{m} x_{ij} - 2\lambda \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$$

which is equation (5) of [1], for the $(m+n-1)$ independent constraints, which are equations (4) of [1]:

(o) (ii) $\sum_{j=1}^{n} x_{ij}=x_{i\cdot}$ for $i=1, 2, \ldots, m-1$

(o) (iii) $\sum_{i=1}^{m} x_{ij}=x_{\cdot j}$ for $j=1, 2, \ldots, n-1$

(o) (iv) $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}=0$

the last constraint having a right-hand value of zero due to Geary scaling up of ξ_{ij} values to match the grand total of the x'_{ij}. The typical solution-value for x'_{ij} is

(o) (v) $x_{ij}' = \xi_{ij}+\lambda_{i\cdot}+\lambda_{\cdot j}+\lambda$

but with both $\lambda_{m\cdot}$ and $\lambda_{\cdot n}$ effectively zero.

The Geary LS Method, modified by E. W. Henry

Some reasons for the modifications will be given below, at the end of this section. There are $(m+n-1)$ independent constraints on the system and these can be specified as

(1) $$\sum_{j=1}^{n} x_{ij} = x_{i\cdot}, \qquad i=1, 2, \ldots, m$$

(2) $$\sum_{i=1}^{m} x_{ij} = x_{\cdot j}, \qquad j=1, 2, \ldots, n-1$$

It will be observed that column n constraint as such has been omitted, for (1) and (2) as specified. The number of degrees of freedom in the system is $(N-m-n+1)$. The first modification of the Geary method is in taking row m for (1) above instead of the grand total of all transaction changes.

The problem is to find the values of all variables which minimise z, given by the Lagrange multiplier technique:

(3) $$2z = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{(x_{ij})^2}{\xi_{ij}}-2\sum_{i=1}^{m}\lambda_{i\cdot}\left(\sum_{j=1}^{n} x_{ij}-x_{i\cdot}\right)-2\sum_{j=1}^{n-1}\lambda_{\cdot j}\left(\sum_{i=1}^{m} x_{ij}-x_{\cdot j}\right)$$

The variables are the x_{ij}, numbering N and the λs, numbering $(m+n-1)$, to be found from the same number of simultaneous linear equations, $(m+n-1)$ from (1) and (2) and N from equating the partial derivitives of (3) with zero. It will be observed that in (3) the squared deviations x_{ij} are divided by ξ_{ij}, whereby large values of the latter will carry large x_{ij} and small x_{ij} will be associated with small values of ξ_{ij}. The second modification* of the Geary method is in taking $\Sigma(x_{ij})^2/\xi_{ij}$ instead of (x_{ij}).[2]

For the N non-zero x_{ij} (at least one of which is in each row and column), the partial derivitives equated with zero give:

$$(4) \qquad 0 = \frac{\delta z}{\delta x_{ij}} = x_{ij}/\xi_{ij} - \lambda_{i\cdot} - \lambda_{\cdot j}$$

for $i = 1, 2, \ldots, m$

$j = 1, 2, \ldots, n-1$

there being N such equations, which may be re-written:

$$(5) \qquad x_{ij} = \xi_{ij}(\lambda_{i\cdot} + \lambda_{\cdot j})$$

These are now substituted in (1) and (2) above and give $(m+n-1)$ simultaneous linear equations for the m values of $\lambda_{i\cdot}$ and the $(n-1)$ values of $\lambda_{\cdot j}$, with $\lambda_{\cdot n}$ for column n effectivey zero, being omitted from the scheme.

The typical λ-equation for each of the m rows, by substitution in (1), can be written:

$$(6) \qquad \xi_{i\cdot}\lambda_{i\cdot} + \sum_{j=1}^{n-1} \xi_{ij}\lambda_{\cdot j} = x_{i\cdot}$$

Likewise, for the typical column of the $(n-1)$ columns, the λ-equation via (2) is:

$$(7) \qquad \xi_{\cdot j}\lambda_{\cdot j} + \sum_{i=1}^{m} \xi_{ij}\lambda_{i\cdot} = x_{\cdot j}$$

After solving for the $\lambda_{i\cdot}$ and $\lambda_{\cdot j}$ and substituting in (5), the new matrix has as its typical entry

$$(8) \qquad x'_{ij} = \xi_{ij} + x_{ij} = \xi_{ij}(1 + \lambda_{i\cdot} + \lambda_{\cdot j})$$

with $\lambda_{\cdot n}$ in the scheme chosen above emerging as zero.

It will be observed that the x'_{ij} estimates appear as multiples of the original ξ_{ij}, with a row effect via the factor $\lambda_{i\cdot}$ and a column effect via the factor $\lambda_{\cdot j}$. Since the ξ_{ij} have been set to the scale of the current total inputs it is unlikely that

*Geary himself considered any positive coefficients (even $1/\xi_{ij}$) of the x^2_{ij} in z in the first draft of his paper but (as he now thinks wrongly in view of this paper) preferred coefficients of unity in the published version.

these multiplier effects can be so drastic as to change the sign of the ξ_{ij}, i.e. their combined effect is much less than that of the unit in the right hand factor of (8).

It will also be observed that there is no restriction upon the sign of the ξ_{ij}, since (3), (4) and (5) are valid for any non-zero ξ_{ij}.

