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# Biholomorphic mappings and Banach function modules

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In [12] the authors showed that if $E$ is a Banach space which does not contain $c_0$ then every bounded domain in $E$ is biholomorphically equivalent to a finite product of irreducible domains (which are unique up to permutation and biholomorphic equivalence for a convex balanced domain). In this article we continue this direction of research and consider domains in arbitrary Banach spaces. We confine ourselves to the open unit ball of a Banach space and consequently, by a result of Kaup and Upmeier [22], biholomorphic equivalence is the same as linear isometric equivalence. Thus we seek a method of expressing an arbitrary Banach space as a unique product of irreducible Banach spaces i.e. as $c_0(\{E_i\}_{i\in I})$ or as $l^\infty(\{E_i\}_{i\in I})$ for some collection of irreducible Banach spaces $E_i$, $i\in I$. This is not always the case. To find examples in which it is true we turn to the well developed theory of $M$-ideals, $M$-summands and function module representation of Banach spaces. This theory is reasonably well suited to our purposes since a Banach space is irreducible if and only if it has only trivial $M$-summands and a function module representation of a Banach space may be regarded as a decomposition of the Banach space into component spaces. On the other hand we note, the indexing set is a compact space rather than a discrete set and so we do not have what we might ordinarily consider as a product, the component spaces may not be irreducible and an irreducible space may have more than one component space.

In Section 1 we recall some notions of irreducibility. We compare and contrast them with similar concepts which are also discussed in the literature e.g. Banach bundles and continuous products of Banach spaces. In Section 2 we obtain a general decomposition theorem for Banach spaces into atomic and nonatomic parts. Using this result we obtain an irreducible product decomposition of a Banach space $X$ in the following cases:

(i) $X=Y'$ and $Y$ has RNP (= the Radon-Nikodym Property),

(ii) $X=Y''$ and $Y$ is an $M$-ideal in $X$,

(iii) $X$ has a 1-unconditional finite dimensional decomposition.



Moreover, our decomposition leads to a fairly transparent proof of the Friedman-Russo decomposition of a JBW*-triple [13] and to a characterisation of preduals of JBW*-triples having RNP [3], [7].

In Section 3 we discuss biholomorphic automorphisms of the unit ball of a function module and show that they can be recovered from the biholomorphic automorphisms of the component spaces and from the homeomorphisms of the compact indexing set. We also show that any Banach space $X$ can be isometrically embedded as a weak*-dense subspace of an $l^\infty$-product $Y$ of irreducible dual Banach spaces such that each biholomorphic automorphism of the ball of $X$ is the restriction of a biholomorphic automorphism of the unit ball of $Y$.

**§ 1.** All Banach spaces we consider are over the complex numbers although a number of our results are easily seen to be true for real Banach spaces. We let $\mathscr{L}(E)$ denote the set of all continuous linear operators from the Banach space $E$ into itself and $h(E)$ will denote the real subspace of $\mathscr{L}(E)$ consisting of all hermitian operators ($T\in h(E)$ if and only if the numerical range of $T$ is real).

The concepts of $M$-summand, function module etc. are primarily due to Cunningham [8] and have also been extensively developed by Alfsen and Effros [1]. We refer to [5], unless otherwise indicated, for all unexplained definitions.

Throughout the paper we shall investigate various properties of function modules. There exist several notions similar to that of a function module and related to the problem of reducibility of Banach spaces. The concepts of a Banach space over a topological space and that of a continuous product of a family of Banach spaces, have been introduced and studied by Vigué [28]. Another similar concept is that of a bundle of Banach spaces. The theory of bundles of Banach spaces is well developed and its comprehensive exposition can be found in [15] and [18]. Also in [18], there is a detailed account of the relationship between bundles of Banach spaces, sheaves and Banach $C(X)$-modules. The relationship between the other notions mentioned above and that of a function module is explained in the next two propositions.

**Proposition 1.** (i) *Let $(E,S,p)$ be a bundle of Banach spaces over a completely regular base space $S$ with a continuous norm $q$ on $E$. Then $E$ is a Banach space over $S$ (which is reduced if the bundle is).*

(ii) *If $(E,S,p,q)$ is a Banach space over a topological space then there is a unique coarsest topology on $E$ under which $(E,S,p,q)$ is a Banach space over a topological space. Under this topology $(E,S,p)$ is a bundle of Banach spaces. Moreover if $S$ is compact the change of topology of $E$ does not affect the space of sections of $p$.*

*Proof.* (i) Follows from [15], Theorem 2.9.

(ii) If $\sigma: U\to E$ is a local section of $p$ then continuity of $\sigma$, $p$, $q$ implies that for any $\varepsilon>0$ the sets

$$T(U,\sigma,\varepsilon)=\{y\in E: p(y)\in U \text{ and } q(y-\sigma(p(y)))<\varepsilon\}$$

are open. Hofmann's construction (as described in Sections 5.3–5.5 of [15]) can be used to show that $\{T(U,\sigma,\varepsilon)\}_{U,\sigma,\varepsilon}$ is a base for a topology on $E$ (which is obviously coarser than the original one) and that the algebraic operations on $E$ are continuous. It is clear that $p$ is continuous with respect to the new topology. Now we shall prove that $q$ is also continuous in the new topology. Fix a point $x_0 \in E$ and a positive number $\varepsilon$. Take a local section $\sigma$ such that $x_0 = \sigma(p(x_0))$. Continuity of $q \circ \sigma$ implies the existence of a neighbourhood $U$ of $p(x_0)$ such that $U$ is contained in the domain of $\sigma$ and $|q(\sigma(p(x_0))) - q(\sigma(s))| < \varepsilon/2$ for all $s \in U$. Therefore for all $x \in T(U, \sigma|U, \varepsilon/2)$ we have

$$|q(x) - q(x_0)| \leqq |q(x) - q(\sigma(p(x)))| + |q(\sigma(p(x))) - q(\sigma(p(x_0)))|$$
$$\leqq q(x - \sigma(p(x))) + \varepsilon/2 < \varepsilon.$$

This completes the proof of the first statement in (ii). The second conclusion of (ii) follows directly from the generalized Stone-Weierstrass Theorem (see [15], Theorem 4.2, Corollary 4.3).

If $(E, S, p, q)$ is a Banach space over a topological space, we shall denote by $\Gamma(p)$ the Banach space of all $q$-bounded global sections of $p$.

**Proposition 2.** *Let $(K, (X_k)_{k \in K}, X)$ be a function module such that for every $x \in X$ the mapping $k \to \|x(k)\|$ is continuous. Define*

$$E = \bigcup_{k \in K} \{k\} \times X_k,$$
$$p : E \to K,\ p|\{k\} \times X_k \equiv k,$$
$$q : E \to R_+,\ q|\{k\} \times X_k \text{ is the norm on } X_k,$$
$$T(U, x, \varepsilon) = \{y \in E : p(y) \in U \text{ and } q(y - x(p(y))) < \varepsilon\}$$

*for $\varepsilon > 0$, $x \in X$ and an open set $U \subset K$. Then the sets $\{T(U, x, \varepsilon)\}_{U, x, \varepsilon}$ form a base for a topology on $E$ under which $(E, K, p, q)$ becomes a Banach space over a topological space. Moreover, $X = \Gamma(p)$ i.e. $X$ is a continuous product of $(X_k)_{k \in K}$.*

*Proof.* By Theorem 5.9 in [15], $(E, K, p)$ is a bundle of Banach spaces such that $X = \Gamma(p)$. The base space of the bundle is compact, hence for each $a \in E$, there is a local section $\sigma$ of $p$ such that $a = \sigma(p(a))$ ([15], Theorem 2.9). So, it is enough to prove that $q$ is continuous, and this can be done exactly as in the proof of the second part of Proposition 1.

**Remark.** It is obvious that if $(E, S, p, q)$ is a reduced Banach space over a topological space and $S$ is compact then $(S, (E_s)_{s \in S}, \Gamma(p))$ is a function module.

Let $K$ be a compact Hausdorff space and let $X$ be a Banach space which is a $C(K)$-module (see [15], Definitions 7.1, 7.18 or [18]). We call $X$ *reduced* if the only $f \in C(K)$ with $f \cdot x = 0$ for each $x \in X$ is $f = 0$.

**Proposition 3** ([15], Corollary 7.19, Theorem 5.9). *Let $K$ be a compact Hausdorff space. Then the following are equivalent for a Banach space $X$:*

(i) *$X$ is a reduced locally $C(K)$-convex $C(K)$-module* (see [15], [18]).

(ii) *$X$ is isometrically isomorphic to the space of bounded sections of a reduced bundle of Banach spaces with base space $K$.*

(iii) *$X$ has a function module representation over $K$.*



**Corollary 4.** *Let $X$ be a Banach space with maximal function module representation $(K, (X_k)_{k \in K}, \tilde{X}, \varrho)$ and $K_0$ a compact Hausdorff space. Then:*

(i) *$X$ is a reduced locally $C(K_0)$-convex $C(K_0)$-module if and only if there is a continuous surjection of $K$ onto $K_0$.*

(ii) *If $X$ is a continuous product of Banach spaces over a completely regular base space $S$, then there is a continuous surjection from $K$ onto the Stone-Čech compactification $\beta S$.*

(iii) *If $\tilde{K}$ is the compact Hausdorff space occurring in the maximal function module representation of $X''$, then there is a continuous surjection of $\tilde{K}$ onto $K$.*

*Proof.* (i) follows from Proposition 3 and [5], Theorem 4.16.

(ii) It is easy to check that if $X$ is a continuous product over $S$, then $X$ is a locally $C(\beta S)$-convex $C(\beta S)$-module. Thus (ii) follows from (i).

(iii) It is not difficult to check that $X''$ is a locally $C(K)$-convex $C(K)$-module.

The following result compares various definitions of irreducibility of Banach spaces. $Z(X)$ denotes the centralizer of $X$.

**Proposition 5.** *Let $X$ be a Banach space. Consider the following conditions:*

(a) *$Z(X'')$ is one-dimensional.*

(b) *$X$ has no nontrivial $M$-ideals.*

(c) *$Z(X)$ is one-dimensional.*

(d) *$X$ cannot be represented as a function module over a compact set containing more than one point.*

(e) *The trivial function module representation of $X$ $(\{1\}, X_1, X)$ with $X_1 = X$) is maximal.*

(f) *$X$ cannot be represented as a continuous product of Banach spaces over a base space $S$ with more than one point.*

(g) *$X$ cannot be represented as a continuous product of Banach spaces over a compact base space with more than one point.*

(h) *$X$ does not contain non-trivial $M$-summands.* (In this case we say $X$ is irreducible.)

