DAMTP 96-32, UKC/IMS/96-24

To appear in Nonlinearity

Inversion symmetric 3-monopoles and the Atiyah-Hitchin manifold

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March 1996

Abstract

We consider 3-monopoles symmetric under inversion symmetry. We show that the moduli space of these monopoles is an Atiyah-Hitchin submanifold of the 3-monopole moduli space. This allows what is known about 2-monopole dynamics to be translated into results about the dynamics of 3-monopoles. Using a numerical ADHMN construction we compute the monopole energy density at various points on two interesting geodesics. The first is a geodesic over the two-dimensional rounded cone submanifold corresponding to right angle scattering and the second is a closed geodesic for three orbiting monopoles.

*This work was supported in part by the Nuffield Foundation

1 Introduction

The moduli space of 2-monopoles, M_2, is well understood. Its non-trivial structure is contained in the totally geodesic submanifold of strongly centred 2-monopoles, M_2^0. M_2^0 is the space of gauge inequivalent 2-monopoles with fixed centre of mass and fixed overall phase. It is a hyper-Kähler 4-manifold and has an $SO(3)$ action which permutes the almost complex structures I, J and K. These properties allowed Atiyah and Hitchin to calculate the metric on M_2^0 [1] and M_2^0 is now known as the Atiyah-Hitchin manifold.

Similarly, the non-trivial structure of the moduli space of 3-monopoles in contained in the totally geodesic submanifold of strongly centred 3-monopoles M_3^0. M_3^0 is also a hyper-Kähler manifold with an $SO(3)$ action. However, M_0^3 is an 8-dimensional manifold and these properties are not sufficient to calculate its metric, which still remains unknown. Although we do not know how to compute this metric, we present a more modest result in this paper by proving that there is a 4-dimensional submanifold whose metric is the Atiyah-Hitchin one. This submanifold is the submanifold of strongly centred 3-monopoles which are symmetric under the inversion

$$I : (x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3). \tag{1.1}$$

This gives a group action on the moduli space commuting with the $SO(3)$ action. We consider the 4-dimensional fixed point set of I in the moduli space.

2 Monopoles

Here we are concerned with Bogomolny-Prasad-Sommerfield (BPS) monopoles. A BPS monopole is a pair (A, Φ) where A is a 1-form on $\mathbb{R}^3$ with values in $su(2)$ and Φ, the Higgs field, is an $su(2)$ valued function. They satisfy the Bogomolny equation

$$\nabla_A \Phi = \star F_A, \tag{2.1}$$

the finite energy condition

$$\int |F_A|^2 < \infty \tag{2.2}$$

and the boundary condition

$$|\Phi| \stackrel{r\to\infty}{\longrightarrow} 1. \tag{2.3}$$

The $\star$ is the Hodge star on $\mathbb{R}^3$ and $\nabla_A \Phi = d\Phi + [A, \Phi]$ is the covariant derivative of Φ. An element g of the gauge group $SU(2)$ acts by

$$(A, \Phi) \mapsto (gAg^{-1} - dgg^{-1}, g\Phi g^{-1}) \tag{2.4}$$

and monopoles are considered equivalent if they are related by a gauge transformation. The moduli space is the space of gauge inequivalent monopoles.

It can be demonstrated [13] that the Higgs field of a monopole automatically satisfies a stronger boundary condition than (2.3). In fact,

$$|\Phi| = 1 - \frac{n}{2r} + O(r^{-2}) \tag{2.5}$$

where n is an integer topological charge so that the moduli space is divided up into topological sectors. We shall call a monopole configuration with charge n an n-monopole. The 1-monopole is spherically symmetric and its moduli space is $\mathbb{R}^3 \times \mathrm{S}^1$, corresponding to a position and internal phase. The phase can be gauge transformed to unity, but it is convenient to consider it as a degree of freedom for the 1-monopole. The position of a 1-monopole is well defined and can be taken to be the position of the unique zero of the Higgs field.

To precisely define the moduli space of n-monopoles we follow [9] and first define a framed monopole [1]. We say a monopole (A, Φ) is framed if

$$\lim_{x_3 \to \infty} \Phi(0,0,x_3) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.6}$$

Every monopole can be gauge transformed into a framed one. We define a framed gauge transformation as one for which

$$\lim_{x_3 \to \infty} g(0,0,x_3) = 1. \tag{2.7}$$

The n-monopole moduli space M_n is now the quotient of the space of framed monopoles by the space of framed gauge transformations.

It is more difficult to define the moduli space M_n^0 of strongly centred n-monopoles. To do this precisely we must first discuss rational maps. We will discuss rational maps in the next Section. Roughly the space of strongly centred n-monopoles is the submanifold of M_n of monopoles with the centre of mass fixed at the origin and the overall internal phase factor fixed at one. This manifold is $(4n-4)$-dimensional. Over most of the manifold the $4n-4$ coordinates can be understood as corresponding to the relative positions and relative phases of n well-separated monopoles.

There is a natural metric on M_n^0. It is derived from the L^2-norm on the fields (A, Φ). This space is known to be hyper-Kähler for all n. This means that there is a trio of complex structures on M_n^0 satisfying the Hamilton relation $IJ = K$. The full moduli space M_n is also hyper-Kähler and, in fact, there is an isometric splitting

$$\widetilde{M}_n = \mathbb{R}^3 \times \mathrm{S}^1 \times M_n^0 \tag{2.8}$$

where $\widetilde{M}_n$ is an n-fold covering of M_n.