There have been two main reasons for the modifications of the Geary method, as shown above. The first modification, namely the use of the row m condition instead of the condition for the grand total of the inter-industry changes, removes the constant λ from every non-zero transaction of the solution. It gives in general $(\lambda_i.+\lambda._j)$, instead of $(\lambda_i.+\lambda._j+\lambda)$, as expressing the change. Thus it brings the form of the solution closer to that of the RAS. The latter expresses the change for the typical transaction via the multipliers r_i and s_j.

The second modification, namely to obtain the changes as proportions of the basic transaction ξ_{ij} (rather than as absolute changes which ignore the magnitudes of the ξ_{ij}), has been found necessary in order to avoid negative entries among the solutions of a set of ξ_{ij} displaying an extreme range of values. The author found that application of the Geary LS method to the data shown in Tables 1 and 2 produced 1968 estimates which were negative in sign for some one in three of the smaller transactions. But the modified form of solution, as described by equation (8), gave all positive estimates.

The Uniqueness of the λ-Solution

While there is no question of the number of independent constraints being other than $(n+m-1)$, nor of the number of non-zero entries differing from N as specified, a certain arbitrariness appears to arise as to the choice of independent constraints and corresponding λ-variables. Does one get a different solution for each selection of $(m+n-1)$ equations?

Equations (4) and (5) show the contrary to be the fact. The minimum distribution of x_{ij}^2/ξ_{ij} is by definition unique, which means that each x_{ij} is unique. With ξ_{ij} a non-zero constant, equation (5) shows that each unique x_{ij} must be equal to $\xi_{ij}(\lambda_i.+\lambda._j)$, regardless of which $(n+m-1)$ constraints we select as independent. Thus each of the N combinations of $\lambda_i.$ with $\lambda._j$ is unique and the set of x_{ij} values, obtained by solution of (1) and (2) for whatever $(m+n-1)$ equations are selected for the same number of $\lambda_i.$ and $\lambda._j$ variables, is a unique set.

Comparison with the Deming and Stephen LS Method

What seems to be the earliest published paper on constrained LS adjustment of entries in a table, so as to match specified row and column totals, was by Deming and Stephen [11] and appeared in 1940.

Their formulation of constraints and algebraic method of solution is effectively that given above for the modified LS method and their equation (19) is indeed the same formula for x'_{ij} as equation (8) above, the new value being proportional to the old value, by a factor $(1+\lambda_i.+\lambda._j)$ with $\lambda._n$, for the last column of the table, taken as zero.

Having used the λ-solution to estimate the new values in a table of 6 rows and 4 columns (Table 1 of [11]), they proceed to obtain what appear to be almost identical results by "A simplified procedure—iterative proportions" (section 5 of [11]). The method of iteration seems to be the same as that used for the RAS, namely successive scaling of rows and columns. Only three iterations were needed to give their Table V results, having apparent close agreement with results via the λ-method.

They conclude that "The final results coincide with the least squares solution, which is thus accomplished without the use of the normal equations" (page 440). A careful reading of their approach to such a conclusion reveals that it is based on intuitive appeal, rather than on mathematical proof. The apparent lack of discrepancies between the genuine LS solution and the iterative RAS solution can be explained by the smallness of the adjustments, as follows.

Table 1 of [11] shows that the same grand total is used for old and new entries and that the change within each cell is nowhere in excess of some 4 per cent of the original entry. The numerically largest value of $(1+\lambda_{i}.+\lambda._{j})$, via figures shown in the paper, is 1·0383 for column (2) of row (3). The r_i and s_j values can be calculated by comparing Table V entries with corresponding entries of Table 1. For s_4 taken as unity, r_3 emerges as 1·02586, given by 119/116. For column (2) of row (3), $r_3 s_2$ is 1·03819, given by 435/419. The value of s_2 is 1·01201, via $r_3 s_2/r_3$.

The sum of the decimal fraction parts of r_3 and s_2 is therefore 0·03787 and the product of these parts is roughly 0·0003. Thus for such small deviations of r_i and s_j from unity, $r_i s_j$ is a close approximation to $(1+\lambda_{i}.+\lambda._{j})$. A further strong indication of the small changes involved is that only 3 iterations were required to produce satisfactory agreement both for row and column aggregates.

Thus the empirical evidence of equivalent results (via the two methods of solution) for changes nowhere exceeding 4 per cent, is no proof of a like outcome for changes of much greater magnitude.

Comparison of LS Solution with RAS Solution

The typical transaction of the LS solution is given by

$$\xi_{ij}(1+\lambda_{i}.+\lambda._{j}) \tag{9}$$

and that of the RAS by

$$\xi_{ij}(r_i s_j) \tag{10}$$

If r_i the row multiplier be equated with $(1+\lambda_{i}.)$ everywhere, then the constant column multiplier s_j cannot be equated with the variable multiplier $[1+\lambda._{j}/(1+\lambda_{i}.)]$, which varies according to row i as well as according to column j. Thus the two forms cannot be identified, although each incorporates a row and column effect.

The logarithmic transform of the RAS

This section of Part 1 and the following section, which discusses the impossibility of solving the RAS by means of logarithms, form part of the author's published paper [2], with certain modifications. The comparison of the logarithmic transformation of the RAS with a LS solution may be original, but would anyhow seem to give a deeper insight into the RAS process.

