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# Time-dependent Reliability Analysis Method Based on An Explicit Outcrossing Rate in PHI2

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ABSTRACT: In practical engineering, existing aging structures and infrastructure may suffer from aggressive external load and environmental conditions during their service life, which conditions may cause the deterioration of structural resistance to fall below an acceptable level as specified ones. To analyze and predict the reliability of structure in the whole service period, the development of a time-dependent reliability analysis method is particularly important. PHI2 is one of the most commonly used time-dependent reliability analysis methods, where outcrossing rate is determined by the joint probability with reliability indexes at successive time instants. However, the joint probability is numerical integration essentially, the computational efficiency of which is lower than that of explicit one. To improve efficiency, an explicit outcrossing rate of PHI2 is proposed in this study. Firstly, ourcrossing rate in PHI2 is expressed on the geometric plane with the aid of the definition of outcrossing rate. Secondly, the geometrical relationship of each component is derived from the geometric expression of ourcrossing rate in PHI2. Finally, the explicit outcrossing rate in PHI2 is constructed. The accuracy and efficiency of explicit ourcrossing rate in PHI2 are studied by three numerical examples. The results show that the proposed explicit ourcrossing rate in PHI2 improves the efficiency and keeps accuracy in structural time-dependent reliability analysis.

## 1. INTRODUCTION

During the service life, the material properties, environmental conditions and loads on the structure are random. And the structural parameters are uncertain and time-varying [1]. Therefore, the development of time-dependent reliability analysis (TRA) methods for structures is of great importance.

In recent years, a number of structural TRA methods have been developed, which fall into two main categories: simulation-based methods and outcrossing-based methods. Among the simulation-based methods, the most straightforward and easy-to-understand method is the Monte Carlo simulation (MCS) [2]. MCS method is a relatively accurate method, and a large number of sampling operations are required which is time-consuming. To improve computational efficiency, extreme value-based methods [3, 4] have been developed in recent years.

Another approach is the outcrossing-based method, which was first proposed by Rice [5] in the 1940s. Ditlevsen [6] considered the structural time-dependent reliability problem as the weakest link problem and studied the outcrossing model for Gaussian processes. Madsen and Tvedt [7] proposed a structural TRA method for Gaussian processes or Gaussian vector processes. Li and Melchers [8] developed an analytical solution for the outcrossing rate of non-stationary Gaussian processes. Hagen and Tvedt [9] proposed a TRA method based on sensitivity analysis of parallel systems. In 2004, Andrieu-Renaud proposed the PHI2 method [10], which treats the outcrossing event as a time-invariant reliability problem and estimates the joint probability of two successive time instants. The PHI2 method is one of the more commonly used methods nowadays [11] since its theoretical concept is simple and easy to implement. In PHI2 method, the joint probability is a two-dimensional normal cumulative distribution function that has no explicit expression and still requires numerical integration. Although the difference in computational time between numerical integration and explicit calculation at each instant is not obvious, it may be a non-negligible problem for structural TRA, since the evaluation of the outcrossing rate of PHI2 method depends on the divided length of the time interval (usually greater than 100 times). If the original numerical integration is used, the calculation may not be efficient. Therefore, it is essential to develop an explicit expression for outcrossing rate of PHI2 method.

An analytical method for calculating the joint probability of a series system [12] provides a channel for the explicit expression of outcrossing rate of PHI2 method. and it has been effectively applied in the time-invariant reliability analysis. However, outcrossing rate of PHI2 method is based on the joint probability of a parallel system, which cannot be directly applied by existing methods. Therefore, this paper proposes an accurate explicit expression of outcrossing rate of PHI2 method by geometric construction. First, the original PHI2 method is briefly reviewed. Then, an explicit expression for outcrossing rate of PHI2 method is derived. Then, the structural TRA procedure for the proposed model is summarized. Finally, the accuracy and efficiency of the explicit model of outcrossing rate are analyzed by two examples.