The Bogomolny equation is time independent and its solutions are static solutions to a Yang-Mills Higgs field theory in (3+1)-dimensions. The dynamics of monopoles in this theory is of interest and the low energy dynamics can be approximated by geodesic flow in the moduli space of BPS monopoles [14, 17].

3 Spectral curves, Nahm data and rational maps

It is difficult to study monopoles directly by solving the Bogomolny equation (2.1) for the fields A and Φ. However, there are powerful mathematical correspondences which allow us to study monopoles indirectly. To be precise, there is a correspondence between n-monopoles and the following [10, 15, 6]

A. Spectral curves.

A spectral curve is an algebraic curve $S \subset T\mathbb{P}_1$ which has the form

$$\eta^n + \eta^{n-1}a_1(\zeta) + \ldots + \eta^r a_{n-r}(\zeta) + \ldots + \eta a_{n-1}(\zeta) + a_n(\zeta) = 0 \tag{3.1}$$

where, for $1 \leq r \leq n$, $a_r(\zeta)$ is a polynomial in ζ of degree not greater than $2r$. Here ζ is the inhomogeneous coordinate over $\mathbb{P}_1$ the Riemann sphere, and (ζ, η) are the standard local coordinates on $T\mathbb{P}_1$ defined by $(\zeta, \eta) \rightarrow \eta\frac{d}{d\zeta}$. It must be real, with respect to the standard real structure on $T\mathbb{P}_1$

$$\tau : (\zeta, \eta) \mapsto (-\frac{1}{\bar{\zeta}}, -\frac{\bar{\eta}}{\bar{\zeta}^2}) \tag{3.2}$$

and satisfy some non-singularity conditions [10].

The spectral curve of a 1-monopole positioned at (x_1, x_2, x_3) is called a star. It is

$$\eta - (x_1 + ix_2) + 2x_3\zeta + (x_1 - ix_2)\zeta^2 = 0. \tag{3.3}$$

B. Nahm data.

Nahm data are anti-hermitian $n \times n$ matrices (T_1, T_2, T_3) depending on a real parameter $s \in [0, 2]$ and satisfying the Nahm equation

$$\frac{dT_i}{ds} = \frac{1}{2}\epsilon_{ijk}[T_j, T_k]. \tag{3.4}$$

The $T_i(s)$ are regular for $s \in (0, 2)$ and have simple poles at $s = 0$ and $s = 2$. The matrix residues of (T_1, T_2, T_3) at each pole form the irreducible n-dimensional representation of $su(2)$.

Nahm's equations (3.4) are equivalent to a Lax pair. Hence, there is an associated algebraic curve, which is, in fact, the spectral curve. Explicitly, the spectral curve may be read off from the Nahm data as the equation

$$\det(\eta + (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2) = 0. \tag{3.5}$$

C. Based rational maps.

A based rational map of degree n from $\mathbb{P}_1$ into $\mathbb{P}_1$ sending $z = \infty$ to $z = 0$ can be given by

$$r(z) = \frac{p(z)}{q(z)} \tag{3.6}$$

where $q(z)$ is a monic polynomial in z of degree n and $p(z)$ is a polynomial in z of degree less than n, which has no factor in common with $q(z)$. We let R_n denote the space of such rational maps.

It was conjectured in [9] and proved by Bielawski [3], that for a rational map $p(z)/q(z)$ with well-separated poles $\beta_1, \ldots, \beta_n$ the corresponding monopole is approximately composed of well-separated 1-monopoles with phases $p(\beta_i)/|p(\beta_i)|$ located at the points (x_1, x_2, x_3), where $x_1 + ix_2 = \beta_i$ and $x_3 = \frac{1}{2}\log|p(\beta_i)|$. This approximation applies only when the values of the numerator at the poles is small compared to the distance between the poles.

The rational map provides a convenient parameterization of the moduli space and with it it is possible to define the moduli space of strongly centred monopoles precisely. A monopole with rational map $p(z)/q(z)$ is strongly centred if the roots of $q(z)$ sum to zero and the product of $p(z)$ evaluated at each of the roots of $q(z)$ is equal to unity. Thus, if we label the roots of $q(z)$ as $\beta_1, \ldots, \beta_n$, as above, a monopole is strongly centred if

$$\sum_i \beta_i = 0 \tag{3.7}$$

$$\prod_i p(\beta_i) = 1. \tag{3.8}$$

The spectral curve of a strongly centred monopole has $a_1(\zeta) = 0$. The Nahm data of a strongly centred monopole are traceless.

Further asymptotic information can be derived from the rational map. In [1] pp. 25-26, it is argued that for monopoles strung out in well-separated clusters along, or nearly along, the x_3-axis, the first term in a large z expansion of the rational map $r(z)$ will be $e^{2x+i\chi}/z^L$ where L is the charge of the topmost cluster and x is its elevation above the plane.

It is possible to understand the action of the group $O(3)$ on the spectral curves. The group of improper and proper rotations $O(3)$ has a natural action on the Riemann sphere. This is just the natural action on the sphere in $\mathbb{R}^3$ written in terms of the inhomogeneous coordinate. This lifts to an action on all of $T\mathbb{P}_1$ and so gives an action on spectral curves.