If in (3) above the ξ_{ij} is replaced by unity, i.e. there is no specific weighting of each x^2_{ij} in inverse proportion to each ξ_{ij}, then equation (5) is replaced by

$$x_{ij} = \lambda_{i\cdot} + \lambda_{\cdot j} \tag{11}$$

which makes each adjustment x_{ij} to the original ξ_{ij} the simple sum of $\lambda_{i\cdot}$ and $\lambda_{\cdot j}$. Let us now think of all ξ_{ij} and x_{ij} values for the LS solution given by (11) as logarithms.

The RAS solution for the original data has as its typical element the product given by (10). By taking logarithms of the latter, but by keeping blank entries as blanks and showing separately the matrix of logarithms of the ξ_{ij}, the logarithmic transform is as follows:

$$[\log \xi_{ij}] + [\log r_i + \log s_j] \tag{12}$$

Compare the typical element of the second matrix of (12) with the right hand side of (11). There is a one-one correspondence between $\lambda_{i\cdot}$ and $\log r_i$, likewise between $\lambda_{\cdot j}$ and $\log s_j$. In RAS numerical solutions having m values r_i and n values s_j only $(m+n-1)$ of these multipliers are independent, as is well known, and any one of them taken as unity sets the scale which determines all the other values of r_i and s_j, with $r_i s_j$ being invariant regardless of scale. It has also been seen above that one value of $\lambda_{i\cdot}$ or of $\lambda_{\cdot j}$ must be zero, in the full set of $(m+n)$, with $(m+n-1)$ of these values differing from zero, in the LS solution.

If $\lambda_{\cdot n}$ be chosen as zero and be identified with $\log s_n$ then

$$s_n = 1 \tag{13}$$

Consequently, from comparison of element (m, n) of both matrices, $\log r_m$ must be identified with $\lambda_{m\cdot}$ which gives

$$r_m = \exp(\lambda_{m\cdot}) \tag{14}$$

For consistency along row m,

$$s_1 = \exp(\lambda_{\cdot 1}),\ s_2 = \exp(\lambda_{\cdot 2}),\ \ldots,\ s_{n-1} = \exp(\lambda_{\cdot n-1}) \tag{15}$$

Row-wise, there is complete consistency for

$$r_1 = \exp(\lambda_{1\cdot}),\ r_2 = \exp(\lambda_{2\cdot}),\ \ldots,\ r_{m-1} = \exp(\lambda_{m-1\cdot}) \tag{16}$$

The s_n is the only unit multiplier, all the rest being different from unity.

Since logarithmic transformation must be applied to the ξ_{ij} as well as to the r_i and s_j multipliers in order to permit identification of the latter with the λ_i. and $\lambda._j$ of (11) in the comparison of forms given above, each of these ξ_{ij} must be positive, but their logarithms may be negative. The LS method is not confined to positive entries. Since the logarithm of a negative number is of the form $p+i\pi$, where i is $\sqrt{-1}$, the logarithmic transform of the RAS, for any matrix having one of more negative elements, is in the domain of complex variable and is not fully accounted for by the real-variable part alone. Hence a possible explanation of the empirical experience of non-convergence, in attempted iterative solution of problems having one or more negative transactions in the inter-industry matrix.

The impossibility of solving the RAS by logarithms

Since for row i of the RAS solution of the original data,

$$r_i(\xi_{i1}s_1 + \xi_{i2}s_2 + \ldots + \xi_{in}s_n) = x'_i. \tag{17}$$

this equality can be written

$$r_i\xi_i.\bar{s}_i = x'_i. \tag{18}$$

where $\xi_i.$ is the row sum of ξ_{ij} for row i and $\bar{s}_i$ is a weighted average of the s_j for row i, the weights being the $(\xi_{ij}/\xi_i.)$, some of them zero, for blank entries. It follows that

$$\log r_i + \log \bar{s}_i = \log (x'_i./\xi_i.) \tag{19}$$

with the right hand side of (19) known, being derived from the base and current row sums.

For the LS solution of the related problem having values expressed in logarithms as stated in (12), the row sum for row i of the matrix of elements ($\log r_i + \log s_j$) is

$$n_i \log r_i + \Sigma'_j \log s_j \tag{20}$$

where n_i is the number of non-blank entries in row i and the sum of $\log s_j$ is only for such non-blank entries. Division of row sum i by n_i gives the average effect per non-blank entry of row i

$$\log r_i + (1/n_i)\Sigma'_j \log s_j \tag{21}$$

Since formula (21) is not the same as the left hand side of (19), with (19) and (21) giving the closest approach of the two related problems to each other, there is no known way of using the numerical data available for the right hand side of (19) in order to set up equations for the variables in formula (21).

Conclusions on the Comparison of Forms for the RAS and LS

(1) In the untransformed variables the LS and RAS forms are distinct although each incorporates a row and column effect.

(2) The logarithmic transformation of the RAS solution of one problem may be regarded as the LS solution of a related problem having all data expressed as logarithms. As such it may explain the empirical evidence of convergence via iteration to any required degree of precision, for the original RAS solution, leading to precise estimation of the r_i and s_j multipliers, provided no negative transactions are included in the original matrix.

(3) It is not possible to state the LS related logarithmic problem in numerical terms, from the data of the base matrix and the row and column sums of the current (RAS) matrix.