*Then* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h). *Moreover, if $X$ is a dual space* (h) $\Rightarrow$ (c). *If $X$ is reflexive, all the conditions are equivalent.*

*Proof.* (a) $\Rightarrow$ (b). By [5], Proposition 5.2, $X''$ has only trivial $M$-summands. Hence $X'$ has only trivial $L$-summands, which implies (b).

(b) $\Rightarrow$ (c) by [5], Proposition 5.1 (iii).

(c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) by properties of the maximal function module representation.

(e) $\Rightarrow$ (f) by Corollary 4.

(f) $\Rightarrow$ (g) and (g) $\Rightarrow$ (h) are trivial.

For $X$ a dual space (h) $\Rightarrow$ (c). Indeed, if $X$ is a dual space and $(K, (X_k)_{k\in K}, \tilde{X}, \varrho)$ is a maximal function module representation of $X$, then $K$ is extremally disconnected (see [5]). Also, every $M$-summand in $X$ is of the form $X_L = \{x \in X : \varrho(x)(k) = 0 \text{ for all } k \in L\}$, when $L$ is a closed and open subset of $K$. Hence (h) implies that $K$ consists of a single point. This yields (c).

**Remark.** A Banach space $X$ is said to be *strongly irreducible* if it does not contain non-trivial $M$-ideals. It has been shown in [4] that a closed subspace of a JB*-triple system is a JB*-ideal if and only if it is an $M$-ideal. Hence the above definition agrees with that given by Vigué [29] in the context of JB*-triples.

A Banach space $X$ is said to be *irreducible* in the sense of Vigué [29] if it cannot be represented as a continuous product of a family of more than one Banach spaces and if the family of all biholomorphic automorphisms of the unit ball of $X$ behaves as in Theorem 48 in the last section of this paper.

Proposition 5 and Theorem 48 yield the following

**Corollary 6.** *Strong irreducibility implies irreducibility in the sense of Vigué. In dual spaces irreducibility in the sense of Vigué yields irreducibility. All three concepts coincide for reflexive spaces.*

Corollary 6 generalises Proposition 2.9 and Theorem 5.1 of [29].

There exist Banach spaces which have one-dimensional centralizer but admit non-trivial $M$-ideals. Indeed, if $H$ is an infinite dimensional Hilbert space then the compact operators on $H$ furnish a non-trivial $M$-ideal in the space $B(H)$ of all bounded operators on $H$ and $B(H)$ has one-dimensional centralizer. Consequently there are Banach spaces which are irreducible in the sense of Vigué but not strongly irreducible. Also, there exist irreducible Banach spaces which are not irreducible in the sense of Vigué. A simple example is given by $C([0, 1])$.

Harris [17] discusses irreducibility of J*-algebras. His concept of indecomposable J*-algebra [17], p. 340, is the same as our notion of irreducible J*-algebra and a J*-algebra is simple, [17], p. 347, in the terminology of Harris if and only if it contains no $M$-ideals. Harris also discusses further types of irreducibility which are only relevant for spaces of operators and thus only to J*-algebras (= special JB*-triple systems).

We now describe briefly the basic definitions from the theory of bounded symmetric domains and refer to [26], [27] for further details.

**Definition 7.** A JB*-*triple system* is a Banach space $E$ endowed with a continuous triple product $\{\cdot,\cdot,\cdot\} : E^3 \to E$ such that

(i) $\{\cdot,\cdot,\cdot\}$ is linear in the first variable, antilinear in the second and symmetric in the first and third variables.

(ii) $\{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}vz\} - \{u\{vxy\}z\}$ for all $x, y, z, u, v \in E$ (Jordan triple identity).

(iii) if $x \square y \in \mathscr{L}(E)$ is defined by $x \square y(z) = \{xyz\}$ then $z \square z \in h(E)$ for all $z \in E$ and $\sigma(z \square z) \subset [0, \infty)$.

(iv) $\|z \square z\| = \|z\|^2$ for all $z \in E$.



A deep result of Kaup [20] says that there is a one to one correspondence between bounded symmetric domains and JB*-triple systems.

For a domain $\mathscr{D}$ we let $G(\mathscr{D})$ and $V(\mathscr{D})$ denote the group of all biholomorphic mappings of $\mathscr{D}$ onto itself and the (real) vector space of all complete holomorphic vector fields on $\mathscr{D}$ respectively.

**Proposition 8.** [22]. (a) *If $E$ is a Banach space then there exists a closed subspace $F$ of $E$ such that*

$$G(B_E)(0) = B_E \cap F = B_E \cap \{X(0) | X \in V(B_E)\}.$$

(b) *$E$ is a JB*-triple system if and only if $E = \{X(0) | X \in V(B_E)\}$.*

A subspace $F$ of a JB*-triple system is a JB*-*ideal* if $\{xyz\} \in F$ wherever at least one of $x, y, z \in F$. If $F$ is a closed JB*-ideal then $E/F$ is a JB*-triple system in the canonical fashion.

**Proposition 9.** [4, 19]. *A closed subspace $F$ of a JB*-triple system is a JB*-ideal if and only if it is an $M$-ideal.*

**§ 2.** In this section we establish a general decomposition theorem which applies in a variety of situations and which is based on the maximal function module representation.

**Definition 10.** An *admissible class*, $\mathscr{C}$, is a collection of Banach spaces such that

(a) if $E \in \mathscr{C}$ and $F \in \mathscr{C}$ then $E \oplus_\infty F \in \mathscr{C}$,

(b) if $E \in \mathscr{C}$ and $E = F \oplus_\infty G$ then $F \in \mathscr{C}$ and $G \in \mathscr{C}$.

**Example 11.** The following are examples of admissible classes:

(a) $\mathscr{B}$ — the collection of all Banach spaces,

(b) $\mathscr{B}^*$ — the collection of all dual Banach spaces,

(c) $\mathscr{J}\mathscr{B}$ — the collection of all JB*-triple systems,

(d) $\mathscr{J}\mathscr{B}\mathscr{W}^*$ — the collection of all JB*-triple systems which are dual spaces ($\mathscr{J}\mathscr{B}\mathscr{W}^* = \mathscr{J}\mathscr{B} \cap \mathscr{B}^*$),

(e) $\mathscr{E}$ — the collection of all Banach spaces for which the compact set occurring in their maximal function module representation is extremally disconnected,

(f) $\mathscr{R}$ the collection of all Banach spaces such that the mapping $k \in K \to \|\varrho(x)(k)\|$ is continuous, $K$ and $\varrho$ as in the maximal function module representation (see [5]).

We have already seen that $\mathscr{B}^* \subset \mathscr{E} \cap \mathscr{R}$. If $K$ denotes a compact set then $C(K) \in \mathscr{R}$. If $K$ is extremely disconnected but not hyperstonean, e.g. the classical Cantor set, then $C(K) \in \mathscr{E} \cap \mathscr{R}$ but $C(K) \notin \mathscr{B}^*$.

**Definition 12.** A collection of operators $\mathcal{O}$ is called *admissible* for any admissible class $\mathscr{C}$ if the following hold.

(a) If $T: E \to F \in \mathcal{O}$, $E \in \mathscr{C}$ and $E = M \oplus_\infty M^\perp$ then there exists $T_1: M \to G \in \mathcal{O}$ and an isometric embedding $i: G \to F$ such that the following diagram commutes

$$(*) \qquad \begin{array}{ccc} M & \xrightarrow{\quad T_1 \quad} & G \\ \downarrow & & \downarrow i \\ M \oplus_\infty M^\perp = E & \xrightarrow{\quad T \quad} & F \end{array}$$

(This condition says essentially that an admissible collection of operators is closed with respect to restriction to $M$-summands.)

(b) If $S: E_1 \to E_2 \in \mathcal{O}$, $E_1 \in \mathscr{C}$ and $F_1 \in \mathscr{C}$ then there exists $F_2$ such that

$$(**) \qquad \begin{aligned} (S, 0): E_1 \oplus_\infty F_1 &\longrightarrow E_2 \oplus_\infty F_2, \\ (x, y) &\longrightarrow S(x) + 0 \end{aligned}$$

belongs to $\mathcal{O}$.

**Example 13.** The weak*-continuous linear functionals form an admissible class. It suffices in $(*)$ to take $G = C$, $i = I_C$ and $T_1$ the restriction to $M$. In $(**)$ let $F_2 = \{0\}$.

Our next example depends on the following lemma. $\mathscr{E}_E$ denotes the set of extreme points of the unit ball of $E$.

**Lemma 14.** *If $T$ is an isometry of the Banach space $E$ onto itself, $\|I - T\| < 2$ and $M$ is a closed $M$-ideal in $E$ then $T(M) = M$.*

*Proof.* Since $\|I - T^{-1}\| = \|TT^{-1} - IT^{-1}\| \leq \|I - T\| \cdot \|T^{-1}\| < 2$ it suffices to show $T(M) \subset M$.

Let ${}^tT: E' \to E'$ denote the adjoint of $T$. Since $M$ is an $M$-ideal we have $E' = M^0 \oplus_1 (M^0)^\perp$. Let $e = e_1 + e_2 \in \mathscr{E}_{E'}$ where $e_1 \in M^0$ and $e_2 \in (M^0)^\perp$. We claim that either $e_1$ or $e_2$ is zero. If not we can choose positive real numbers such that $\alpha\|e_1\| = \beta\|e_2\|$. Let $\delta = \min(\frac{1}{\alpha}, \frac{1}{\beta})$. Then, for $|\lambda| \leq 1$,

$$\begin{aligned} \|e_1 + e_2 + \lambda(\delta\alpha e_1 - \delta\beta e_2)\| &= \|(1 + \lambda\delta\alpha) e_1\| + \|(1 - \lambda\delta\beta) e_2\| \\ &= (1 + \lambda\delta\alpha)\|e_1\| + (1 - \lambda\delta\beta)\|e_2\| = 1 + \lambda\delta(\alpha\|e_1\| - \beta\|e_2\|) \leq 1 \end{aligned}$$

This is a contradiction and proves our claim. If $e_1 \in \mathscr{E}_{M^0}$ and $e_2 \in \mathscr{E}_{(M^0)^\perp}$ then

$$\|e_1 + e_2\| = 2.$$

Hence

$${}^tT(\mathscr{E}_{M^0}) \subset \mathscr{E}_{M^0} \text{ and } {}^tT(\mathscr{E}_{(M^0)^\perp}) \subset \mathscr{E}_{(M^0)^\perp}.$$

Since $M^0 \in \mathscr{B}^*$ and $(M^0)^\perp \in \mathscr{B}^*$ it follows that ${}^tT(M^0) \subset M^0$ and ${}^tT((M^0)^\perp) \subset (M^0)^\perp$. Hence ${}^{tt}T(M^{00}) \subset M^{00}$. Let $J: E \to E''$ denote the canonical embedding. Since

$$\begin{aligned} T(M) &= J^{-1}({}^{tt}T(JM)) \subset J^{-1}({}^{tt}T(M^{00}) \cap J(E)) \\ &\subset J^{-1}(M^{00} \cap J(E)) = J^{-1}(J(M)) = M. \end{aligned}$$

This completes the proof.