It is also possible to understand the action of the rotation group on the Nahm data. The strongly centred Nahm data are an $\mathbb{R}^3 \otimes sl(n, \mathbb{C})$ valued function of s, which transform under the rotation group $SO(3)$ as

$$\begin{aligned} \underline{3} \otimes sl(\underline{n}) &\cong \underline{3} \otimes (\underline{2n-1} \oplus \underline{2n-3} \oplus \ldots \oplus \underline{3}) \\ &\cong (\underline{2n+1} \oplus \underline{2n-1} \oplus \underline{2n-3}) \oplus \ldots \oplus (\underline{5} \oplus \underline{3} \oplus \underline{1}). \end{aligned} \tag{3.9}$$

where $\underline{r}$ denotes the unique irreducible r-dimensional representation of $su(2)$. Thus, for example, for 3-monopoles the Nahm data transform as the representation

$$(\underline{7} \oplus \underline{5} \oplus \underline{3}) \oplus (\underline{5} \oplus \underline{3} \oplus \underline{1}). \tag{3.10}$$

The construction of the rational map of a monopole requires a decomposition of space as $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$ and only those spatial rotations and reflections which respect this decomposition act on the space of rational maps. This means that a rational map can be rotated about the x_3-axis and can be reflected in the x_1x_2-plane, but other $O(3)$ transformations are unknown.

4 Symmetric rational maps

We can easily calculate the action of I on the space of rational maps, R_n. From [9] the action of the reflection[1] in the x_1x_2-plane,

$$\sigma : (x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3), \tag{4.1}$$

on R_n is

$$\sigma : \frac{p(z)}{q(z)} \mapsto \frac{\sigma(p)(z)}{q(z)} \tag{4.2}$$

where $\sigma(p)(z)$ is the unique degree $n-1$ polynomial such that

$$\sigma(p)(z)p(z) \equiv 1 \qquad \mod q(z). \tag{4.3}$$

If the roots, β_i, of $q(z)$ are distinct a useful alternative way of obtaining $\sigma(p)(z)$ is to notice that it is the unique polynomial of degree less than n such that

$$\sigma(p)(\beta_i)p(\beta_i) = 1 \tag{4.4}$$

for all i.

The action of a rotation, Rot_θ, about the x_3-axis is also known,

$$Rot_\theta : \frac{p(z)}{q(z)} \mapsto \frac{p(\lambda z)}{\lambda^{-n}q(\lambda z)} \qquad \lambda = e^{-i\theta}. \tag{4.5}$$

The factor of λ^{-n} ensures that $q(z)$ remains monic under the rotation and so guarantees that the rotated rational map is strongly centred. We will be interested in Rot_π;

$$Rot_\pi : \frac{p(z)}{q(z)} \mapsto \frac{p(-z)}{-q(-z)}. \tag{4.6}$$

Since $I = \sigma \circ Rot_\pi$ we can act with inversion on an element of R_n. This allows us to calculate the form of an inversion invariant element of R_3. First we calculate the form of the numerator, $q(z)$, of such an element. The action of the reflection does not affect $q(z)$. This means that it must be invariant under Rot_π. Since Rot_π acts on $q(z)$ by changing the

[1] In [9] (and some of our previous papers) this reflection is referred to as inversion, but in this paper we follow the more standard notation and reserve the term inversion for the operation (1.1).

sign of z and the overall sign $q(z) = z(z^2 - \beta^2)$ for some $\beta \in \mathbb{C}$. The candidate inversion symmetric rational map is thus

$$r(z) = \frac{az^2 + bz + c}{(z-\beta)z(z+\beta)} \tag{4.7}$$

for complex a, b and c. From (4.3) and (4.6) this is inversion invariant iff

$$p(-z)p(z) \equiv 1 \qquad \mathrm{mod}\ z^3 - \beta^2 z \tag{4.8}$$

or equivalently, by (4.4), we require

$$p(\beta)p(-\beta) = 1, \tag{4.9}$$
$$p(0)^2 = 1. \tag{4.10}$$

The first strong-centring condition (3.7) is automatically satisfied by $q(z)$. The second strong-centring condition (3.8) is given by

$$p(\beta)p(0)p(-\beta) = 1. \tag{4.11}$$

When combined with (4.9) and (4.10) this gives $p(0) = 1$ and thus $c = 1$. Explicitly substituting $p(z)$ into (4.9) gives the condition

$$b^2 - a^2\beta^2 = 2a. \tag{4.12}$$

This defines a surface in $\mathbb{C}^3$ of two complex dimensions corresponding to inversion symmetric strongly centred 3-monopoles. The argument above assumes that the roots are distinct, that is that $\beta \neq 0$. By using (4.3) it can be demonstrated that (4.12) applies in the $\beta = 0$ case as well. We denote the space of inversion symmetric, strongly centred rational maps by R_I.

$$R_I = \left\{ \frac{az^2 + bz + 1}{z(z^2 - \beta^2)} \ : \ b^2 - a^2\beta^2 = 2a \right\}. \tag{4.13}$$

We can find some particularly symmetric 1-parameter families of maps in R_I. We impose C_2 symmetry around the x_3-axis by requiring invariance under $Rot_\pi : z \mapsto -z$. This means that $b = 0$ and, hence, either $a = 0$ or $a = -2/\beta^2$. The C_2-symmetric rational maps in R_I are then

$$r_1(z) = \frac{1}{z(z^2 - \beta^2)} \tag{4.14}$$

$$r_2(z) = \frac{-\frac{2}{\beta^2} z^2 + 1}{z(z^2 - \beta^2)} \tag{4.15}$$

where $\beta \in \mathbb{C}$ in r_1 and $\beta \in \mathbb{C}^\times$ in r_2. They correspond to monopoles symmetric under inversion and rotation by π about the x_3-axis. If we impose a further symmetry, that of reflection in the x_1x_3-plane, β^2 is forced to be real and thus β either purely real or purely imaginary.