(4) The RAS is not necessarily the most efficient way of distributing proportionate changes in transactions, since it is the logarithms of the r_i and s_j multipliers, and not these multipliers themselves, which satisfy the LS criterion of distribution.

2. NUMERICAL APPLICATION OF LS AND RAS FOR COMPARISON OF EFFICIENCY OF ESTIMATION

Basic Data for 1964 and 1968

Tables 1 and 2 show Irish 17-sector inter-industry transactions, at current prices, for 1964 and 1968. These are used for the two tests described below. *The 1964 values have been scaled up in each column to give 1964 levels corresponding to 1968 total inputs.* These might be considered to be the best prior estimates for 1968, before distribution of changes in row and column totals arising from the different 1968 structure. For each non-zero 1964 entry there is a corresponding 1968 entry.

The full set of non-zero transactions number 174 in both years. Each table has 21 values marked with an asterisk and omission of asterisked values gives a table of transactions each of which is less than £10 million in value for 1964. For both the full set of 174 transactions and the reduced set of 153 (by omission of the 21 asterisked values) row and column totals are shown for 1964 and 1968.

For the purposes of the second test it will be supposed that the 21 largest 1964 transactions might be independently estimated for 1968 and entered after RAS or LS estimation of the 1968 values of the other 153 transactions. These 21 largest values in fact absorb roughly 74 per cent of the aggregate value of transactions in both years.

The data which appear in Tables 1 and 2 are to be regarded as illustrative rather than of high precision, for the following reasons. Similar imports, which include competitive imports as a subset, have been excluded from both tables and the combined distribution of domestic output and similar imports is of much higher reliability than the estimated distribution of either, because separate data for similar imports are not available in detail for individual sector inputs. Some of the small transaction values may be unreliable. The 1968 tables, from which Table 2 is derived, result from a commodity analysis in less detail and with less attention to balancing checks on detailed commodity flows than those of 1964.

Tables 1 and 2 have been chosen, however, because they are considered to impose a severe test on the robustness of the LS procedure. Robustness here means not changing positive 1964 entries into negative 1968 estimated and the RAS method itself cannot produce negative estimates, by its nature. The 1964 transaction values of Table 1 range from 0·003units to 162·314 units for all 174 entries and from 0·003 to almost 10·000 units for the 153 smaller entries. That the 1968 Table 2 is highly irregular by comparison with that of 1964 will be indicated by the sizeable errors of estimation for results of either method. There has in fact been a considerable change in structure between 1964 and 1968, at the level of 33 productive sectors, including changes in the estimated pattern of the distribution of similar imports, as described in the author's paper [9]. How the two methods compare in estimating the transactions for 1968 will appear below.

Results for 174 transactions

The RAS estimates are the outcome of 19 double iterations by the author's computer programme, at which stage of computing each correction factor deviated from unity by less than one part in 10,000. There is one correction factor for each row and one for each column, a row correction factor being the quotient of the specified row total and the current row sum (the latter resulting from the most recent column adjustments) and a column correction factor having a corresponding definition. The computer programme continues the iterations until each correction factor passes the test of deviating by less than $|\epsilon|$ from unity, where $|\epsilon|$ here is set at 1/10,000.

The LS estimates result from application of the $\lambda_{i}.$ and $\lambda._{j}$ multipliers, appearing in Table 3, to the 1964 ξ_{ij} data given in Table 1. In order to find the numerical values of $\lambda_{i}.$ and $\lambda._{j}$, it was decided to set $\lambda._{17}$ (for column 17) at zero and to solve the 33 linear simultaneous equations needed for 17 values of $\lambda_{i}.$ and 16 values of $\lambda._{j}$, the latter 16 unknowns beng variables (18) to (33) for the equations The numerical application of equation (6) above to row (1) gave

$$(22)\quad 268{\cdot}816\lambda_{1}.+100{\cdot}180\lambda._{1}+162{\cdot}314\lambda._{3}+2{\cdot}475\lambda._{4}+1{\cdot}501\lambda._{5}+0{\cdot}711\lambda._{7} +1{\cdot}635\lambda._{8}=42{\cdot}456$$

Likewise, the application of equation (7) to column (14) yielded

$$(23)\quad 3{\cdot}355\lambda_{2}.+2{\cdot}154\lambda_{12}.+2{\cdot}069\lambda_{14}.+0{\cdot}463\lambda_{15}.+3{\cdot}986\lambda_{17}.+12{\cdot}027\lambda._{14}=0{\cdot}528$$

A computer programme written by the author took the 1964 transactions and the 1968 row and column specified totals and carried out all further processing required to calculate the λ-values and then apply them, finally printing out the 1968 estimated inter-industry transactions.