## 2. PHI2 METHOD

A limit state function $G(\mathbf{X}, \mathbf{Y}(t), t)$ is considered in the structural TRA, where $\mathbf{X}=[x_1, x_2, \ldots, x_n]$ is a $n$-dimensional random variable; $\mathbf{Y}(t)=[y_1(t), y_2(t), \ldots, y_m(t)]$ is a $m$-dimensional random process; and $t$ is time instant. The failure of the structure in the time period $[0, T]$ is defined as $\{\exists\ t\in[0, T]: G(\mathbf{X}, \mathbf{Y}(t), t)\le 0\}$. In the outcrossing-based methods, the failure event refers to the set of failure events at the initial instant and subsequent outcrossing events from the safety domain to the failure domain within the time period $[0, T]$, and the corresponding failure probability $P_{f,c}(0, T)$ is [13]

$$P_{f,c}(0,T) = \text{Prob}(\{G(\mathbf{X},\mathbf{Y}(0),0)\le 0\}\bigcup\{N^+(0,T)>0\}) \quad (1)$$

where $N^+(0, T)$ is the number of outcrossings. Since outcrossings rarely occur, they can be followed by a Poisson process [13]. Thus, the upper bound of failure is [13]

$$P_{f,c}(0,T)\le P_{f,i}(0)+\int_0^T v^+(t)dt \quad (2)$$

where $P_{f,i}(0)$ is the failure probability at the initial instant; $v^+(t)$ is outcrossing rate at $t$, which is considered as the system probability of the limit state functions $G(\mathbf{X}, \mathbf{Y}(t), t)$ and $G(\mathbf{X}, \mathbf{Y}(t+\Delta t), t+\Delta t)$ at two successive time instants, and can be expanded as [9]

$$v^+(t)=\lim_{\Delta t\to 0,\Delta t>0}\frac{\text{Prob}\{G(t)>0\bigcap G(t+\Delta t)\le 0\}}{\Delta t} \quad (3)$$

where $G(t) = G(\mathbf{X}, \mathbf{Y}(t), t)$, $G(t+\Delta t) = G(\mathbf{X}, \mathbf{Y}(t+\Delta t), t+\Delta t)$. Based on Eq.(3), $v^+(t)$ can be formulated with Gaussian model as [10]

$$v^+_{\text{PHI2}}(t)=\frac{\Phi_2\left[\beta(t),-\beta(t+\Delta t);\rho_G(t,t+\Delta t)\right]}{\Delta t} \quad (4)$$

where

$$\Phi_2\left[\beta(t),-\beta(t+\Delta t);\rho_G(t,t+\Delta t)\right]= \int_{-\infty}^{\beta(t)}\int_{-\infty}^{-\beta(t+\Delta t)}\phi_2[x(t),x(t+\Delta t),\rho_G(t,t+\Delta t)]dx(t)dx(t+\Delta t) \quad (5)$$

where $\Phi_2[\cdot,\cdot;\cdot]$ is a two-dimensional normal cumulative distribution function; $\beta(t)$ and $\beta(t+\Delta t)$ are reliability indexes of $G(\mathbf{X}, \mathbf{Y}(t), t)$ and $G(\mathbf{X}, \mathbf{Y}(t+\Delta t), t+\Delta t)$ at $t$ and $t+\Delta t$; $\rho_G(t, t+\Delta t)$ is correlation coefficient of $G(\mathbf{X}, \mathbf{Y}(t), t)$ and $G(\mathbf{X}, \mathbf{Y}(t+\Delta t), t+\Delta t)$ in the Gaussian space, which can be expressed as [10]

$$\rho_G(t,t+\Delta t) = -\boldsymbol{\alpha}(t)^T \cdot \boldsymbol{\alpha}(t+\Delta t) \quad (6)$$

where $\boldsymbol{\alpha}(t) = \mathbf{u}^*(t)/\beta(t)$ and $\boldsymbol{\alpha}(t+\Delta t) = \mathbf{u}^*(t+\Delta t)/\beta(t+\Delta t)$; $\mathbf{u}^*(t)$ and $\mathbf{u}^*(t+\Delta t)$ are design points at limit state surface. Eqs. (4)-(6) show that the outcrossing rate model can be constructed using a two-dimensional normal cumulative distribution function, which is implicit and requires numerical integration. Therefore, it is necessary to express the outcrossing rate model explicitly using an explicit two-dimensional normal cumulative distribution function.