**Corollary 15.** *If $T$ is a Hermitian operator on $E$ and $M$ is an $M$-ideal in $E$ then $T(M) \subset M$.*

*Proof.* Since $T$ is Hermitian $e^{itT}$ is an isometry for all $t \in R$. For $t$ sufficiently small $\|e^{itT} - I\| < 2$ and hence $e^{itT}(M) \subset M$. Hence for $t$ sufficiently small and non-zero and $x \in M$

$$\frac{e^{itT} - I}{t}(x) \in M.$$

Since $\lim_{t \to \infty} \frac{e^{itT} - I}{t}(x) = iT(x)$ for all $x \in E$, this completes the proof.

**Example 16.** The Hermitian operators form an admissible class.

If $T: E \to E$ is Hermitian and $E = M \oplus_\infty M^\perp$, then Corollary 15 implies that $T(M) \subset M$.

Let $T_1: M \to M$ be the restriction of $T$ to $M$. Using the Hahn-Banach Theorem and the fact that an operator is Hermitian if and only if its numerical range is real we see that $T_1$ is Hermitian. Let $G = M$ and let $i$ be the canonical embedding of $M$ in $E$ then condition $(*)$ of Definition 12 is satisfied and the Hermitian operators are admissible.

**Example 17.** The Hermitian operators $T: E \to E$ with $\dim T(E) < \infty$ form an admissible class. So does the subclass with $\dim T(E) \leq n$ for some fixed $n$.

**Example 18.** Let $\mathscr{E}^*$ consist of all weak*-continuous linear functionals arising from extreme points of the unit ball of a predual or which are identically zero. We claim $\mathscr{E}^*$ is an admissible class.

Let $x: E \to C \in \mathscr{E}^*$ and suppose $E = M \oplus_\infty M^\perp$. Then (see [5], p. 114) if $F$ is a predual of $E$ such that $x \in \mathscr{E}_F$ then $F = M_0 \oplus_1 M_0^\perp$ where

$$M_0 = \{y \in F;\ \phi(y) = 0 \text{ for all } \phi \in M\}.$$

We have $(M_0)' \cong E/(M_0)^0 \cong M^\perp$ and $(M_0^\perp)' \cong E/(M_0^\perp)^0 \cong M$. Since $x \in \mathscr{E}_F$ (the case where $x = 0$ is trivial) we have, as in Lemma 14, that if $x = x_1 + x_2$, $x_1 \in M_0$ and $x_2 \in M_0^\perp$ then $x_1 = 0$ or $x_2 = 0$. If $x_1 = 0$ we take $T_1 = 0$ in Definition 12. If $x_1 \neq 0$ then $x_1 \in \mathscr{E}_{M_0}$ and let $T_1 = x|_M$ in Definition 12. This shows that $\mathscr{E}^*$ is admissible.

**Example 19.** An element $e$ of a JB*-triple $X$ is called a tripotent if $\{e, e, e\} = e$. If $X = M \oplus_\infty M^\perp$ then $e = e_1 + e_2$ where $e_1 \in M$ and $e_2 \in M^\perp$. It is known (see for instance [27] or Lemma 47) that $e_1$ is a tripotent in $M$ and $e_2$ is a tripotent in $M^\perp$. Hence the set of all operators of the form $e \square e$ where $e$ is a tripotent form an admissible collection.

**Example 20.** The subclass of operators $e \square e$ (on the JB*-triples $X$) with the dimension of the 1-eigenspace of $e \square e$ at most $n$ (for some fixed $n$) form an admissible class. To see this, with the notation of Example 19, write $e = e_1 + e_2$ and notice that for $x = x_1 + x_2 \in M \oplus_\infty M^\perp = X$, $\{e, e, x\} = \{e_1, e_1, x_1\} + \{e_2, e_2, x_2\}$ with $\{e_1, e_1, x_1\} \in M$, $\{e_2, e_2, x_2\} \in M^\perp$.

**Definition 21.** Let $T: E \to F$ be a continuous linear operator between the Banach spaces $E$ and $F$. Let $(K, (X_k)_{k\in K}, X, \varrho)$ denote the maximal function module representation of $E$. $T$ is an *atomic operator* if $T$ is nonzero and there exists an isolated point $k$ in $K$ such that

$$\varrho(\ker(T)) \supset M_k := \{x \in X;\, x(k) = 0\}.$$

We refer to the point $k$ as the *support* of the atomic operator.

Notice that if $K$ has only one point (which is equivalent to the centralizer $Z(E)$ being one dimensional) then all nonzero operators on $E$ are atomic.

Also, it is easily seen that if $k_1 \neq k_2$ then $M_{k_1} + M_{k_2} = X$ and thus that there exists at most one $k$ with the above property. If $X$ is a dual space then the property is equivalent to the existence of a maximal $M$-summand contained in $\ker T$.

We now give a number of examples of atomic operators.

**Proposition 22.** *If $X$ is a dual space with predual $Y$ and $(K, (X_k)_{k\in K}, \tilde{X}, \varrho)$ is the maximal function module representation of $X$ then the following are equivalent for $k \in K$:*

(i) *$k$ is isolated in $K$,*

(ii) *$M_k$ is an $M$-summand in $\tilde{X}$,*

(iii) *$M_k$ is weak*-closed in $\tilde{X}$,*

(iv) *$M_k$ is not weak*-dense in $\tilde{X}$,*

(v) *there exists a non-zero $y \in Y$ such that*

(*) $\qquad \varrho(\ker(y)) := \{\varrho(x) \in \tilde{X} : x \in X \text{ and } x(y) = 0\} \supset M_k;$

(vi) *$\{y \in Y : \varrho(\ker(y)) \supset M_k\}$ is a (non-zero) minimal $L$-summand in $Y$.*

*Proof.* (i) ⇒ (ii) by [5], Corollary 4.10; (ii) ⇒ (iii) by [5], p. 114; (iii) ⇒ (iv) by [5], p. 120 and (iv) ⇒ (v) by definition; (vi) ⇔ (ii) by [5], p. 114 and maximality of $M_k$. So it is enough to show (v) ⇒ (i).

Suppose that (*) is satisfied by $y$ and $k$. Let $(V_\alpha)_\alpha$ be the set of neighbourhoods of $k$ ordered by set inclusion. For each $\alpha$ choose $\phi_\alpha \in C(K)$ such that $\|\phi_\alpha\|_k \leq 1$, $\phi_\alpha(k) = 1$ and support $(\phi_\alpha) \subset V_\alpha$.

Let $x_0 \in X$ be chosen so that $\varrho(x_0)(k) \neq 0$. The net $(\phi_\alpha \varrho(x_0))_\alpha$ is a bounded net in $\tilde{X}$ and hence contains a weak*-convergent subnet. Let $w$ be a limit point of some such subnet. Since $(\phi_\alpha \varrho(x_0) - \varrho(x_0))(k) = 0$ for all $\alpha$ we have $\varrho^{-1}(\phi_\alpha \varrho(x_0))(y) = x_0(y)$ for all $\alpha$ and hence $\varrho^{-1}(w)(y) = x(y) \neq 0$. By (*) it follows that $w(k) \neq 0$.

If $k \neq l$ then by [5], Corollary 5.10, there exists a clopen neighbourhood of $k$, $V$, which does not contain $l$. The set $W$ of all $x \in X$ such that $\varrho(x)(k') = 0$ for all $k' \notin V$ is an $M$-summand which contains $\phi_\alpha \varrho(x_0)$ for all $\alpha$ sufficiently large. By [5], p. 114, $W$ is weak*-closed and hence $w \in W$. Hence $w(l) = 0$ for all $l \neq k$. By [5], Theorem 5.13, the mapping $k' \in K \to \|w(k')\|$ is continuous and since $k$ is the only point at which $w$ is non-zero it follows that $k$ must be isolated. This completes the proof.



**Proposition 23.** *Let $X$ be the dual of a Banach space $Y$.*

(i) *A non-zero element of $Y$ is an atomic operator on $X$ if and only if it is contained in a minimal $L$-summand of $Y$. In particular, every extreme point of the unit ball of $Y$ is an atomic operator on $X$.*

(ii) *Suppose $T: X \to Z$ is a non-zero bounded operator which is continuous with respect to the weak*-topology on $X$ and some Hausdorff locally convex topology on $Z$. Then $T$ is atomic if and only if, for every decomposition $X = M \oplus_\infty M^\perp$, $\ker T$ contains either $M$ or $M^\perp$.*

*Proof.* (i) Assume that $y \neq 0$ is contained in a minimal $L$-summand $L$ of $Y$. It follows that $L^0$ is a maximal $M$-summand of $X$ (by [5], Theorem 5.7 (i)). Let $(K, (X_k)_{k\in K}, \tilde{X}, \varrho)$ denote the maximal function module representation of $X$. Then $L^0 = \{x \in Z;\, \varrho(x)(k) = 0 \text{ all } k \in H\}$ for some minimal clopen subset $H$ of $K$. Since $K$ is extremally disconnected, minimality of $H$ implies that $H = \{k\}$ is a singleton (and $k$ is isolated). We have $\varrho(\ker(y)) \supset \varrho(L^0) \supset M_k$, which implies that $y$ is atomic.

If $y \neq 0$ and $y$ is atomic, then by Proposition 22 $y$ is contained in a minimal $L$-summand. Finally, if $e \in \mathscr{E}_Y$ then the intersection $E$ of all $L$-summands containing $e$ is an $L$-summand [5], Theorem 1.11 (ii). Moreover $E$ is minimal. Indeed, if $Y = F \oplus_1 G$ then one can use the same reasoning as in Lemma 14 to prove that either $e \in F$ or $e \in G$. This means that either $E \subset F$ or $E \subset G$ and hence $E$ is minimal.

(ii) By Corollary 4.10 in [5], it is clear that an atomic operator $T$ must satisfy the condition. For the converse, let $K_0$ denote the intersection of all clopen subsets $H$ of $K$ satisfying

$$\varrho(\ker T) \supset \{x \in \tilde{X};\, x(k) = 0 \text{ all } k \notin H\}.$$

Every such $H$ is non-empty and the intersection of two of them will have the same property. Thus, it follows from compactness of $K$ that $K_0$ is non-empty. Since $K$ is extremally disconnected, the condition on $T$ implies that $K_0$ must be a singleton $\{k_0\}$. By weak*-continuity of $T$ and Proposition 22, $k_0$ must be an isolated point of $K$.