TABLE 1: *Irish 1964 17–Sector Inter-Industry Transactions, all Imports*

Source of Inputs ↓ / Distribution of Outputs →		Agriculture, forestry, fishing	Mining and peat processing	Food manufacturing	Drink tobacco	Textiles except hosiery	Clothing, hosiery, shoes, leather	Wood products, furniture	Paper, printing	Chemicals
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Agriculture etc.	(1)	*100·180		*162·314	2·475	1·501		0·711		
Mining etc.	(2)	1·190	0·230	0·808	0·280	0·022	0·048	0·017	0·133	0·069
Food manufacturing	(3)	*29·615		*52·188	0·446		2·553	0·109		0·498
Drink, tobacco	(4)			0·141	4·532					0·014
Textiles except hosiery	(5)	0·602				9·108	*11·527	0·521	0·053	
Clothing etc.	(6)						6·117		0·013	
Wood, furniture	(7)		0·013					2·337	0·132	
Paper, printing	(8)		0·039		0·085		0·148		6·276	
Chemicals	(9)	*11·206		1·454		0·019		0·205	0·430	8·638
Clay products etc.	(10)	0·150	0·088					0·049		
Metal etc.	(11)	3·409	0·245				0·326	0·379	0·167	
Other manufacturing	(12)	2·247	0·589	0·941	0·199	0·220	0·505	0·153	0·185	0·233
Construction	(13)									
Electricity etc.	(14)	1·024	0·648	1·987	0·262	0·558	0·457	0·262	0·709	0·344
Services except government	(15)	*18·106	0·576	6·111	0·220	0·164	0·835	0·048	0·207	0·136
Government services	(16)									
Artificial sectors n.e.s.	(17)	5·962	4·533	33·579	9·265	5·453	8·167	0·758	5·277	7·742
All Entries:										
Number of Entries		11	9	9	9	8	10	12	11	8
Value £ million		173·691	6·961	259·523	17·764	17·045	30·683	5·549	13·582	17·674
Excluding Those Marked *										
Number of Entries		7	9	6	9	8	9	12	11	8
Value £ million		14·584	6·961	11·442	17·764	17·045	19·156	5·549	13·582	17·674
Column code		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)

excluded, scaled up to fit 1968 Total Inputs. £ million. Current Prices.

								All Entries		Excluding those marked*		
Clay products, cement, glass, pottery	Metal, engineering vehicles	Other manufacturing	Construction, new and repair	Electricity, gas, water	Services except government	Government services	Artificial sectors not elsewhere specified	Number of entries	Value £ million	Number of entries	Value £ million	Row code
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)					
						1·635		6	268·816	4	6·322	(1)
1·327	0·179	0·048	9·186	3·355	0·042	1·255		16	18·189	16	18·189	(2)
					0·946	2·275		8	88·630	6	6·827	(3)
						0·018		4	4·705	4	4·705	(4)
0·008	0·003	0·835	0·257		1·008	0·371	1·229	12	25·522	11	13·995	(5)
					0·079	0·222	0·021	5	6·452	5	6·452	(6)
0·040	0·484		3·205		0·040	0·318	0·202	9	6·771	9	6·771	(7)
0·148			0·352		4·935	1·094	*12·659	9	25·736	8	13·077	(8)
0·195	0·791	0·028	1·133		2·989	0·678	0·558	13	28·324	12	17·118	(9)
1·678	0·069		*15·886		0·654	0·056	1·949	9	20·579	8	4·693	(10)
0·073	*11·075		*10·755		*13·960	0·520	*11·678	11	52·587	7	5·119	(11)
0·630	6·930	2·883	2·714	2·154	3·675	1·293	3·690	17	29·241	17	29·241	(12)
			*15·676		8·295	2·855	0·112	4	26·938	3	11·262	(13)
1·001	1·295	0·725	0·482	2·069	3·994	1·408		16	17·225	16	17·225	(14)
0·949	0·245	0·108	*17·195	0·463	*42·363	*14·634	*76·153	17	178·513	12	10·062	(15)
							6·443	1	6·443	1	6·443	(16)
3·654	*10·909	4·844	9·515	3·986	*35·767	1·647	6·385	17	157·443	14	77·188	(17)
11	10	7	12	5	14	16	12	174				No.
9·703	31·980	9·471	86·356	12·027	118·747	30·279	121·079		962·114			Value
11	8	7	8	5	11	15	9			153		No.
9,703	9·996	9·471	26·844	12·027	26·657	15·645	20·589				254·689	Value
(10)	(11)	(12)	(13)	(14)	15)	(16)	(17)	No.	Value	No.	Value	

TABLE 2: *Irish 1968 17-Sector Inter-Industry Transactions,*

Distribution of Outputs → / Source of Inputs ↓		Agriculture, forestry, fishing	Mining and peat processing	Food manufacturing	Drink tobacco	Textiles except hosiery	Clothing, hosiery, shoes, leather	Wood products, furniture	Paper, printing	Chemicals
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Agriculture etc.	(1)	*123·368		*183·093	3·017	0·264		0·029		
Mining etc.	(2)	1·273	0·122	0·300	0·098	0·019	0·020	0·001	0·046	0·095
Food manufacturing	(3)	*33·863		*34·216	0·637		2·494	0·006		0·426
Drink, tobacco	(4)			0·208	4·024					0·000
Textiles except hosiery	(5)	0·500				11·309	*5·489	0·329	0·014	
Clothing etc.	(6)						1·364		0·013	
Wood, furniture	(7)		0·008					3·135	0·031	
Paper, printing	(8)		0·004		0·374		0·377		9·353	
Chemicals	(9)	*14·422		2·404		0·024		0·461	0·389	2·917
Clay products etc.	(10)	0·020	0·150					0·031		
Metal etc.	(11)	4·746	0·037				0·628	0·236	0·336	
Other manufacturing	(12)	1·770	0·256	1·019	0·218	0·234	0·751	0·129	0·426	0·033
Construction	(13)									
Electricity etc.	(14)	1·152	0·753	2·039	0·326	0·577	0·503	0·241	0·702	0·630
Services except government	(15)	*24·006	0·902	4·052	0·161	0·360	0·346	0·042	0·163	0·109
Government services	(16)									
Artificial sectors n.e.s.	(17)	0·080	5·418	*34·347	11·075	7·189	9·543	2·395	6·982	12·677
All entries:										
Number of Entries		11	9	9	9	8	10	12	11	8
Value £ million		205·200	7·650	261·678	19·930	19·976	21·515	7·035	18·455	16·887
Excluding those Marked*										
Number of Entries		7	9	6	9	8	9	12	11	8
Value £ million		9·541	7·650	10·022	19·930	19·976	16·026	7·035	18·455	16·887
Column code		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)

all Imports excluded. £ million. Current Prices.