## 3. PROPOSED EXPLICIT MODEL OF OUTCROSSING RATE

Theoretically, $v^+(t)$ by Eq.(4) can be denoted as

$$v^+(t) = \frac{\text{Prob}[G(t+\Delta t) \le 0] - \text{Prob}[G(t) \le 0 \cap G(t+\Delta t) \le 0]}{\Delta t} = \frac{\Phi[-\beta(t+\Delta t)] - \text{Prob}[G(t) \le 0 \cap G(t+\Delta t) \le 0]}{\Delta t} \quad (7)$$

where $\Phi(\cdot)$ is the normal cumulative distribution function. Eq.(7) can be formulated with Gaussian model as

$$v^+(t) = \frac{\Phi[-\beta(t+\Delta t)] - \Phi_2\left[-\beta(t), -\beta(t+\Delta t); \rho_G(t,t+\Delta t)\right]}{\Delta t} \quad (8)$$

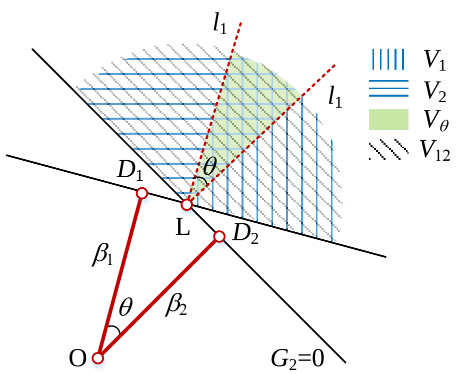


*Figure 1: Geometrical relationship of $V_1$、$V_2$ and $V_{12}$ .*

To derive conveniently, $G(\mathbf{X}, \mathbf{Y}(t), t)$ and $G(\mathbf{X}, \mathbf{Y}(t+\Delta t), t+\Delta t)$ are seen as $G_1$ and $G_2$. $\beta(t)$ and $\beta(t+\Delta t)$ are seen as $\beta_1$ and $\beta_2$. $\rho_G(t, t+\Delta t)$ are seen as $\rho_G$. To derive an explicit expression for outcrossing rate, this section first discusses $\Phi_2[-\beta(t),-\beta(t+\Delta t); \rho_G(t, t+\Delta t)]$. From Eqs.(3)-(4), $\Phi_2[-\beta(t),-\beta(t+\Delta t); \rho_G(t, t+\Delta t)]$ is equivalent to the probability corresponding to the region $\{G_1 \le 0 \cap G_2 \le 0\}$, so an explicit expression for outcrossing rate can be derived by the geometric relationship between $G_1$ and $G_2$. After $\beta_1$ and $\beta_2$ are obtained, $\mathbf{u}^*(t)$ and $\mathbf{u}^*(t+\Delta t)$ can be determined. The lines are conducted by crossing $\mathbf{u}^*(t)$ and $\mathbf{u}^*(t+\Delta t)$ and perpendicular to $\beta_1$ and $\beta_2$. It can be seen as limit state surface $G_1$=0 and $G_2$=0 in Gaussian space. The geometric representation of $G_1$=0 and $G_2$=0 is shown in Figure 1, where O is the origin point of the Gaussian space; $D_i$ ($i$=1, 2) is the design point of $G_i$; $L$ is the intersection point of $G_1$=0 and $G_2$=0; $\beta_1$ and $\beta_2$ are the shortest distance of $G_1$=0 and $G_2$=0; $l_1$ is the line perpendicular to $G_1$=0 through the point $L$; $l_2$ is the line perpendicular to $G_2$=0 through the point $L$; $V_1$ and $V_2$ are the regions enclosed by $G_1$=0 and $G_2$=0 with $l_1$ and $l_2$, respectively; $V_\theta$ is the region enclosed by the angle $\theta$; $V_{12}$ is the region $\{G_1 \le 0 \cap G_2 \le 0\}$. From Figure 1, it can be found that

