Relative Efficiency of Ras Versus Least Squares Methods of Updating Input-Output Structures: An Addendum

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THE following note is intended to clarify both the mathematical derivation of the Least-Squares minimum and its interpretation, as given in the author's paper [1], published in the October 1973 issue of this Review. I am grateful to Dr Richard Lecomber of the Department of Economics, University of Bristol, for drawing my attention to the point at issue, and for his paper [2] having some new proposals for updating transactions.

In equation (3) of [1], the problem is to minimise z given by

$$(1) \qquad 2z = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij})^2 / \xi_{ij} - 2\sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} x_{ij} - x_{i.}\right) - 2\sum_{j=1}^{n-1} \lambda_{.j} \left(\sum_{i=1}^{m} x_{ij} - x_{.j}\right)$$

For N non-zero x_{ij} (at least one of which is in each of the m rows and n columns), the partial derivatives equated with zero give

$$(2) \qquad (i) \qquad \partial z / \partial x_{ij} = x_{ij} / \xi_{ij} - \lambda_{i.} - \lambda_{.j} = 0$$

there being N such equations, together with the conditions

(ii) $$\sum_{j=1}^{n} x_{ij} - x_{i.} = 0$$

with m such equations,

(iii) $$\sum_{i=1}^{m} x_{ij} - x_{.j} = 0$$

with $(n-1)$ such equations.

The $x_{i.}$ and $x_{.j}$ of (ii) and (iii) are the row and column aggregates for the changes x_{ij} to be super-imposed on the original non-zero transactions ξ_{ij}. The set (2) (i) can be re-written

(3) $$x_{ij} = \xi_{ij} (\lambda_{i.} + \lambda_{.j})$$

The $(N+m+n-1)$ simultaneous linear equations for the same number of unknowns, namely x_{ij}, $\lambda_{i.}$ and $\lambda_{.j}$ (and with $\lambda_{.n}$ pre-set at zero) enable the system to be solved, first for $\lambda_{i.}$ and $\lambda_{.j}$ and then for x_{ij}, by using (3). The above, formulation and method of solution is contained in Part 1 of [1].

Lecomber raises the query as to whether a negative ξ_{ij} should be replaced by $|\xi_{ij}|$, its positive numerical value, in seeking the minimum. There are three aspects of this possibility to be considered, namely

(A) the mathematical aspect, regarding a true minimum,

(B) whether a negative x^1_{ij} associated with an original negative ξ_{ij} makes input-output sense,

(C) how does $|\xi_{ij}|$ in place of a negative ξ_{ij} affect the treatment and results.

(A) *Mathematical Aspects.*

In textbooks on the calculus, for instance in Chapter XX of Parry Lewis [3], the conditions for a Minimum are two-fold:

(*a*) The first-order partial derivatives are all zero, that is

(4) $$\partial z / \partial x_{ij} = 0 \qquad \text{for all } x_{ij} \text{ ;}$$

(*b*) The principal minor determinants of the second-order partial derivatives forming the Hessian are all positive, as their dimension is increased. In the context of z, the only non-zero second-order partial derivatives are the $\partial^2 z \,/\, \partial x_{ij}^2$, forming the diagonals of the minor determinants in question. Thus (*b*) reduces to

$$(5) \qquad \partial^2 z \,/\, \partial x^2_{ij} = 1 \,/\, \xi_{ij} > 0$$

for a true minimum.

Thus for all ξ_{ij} positive the procedure used in [1] gives a perfect minimum. If however at least one of the ξ_{ij} is negative, then the solution of (2) yields a so-called "Saddle-Point", which is a minimum for variables having $1/\xi_{ij}$ positive and a maximum for those having $1/\xi_{ij}$ negative. Thus the latter condition does not yield a perfect minimum.

It is of relevance at this point to consider the sign of x'_{ij}, the estimated updated transaction given by $(x_{ij}+\xi_{ij})$, associated with a negative ξ_{ij}. It can be taken that $(1+\lambda_{i.}+\lambda_{.j})$ is rarely negative, this event occurring only once in almost 600 combinations of $\lambda_{i.}$ with $\lambda_{.j}$ in the numerical examples given in [1] and coinciding (fortunately) with a zero-level ξ_{ij}. Thus via (3) above the sign of x'_{ij} is usually that of ξ_{ij} and thus a negative ξ_{ij} will very likely yield a negative x'_{ij}.