Clay products, cement, glass, pottery	Metal, engineering vehicles	Other manufacturing	Construction, new and repair	Electricity, gas, water	Services except government	Government services	Artificial sectors not elsewhere specified	All Entries: Number of entries	All Entries: Value £ million	Excluding those marked*: Number of entries	Excluding those marked*: Value £ million	Row code
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)					
						1·501		6	311·272	4	4·811	(1)
2·231	0·065	0·027	6·190	5·197	1·391	0·917		16	17·992	16	17·992	(2)
					0·526	2·431		8	74·599	6	6·520	(3)
						0·020		4	4·252	4	4·252	(4)
0·043	0·069	0·360	0·571		0·743	0·152	0·345	12	19·924	11	14·435	(5)
					0·209	0·338	0·000	5	1·924	5	1·924	(6)
0·025	0·199		2·281		0·000	0·206	2·272	9	8·157	9	8·157	(7)
0·291			0·000		11·187	1·621	*7·183	9	30·390	8	23·207	(8)
0·074	1·185	0·037	1·653		2·087	0·721	0·075	13	26·449	12	12·027	(9)
3·011	0·488		*11·372		0·331	0·020	3·622	9	19·045	8	7·673	(10)
0·219	*0·681		*9·276		*11·581	0·935	*25·783	11	54·458	7	7·137	(11)
0·833	4·601	0·119	2·423	2·398	3·930	0·800	9·229	17	29·169	17	29·169	(12)
			*29·326		9·175	3·805	0·000	4	42·306	3	12·980	(13)
1·012	1·276	0·566	0·695	0·894	4·089	1·485		16	16·940	16	16·940	(14)
0·368	1·007	0·056	*11·878	1·204	*64·175	*21·697	*77·610	17	208·136	12	8·770	(15)
							5·000	1	5·000	1	5·000	(16)
5·042	*16·019	9·147	12·667	2·862	*31·392	3·289	0·000	17	170·124	14	88·366	(17)
11	10	7	12	5	14	16	12	174				No.
13·149	25·590	10·312	88·332	12·555	140·816	39·938	131·119		1,040·137			Value
11	8	7	8	5	11	15	9			153		No.
13·149	8·890	10·312	26·480	12·555	33·668	18·241	20·543				269·360	Value
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	No.	Value	No.	Value	

TABLE 3: *For all 174 transactions 1968, the Least Squares row and column multipliers $\lambda_{i.}$ and $\lambda_{.j}$*

Row or Column Number Code	*Row Multiplier $\lambda_{i.}$*	*Column Multiplier $\lambda_{.j}$*
(1)	0·131618	0·129966
(2)	−0·002800	0·063081
(3)	−0·181578	−0·044873
(4)	−0·233831	0·143332
(5)	−0·334016	0·326053
(6)	−0·673330	−0·039573
(7)	0·194975	0·199977
(8)	0·079496	0·310220
(9)	−0·127310	0·003686
(10)	−0·006923	0·336191
(11)	0·077037	−0·241493
(12)	0·011384	0·097462
(13)	0·596692	−0·128448
(14)	−0·092534	0·038557
(15)	0·126670	0·083535
(16)	−0·223964	0·215288
(17)	0·045649	Zero, by choice

In obtaining the λ-results given in Table 3 the computer programme first scaled each linear equation (by dividing through by the diagonal coefficient) so as to give unity in the diagonal position and smaller coefficients elsewhere. This matrix of coefficients was then inverted and after inversion rescaled via division of each column by the original unscaled diagonal entry. After rescaling, the inverse premultiplied into the right-hand-side vector of constants (in original values) yielded the Table 3 results.

The purpose of the scaling before matrix inversion is to improve the precision of the inverse and to avoid astronomical magnitudes of products of elements, which may cause "overflow" computer trouble. The Table 3 results, when applied in the form of $(1+\lambda_{i.}+\lambda_{.j})$ multipliers to Table 1 base data gave estimates adding to the specified 1968 row and column totals with high precision in all 17 rows and columns.

The first satisfactory outcome of the LS test is that the estimates contain no negative transactions—all are positive entries. In the context of the multiplier $(1+\lambda_{i.}+\lambda_{.j})$, it is clear from Table 3 that these multipliers are in all cases positive, the minimum being for row (6) combined with column (11) and having a value of 0·0852, which in fact relates to a blank element. Thus the robustness of the LS proportionate distribution of change has been demonstrated.