$$V_{12} = V_1 + V_2 - V_\theta \quad (9)$$

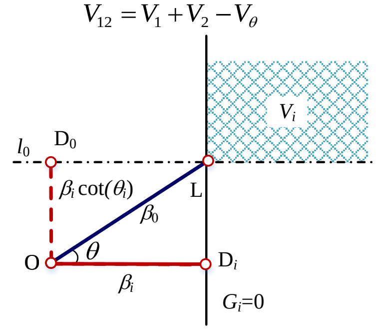


*Figure 2: Geometrical expression of $V_i$ (i=1, 2).*

The probability $P_{12}$ corresponding to the region $V_{12}$ is

$$P_{12} = P_1 + P_2 - P_\theta \quad (10)$$

where $P_1$、$P_2$ and $P_\theta$ are the probability of $V_1$、$V_2$ and $V_\theta$. $P_1$ and $P_1$ are calculated as shown in Figure 2, in which $l_0$ is the line perpendicular to $G_i = 0$ ($i$=1, 2) through the point L; $D_0$ is the point perpendicular to the line $l_0$ and through the point O; and $\beta_0$ is the length of the line segment OL.

According to the sine theorem, $\beta_0$ can be formulated as

$$\beta_0 = \frac{\beta_i}{\sin\theta_i} \quad (11)$$

The length of $OD_0$ is $\beta_i \cot(\theta_i)$, and then $P_i$ is

$$P_i = \Phi_2\left[-\beta_i, -\beta_i \cot(\theta_i), \cos\left(\frac{\pi}{2}\right)\right] = \Phi(-\beta_i)\Phi[-\beta_i \cot(\theta_i)] \quad (12)$$

Substituting Eq.(12) into Eq.(10)

$$P_i = \Phi(-\beta_i)\Phi(-a_i\beta_j) \quad (13)$$

$$a_i = \frac{1-\rho_G\beta_i/\beta_j}{\sqrt{1-\rho_G^2}}, \quad \cos\theta = \rho_G \quad (14)$$

where $j=3-i$ ($i$=1, 2). $P_\theta$ can be approximated as being proportional to angle $\theta$, while $P_1+P_1$ is approximated as being proportional to angle $\pi$. Then it can be obtained as

$$P_\theta = \frac{P_1+P_2}{\pi}\cdot\theta \quad (15)$$

Substituting Eq.(15) into Eq.(8), explicit outcrossing rate $v_{\mathrm{E}}^{+}(t)$ as

$$v_{\mathrm{E}}^{+}(t) = \left[(1-\alpha_2)\Phi(-\beta_2) - \alpha_1\Phi(-\beta_1)\right]/\Delta t \quad (16)$$

where $\alpha_i = (1-\theta/\pi)P_i$($i$=1, 2).

From Eq.(17), the proposed $v_{\mathrm{E}}^{+}(t)$ is defined by $\beta_1$, $\beta_2$ and $\rho_G$, which are the same parameters as the original model in Eq.(4). It indicates that the proposed explicit model does not require the additional calculation of other parameters. The proposed explicit model is an explicit expression that allows the outcrossing rate to be evaluated using an explicit method, whereas the original model can only be evaluated using numerical integration, which is more computationally efficient than the original model.