We now consider Hermitian operators.

**Proposition 24.** *Let $(K, (X_k)_{k\in K}, \tilde{X}, \varrho)$ be the maximal function module representation of $X$, $T$ an isometry of $X$ such that $\|I - T\| < 2$. There exists for all $k$ an isometry of $X_k$, $T_k$, such that*

(*) $\qquad (\varrho \circ T \circ \varrho^{-1})((x(k))_{k\in K}) = (T_k(x(k)))_{k\in K}$

*for all $x \in \tilde{X}$.*

*Proof.* For each $k \in K$ and $x_k \in X_k$ choose $x \in X$ such that $\varrho(x)(k) = x_k$. Define $T_k: X_k \to X_k$ by $T_k(x_k) = (\varrho(Tx))(k)$.

(a) $T_k$ is well defined. If $x, y \in X$ and $\varrho(x)(k) = \varrho(y)(k)$, then $\varrho(x - y) \in M_k$ and hence by Lemma 14 $\varrho(Tx - Ty) \in M_k$ i.e. $(\varrho(Tx))(k) = (\varrho(Ty))(k)$.

(b) $\|T_k\| \leq 1$. Since $T_k$ is obtained by factoring out the kernel from the composition of an isometry and a quotient mapping it follows that each $T_k$ is a continuous linear mapping and that $\|T_k\| \leq 1$.

(c) $T_k$ is an isometry for all $k \in K$. On applying the above construction to $T^{-1}$ it is easily seen that $T_k$ is invertible and that $T_k^{-1} = (T^{-1})_k$. Hence $\|T_k^{-1}\| \leq 1$. This, together with (b), implies that $T_k$ is an isometry.

(d) (*) is valid. Let $\tilde{T}((x(k))_{k\in K}) = (T_k(x(k)))_k$. Since

$$(\varrho(Tx))(k) = T_k(\varrho(x)(k)) = (\tilde{T}(\varrho(x)))(k) \text{ for all } k \in K \; (*) \text{ holds.}$$

This completes the proof.

**Proposition 25.** *If $(K, (X_k)_{k\in K}, \tilde{X}, \varrho)$ is a maximal function module representation of $X$ and $T$ is a Hermitian operator on $X$ then there exists for all $k \in K$ a Hermitian operator on $X_k$, $T_k$, such that*

$$(**) \qquad (\varrho \circ T \circ \varrho^{-1})((x(k))_{k\in K}) = (T_k(x(k)))_{k\in K}.$$

*Proof.* On applying Corollary 15 and the method of the preceding lemma we can define $T_k$ such that (**) holds. Once more we apply the method of Proposition 14 and we find that $(\varrho \circ T^n \circ \varrho^{-1})((x(k))_{k\in K}) = (T_k^n(x(k)))_{k\in K}$ for all positive integers $n$. Hence, for all $t$,

$$(\varrho \circ e^{itT} \circ \varrho^{-1})((x(k))_{k\in K}) = (e^{itT_k}((x(k))))_{k\in K}.$$

Since $e^{itT}$ is an isometry it follows that $e^{itT_k}$ is an isometry for all $k$ and all $t$. Hence $T_k$ is Hermitian. This completes the proof.

The following known proposition characterizes Hermitian projections and Hermitian operators with one-dimensional range (see for example Berkson [6].)

**Proposition 26.** *Let $T: X \to X$ be an operator.*

(a) *If $T$ is Hermitian then $\ker T \cap T(X) = \{0\}$.*

(b) *If $X = E \oplus F$ (where $E$, $F$ are closed subspaces of $X$) then the projection $E \oplus F \to F$ is Hermitian if and only if for each $\lambda \in C$, $|\lambda| = 1$ and $x_1 + x_2 \in E \oplus F$*

$$\|x_1 + \lambda x_2\| = \|x_1 + x_2\|.$$

*In particular all M- and L-projections are Hermitian.*

(c) *If $\dim(X) = 1$ and $T$ is Hermitian then $X = \ker T \oplus T(X)$ and $T$ is a real scalar multiple of the projection $\ker T \oplus T(X) \to T(X)$.*

*Proof.* (a) If $y \in \ker T \cap T(X)$ then $y = T(x)$ for some $x \in X$ and $T^2(x) = 0$. Since $T$ is Hermitian we have $\|x\| = \|x + itT(x)\| \geq \big| \|x\| - |t| \, \|T(x)\| \big|$ for all $t \in R$. Hence $y = T(x) = 0$.



(b) If $T$ is the projection $E \oplus F \to F$ and $x_1 + x_2 \in E \oplus F$ then

$$e^{itT}(x_1 + x_2) = x_1 + x_2 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} T^n(x_1 + x_2) = x_1 + x_2 + x_2(e^{it} - 1)$$
$$= x_1 + e^{it} x_2 \quad \text{for all } t \in R,$$

which yields (b).

(c) By (a), $X = \ker T \oplus T(X)$. Take $x_0 \in T(X) \setminus \{0\}$. Then $T(x_0) = s x_0$ for some $s \in C \setminus \{0\}$. Thus $T^n(x_0) = s^n x_0$ and hence $\|x_0\| = \|e^{iT}(x_0)\| = \|e^{is} x_0\|$. Consequently $|e^{is}| = 1$ which implies that $s \in R \setminus \{0\}$. This means that $T$ is the projection $\ker T \oplus T(X) \to T(X)$ multiplied by $s$.

**Corollary 27.** *If $P$ is a Hermitian projection on $X$ then for every $x_1 + x_2 \in \ker P \oplus P(X)$*

$$\max\{\|x_1\|, \|x_2\|\} \leq \|x_1 + x_2\|.$$

*Proof.* Use convexity of $\lambda \to \|x_1 + \lambda x_2\|$, $\lambda \to \|\lambda x_1 + x_2\|$. (Both functions are equal to $\|x_1 + x_2\|$ on the unit circle.)

**Proposition 28.** *Let $X \in \mathscr{R} \cap \mathscr{E}$ and let $T: X \to X$ be a non-zero operator. Let $(K, (X_k)_{k\in K}, \tilde{X}, \varrho)$ be the maximal function module representation of $X$. The following conditions are equivalent.*

(a) *$T$ commutes with $Z(X)$ and if $X = M \oplus_\infty M^\perp$ then either $M$ or $M^\perp$ is contained in $\ker T$.*

(b) *$T$ commutes with $Z(X)$ and is atomic.*

(c) *There exists an isolated point $k_0 \in K$ and an operator $S: X_{k_0} \to X_{k_0}$ such that for every $x \in X$*

$$(\varrho \circ T \circ \varrho^{-1})(x)(k) = \begin{cases} S(x(k_0)) & \text{if } k = k_0, \\ 0 & \text{if } k \neq k_0. \end{cases}$$

*Proof.* First observe that $T$ commutes with $Z(X)$ if and only if for each $k \in K$ there is an operator $T_k: X_k \to X_k$ such that $(\varrho \circ T \circ \varrho^{-1})(x)(k) = T_k(x(k))$ (see [5], Prop. 4.7 (ii) and Theorems 4.14, 4.16 (ii)). Let $\tilde{T} = \varrho \circ T \circ \varrho^{-1}$. It is clear that (b) and (c) are equivalent and that they imply (a) (see [5], Corollary 4.10). If (a) holds, arguing as in Proposition 23 (ii), we deduce that there is a point $k_0 \in K$ with $M_{k_0} \subset \ker \tilde{T}$. Hence $T_k = 0$ for $k \neq k_0$. If $y = \tilde{T}(x) \neq 0$ then $y(k_0) \neq 0$ but $y(k) = 0$ for all $k \neq k_0$. Since $X \in \mathscr{R}$, the mapping $k \to \|y(k)\|$ is continuous on $K$ and thus $k_0$ must be isolated.

**Proposition 29.** *If $X \in \mathscr{R}$ and $T: X \to X$ is an operator that commutes with $Z(X)$ and has one dimensional range then $T$ is atomic.*

*Proof.* Let $\tilde{T}$ and $T_k$ be as in the proof of Proposition 28. Since $\tilde{T} \neq 0$ there exist $x, y \in \tilde{X}$ and $k_0 \in K$ such that $y = T(x)$ and $y(k_0) \neq 0$. Let $k \in K \setminus \{k_0\}$. Choose $\phi \in C(K)$ such that $\phi(k_0) = 1$ and $\phi(k) = 0$. Then $\tilde{T}(\phi x) = \phi \tilde{T}(x) = \phi y$. Since $(\phi y)(k_0) \neq 0$, $\phi y$ is a non-zero element of the range of $\tilde{T}$. Hence $\lambda \phi y = y$ for some complex number $\lambda$. Since $\phi(k_0) = 1$ and $y(k_0) \neq 0$ we have $\lambda = 1$. Thus $(\phi y)(k) = 0 = y(k)$. Therefore $y = 0$ on the set $K \setminus \{k_0\}$. As $X \in \mathscr{R}$, the mapping $k \to \|y(k)\|$ is continuous on $K$ and thus $k_0$ is isolated.

**Corollary 30.** *If $X \in \mathcal{R}$ and $T: X \to X$ is a Hermitian operator with one-dimensional range then $T$ is atomic.*

*Proof.* By Proposition 25, $T$ commutes with $Z(X)$ and hence is atomic by Proposition 29.

**Remark.** In view of Proposition 28 it is clear that there are atomic operators which do not commute with the centralizer (a simple example is furnished by $X = l_2^\infty$ and $T(z, w) = (z, z)$).

**Corollary 31.** *If $T$ is an $M$- or $L$-projection with one-dimensional range then $T$ is atomic.*

**Proposition 32.** *Let $X \in \mathcal{J}\mathcal{B}\mathcal{W}^*$ and let $e$ be a minimal tripotent in $X$ (i.e. $e \square e$ has one-dimensional eigenspace for the eigenvalue 1). Then $e \square e$ is atomic.*

*Proof.* Since $X$ is a dual Banach space, $X \in \mathcal{R} \cap \mathcal{E}$. We check that condition (a) of Proposition 28 holds for $e \square e$. If $X = M \oplus_\infty M^\perp$ write $e = e_1 + e_2$, $e_1 \in M$, $e_2 \in M^\perp$ as in Example 20. Then $(e \square e) e_1 = e_1$ and $(e \square e) e_2 = e_2$. Thus $e_1$ or $e_2$ is zero and the kernel of $e \square e$ contains either $M$ or $M^\perp$. By Proposition 25 $e \square e$ commutes with $Z(X)$.

If an operator $T$ is atomic and belongs to the admissible class $\mathcal{O}$ we call $T$ an atomic $\mathcal{O}$-operator.