The fact that almost always x'_{ij} has the same sign as ξ_{ij} means that with very high probability (x'_{ij}/ξ_{ij}) turns out to be positive, regardless of the sign of the non-zero ξ_{ij}. This means that the changes are distributed, via (3), in proportion to the magnitudes of the ξ_{ij}, in the solutions which emerge from equation-set (2), and there is no unusual or significantly different behaviour in allocation, for negative ξ_{ij}, other than the negative sign of the corresponding x'_{ij}.

(B) *The meaning of a negative x'_{ij} for a corresponding negative ξ_{ij}.*

The negative ξ_{ij} transactions occur for artificial transfers of by-products and thus the estimated updated value should also be negative. This makes better input-output sense than a new positive estimate replacing an original negative transaction. Thus there is as much justification for the new estimate $(x_{ij}+\xi_{ij})$ negative, as for the original negative ξ_{ij}.

(C) *How $|\ \xi_{ij}\ |$ versus a negative ξ_{ij} affects the treatment and solution.*

This is best illustrated by a small numerical example, first solved by the method of (2) above and set out as follows:

Rows	*Transactions*		*Original Row Sums*	*New Row Sums*	*Change*
Row (1)	7	10	17	34	17
Row (2)	−3	5	2	4	2
Original Column Sums	4	15	19	—	—
New Column Sums	8	30	—	38	—
Change	4	15	—	—	19

It will be observed that the new row and column sums are twice the corresponding original aggregates and that one of the four original entries is negative.

For $\lambda_{.2}$, column (2), pre-set at zero there are three unknowns, $\lambda_{1.}$, $\lambda_{2.}$ and $\lambda_{.1}$. The three equations, via system (2), are:

$$(6) \quad \begin{cases} \text{Row (1):} & 7(\lambda_{1.} + \lambda_{.1}) + 10\lambda_{1.} = 17 \\ \text{Row (2):} & -3(\lambda_{2.} + \lambda_{.1}) + 5\lambda_{2.} = 2 \\ \text{Column (1):} & 7(\lambda_{1.} + \lambda_{.1}) - 3(\lambda_{2.} + \lambda_{.1}) = 4 \end{cases}$$

The solutions are

(7) $\lambda_{1.} = \lambda_{2.} = 1$; $\lambda_{.1} = 0$ and $\lambda_{.2}$ pre-set at 0.

The matrix of change is given by

$$(8) \quad \left[\begin{array}{c|c} 7(\lambda_{1.} + \lambda_{.1}) & 10(\lambda_{1.} + \lambda_{.2}) \\ -3(\lambda_{2.} + \lambda_{.1}) & 5(\lambda_{2.} + \lambda_{.2}) \end{array}\right] = \begin{bmatrix} 7 & 10 \\ -3 & 5 \end{bmatrix}$$

This result shows that what we might have suspected has happened, i.e., the new matrix, given by the sum of the original and the change, has each element or transaction twice that of the original and the negative entry has received fair and equitable treatment.

Now if we try to introduce $|\xi_{21}|$ which is $+3$, it is clear that the old row (2) and column (1) aggregates must be modified, by an increase of 6 units. In order to give the correct changes from the original aggregates to the new aggregates, for this row and column, it is clear that the new row sum and column sum in question must also be changed. The latter condition means that the new value of $| \xi_{21} |$ is specified by the user before any least squares calculations have been made. In this case the best treatment is to replace it by zero in the original transactions, omit it from the old and new row and column sums and put in its new value after the other estimates have been found.

Conclusions

(*a*) For all non-zero ξ_{ij} transactions positive, the least squares process gives a true minimum.

(*b*) For some ξ_{ij} negative, the least squares process gives a stationary value of the Saddle Point type, which is not a perfect minimum. The solution however appears to give equitable numerical distribution of the incremental change to both positive and negative ξ_{ij}, with the latter usually yielding a negative updated estimate.

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[2] Richard Lecomber "A Critique of Methods of Adjusting, Updating and Projecting Matrices, together with some New Proposals". *Discussion Paper in Economics*, No. 40, Department of Economics, University of Bristol, August 1971.

[3] J. Parry Lewis *An Introduction to Mathematics for Students of Economics*. London: Macmillan and Co. Ltd., 1964.