In Section 3, we discussed how to relate the rational maps of well-separated monopoles to the positions of those monopoles. In this spirit, we can examine the rational maps r_1 and r_2 for extreme values of β. In the case of r_1, if β is large and real, it corresponds to three monopoles along the x_1-axis positioned at the origin and $\pm\beta$. If β is zero, the rational map corresponds to the 3-monopole axisymmetric about the x_3-axis. If it is large and imaginary, it corresponds to monopoles along the x_2-axis positioned at the origin and $\pm i\beta$.

In the case of r_2 there are two distinct geodesics, one with real β and the other with imaginary β. If β is large and real, it corresponds to three monopoles along the x_1-axis positioned at the origin and $\pm\beta$. For β small and real it corresponds to monopoles along the x_3-axis positioned at the origin and $\pm \log\sqrt{\frac{2}{\beta^2}}$. The imaginary β geodesic is similar except the x_2-axis replaces the x_1-axis.

Of course it is possible to interpret these geodesics as corresponding to low energy scattering events. They are events in which two monopoles approach a third monopole positioned half way between them, instantaneously form a torus and then separate out and move along a line at right angles to the one along which they previously moved.

5 Symmetric spectral curves

On $T\mathbb{P}_1$ reflection is given by

$$\sigma : (\zeta, \eta) \mapsto (\frac{1}{\bar{\zeta}}, -\frac{\bar{\eta}}{\bar{\zeta}^2}) \tag{5.1}$$

and the rotation Rot_π by

$$Rot_\pi : (\zeta, \eta) \mapsto (-\zeta, -\eta). \tag{5.2}$$

Thus, since $I = \sigma \circ Rot_\pi$,

$$I : (\zeta, \eta) \mapsto (-\frac{1}{\bar{\zeta}}, \frac{\bar{\eta}}{\bar{\zeta}^2}). \tag{5.3}$$

Reality requires a spectral curve to be invariant under τ, (3.2). Since $I \circ \tau : (\zeta, \eta) \mapsto (\zeta, -\eta)$ a charge n spectral curve is inversion symmetric if all its terms are of even degree in η if n is even and of odd degree if n is odd.

The spectral curve of an inversion symmetric 3-monopole must be of the general form

$$\eta^3 + (c_1 + c_2\zeta + r\zeta^2 - \bar{c}_2\zeta^3 + \bar{c}_1\zeta^4)\eta = 0 \tag{5.4}$$

where $c_i \in \mathbb{C}$ and $r \in \mathbb{R}$. The values of the c_i and r are constrained by the non-singularity conditions satisfied by the spectral curves of monopoles. This spectral curve is almost identical in form to the one presented by Hurtubise in [12]. It differs in the overall factor of η. We follow Hurtubise in rotating the general spectral curve so that it is in the form

$$\eta^3 + (a_1 + a_2\zeta^2 + a_1\zeta^4)\eta = 0 \tag{5.5}$$

where $a_i \in \mathbb{R}$. By choosing this standard orientation, we can observe the extra symmetries automatically satisfied by inversion symmetric 3-monopoles. For $a_1 \neq 0$ the spectral curve has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry corresponding to rotations of π about the three cartesian axes. This means the inversion symmetric 3-monopole has a D_2 symmetry. For $a_1 = 0$ there is a $\mathrm{S}^1 \times \mathbb{Z}^2$ symmetry. For the particular value of a_2 determined by the non-singularity conditions this is the spectral curve of the axisymmetric monopole.

The spectral curve (5.5) is symmetric under inversion, rotation of π about the x_3-axis and reflection in the x_1x_3-plane. It has the same symmetries as the rational maps $r_1(z)$ and $r_2(z)$. This means that the non-singularity constraints satisfied by a_1 and a_2 restrict them to, at most, 1-parameter families.

Even without considering the non-singularity conditions, we have learned a lot about inversion symmetric monopoles by considering the spectral curve. The rational maps $r_1(z)$ and $r_2(z)$ indicate that there are inversion symmetric 3-monopoles which are C_2 and reflection symmetric. Because all inversion symmetric $n = 3$ spectral curves can be rotated to the form (5.5), the spectral curve demonstrates that all inversion symmetric 3-monopoles are D_2-symmetric about some triplet of orthogonal axes. It is also true that all 2-monopoles are D_2-symmetric about some triplet of orthogonal axes.