We now prove our decomposition theorem.

**Theorem 33.** *Let $\mathcal{C}$ and $\mathcal{O}$ denote admissible classes of spaces and operators respectively. Suppose $\mathcal{C} \subset \mathcal{R} \cap \mathcal{E}$. If $X \in \mathcal{C}$ then there exists $E, F, (X_i)_{i \in I}$ all belonging to $\mathcal{C}$ and an isometry $\varrho: E \to \varrho(E) \subset l^\infty(\{X_i\}_{i \in I})$ such that*

(i) $X = E \oplus_\infty F$,

(ii) $c_0(\{X_i\}_{i \in I}) \subset \varrho(E)$,

(iii) *each $X_i$ is irreducible (in fact $Z(X_i)$ is one-dimensional) and admits a non-zero (hence atomic) $\mathcal{O}$-operator,*

(iv) *$F$ does not admit an atomic $\mathcal{O}$-operator.*

*Moreover, if $X \in \mathcal{B}^*$ and $Y$ is a predual of $X$ then there exist Banach subspaces of $Y$, $E_1$, $F_1$ and $(Y_i)_{i \in I}$ such that $(E_1)' = E$, $(F_1)' = F$, $Y_i' = X_i$, $Y_i$ does not contain any non trivial L-summands and*

(v) $Y = l^1(\{Y_i\}_{i \in I}) \oplus_1 F_1$,

(vi) $\varrho(E) = l^\infty(\{X_i\}_{i \in I})$.

*Proof.* Let $(K, (X_k)_{k \in K}, \tilde{X}, \varrho)$ denote the maximal function module representation of $X$.

We shall assume $K$ contains more than one point as otherwise the theorem is trivial.

Let $I = \{k \in K; \exists$ an atomic $\mathcal{O}$-operator $T_k$ on $X$ with $\varrho(\ker(T_k)) \supset M_k\}$.

$I$ is a set of isolated points of $K$ and hence $K_1 := \bar{I}$ is clopen. Let $K_2 = K \setminus K_1$. Then $K_2$ is also clopen. Let $E = \{x \in X; \varrho(x)|_{K_2} = 0\}$ and $F = \{x \in X; \varrho(x)|_{K_1} = 0\}$. By [5], Corollary 4.10, $X = E \oplus_\infty F$ and since $\mathcal{C}$ is an admissible class, $E$ and $F$ belong to $\mathcal{C}$.



Now $(K_1, (X_k)_{k \in K_1}, \varrho(E), \varrho|_E)$ and $(K_2, (X_k)_{k \in K_2}, \varrho(F), \varrho|_F)$ are the maximal function module representations of $E$ and $F$ respectively. Suppose $F$ admits an atomic $\mathcal{O}$-operator $T: F \to G_1$. Since $\mathcal{O}$ is an admissible class there exists a Banach space $G_2$ such that the mapping $\tilde{T}: E \oplus_\infty F \to G_2 \oplus_\infty G_1$, $\tilde{T}(x, y) = T(y)$ belongs to $\mathcal{O}$. If $K_2$ consists of a single point $k_2$ then $k_2$ is an isolated point of $K$ and $\ker(\tilde{T}) \supset M_{k_2}$. Hence in this case $\tilde{T}$ is an atomic $\mathcal{O}$-operator.

If $K_2$ contains more than one point then there exists an isolated point of $K_2$, $k_3$; such that $\ker(T) \supset M_{k_3} \cap F$. Since $K_2$ is clopen $k_3$ is also an isolated point of $K$ and $\ker(\tilde{T}) \supset M_{k_3}$. Hence in this case $\tilde{T}$ is also an atomic $\mathcal{O}$-operator.

By our definition of $I$ this would imply that $k_2$ in the first case and $k_3$ in the second case would belong to $I$ and hence to $K_1$. This is a contradiction and hence $F$ does not admit an atomic $\mathcal{O}$-operator. This proves (iv).

Since we are dealing with a maximal function module representation it is clear that each $X_i$ is irreducible (in fact $Z(X_i)$ is one-dimensional). Since $\mathcal{O}$ is an admissible class and $\varrho(X) = X_i \oplus_\infty M_i$ for all $i \in I$ there exists for each $i \in I$ a Banach space $Z_i$, an isometric embedding $\theta_i$ of $Z_i$ into the range of $T_i$, $W_i$, and $\tilde{T}_i: X_i \to Z_i$ such that the following diagram commutes

$$\begin{array}{ccc} X_i & \xrightarrow{\tilde{T}_i} & Z_i \\ \big\uparrow{\scriptstyle \varrho^{-1}} & & \big\downarrow{\scriptstyle \theta_i} \\ X & \xrightarrow{T_i} & W_i \end{array}$$

Since $T_i \neq 0$ and $T_i|_{\varrho^{-1}(M_i)} = 0$ it follows that $\tilde{T}_i$ is non-zero. Since $Z(X_i)$ is one dimensional it follows that $\tilde{T}_i$ is an $\mathcal{O}$-operator. This proves (iii).

Now consider the mapping

$$\Phi: E \to l^\infty(\{X_i\}_{i \in I}),$$
$$x \to (\varrho(x)(k))_{k \in I}.$$

$\Phi$ is linear and since $I$ is dense in $K_1$

$$(*) \qquad \|(x(k))_{k \in K_1}\| = \sup_{k \in K_1} \|x(k)\| = \sup_{k \in I} \|x(k)\|$$

and we have that $\Phi$ is an isometry onto its range.

By the definition of a function module, if $I_1$ is a finite subset of $I$, $x_i \in X_i$ for $i \in I_1$, then there exists $x \in X$ such that

$$(**) \qquad \varrho(x)(k) = \begin{cases} x_k & \text{if } k \in I_1, \\ 0 & \text{otherwise.} \end{cases}$$

Using (*) and (**) and the fact that $\Phi(E)$ is complete it follows that (ii) holds.

For the remainder of the proof we suppose that $X$ is a dual space with predual $Y$.

By [5], Propositions 5.6 and 5.7, if $E_1=\{y\in Y; x(y)=0$ for all $x\in F\}$ and $F_1=\{y\in Y; x(y)=0$ for all $x\in E\}$ then $(E_1)'=X/E_1^0=E$, $(F_1)'=X/F_1^0=F$ and $Y=E_1\oplus_1 F_1$.

For $i\in I$ we have $\tilde{X}=X_i\oplus_\infty M_i$.

Let

$Y_i=\{y\in Y; x(y)=0$ for all $x\in\varrho^{-1}(M_i)\}$ and $Z_i=\{y\in Y; x(y)=0$ for all $x\in\varrho^{-1}(X_i)\}$.

Then $Y_i'=\varrho^{-1}(X_i)\cong X_i$, $Z_i'=\varrho^{-1}(M_i)$ and $Y_i\oplus_1 Z_i=Y$ for all $i$. We now show that the mapping $\Phi$ is surjective.

Let $x=(x_k)_{k\in I}\in l^\infty(\{X_i\}_{i\in I})$. For each finite subset $J$ of $I$ we use (**) to obtain $x_J\in X$ such that

$$\varrho(x_J)(k)=\begin{cases}x_k & \text{if } k\in J,\\ 0 & \text{otherwise.}\end{cases}$$

This implies that $\|x_J\|=\|\varrho(x_J)\|=\sup_{k\in J}\|x_k\|\leq\|x\|$. We order the finite subsets of $I$ by set inclusion. Then $(x_J)_J$ is a bounded net in $X$ and hence contains a $\sigma(X,Y)$ convergent subnet. Let $w$ be a limit point of some such convergent subnet. Since $x_J\in E$ for all $J$ and $E$ is a weak*-closed $M$-summand it follows that $w\in E$. Let $k_0$ be a fixed point in $K$. Then $(x_{J'}-x_{\{k_0\}})_{J'}$ is weak*-convergent if $(x_{J'})_{J'}$ is weak*-convergent. Since $\varrho(x_J-x_{\{k_0\}})(k_0)=0$ for all $J$ which contain $k_0$ and since $\varrho^{-1}(M_{k_0})$ is weak*-closed it follows that $\varrho(w-x_{\{k_0\}})(k_0)=0$. Hence $\varrho(w)(k_0)=\varrho(x_{\{k_0\}})(k_0)=x_{k_0}$.

Since $k_0$ was arbitrary it follows that $\Phi(w)=(x_k)_{k\in I}$ and hence $\Phi$ is surjective. Hence we have proved (vi).

To complete the proof it suffices to show

$$E_1=l^1(\{Y_i\}_{i\in I})$$

If $\{i_1,\dots,i_s\}$ is a finite subset of $I$ and $z_{i_j}\in Y_{i_j}$ for $j=1,\dots,s$ then

$$\left\|\sum_1^s z_j\right\|=\sup_{x\in X,\|x\|\leq 1}\left|\sum_{j=1}^s x(z_j)\right|=\sum_{j=1}^s\|z_j\|.$$

Hence $W_1:=l^1(\{Y_i\}_{i\in I})$ is a closed subspace of $Y$.

Now

$$W_1\subset\{y\in Y; x(y)=0 \text{ for all } x\in X \text{ such that } \varrho(x)\in M_i \text{ for all } i\in I\}$$
$$=\{y\in E_1; x(y)=0 \text{ for all } x\in F\}=E_1.$$

Suppose $l^1(\{Y_i\}_{i\in I})\neq E_1$. Then there exists a non-zero $x\in E$ such that $x(W_1)=0$. Since $x\neq 0$ there exists $i\in I$ such that $\varrho(x)(i)\neq 0$. Hence there exists $y\in Y_i$ such that $x(y)\neq 0$. This is a contradiction and completes the proof.

We shall write the spaces $E$ and $F$ occurring in Theorem 33 as $A_\Theta$ and $N_\Theta$ and call them the atomic and nonatomic $\Theta$ subspace of $X$ respectively.



We now return to our original question i.e. when can we write a Banach space as a product of irreducible Banach spaces? We see immediately, either by inspection of the function module construction or by considering the admissible class of operators consisting of all linear operators, that any dual Banach space $X$ can be written as

$$l^\infty(\{X_i\}_{i\in I})\oplus N$$

where each $X_i$ is an irreducible weak*-closed subspace of $X$ and where $N$ is also weak*-closed and contains no minimal $M$-ideals and that any predual of $X$, $Y$, has the form

$$l^1(\{Y_i\}_{i\in I})\oplus_1 Z$$

where each $Y_i$ contains no non-trivial $L$-summands, $Z$ contains no maximal $L$-summands, $Y_i'=X_i$ and $Z'=N$.

Hence we are interested in identifying situations and admissible classes of operators $\Theta$ such that $N_\Theta=\{0\}$. This motivates the following definition.