It is a further satisfactory outcome that the LS and RAS estimates are close together, but still distinct. For example, the largest transaction estimate, row (1) column (3), has 176·394 via LS and 175·530 via RAS. The RAS multipliers r_i and s_j

have not been calculated, as there seemed little point in doing so. We may, however, compare r_i for rows (8) and (15) with $(1+\lambda_{i\cdot})$ for these rows, by using column (17) of Table 1 and of the RAS estimate and forming the quotients of the two largest entries, namely row (8) 1968 RAS estimate divided by row (8) 1964 actual value and likewise for row (15). Since the column multiplier s_{17} is being taken as unity and 1964 data are in the scale of 1968 total inputs, the two quotients for column (17) matching elements give RAS multipliers $r_8 = 1{\cdot}0789$ and $r_{15}=1{\cdot}1259$ in conjunction with $s_{17}=1{\cdot}0$. These may be compared with the LS multipliers $(1+\lambda_{8\cdot})=1{\cdot}0795$ and $(1+\lambda_{15\cdot})=1{\cdot}1267$ in conjunction with $\lambda_{\cdot 17}=0{\cdot}0$. Thus the numerical results veify the distinct algebraic forms given above in formulae (9) and (10).

A comparison of errors of estimation of 1968 transactions, for the two methods, is given in Table 4. The error is measured by the numerical value of the 1968 actual transaction minus its estimate. For all 174 transactions the LS method produces an aggregate error of 219·424 units which is some 97·5 per cent of the aggregate RAS error, 225·130 units. Thus the LS shows a mild improvement of estimation over that of the RAS, and gives numerically smaller aggregate errors for 9 rows and for 13 columns, i.e. for about 2/3 of the 34 rows and columns. The LS method would appear to cater better for the column-wise changes in the data than does the RAS, in view of being better for 13 of the 17 columns. Although one must be cautious in generalising results of one test, it seems safe to conclude that in this particular instance the LS method provides more satisfactory estimates.

Results for 153 transactions

The RAS estimates are the outcome of 14 double iterations, to reach row and column correction factors within the specified range of precision, 1/10,000. The LS estimates result from the application of the $\lambda_{i\cdot}$ and $\lambda_{\cdot j}$ multipliers, shown in Table 5, to the 1964 ξ_{ij} data (omitting asterisked entries) given in Table 1.

The LS estimates produce no negative values, with results containing only positive values. The only possible negative entry is for row (1) column (1), for which $(1+\lambda_{i\cdot}+\lambda_{\cdot j})$ has the value $-0{\cdot}0167$, but this element is blank. Thus the LS outcome is satisfactory, in producing acceptable transactions for the numerical problem being considered.

As is to be expected, the LS and RAS estimates are close in value, but distinct. For example, the row (2) column (1) estimate is 0·549 via LS and 0·627 via RAS. For row (17) of column (17), the RAS multipler r_{17} is 1·1114 in conjunction with s_{17} taken as unity, whereas the LS row (17) multiplier $(1+\lambda_{17\cdot})$ is 1·1121, for column (17) multiplier $\lambda_{\cdot 17}$ taken as zero.

A comparison of the errors of estimation, for the 153 smaller transactions of 1968, is given in Table 6. The aggregate error for all transactions is only very slightly less for the LS estimates than for those of RAS, the comparative aggregates being 88·564 and 88·588, respectively. The LS shows smaller aggregate errors for 5 rows and 10 columns, which is less than half the 34 rows and columns, but here again the LS method seems to be the better for columns since it performs better

TABLE 4: *For all 174 transactions 1968, the numerical values of the errors of the estimates, aggregated by rows and columns, and the average numerical value of the error per transaction. £ million.*

Row or Column Number Code	By Rows: Number of transaction entries in row	By Rows: Numerical Values of LS Errors		By Rows: Numerical Values of RAS Errors		By Columns: Number of transaction entries in column	By Columns: Numerical Values of LS Errors		By Columns: Numerical Values of RAS Errors	
		Total per row	*Average per entry in row*	*Total per row*	*Average per entry in row*		*Total per column*	*Average per entry in column*	*Total per column*	*Average per entry in column*
(1)	6	13·398	2·233	15·126	2·521	11	21·914	1·992	23·276	2·116
(2)	16	7·110	0·444	7·202	0·450	9	1·580	0·176	1·572	0·175
(3)	8	13·172	1·647	14·788	1·848	9	18·366	2·041	19·928	2·214
(4)	4	0·216	0·054	0·206	0·051	9	1·174	0·130	1·202	0·134
(5)	12	5·660	0·472	6·356	0·530	8	4·792	0·599	5·506	0·688
(6)	5	0·798	0·160	0·888	0·178	10	5·424	0·542	6·088	0·609
(7)	9	4·062	0·451	4·062	0·451	12	3·384	0·282	3·404	0·284
(8)	9	13·716	1·524	13·708	1·523	11	1·844	0·168	1·776	0·161
(9)	13	12·268	0·944	12·272	0·944	8	9·824	1·228	9·914	1·239
(10)	9	5·922	0·658	6·066	0·674	11	2·950	0·268	3·062	0·278
(11)	11	29·220	2·656	29·188	2·653	10	19·280	1·928	19·132	1·913
(12)	17	12·422	0·731	12·396	0·729	7	7·242	1·035	7·226	1·032
(13)	4	12·620	3·155	13·908	3·477	12	23·674	1·973	24·446	2·037
(14)	16	3·752	0·235	3·620	0·226	5	5·044	1·009	5·096	1·019
(15)	17	36·024	2·119	35·936	2·114	14	40·014	2·858	40·242	2·874
(16)	1	Nil	Nil	Nil	Nil	16	8·076	0·505	8·414	0·526
(17)	17	49·064	2·886	49·408	2·906	12	44·842	3·737	44·846	3·737
Total	174	219·424	1·261	225·130	1·294	174	219·424	1·261	225·130	1·294