6 Symmetric Nahm data

To find the 1-parameter family of values of a_1 and a_2 we rely on the Nahm data formulation and construct D_2 invariant Nahm data. The construction for Nahm data invariant under a finite rotational symmetry group was introduced in [9] and is discussed in [11]. Nahm data invariant under the D_2 transformation is given by

$$T_1(s) = \frac{f_1(s)}{2}\begin{bmatrix} 0 & \sqrt{2}i & 0 \\ \sqrt{2}i & 0 & \sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{bmatrix}, \quad T_2(s) = \frac{f_2(s)}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{bmatrix},$$

$$T_3(s) = \frac{f_3(s)}{2}\begin{bmatrix} -2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{bmatrix}. \tag{6.1}$$

In the notation of (3.10) these invariant Nahm data correspond to the $SO(3)$ invariant $\underline{1}$ and the $SO(2)$ and D_4 invariant vectors in $\underline{5}$.

Nahm's equations for these data become

$$\frac{df_1(s)}{ds} = f_2(s)f_3(s) \tag{6.2}$$

and its cyclic permutations. The corresponding spectral curve is

$$\eta^3 + \left((f_1^2 - f_2^2) + (2f_1^2 + 2f_2^2 - 4f_3^2)\zeta^2 + (f_1^2 - f_2^2)\zeta^4\right)\eta = 0 \tag{6.3}$$

and so we can identify the constants $a_1 = f_1^2 - f_2^2$ and $a_2 = 2(f_1^2 + f_2^2 - 2f_3^2)$. Equation (6.2) and its cyclic permutations are the Euler top equations, which are well known to be solvable in terms of elliptic functions. We take the solution in the form given by Dancer when examining symmetric Nahm data for $SU(3)$ monopoles [5]

$$f_1(s) = -\frac{D\mathrm{dn}_k(Ds)}{\mathrm{sn}_k(Ds)}, \quad f_2(s) = -\frac{D}{\mathrm{sn}_k(Ds)}, \quad f_3(s) = -\frac{D\mathrm{cn}_k(Ds)}{\mathrm{sn}_k(Ds)}. \tag{6.4}$$

where D is a constant. The $\mathrm{sn}_k(s)$, $\mathrm{cn}_k(s)$ and $\mathrm{dn}_k(s)$ are, of course, Jacobi elliptic functions. The parameter k is the modulus of the Jacobi elliptic functions and $0 \leq k < 1$. Details of the Jacobi elliptic functions can be found in, for example, Whittaker and Watson [21].

To determine D we require that the data satisfy the boundary condition given in section 3. We must examine the elliptic functions near $s = 0$ and $s = 2$. Near $s = 0$ we have $\mathrm{sn}_k(Ds) \sim Ds$ whereas $\mathrm{cn}_k(0) = 1$ and $\mathrm{dn}_k(0) = 1$. The functions $f_1(s)$, $f_2(s)$ and $f_3(s)$ all have simple poles at $s = 0$ with residues -1. It is now easy to verify that the residue matrices are an irreducible 3-dimensional representation of $su(2)$. The functions have another simple pole at $Ds = 2K$ where K is the complete elliptic integral of the first kind with modulus k. Again it is simple to check that the irreducible representation boundary conditions are satisfied at this pole also. The functions are analytic for $0 < s < 2K/D$. If we set $D = K$, the data are valid Nahm data.

By substituting into the expressions for a_1 and a_2 and using the standard elliptic function identities $\mathrm{sn}_k^2(u) + \mathrm{cn}_k^2(u) = 1$ and $k^2\mathrm{sn}_k^2(u) + \mathrm{dn}_k^2(u) = 1$, we obtain

$$a_1 = -K^2k^2, \qquad a_2 = -2K^2(k^2 - 2). \tag{6.5}$$

The spectral curve (5.5) is now

$$\eta^3 - K^2\left(k^2 + 2(k^2 - 2)\zeta^2 + k^2\zeta^4\right)\eta = 0. \tag{6.6}$$

Using the standard formula [21]

$$K = \int_0^{\frac{1}{2}\pi} (1 - k^2\sin^2\phi)^{-\frac{1}{2}} d\phi \tag{6.7}$$

gives that when $k = 0$, $K = \pi/2$ and so the spectral curve is

$$\eta^3 + \pi^2\zeta^2\eta = 0, \tag{6.8}$$

which is the spectral curve of a 3-monopole symmetric about the x_3-axis. As $k \to 1$, $K \to \infty$ and the spectral curve is asymptotic to the product of stars

$$\eta(\eta - K(1 - \zeta^2))(\eta + K(1 - \zeta^2)) = 0. \tag{6.9}$$

This describes three well-separated monopoles located at positions $(\pm K, 0, 0)$ and $(0, 0, 0)$. We note that the Nahm data correspond to monopoles moving along the x_1-axis. We have explicitly plotted surfaces of constant energy density for these monopoles and they are discussed in Section 9.

Recently one of us has used an n-dimensional generalization of the D_2-symmetric Nahm data discussed in this Section to produce 1-parameter families of spectral curves with interesting properties. Details may be found in [18].

The spectral curve (6.6) is very similar to the 2-monopole spectral curve derived by Hurtubise [12]. The easiest way to derive this spectral curve is to consider 2-monopole Nahm data. Since a 2-monopole is always D_2-symmetric about some triplet of orthogonal axes we choose an orientation and construct invariant Nahm data as above. They are given by

$$T_1(s) = \frac{f_1(s)}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ T_2(s) = \frac{f_2(s)}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ T_3(s) = \frac{f_3(s)}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{6.10}$$

where the f_1, f_2, f_3 are the same as those defined earlier. By (3.9), 2-monopole Nahm data transforms under $SO(3)$ as $\underline{5}\oplus\underline{3}\oplus\underline{1}$. As with the 3-monopole Nahm data the D_2-symmetric Nahm data correspond to the $SO(3)$ invariant $\underline{1}$ and the $SO(2)$ and D_4 invariant vectors in $\underline{5}$.