**Definition 34:** If $X$ is a Banach space and $\Theta$ is an admissible class of operators then $\Theta$ is *X-determining* if

$$\bigcap\{\ker(T);\ T \text{ an atomic operator with domain } X \text{ and } T\in\Theta\}=\{0\}.$$

The following is easily proved.

**Proposition 35.** *If $X\in\mathcal{R}\cap\mathcal{E}$ and $\Theta$ is $X$-determining then $N_\Theta=\{0\}$.*

It is well known ([9]) that the closed unit ball of a Banach space with RNP is the closed convex hull of its extreme points and hence the following is immediate from Theorem 33.

**Example 36.** If a Banach space $X$ has RNP then $X=l^1(\{X_i\}_{i\in I})$ where each $X_i$ has RNP and contains no non-trivial $L$-summands. Moreover, $X'=l^\infty(\{Y_i\}_{i\in I})$ where each $Y_i$ is an irreducible dual Banach space.

**Example 37.** If $X$ is a Banach space then $X$ is isometrically isomorphic to a weak*-dense subspace of $l^\infty(\{X_i\}_{i\in I})$ where each $X_i$ is irreducible and a dual Banach space.

*Proof.* Let $l^\infty(\{X_i\}_{i\in I})\oplus_\infty N$ denote the decomposition of $X''$ arising from the extreme points of $B_{X'}$.

Let $\mathscr{C}$ be the class of all finite $l_\infty$ direkt sums

$$M_1\oplus_\infty M_2\oplus_\infty\cdots\oplus_\infty M_n$$

with $M_j$ an $M$-summand of $X''$. Let $\Theta$ be the class of operators of the form

$$(x_1,x_2,\dots,x_n)\mapsto x_j(\phi): M_1\oplus_\infty M_2\oplus_\infty\cdots\oplus_\infty M_n\to\mathbb{C}$$

with $\phi\in\mathscr{E}_{X'}$ or $\phi=0$, $1\leq j\leq n$. Let $J: X\to X''$ denote the canonical embedding. If $x\in X$ and $\phi(x)=0$ for all $\phi\in\mathscr{E}_{X'}$ then $x$ and hence $Jx$ are zero. Hence $PJ$ embeds $X$ as a closed subspace of $l^\infty(\{X_i\}_{i\in I})$, where $P$ is the $M$-projection of $X''$ with kernel $N$. $PJ(X)$ is weak*-dense in $l^\infty(\{X_i\}_{i\in I})$ since $P$ is weak*-continuous and $JX$ is weak*-dense in $X''$. Since $\|x\|=\sup\{|\phi(x)|;\ \phi\in\mathscr{E}_{X'}\}$ and for each $\phi\in\mathscr{E}_{X'}$ we can find $i\in I$ with $M_i\subset\ker\phi$, it follows that $PJ$ is an isometry.

**Example 38.** If $X$ is a JBW*-triple then $X \cong l^\infty(\{X_i\}_{i\in I}) \oplus N$ where each $X_i$ is an irreducible JBW*-triple which admits a minimal tripotent and where $N$ does not admit a minimal tripotent. To see this take $\mathscr{C}$ to be the class of operators $e\,\square\, e$ with $e$ a minimal tripotent or zero (see Example 20 and Proposition 32). This example is part of a much stronger result due to Friedman and Russo [13].

**Remarks.** Let $X$ be a JBW*-triple with predual $Y$. By a result of Friedman and Russo [13], Proposition 4, there is a one-to-one correspondence between "atoms" of $Y$ (i.e. extreme points of $B_Y$) and minimal tripotents of $X$. If $f$ is an atom of $Y$, the minimal tripotent $e$ associated with $f$ satisfies $e(f)=1$. This implies that the decomposition of $X$ in Example 38 is, in fact, identical to the decomposition resulting from the extreme points of $B_Y$. To see this let $(K, (X_k)_{k\in K}, \tilde{X}, \varrho)$ denote the maximal function module representation of $X$. By Proposition 23 all extreme points of $B_Y$ are atomic. If $e$ is a minimal tripotent, $e\,\square\, e$ is atomic by Proposition 32. If $M_{k_0} \subset \varrho(\ker(e\,\square\, e))$ ($k_0 \in K$ isolated) then $e=\{e,e,e\}=(e\,\square\, e)(e)$ must satisfy $\varrho(e)(k)=0$ for all $k \neq k_0$ (see Propositions 32 and 25). If $f$ is the corresponding extreme point of the unit ball of $Y$ then $e(f)\neq 0$ implies that $\varrho(\ker f)$ must contain $M_{k_0}$ (for the same $k_0$).

If we use Example 37, the fact that the second dual of a JB*-triple is again a JB*-triple [10], [11] and the relationship between extreme points of the predual and minimal tripotents [13] we see that any JB*-triple system can be embedded in an $l^\infty$ product of irreducible JBW*-triples each of which admits a minimal tripotent ([14]).

Now Horn [19] has classified all irreducible JBW*-triples which admit minimal tripotents and using the above examples and this classification we get immediately:

(a) The Gelfand-Naimark theorem for JB*-triples ([14]).

(b) The classification of preduals of JBW*-triples having RNP ([3], [7]).

Finally we establish uniqueness of the decomposition (a more general result is given in § 3).

**Proposition 39.** *If $X \cong l^\infty(\{X_i\}_{i\in I}) \cong l^\infty(\{Y_j\}_{j\in J})$ where each $X_i$ and $Y_j$ is irreducible (isometrically) then $|I|=|J|$ and there exists a bijective mapping $\sigma: I \to J$ such that*

$$X_i \cong Y_{\sigma(i)}.$$

*Proof.* If $\phi: l^\infty(\{X_i\}_{i\in I}) \to l^\infty(\{Y_j\}_{j\in J})$ is an isometric isomorphism then $\phi$ maps minimal $M$-summands onto minimal $M$-summands. Hence $\forall i\in I\,\exists$ a unique $j$, $\sigma(i)$, such that $\phi(X_i)=Y_{\sigma(i)}$. This completes the proof.

Using example 37 we find another situation in which a Banach space can "more or less" be written as a product of irreducible domains.

**Proposition 40.** *If $X$ is an $M$-ideal in $X''$ then $X=c_0(\{X_i\}_{i\in I})$ where each $X_i$ contains only trivial $M$-ideals and $X_i''$ is irreducible and, moreover, $X''\cong l^\infty(\{X_i''\}_{i\in I})$.*

*Proof.* Let $(K, (Z_k)_{k\in K}, \tilde{Z}, \varrho)$ be the maximal function module representation of $X''$, let $J: X\to X''$ be the canonical embedding and let $I$ denote the set of isolated points in $K$. By [5], p. 86, we can identify $J(X)$ with $(K, (X_k)_{k\in K}, \widetilde{JX}, \varrho)$ where each $X_k$ is an $M$-ideal in $Z_k$ and, moreover, $\widetilde{JX}$ is a $C(K)$-module.



By Example 37 and the $C(K)$-module property we see that $X_k=0$ for $\forall k\notin I$. If $\phi\in X'$ and $\phi(z)=0$ for all $z\in\tilde{Z}$ such that $z(k)=0$ for all $k\in I$, then $\phi(JX)=0$ and hence $\phi\equiv 0$. Hence $\bar{I}=K$, $\tilde{Z}\cong l^\infty((Z_i)_{i\in I})$. Hence there exists, for each $i$, preduals of $Z_i$, $Y_i$, such that $X'=l^1(\{Y_i\}_{i\in I})$ and the $(X', X'')$ duality is given by

$$(*)\qquad \langle (z(k))_{k\in K}, (y_i)_{i\in I}\rangle = \sum_I \langle z(i), y_i\rangle.$$

We claim that $X_k=\{0\}$ for all $k\in K\setminus I$. Suppose otherwise. Then there exists $k\in K\setminus I$ and $\phi_k\neq 0\in X_k'$. Let $\tilde{\phi}_k((x(k))_{k\in K})=\phi_k(x(k))$ for all $(x(k))_{k\in K}\in X''$. Since $\tilde{\phi}_k(Z_i)=0$ for all $i\in I$ it follows by $(*)$ that $\tilde{\phi}_k=0$. This is a contradiction and shows $X_k=0$ for all $k\notin I$.

By the upper semicontinuity of the mapping $k\to\|x(k)\|$ it follows that for every $\varepsilon>0$ there exists a finite subset $I_1$ of $I$ such that $\|x(k)\|\leq\varepsilon$ if $k\in I\setminus I_1$ i.e. $JX\subset c_0(\{X_i\})$. Since $I$ is discrete, $JX\supset c_0(\{X_i\}_{i\in I})$ and hence we have shown $JX=c_0(\{X_i\}_{i\in I})$. It now easily follows that $X_i''=Z_i$. Since $Z_i$ is irreducible it follows that $X_i$ is also irreducible and by [16], Theorem 4.1, only contains trivial $M$-ideals. By [16], Theorem 4.4, $X_i$ is the unique minimal $M$-ideal in $X_i''$ for all $i$. This completes the proof.

**Remark.** Proposition 40 could also be proved in a less elementary but shorter fashion by using Example 36 and [24], Theorem 2.6, which states that if $X$ is an $M$-ideal in $X''$ then $X'$ has RNP.

**Corollary 41** ([23]). *If $X'\cong L^1(\mu)$ then $X$ is an $M$-ideal in $X''$ if and only if $X\cong c_0(I)$.*

*Proof.* In this case the $X_k$'s are all one-dimensional and the result follows immediately.

**Definition 42.** By a *one-unconditional finite dimensional decomposition* (1-UFDD) of a Banach space $X$ we mean a sequence $P_j: X\to X$ of finite rank *mutually orthogonal* (i.e. $P_iP_j=0$ if $i\neq j$) projections on $X$ satisfying

(i) $\forall x\in X,\quad x=\sum_{j=1}^{\infty} P_j x,$

(ii) if $x\in X$ and $|\lambda_j|=1\quad\forall j$, then $\sum_{j=1}^{\infty}\lambda_j P_j x\in X$ and $\left\|\sum_{j=1}^{\infty}\lambda_j P_j x\right\|=\|x\|$.

**Remarks.** Notice that the projections $P_j$ in a 1-UFDD must be Hermitian (by Proposition 36(b)) and hence $\|P_j\|\leq 1$ (Corollary 37). Conversely it is easy to check that a sequence $\{P_n\}_{n=1}^{\infty}$ of mutually orthogonal finite rank Hermitian projections on $X$ is a 1-UFDD if and only if

(i)' $X$ is the closed linear span of $\bigcup_{n=1}^{\infty} P_n(X)$.