TABLE 5: *For the 153 smaller transactions 1968, the Least Squares row and column multipliers* $\lambda_{i.}$ *and* $\lambda_{.j}$

Row or Column Number Code	*Row Multiplier* $\lambda_{i.}$	*Column Multiplier* $\lambda_{.j}$
(1)	−0·460709	−0·556014
(2)	0·017241	0·012850
(3)	−0·185636	0·048476
(4)	−0·300223	0·209076
(5)	−0·144772	0·259049
(6)	−0·757233	0·048329
(7)	0·149344	0·232311
(8)	0·664452	0·019105
(9)	−0·401330	0·110996
(10)	0·533827	0·237779
(11)	0·719075	−0·080670
(12)	0·009493	0·051292
(13)	−0·057282	−0·066810
(14)	−0·086166	0·022638
(15)	−0·196440	0·215382
(16)	−0·223964	0·201923
(17)	0·112059	Zero, by choice

than RAS for 10 columns out of 17. The conclusion offered is that these particular 153 transactions, to be estimated for 1968, are equally well (or badly) described by LS and by RAS and either method has negligible advantages over the other.

Conclusions and Summary for Part 2

It is clear from the results, which have been calculated to a high order of precision, that the LS and RAS estimates are distinct, in agreement with the comparison made above between formulae (9) and (10).

It has been demonstrated that the LS method is usable and can produce acceptable results, which form an interesting contrast with those of the RAS. The robustness of the LS method, in not producing negative estimates, has been shown.

For the data the two kinds of estimate agree closely, thus the magnitudes of errors are close. The small apparent increase in efficiency for the LS method of estimating all 174 transactions is not to be taken as conclusive. One can argue that the LS method should be more efficient, in that it directly minimises the weighted squares of deviations over the grid of transactions, whereas the RAS has the corresponding unit-weighted minimum distribution of the logarithms of the multipliers.

One can readily visualise data which are "LS-behaved", so that the LS method gives significantly smaller errors of estimation than does the RAS method and for which, therefore, the LS method would be more appropriate for projections.

TABLE 6: *For the 153 smaller transactions 1968, the numerical values of the errors of the estimates, aggregated by rows and columns, and the average numerical value of the error per transaction. £ million*

	By Rows					By Columns				
		Numerical Values of LS Errors		Numerical Values of RAS Errors			Numerical Values of LS Errors		Numerical Values of RAS Errors	
Row or Column Number Code	*Number of transaction entries in row*	*Total per row*	*Average per entry in row*	*Total per row*	*Average per entry in row*	*Number of transaction entries in column*	*Total per column*	*Average per entry in column*	*Total per column*	*Average per entry in column*
(1)	4	2·908	0·727	2·864	0·716	7	6·724	0·961	6·882	0·983
(2)	16	8·708	0·544	8·620	0·539	9	1·832	0·204	1·924	0·214
(3)	6	1·182	0·197	1·176	0·196	6	3·430	0·572	3·434	0·572
(4)	4	0·212	0·053	0·202	0·050	9	3·186	0·354	3·166	0·352
(5)	11	3·898	0·354	3·892	0·345	8	2·708	0·339	2·916	0·364
(6)	5	0·844	0·169	0·908	0·182	9	1·626	0·181	1·742	0·194
(7)	9	4·080	0·453	4·072	0·452	12	3·336	0·278	3·412	0·284
(8)	8	4·514	0·564	4·098	0·512	11	2·934	0·267	2·592	0·236
(9)	12	7·814	0·651	7·566	0·630	8	7·004	0·876	6·994	0·874
(10)	8	2·146	0·268	2·230	0·279	11	1·766	0·161	1·694	0·154
(11)	7	1·910	0·273	2·718	0·388	8	4·514	0·564	4·382	0·548
(12)	17	13·906	0·818	13·634	0·802	7	7·060	1·009	7·080	1·011
(13)	3	1·074	0·358	1·120	0·373	8	8·856	1·107	8·822	1·103
(14)	16	3·924	0·245	3·650	0·228	5	5·408	1·082	5·414	1·083
(15)	12	4·538	0·378	4·500	0·375	11	6·842	0·622	6·446	0·586
(16)	1	Nil	Nil	Nil	Nil	15	4·984	0·332	5·224	0·348
(17)	14	26·906	1·922	27·338	1·953	9	16·354	1·817	16·464	1·829
Total	153	88·564	0·579	88·588	0·579	153	88·564	0·579	88·588	0·579

Likewise, there may be data which are much more precisely described by the RAS than by LS for changes over time and these are "RAS-behaved" and should be projected via RAS. Thus the argument in the previous paragraph that LS "ought to be" more precise has as its rationale the theory that the actual changes are proportional to the base transactions and form an LS distribution over the grid. The actual changes may have quite a different distribution over the grid, but it is only by trying the alternative LS and RAS methods (or, indeed any other methods) that the "best fit" of the new data may be obtained.

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