The 2-monopole spectral curve (3.5) is then

$$\eta^2 - \frac{K^2}{4}\left(k^2 + 2(k^2-2)\zeta^2 + k^2\zeta^4\right) = 0, \tag{6.11}$$

which is the one obtained by Hurtubise [12] using different methods. If this technology, based on obtaining the spectral curve from Nahm data, had been available to Hurtubise his task in [12] would have been simpler.

7 The metric

We have constructed a 4-dimensional submanifold of the 3-monopole moduli space. It is the fixed set of the inversion action on the entire 3-monopole moduli space. The fixed point set of a finite group action on a Riemannian manifold is always totally geodesic and so our submanifold is a totally geodesic submanifold of M_3^0. Furthermore this inversion action commutes with the action of $SO(3)$ and so, since the entire 3-monopole moduli space is $SO(3)$ invariant, our restricted moduli space has an $SO(3)$ invariant metric. In this section we will show that this metric is the Atiyah-Hitchin metric.

We will do this by considering the metric on the space of Nahm data. It has been proven by Nakajima [16] that the moduli space of Nahm data has the same metric as the corresponding moduli space of monopoles. The moduli space of Nahm data is difficult to define since it involves factoring the space of Nahm data by a set of $SU(n)$ transformations. It also requires the introduction of a fourth Nahm matrix, $T_0(s)$. In any Nahm data calculation this is set to zero by one of the $SU(n)$ transformations.

The metric on Nahm data corresponding to a tangent $Y = (Y_0, Y_1, Y_2, Y_3)$ is given by

$$\|Y_0, Y_1, Y_2, Y_3\|^2 = -\int_0^2 \sum_{i=0...3} \mathrm{tr}(Y_i^2) ds. \tag{7.1}$$

Thus, for example, to calculate the metric on the Nahm data for monopoles moving along the x_1-axis, we would first choose a separation parameter r which may be identified with monopole distance when the monopoles are well-separated. We know, from above, that a suitable choice would be $r = kK$. We define the tangent vectors

$$Y_i = \frac{dT_i}{dr} \tag{7.2}$$

and the metric coefficient is then given by

$$g(r) = -\int_0^2 \sum_{i=1..3} \mathrm{tr}(Y_i^2)\ ds. \tag{7.3}$$

Note that in general the tangent vectors can only be defined by direct differentiation, as in equation (7.2), if certain identities hold between the matrices defining the Nahm data, which ensure that such tangent vectors are orthogonal to the gauge orbits (see [19] for an example). However, since in the present case each Nahm matrix only involves a single function, the orthogonality to gauge orbits is guaranteed (see appendix). By substituting from (6.1) we find that

$$\sum_{i=1..3} \mathrm{tr}(Y_i^2) = -2\left[\left(\frac{df_1}{dr}\right)^2 + \left(\frac{df_2}{dr}\right)^2 + \left(\frac{df_3}{dr}\right)^2\right], \tag{7.4}$$

which allows the metric to be calculated.

Using the rational map formulation a geodesic submanifold of M_2^0 space has been constructed, namely, the space of inversion symmetric 3-monopoles. By constructing a geodesic submanifold of the monopole moduli space a geodesic submanifold of the Nahm moduli space has been constructed. The Nahm data on this submanifold is given along a radial geodesic by (6.1) and, by $SO(3)$ transformation, this geodesic generates the whole space of inversion symmetric, strongly centred 3-monopole Nahm data. We know how the Nahm data transforms under $SO(3)$; it transforms like a vector in $\underline{5}$. Furthermore, we know the Nahm data for a suitably oriented 2-monopole; it has been given above in (6.10). The 2-monopole Nahm data transform the same way under $SO(3)$ as the inversion symmetric 3-monopole Nahm data. This means that the space of 2-monopole Nahm data is identical to the space of 3-monopole Nahm data except the matrices appearing in the 2-monopole case are the basis of $\underline{2}$ and in the 3-monopole case they are the basis of $\underline{3}$. Thus the two spaces are identical apart from an overall factor.

This factor is easy to calculate. It is given by the ratio between the traces of the squares of the matrices in the two cases. This factor is four. The metric for 3-monopoles is four times that for 2-monopoles. Thus, for example, to calculate the 2-monopole metric along the same geodesic as in the 3-monopole case, tangents Y_i can be calculated as before and in this case

$$\sum_{i=1..3} \mathrm{tr}(Y_i^2) = -\frac{1}{2}\left[\left(\frac{df_1}{dr}\right)^2 + \left(\frac{df_2}{dr}\right)^2 + \left(\frac{df_3}{dr}\right)^2\right]. \tag{7.5}$$

Since the 2-monopole metric is Atiyah-Hitchin, so is the metric on the space of inversion symmetric 3-monopoles, except that it is rescaled by a factor of four.