**Lemma 43.** *Let $P$ be a Hermitian projection on a Banach space $X$ and $Q$ an $M$-projection on $X$. Then $PQ$ is a Hermitian projection on $X$ with range $P(X)\cap Q(X)$ (and $PQ=QP$).*

*Proof.* Since $Q$ is an $M$-projection, $Q \in Z(X)$ (the centralizer) and thus $PQ = QP$ (see Proposition 25 and the beginning of the proof of Proposition 28). From Proposition 26(b), it is clear that the restriction

$$P|_{Q(X)} = PQ|_{Q(X)} : Q(X) \to PQ(X)$$

is a Hermitian projection. Again using Proposition 26(b) it is easy to see that a Hermitian projection $R : M \to M$ on an $M$-summand $M$ of $X$ gives rise to a Hermitian projection $\tilde{R}$ on $X$ by taking $\tilde{R} = R$ on $M$ and $\tilde{R} = 0$ on the complementary $M$-summand $M^{\perp}$ to $M$. Taking $R$ to be the restriction of $PQ$ to $Q(X) = M$ yields

$$\tilde{R} = PQ.$$

**Lemma 44.** *Let $X$ be a dual Banach space (or more generally $X \in \mathscr{E} \cap \mathscr{R}$) and $P$ be a non-zero finite rank Hermitian projection on $X$. Then $P$ is a finite sum $P_1 + P_2 + \cdots + P_n$ of mutually orthogonal atomic Hermitian projections.*

*Proof.* Let $(K, (X_k)_{k \in K}, \tilde{X}, \varrho)$ denote the maximal function module representation of $X$. We replace $P$ by the equivalent Hermitian projection $\tilde{P} = \varrho P \varrho^{-1}$ and prove the result for $\tilde{P}$. By Proposition 25 there are Hermitian operators $P_k : X_k \to X_k$ so that

$$\tilde{P}((x(k))_{k \in K}) = (P_k x(k))_{k \in K}.$$

Let $S = \{k \in K : P_k \neq 0\}$. We claim first of all that $S$ has at most $N = \operatorname{rank}(P)$ elements.

If $S$ has $n = N + 1$ or more elements, we can find a partition $K = K_1 \cup K_2 \cup \cdots \cup K_n$ of $K$ into $n$ disjoint clopen subsets with $K_j \cap S$ non-empty for $1 \leqq j \leqq n$. Let $Q_j : \tilde{X} \to \tilde{X}$ be the multiplication operator by the characteristic function of $K_j$. Clearly $Q_j$ is an $M$-projection. Now

$$\tilde{P} = \tilde{P}Q_1 + \tilde{P}Q_2 + \cdots + \tilde{P}Q_n$$

and (by Lemma 43) $P_j = \tilde{P}Q_j$ is a Hermitian projection ($1 \leqq j \leqq n$). Also $P_1, P_2, \ldots, P_n$ are mutually orthogonal and (because of the definition of $S$ and that of a function module) each $P_j$ is non-zero. Thus the rank of $P$ must be at least $n$, a contradiction.

If now $S = \{s_1, s_2, \ldots, s_n\}$ (by repeating the preceding arguments) we can write $\tilde{P}$ as a sum $P_1 + P_2 + \cdots + P_n$ of commuting non-zero Hermitian projections satisfying

$$(P_j x)(k) = 0 \quad \forall k \neq s_j, \ \forall x \in \tilde{X}.$$

Since $X \in \mathscr{R}$, it follows that each of the points $s_j$ must be isolated in $K$ and thus each $P_j$ is atomic.

**Lemma 45.** *If a Banach space $X$ has a 1-UFDD, then it has a 1-UFDD $P_1, P_2, P_3, \ldots$ with $P_j''$ an atomic (Hermitian finite rank) projection on $X''$ for each $j \geqq 1$.*



*Proof.* Suppose $P_1, P_2, \ldots$ is any 1-UFDD of $X$. Then for each $j$

$$P_j'' : X'' \to X''$$

is a Hermitian finite rank projection on $X''$. As in Lemma 44 we write $P_j''$ as a sum

$$P_j'' = Q_{j1} + Q_{j2} + \cdots + Q_{jn}$$

of $n = n(j)$ mutually orthogonal atomic Hermitian projections. Clearly

$$P_j'' Q_{ji} = Q_{ji} = Q_{ji} P_j''.$$

Notice that, if we consider $X$ as contained in $X''$, then the restriction of $P_j''$ to $X$ is $P_j$ and the range of $P_i''$ is the range of $P_j$. Thus

$$Q_{ji}(X'') \subset P_j''(X'') = P_j(X) \subset X.$$

Let $R_{ji}$ denote the restriction of $Q_{ji}$ to $X$. By Proposition 26(b), each $R_{ji}$ is a Hermitian projection on $X$ and $P_j = R_{j1} + R_{j2} + \cdots + R_{jn}$ is a sum of mutually orthogonal projections with $P_j R_{ji} = R_{ji} = R_{ji} P_j (1 \leqq i \leqq n = n(j))$. This last identity and the orthogonality of the projections $P_1, P_2, \ldots$ implies orthogonality of $R_{ji}$ and $R_{lm}$ if $j \neq l$.

Clearly the ranges of the $R_{ji}$ have the same linear span as the ranges of the $P_j$ and thus the $R_{ji}$ (in any order) give a 1-UFDD of $X$.

Finally $R_{ji}'' = Q_{ji}$ (and is therefore atomic). To see this, observe that $P_j''$ is weak*-continuous on $X''$ and $M$-projections on $X''$ are weak*-to weak*-continuous. A review of the proof of Lemma 44 shows that the $Q_{ji}$ are weak*-continuous. Since $R_{ji}''$ is weak*-continuous and agrees with $Q_{ji}$ on $X$, it follows that $R_{ji}'' = Q_{ji}$.

**Theorem 46.** *Suppose a Banach space $X$ has a 1-UFDD. Then $X$ is a countable $c_0$ sum $c_0(\{X_i\}_{i \in I})$ of irreducible Banach spaces $X_i$ each of which have a 1-UFDD and have $Z(X_i'')$ one-dimensional.*

*Proof.* By Lemma 45 $X$ has a 1-UFDD $P_1, P_2, P_3, \ldots$ such that $P_j''$ is atomic for each $j$. Let $(K, (Y_k)_{k \in K}, Y, \varrho)$ denote the maximal function module representation of $X''$. Let

$$I = \{k \in K : k \text{ is the support of some } P_j''\}.$$

We claim first that $I$ is dense in $K$. Notice that the closure $\bar{I}$ of $I$ in $K$ is clopen (since $I$ consists of isolated points) and thus yields an $M$-decomposition $X'' = M \oplus_\infty M^{\perp}$ where

$$M = \{x \in X'' : \varrho(x)(k) = 0 \quad \forall k \notin \bar{I}\}.$$

If $Q$ is the $M$-projection of $X''$ onto $M$ then we have (Lemma 42) $QP_j'' = P_j''Q$. Since the support of $P_j''$ is a point $i \in I$ we have $QP_j'' = 0$ and the range of $P_j''$ is contained in $M^{\perp} (\forall j)$.

Now $X$ (which we consider as being contained in $X''$) is the closed linear span of the subspaces $P_j(X) = P_j''(X)$ which are each contained in $M^\perp$. Thus $X \subset M^\perp$. But $M^\perp$ is weak*-closed (see [5], p. 114) and $X$ is weak*-dense in $X''$. It follows that $M^\perp = X''$, $M = 0$ and $\bar{I} = K$, as claimed.

We now define $X_i$ (for $i \in I$) to be the closed linear span of those $P_j(X)$ such that the support of $P_j''$ is $i$.

Notice that, since each $i \in I$ is isolated,

$$\{x \in X'' : \varrho(x)(i) = 0\}$$

is an $M$-summand of $X''$ and its complementary $M$-summand can be naturally identified with $Y_i$ (see the definition of a function module). Let $Q_i$ be the $M$-projection of $X''$ onto $Y_i$. As in the above proof that $\bar{I} = K$, we can show that, if $P_j''$ has support $i$, then $(1 - Q_i)P_j'' = 0$, $Q_i P_j'' = P_j''$ and $P_j''(X'') \subset Y_i$. Similarly, if $i$ is not the support of $P_j''$,

$$Q_i P_j'' = 0.$$

We now claim that $X_i = X \cap Y_i$ and that $Y_i$ can be naturally identified with $(X \cap Y_i)''$.

As we have just observed, $Q_i P_j'' = P_j''$ if the support of $P_j''$ is $i$. Hence the range of $Q_i$ contains $P_j''(X'') = P_j(X)$ for all such $j$. Since $X_i$ is defined to be the closed span of all such $P_j(X)$ we must have $X_i \subset Q_i(X'') = Y_i$. Consequently $X_i \subset X \cap Y_i$.

Since $Q_i P_j'' = 0$ if the support of $P_j''$ is not $i$, we have $Q_i(P_j''(X'')) = Q_i(P_j(X)) \subset X_i$ for all $j$. It follows that $Q_i(X) \subset X_i \subset X$. Hence $Q_i(X) = X \cap Y_i \subset X_i$. Consequently

$$X_i = X \cap Y_i.$$

If we now use the facts that the $M$-projection $Q_i$ is weak*-weak*-continuous, $X$ is weak*-dense in $X''$, $Q_i(X) \subset X_i \subset X_i$ and $Y_i$ is weak*-closed we see that $Q_i(X) = X \cap Y_i = X_i$ is weak*-dense in $Y_i = Q_i(X'')$. Hence the double dual of $X_i$ can be naturally identified with $Y_i$. This proves the second claim.

Now it is easy to check, since the $Q_i$ are mutually orthogonal $M$-projections, that if $y_{i_r} \in Y_{i_r}$ for $1 \leqq r \leqq n$ and $i_1, i_2, \ldots, i_n$ are distinct elements of $I$, then

$$\|y_{i_1} + y_{i_2} + \cdots + y_{i_n}\| = \max \|y_{i_r}\|.$$

Consequently if $x_{i_r} \in X_{i_r} = Y_{i_r} \cap X$ we have

$$\|x_{i_1} + x_{i_2} + \cdots + x_{i_n}\| = \max \|x_{i_r}\|.$$

From this and the fact that the $X_i (i \in I)$ have dense span in $X$, it follows easily that

$$X = c_0(\{X_i\}_{i \in I}).$$

Finally $X_i'' = Y_i$ has one-dimensional centralizer since $i$ is isolated in $K$. This implies irreducibility of the $X_i$ (Proposition 5).



**Remark.** (i) If we apply Theorem 46 to a JB*-triple $X$ with a 1-UFDD we see that $X$ must be a $c_0$ sum of JB*-triples $X_1, X_2, \ldots$ such that $X_n$ has a 1-UFDD and $X_n''$ is a JBW*-factor (see [19]) for each $n$. This is less than the classification of JB*-triples with 1-UFDD in [2].