## 4. EXAMPLES

### 4.1 CANTILEVER TUBE BEAM

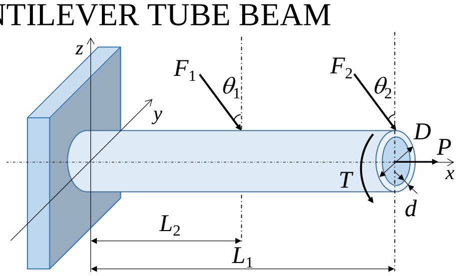


*Figure 3: Cantilever tube beam.*

A cantilever tube beam structure [14] is shown in Figure 3. This cantilever beam is subjected to external forces $F_1$, $F_2$, $P$ and $T(t)$, where $F_1$, $F_2$, $P$ are static loads and $T(t)$ is a time-dependent load. The material strength of this beam is linearly degraded due to corrosion. The yield strength of the material $R(t)$ is $R(t) = R_0(1\text{-}0.02t)$ where $R_0$ is the initial yield strength. The structure failure when the yield strength $R(t)$ exceeds its maximum stress, and the corresponding limit state function $G(\mathbf{X}, \mathbf{Y}(t), t)$ is

$$G(\mathbf{X}, \mathbf{Y}(t), t) = R(t) - \sqrt{\sigma^2 + 3\tau^2(t)} \quad (17)$$

$$\sigma = \frac{P + F_2\sin(\theta_2) + F_1\sin(\theta_1)}{\frac{\pi}{4}[D^2-(D-2d)^2]} + \frac{(F_2L_2\cos(\theta_2) + F_1L_1\cos(\theta_1))D}{\frac{\pi}{32}[D^4-(D-2d)^4]} \quad (18)$$

$$\tau(t) = \frac{T(t)D}{\frac{\pi}{16}[D^4-(D-2d)^4]} \quad (19)$$

where $L_2$=60mm is the length of the half pipe; $L_1$=120mm is the length of the pipe; $\theta_1$ and $\theta_2$ are the angles of $F_1$, $F_2$ respectively. The statistical information of the random variables and processes is listed in Table 1. The autocorrelation function of $T(t)$ is

$$\rho_T(t, t+\Delta t) = \exp\left[-\left(\Delta t/\lambda_T\right)^2\right] \quad (20)$$

where $\lambda_T$ is the autocorrelation length of $T(t)$, with values of 1 day, 1 month and 1 year.

*Table 1 Statistical information of random variables and processes for a tube.*

| *Variable/ process* | *Distribution* | *Mean* | *COV* (%) |
|---|---|---|---|
| *T(t)* | *Gaussian process* | *1900 N·m* | *10* |
| $F_1$ | *Normal* | *1800 N·m* | *10* |
| $F_2$ | *Normal* | *1800 N·m* | *10* |
| *P* | *Gumbel* | *1000 N* | *10* |
| *D* | *Normal* | *0.042 m* | *1.19* |
| *d* | *Normal* | *0.005 m* | *2* |
| $R_0$ | *Normal* | *600 MPa* | *10* |

Note: COV = coefficient of variation.

TRA is performed over the time period [0, 10 years]. For comparison, the original model and the explicit model will be used. The time interval *dt* is taken to be 0.005 years. Figure 4 illustrates the failure probability with different autocorrelation lengths. Table 2 compares the time at different autocorrelation lengths.