8 The asymptotic metric

Recently, Gibbons and Manton [8] have constructed an approximate metric for n well-separated monopoles. It is instructive to construct this metric for 3-monopoles with inversion symmetry. According to Gibbons and Manton, n-monopoles located at $\{\boldsymbol{\rho}_i\}$ with phases $\{\theta_i\}$ have a metric

$$ds^2 = g_{ij} d\boldsymbol{\rho}_i \cdot d\boldsymbol{\rho}_j + g_{ij}^{-1}(d\theta_i + \mathbf{W}_{ik}\cdot d\boldsymbol{\rho}_k)(d\theta_j + \mathbf{W}_{jl}\cdot d\boldsymbol{\rho}_l) \tag{8.1}$$

where $\cdot$ denotes the usual scalar product on $\mathbb{R}^3$ vectors, repeated indices are summed over and

$$\begin{aligned} g_{jj} &= 1 - \sum_{i\neq j}\frac{1}{\rho_{ij}} \qquad \text{(no sum over } j) \\ g_{ij} &= \frac{1}{\rho_{ij}} \qquad (i\neq j) \\ \mathbf{W}_{jj} &= -\sum_{i\neq j}\mathbf{w}_{ij} \qquad \text{(no sum over } j) \\ \mathbf{W}_{ij} &= \mathbf{w}_{ij} \qquad (i\neq j), \end{aligned} \tag{8.2}$$

$\boldsymbol{\rho}_{ij} = \boldsymbol{\rho}_i - \boldsymbol{\rho}_j$ and $\rho_{ij} = |\boldsymbol{\rho}_{ij}|$. The approximation is valid for $\rho_{ij} \gg 1$. The $\mathbf{w}_{ij}$ are Dirac potentials and are defined by

$$\mathrm{curl}\ \mathbf{w}_{ij} = \mathrm{grad}\frac{1}{\rho_{ij}} \tag{8.3}$$

where the curl and grad operators are taken with respect to the ith position coordinate $\boldsymbol{\rho}_i$.

In the case of three monopoles that are symmetric under inversion symmetry $\boldsymbol{\rho}_1 = \boldsymbol{\rho}$, $\boldsymbol{\rho}_2 = 0$ and $\boldsymbol{\rho}_3 = -\boldsymbol{\rho}$, and we write $\rho = |\boldsymbol{\rho}|$. Furthermore we require $d\theta_1 = d\theta$, $d\theta_2 = 0$ and $d\theta_3 = -d\theta$. Denoting $\mathbf{w}_{12} = \mathbf{w}_{23}$ by $\mathbf{w}$ so that $\mathbf{w}_{13} = \frac{1}{2}\mathbf{w}$, we have

$$\begin{aligned} g_{ij} &= \frac{1}{\rho}\begin{pmatrix} \rho-\frac{3}{2} & 1 & \frac{1}{2} \\ 1 & \rho-2 & 1 \\ \frac{1}{2} & 1 & \rho-\frac{3}{2}\end{pmatrix} \\ \mathbf{W}_{ij} &= \frac{1}{2}\begin{pmatrix} -3\mathbf{w} & 2\mathbf{w} & \mathbf{w} \\ 2\mathbf{w} & -4\mathbf{w} & 2\mathbf{w} \\ \mathbf{w} & 2\mathbf{w} & -3\mathbf{w}\end{pmatrix} \end{aligned} \tag{8.4}$$

so

$$ds^2 = 4\left[\frac{1}{2}\left(1-\frac{2}{\rho}\right)d\boldsymbol{\rho}\cdot d\boldsymbol{\rho} + \frac{1}{2}\left(1-\frac{2}{\rho}\right)^{-1}(-d\theta + 2\mathbf{w}\cdot d\boldsymbol{\rho})^2\right]. \tag{8.5}$$

Up to the overall factor of four this is the asymptotic metric for two strongly centred monopoles separated by a distance ρ. Note that in the 2-monopole case ρ is the separation of the two monopoles; whereas in the 3-monopole case it is the distance from the monopole at the origin to either of the other two monopoles.

9 Geodesic scattering

Since the metric on the moduli space of inversion-symmetric 3-monopoles is now known, we can understand their low energy dynamics by studying geodesics. Their moduli space is the Atiyah-Hitchin manifold which allows the known results about 2-monopole dynamics [1, 7] to be translated into results about the dynamics of 3-monopoles.

We have already discussed the right angle scattering geodesics in terms of their rational maps. From the point of view of the Atiyah-Hitchin submanifold such a scattering process is associated with a geodesic which passes over the 2-dimensional rounded cone submanifold [1]. Since we have the Nahm data for these monopoles, we can construct the monopole fields along such a geodesic, using the numerical ADHMN construction we introduced previously [11]. In Figure 1 we plot a surface of constant energy density for various times along the geodesic (corresponding to the elliptic modulus parameter values $k = 0.99, 0.90, 0.80, 0.00$). In Figure 1(a) we see three separated monopoles. As they approach, they deform and merge to form a pretzel shape, Figure 1(b). It is interesting that the pretzel configuration closely resembles the pretzel 3-skyrmion of Walet [20]. Moving along the geodesic, the monopole becomes more ring-like, Figure 1(c). It instantaneously forms the torus, Figure 1(d), before separating out again, through the same configurations, rotated through $\pi/2$, Figure 1(e-g).