(ii) The conclusion (with obvious modifications) of Theorem 46 would still be valid if we assumed only that $X$ has one-unconditional decomposition into reflexive (instead of finite dimensional) subspaces. The same proof works if the fact that reflexive spaces are $M$-finite (i.e. have finite dimensional centralizer) is used.

**§ 3.** In this section we describe the biholomorphic automorphisms of the open unit ball of a Banach space by means of its function module representations. If $A$ is a homogeneous polynomial on $X$ we let $\bar{A}$ denote the associated symmetric $n$-linear form i.e. $\bar{A}(x, x, \ldots, x) = A(x)$ for all $x \in X$. If $A$ is of degree 2 then

$$(*) \qquad \bar{A}(x, y) = \frac{1}{2}[A(x + y) - A(x) - A(y)].$$

If $\xi + p_\xi \in V(B_X)$, $X$ a Banach space and $p_\xi$ a polynomial of degree 2, then the mapping

$$x'' \in B_{X''} \to \xi + \text{weak*-lim}\, p_\xi(x_\alpha)$$

where $(x_\alpha)_\alpha \in J(B_X)$, $x_\alpha \to x''$ in the weak*-topology as $\alpha \to \infty$, is well defined and belongs to $V(B_{X''})$ ([10], [11]).

**Lemma 47.** *If $M$ is a closed $M$-ideal in $X$ and $\xi + p_\xi \in V(B_X)$ then $\bar{p}_\xi(m, x) \in M$ for all $m \in M$ and all $x \in X$.*

**Remark.** This result is immediate if $X$ is a JB*-triple since in that case the $M$-ideals and the JB*-ideals coincide ([4]).

*Proof.* We first suppose that $M$ is an $M$-summand in $X$. Let $\xi = \xi_1 + \xi_2 \in M \oplus_\infty M^\perp$. By the Kaup-Stacho contraction principle [21], [25]

$$\xi_1 + \pi_1 \circ p_\xi|_M \in V(B_M) \quad \text{and} \quad \xi_2 + \pi_2 \circ p_\xi|_{M^\perp} \in V(B_{M^\perp})$$

where $\pi_1$ and $\pi_2$ denote the canonical projections of $X$ onto $M$ and $M^\perp$ respectively. A simple examination of the definition shows that

$$\xi_1 + \xi_2 + \pi_1 p_\xi \pi_1 + \pi_2 p_\xi \pi_2 \in V(B_X)$$

and hence by the uniqueness of complete holomorphic vector fields with the same constant term we have $\pi_1 p_\xi \pi_1 + \pi_2 p_\xi \pi_2 = p_\xi$. Hence if $x \in M$ then $p_\xi(x) = \pi_1 p_\xi(x)$ and $p_\xi(M) \subset M$. By $(*)$ $\bar{p}_\xi(M, M) \subset M$.

If $x_1, x_2 \in M$ and $y_2 \in M^\perp$ then

$$(**) \qquad \bar{p}_\xi(x_1, x_2 + y_2) = \bar{p}_\xi(x_1, x_2) + \bar{p}_\xi(x_1, y_2).$$

Now

$$\tilde{p}_\xi(x_1, y_2) = \frac{1}{2}[p_\xi(x_1+y_2) - p_\xi(x_1) - p_\xi(y_2)]$$
$$= \frac{1}{2}[\pi_1 p_\xi(x_1) + \pi_2 p_\xi(y_2) - \pi_1 p_\xi(x_1) - \pi_2 p_\xi(y_2)] = 0$$

and hence by (**) $\tilde{p}_\xi(x_1, x_2 + y_2) \subset M$. This completes the proof when $M$ is an $M$-summand.

If $M$ is an $M$-ideal in $X$ then $M^{00}$ is an $M$-summand in $X''$. Hence, by the above,

$$\tilde{p}_\xi(M^{00}, X'') \subset M^{00}$$

where we also denote by $\tilde{p}_\xi$ the extension of $\tilde{p}_\xi$ to $X''$. Since $\tilde{p}_\xi(M, X) \subset X$ and $M^{00} \cap JX = M$ this completes the proof.

The main result of this section is the following

**Theorem 48.** *Let $S$ be a completely regular topological space and let $(X_s)_{s\in S}$ be a collection of Banach spaces indexed by the points of $S$. Let $X$ be a closed subspace of $l^\infty(\{X_s\}_{s\in S})$. If $X$ is a $C_b(S)$-module such that for each $x \in X$ the mapping $s \to \|x(s)\|$ is upper semicontinuous and for each $s$ the mapping $x \to x(s)$ is surjective then the following hold.*

(1) *For each $V \in V(B_X)$ and each $s \in S$ there exists $V_s \in V(B_{X_s})$ such that*

$$V(x)(s) = V_s(x(s)), \quad x \in B_X.$$

(2) *For each $\psi$ in the connected component of the identity of $G(B_X)$ and for each $s \in S$ there exists $\theta_s \in G(B_{X_s})$ such that*

$$\psi(x)(s) = \theta_s(x(s)), \quad x \in B_X.$$

In order to prove this theorem we need the following auxiliary result.

**Lemma 49.** *Let $S, (X_s)_{s\in S}, X$ be as in the theorem. If $s_0 \in S$ then*

$$M_{s_0} = \{x \in X : x(s_0) = 0\}$$

*is an $M$-ideal in $X$.*

*Proof.* We may proceed as in the proof of Proposition 13.6 in [15]. It is enough to check that $M_{s_0}$ has the 3-ball property (see [5], [15]). Let $B_j$ denote the open ball in $X$ with centre $y_j$ and radius $\varepsilon_j$, $j = 1, 2, 3$ and let $x \in B_1 \cap B_2 \cap B_3$. Suppose $x_j \in M_{s_0} \cap B_j$ for $j = 1, 2, 3$. Take a positive number $\varepsilon$ such that $\varepsilon < \varepsilon_j - \max\{\|x_j - y_j\|, \|x - y_j\|\}$, $j = 1, 2, 3$. Define $U = \{s \in S : \|x_j(s)\| < \varepsilon/2, j = 1, 2, 3\}$ and choose $f \in C(S, [0, 1])$ such that $f(s_0) = 0$ and $f|S\setminus U \equiv 1$. We claim that $f(x) \in B_1 \cap B_2 \cap B_3 \cap M_{s_0}$. Note that $f(x) \in M_{s_0}$ and $\|f(s)x(s) - y_j(s)\| \leq \|x - y_j\| < \varepsilon_j$ if $s \in S\setminus U$. If $s \in U$ then by the triangle inequality



$$\|f(s)x(s) - y_j(s)\| \leq f(s)\|x(s) - y_j(s)\| + (1 - f(s))(\|y_j(s) - x_j(s)\| + \|x_j(s)\|)$$
$$\leq f(s)(\varepsilon_j - \varepsilon) + (1 - f(s))(\varepsilon_j - \varepsilon + \varepsilon/2) < \varepsilon_j.$$

This completes the proof.

*Proof of Theorem* 48. We have $V = \xi + il + p_\xi$ where $l \in h(X) \subset V(B_X)$ and $\xi + p_\xi \in V(B_X)$. In view of Lemma 49, the required representation for $l$ can be obtained by the method used in Proposition 25. Hence we may assume $V = \xi + p_\xi$.

If $x, y \in X$ and $x(s) = y(s)$ then $x - y \in M_s$, and hence Lemmata 47 and 49 imply that

$$(***) \qquad p_\xi(x) - p_\xi(y) = 2\tilde{p}_\xi(y, x - y) + p_\xi(x - y) \in M_s.$$

If $x_s \in X_s$ and $x \in X$ is chosen so that $x(s) = x_s$ then we define $V_s(x_s)$ by $V(x)(s)$. By (***) $V_s$ is well defined and

$$V(x)(s) = V_s(x(s)), \quad x \in B_X.$$

It is obvious that $V_s$ is a holomorphic vector field. If $\phi_x$ is an integral curve to $V$ with initial point $x$ then for each $s \in S$ define $\phi_{x,s}(t) = \phi_x(t)(s)$, $t \in R$. Obviously each $\phi_{x,s}$ is differentiable. Moreover,

$$\phi'_{x,s}(t) = \phi'_x(t)(s) = V(\phi_x(t))(s) = V_s(\phi_{x,s}(t)).$$

Hence $V_s \in V(B_{X_s})$ and this completes the proof of the first conclusion of the theorem.

To prove the second part we can use the fact ([22]) that the connected component of the identity in $G(B_X)$ is generated by the mappings of the form $\exp(V)$ where $V$ is a complete holomorphic vector field on $B_X$, $V = \xi + p_\xi$, and $p_\xi$ is a 2-homogeneous polynomial. By the first part of the proof $V = (V_s)_{s\in S}$ for any such vector field and hence $\exp V = (\exp V_s)_{s\in S}$. Any composition of elements of $G(B_X)$ which have this form is again of this kind. We let $\theta_s$ denote the $s$-component of the appropriate composition.

If $X$ is a Banach space then $G(B_X) = G_0 G^0$ where $G^0$ denotes the connected component of the identity and $G_0$ all the isometries of $X$ ([22]). Therefore the above theorem combined with [5], Corollary 4.17 implies the following

**Corollary 49.** *If $(K, (X_k)_{k\in K}, \tilde{X}, \varrho)$ is the maximal function module representation of the Banach space $X$ and $\psi \in G(B_X)$ then there exists a homeomorphism $\delta$ of $K$ and for each $k$ there exists $\theta_k \in G(B_{X_k})$ and an isometry $w_k : X_k \to X_{\delta(k)}$ such that*

$$\varrho(\psi(x))(\delta(k)) = w_k(\theta_k(\varrho(x)(k))).$$

*If $\psi$ is contained in the connected component of the identity then $\delta(k) = k$ for all $k \in K$.*

**Remark.** If $(E, S, p, q)$ is a Banach space over a topological space then $X = \Gamma(p)$ satisfies the assumptions of Theorem 48. Hence Theorem 48 can be viewed as a generalisation of [28], Théorème 1.8.

**Theorem 50.** *If $X$ is a Banach space then $X$ may be isometrically embedded as a weak*-dense subspace of $E = l^{\infty}(\{X_i\}_{i \in I})$, where each $X_i$ is an irreducible dual space, such that every $\psi \in G(B_X)$ extends to an element of $G(B_E)$.*

*Proof.* Every $\psi \in G(B_X)$ extends to $G(B_{X''})$ by Dineen [10], [11]. Use Example 37 together with Corollary 49 and the fact that $\delta$ must map isolated points to isolated points.

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