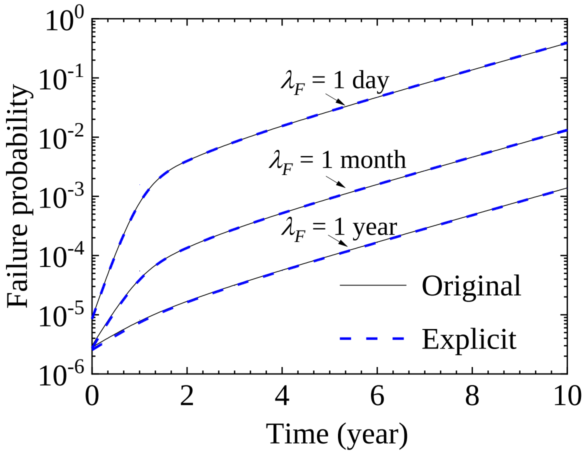


*Figure 4 Failure probability of cantilever tube beam*

*Table 2 Computational time for estimating failure probabilities of a tube.*

| Time period | Computational time (s) | | | | | |
|---|---|---|---|---|---|---|
| | $\lambda_T$ = 1 year | | $\lambda_T$ = 1 month | | $\lambda_T$ = 1 day | |
| | Ori | Ex | Ori | Ex | Ori | Ex |
| [0, 2 years] | 14.7 | 8.9 | 14.1 | 8.6 | 13.8 | 8.6 |
| [0, 5 years] | 37.1 | 22.5 | 35.2 | 21.6 | 35.3 | 22.0 |
| [0, 10 years] | 75.5 | 45.7 | 70.2 | 43.1 | 68.7 | 42.9 |

Note: Ori = Original model; Ex= Explicit model.

From Figure 4 and Table 2, it can be found that:
(1) The results of the explicit model and the original model are basically the same at different autocorrelation lengths, which indicated that the explicit model has a high accuracy for TRA.
(2) Considering different autocorrelation lengths, the computation time of the explicit model is shorter than that of the original one. The difference increases with the length of the time period, which indicates that the explicit model can improve the computation efficiency of TRA.

## 4.2 THREE-SPAN SIX-LAYER FRAME

The frames subjected to loads $P_1(t)$~$P_6(t)$ are shown in Figure 5. The details of the frame members are shown in Table 3. The limit state function $G(\mathbf{X}, \mathbf{Y}(t), t)$ when the frame failure is

$$G(\mathbf{X},\mathbf{Y}(t),t) = \Delta_{\max}(1-0.01t) - \Delta(t) \qquad (21)$$

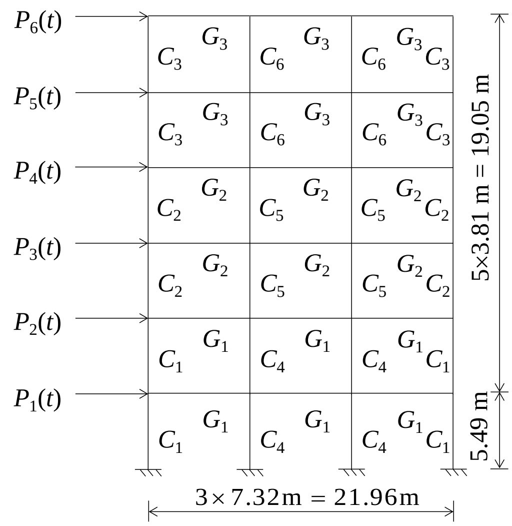


*Figure 5 Three-span six-layer frame*

*Table 3 Detailed information of frame*

| Layer | External columns ($C_1$-$C_3$) | | Internal columns ($C_4$-$C_6$) | | Beam ($G_1$-$G_3$) | |
|---|---|---|---|---|---|---|
| | Area (cm$^2$) | Rotational inertia (cm$^4$) | Area (cm$^2$) | Rotational inertia (cm$^4$) | Area (cm$^2$) | Rotational inertia (cm$^4$) |
| 1, 2 | 301.5 | 79100 | 307.1 | 262600 | 178.7 | 112000 |
| 3, 4 | 250.3 | 63700 | 216.1 | 170000 | 144.5 | 87400 |
| 5, 6 | 187.7 | 46200 | 159.4 | 98600 | 104.5 | 56200 |

where $\Delta_{max}$ = 0.05m is the initial allowable displacement; $\Delta(t)$ is the maximum horizontal displacement of the frame (m) calculated by an implicit function. $P_1(t)$~$P_5(t)$ are external horizontal loads and $E$ is Young's modulus. $P_1(t)$~$P_5(t)$ are random processes with an autocorrelation function $\exp(-\Delta\tau^2)$. The statistical information of the random variables and processes is listed in Table 4.