There is a closed 2-monopole geodesic [2], corresponding to two orbiting monopoles, so we can immediately conclude that a closed 3-monopole geodesic exists. Following [2], the value of the elliptic modulus k for the rotating 3-monopole configuration is determined as the root of the equation

$$\int_0^{\frac{1}{2}\pi} \frac{2k^2\sin^2\phi - 1}{\sqrt{1-k^2\sin^2\phi}}\, d\phi = 0 \tag{9.1}$$

giving $k \approx 0.906$.

In Figure 2 we plot a surface of constant energy density for this monopole. The monopole has been rotated so that the axis of rotation (which is also shown) is in the plane of the page. The monopole motion is a periodic orbit, rotating at constant angular velocity about the shown axis, which is at an angle of approximately $\pi/9$ to the vertical [2].

10 Conclusion

We have shown that the moduli space of inversion symmetric 3-monopoles is an Atiyah-Hitchin submanifold of the 3-monopole moduli space. Using this result we have studied some geodesics in the 3-monopole moduli space and examined the associated monopole dynamics, including displaying energy density plots.

It is possible to apply inversion symmetry to n-monopoles for $n > 3$. However, the resulting geodesic submanifold has more than four dimensions and so cannot be an Atiyah-Hitchin submanifold. We have just learnt, however, that Bielawski [4] has succeeded in finding geodesic Atiyah-Hitchin submanifolds of the n-monopole moduli space for each n. These submanifolds correspond to n-monopoles with inversion symmetry and the individual monopoles equally spaced along an axis. Bielawski derives his result by considering the moment map construction.

One interesting property of these equally spaced monopoles is clear from the asymptotic metric discussion in Section 8. Inversion symmetry requires that $\boldsymbol{\rho}_1 = \boldsymbol{\rho}$, $\boldsymbol{\rho}_2 = 0$ and $\boldsymbol{\rho}_3 = -\boldsymbol{\rho}$. It is then neccessary to fix $d\theta_1 = d\theta$, $d\theta_2 = 0$ and $d\theta_3 = -d\theta$ in order to derive the asymptotic 2-monopole metric. Similarly, for the asymptotic metric for n equally spaced monopoles to be the same, up to a factor, as that for 2-monopoles the monopoles must be given $d\theta$'s proportional to their distance from the origin. Thus, Bielawski's equally spaced monopoles have electric charge proportional to their distance from the centre of mass.

Acknowledgements

We would like to thank Nick Manton for many useful discussions and also Niels Walet, who inspired our interest in D_2 symmetric monopoles. CJH thanks the EPSRC for a research studentship and the British Council for a Chevening award.

A Appendix: Tangent Vectors

In this brief appendix we show that the tangent vectors defined by direct differentiation of the Nahm data are orthogonal to the gauge orbits, providing each Nahm matrix involves only a single function.

Let G be the group of analytic $SU(n)$-valued functions $g(s)$, for $s \in [0,2]$, which are the identity at $s = 0$ and $s = 2$, and satisfy $g^t(2-s) = g^{-1}(s)$. Then gauge transformations $g \in G$ act on $su(n)$-valued Nahm data as

$$T_0 \to gT_0g^{-1} - \frac{dg}{ds}g^{-1}, \qquad T_i \to gT_ig^{-1} \quad i = 1,2,3. \tag{A1}$$

We work in the gauge $T_0 = 0$, and define the tangent vectors Y_i by direct differentiation with respect to the geodesic parameter, r say, ie.

$$Y_i = \frac{dT_i}{dr}. \tag{A2}$$

We want to show that these tangent vectors are orthogonal to the tangent vectors W_i of the gauge orbits ie.

$$< Y_i, W_i >= -\int_0^2 \sum_{i=0..3} \mathrm{tr}(Y_iW_i)\ ds = 0. \tag{A3}$$

To compute W_i we consider the infinitesimal gauge transformation given by

$$g = 1 + \epsilon A \tag{A4}$$

for $A \in su(n)$, and work to first order in ϵ. Since we have set $Y_0 = 0$ then we need only consider W_i for $i = 1,2,3$. These tangent vectors are given by

$$W_i = (gT_ig^{-1} - T_i)/\epsilon = [A, T_i]. \tag{A5}$$

Thus

$$< Y_i, W_i >= -\int_0^2 \sum_{i=1..3} \mathrm{tr}(\frac{dT_i}{dr}[A, T_i])\ ds. \tag{A6}$$

Now we make use of the fact that each of the three Nahm matrices depends on a single function ie

$$T_i = f_iM_i, \quad i = 1,2,3 \tag{A7}$$

where the M_i are constant matrices. Thus

$$< Y_i, W_i >= -\int_0^2 \sum_{i=1..3} \frac{1}{2}\frac{df_i^2}{dr}\mathrm{tr}(M_iAM_i - M_iM_iA)\ ds = 0 \tag{A8}$$

by the cyclic property of the trace. Thus the required result is proved.

Figure Captions

Figure 1(a-g). A surface of constant energy density at increasing times. The corresponding values of the elliptic modulus are (a) $k = 0.99$, (b) $k = 0.90$, (c) $k = 0.80$, (d) $k = 0.00$, (e) $k = 0.80$, (f) $k = 0.90$, (g) $k = 0.99$.

Figure 2. A surface of constant energy density for the rotating 3-monopole, together with the axis of rotation.

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two figures attached as separate gif files.