*Table 4 Statistical information of random variables and processes for frame*

| Parameter | Distribution | Mean | COV |
|---|---|---|---|
| $P_1(t)$ (kN) | Gumbel | 27 | 0.4 |
| $P_2(t)$ (kN) | Gumbel | 45 | 0.4 |
| $P_3(t)$ (kN) | Gumbel | 64 | 0.4 |
| $P_4(t)$ (kN) | Gumbel | 83 | 0.4 |
| $P_5(t)$ (kN) | Gumbel | 101 | 0.4 |
| $P_6(t)$ (kN) | Gumbel | 120 | 0.4 |
| $E$ (GPa) | Lognormal | 200 | 0.1 |

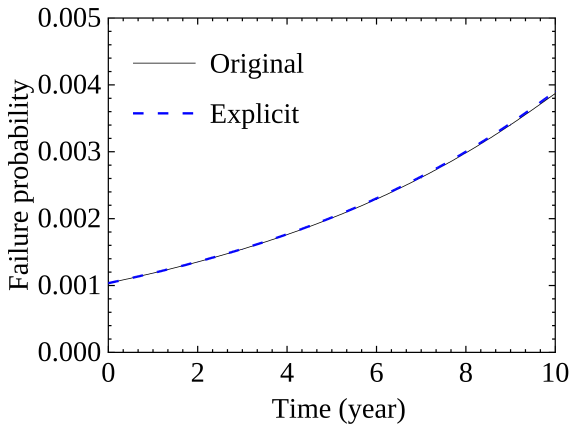


*Figure 6 Failure probability of frame*

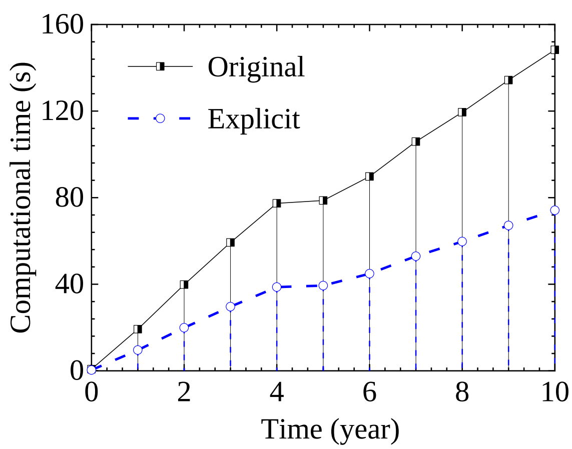


*Figure 7 Computational time for estimating failure probabilities of frame*

TRA for frame within [0, 10 years]. As a comparison, the original model and the explicit model will be used. The time interval *dt* is taken to be 0.05 years. Figure 6 shows the failure probability of frame. Figure 7 compares the computational time of the frame by the different methods.

As can be seen in Figures 6-7, the failure probability of the original model and the explicit model are consistent. The computational time of the explicit model is shorter than the original model around 60%. The results show that the accuracy of the explicit model is enough in the TRA of the implicit limit state function. And its application is feasible and sufficiently accurate for the evaluation of outcrossing rate. The results also demonstrate the efficiency of the explicit model.

## 5. CONCLUSION

In this paper, an explicit expression of outcrossing rate in PHI2 is proposed. A structural TRA method for the explicit model is developed, and the accuracy and efficiency of the method are examined through two examples. The results show that:

(1) The explicit model eliminates numerical integration in PHI2 method. The examples show that the results of the explicit model are consistent with the original model, which demonstrates the accuracy of the proposed explicit model.

(2) The explicit model is easy to understand and can be effectively applied to structural TRA without additional parameter operations. The examples show that the explicit model can reduce the time of TRA and improve efficiency.

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