Hamiltonian lattice gauge models and the Heisenberg double

S.A.Frolov[*]

Section Physik, Munich University
Theresienstr.37, 80333 Munich, Germany [†]

Abstract

Hamiltonian lattice gauge models based on the assignment of the Heisenberg double of a Lie group to each link of the lattice are constructed in arbitrary space-time dimensions. It is shown that the corresponding generalization of the gauge-invariant Wilson line observables requires to attach to each vertex of the line a vertex operator which goes to the unity in the continuum limit.

1 Introduction

In a recent paper [1] I have considered Hamiltonian lattice Yang-Mills theory, based on the assignment of the Heisenberg double of a Lie group to each link of the lattice, in (1+1)- and (2+1)-dimensions. In the present paper having in mind possible applications of the construction to Chern-Simons models and gravity I discuss an analogous formulation of lattice gauge models on an arbitrary lattice or graph. In the case of the regular hyper-cubic space lattice the gauge models proposed are lattice-regularized Yang-Mills models in (d+1)-dimensions.

As is well known there are two possible ways of lattice regularization of gauge theories. In the approach of Wilson [2] one considers Euclidean formulation of a model and

---

[*]Alexander von Humboldt fellow
[†]Permanent address: Steklov Mathematical Institute, Vavilov st.42, GSP-1, 117966 Moscow, RUSSIA

discretizes all space-time, thus replacing the model by some statistical mechanics model. In the Hamiltonian approach of Kogut and Susskind [3] one considers a model in the Minkowskian space-time and introduces only space lattice remaining the time direction continuous. Then one places on each link of the lattice the cotangent bundle of a Lie group and on each vertex lattice Gauss-law constraints which are first-class constraints and generate gauge transformations and finally one finds a gauge-invariant lattice Hamiltonian. Thus in the Hamiltonian formulation the Yang-Mills theory is replaced by some classical mechanics model with first-class constraints, the phase space of the model being the direct product of the cotangent bundle over all links of the lattice: $\prod_{links} T^*G$.

However one could ask oneself whether it is possible to place on each link another phase space and on each vertex other Gauss-law constraints and to get another lattice model which can be reduced to the continuous one in the continuum limit. In [1] I have shown that such a possibility does exist and is based on a phase space which is called the Heisenberg double D_+^γ of a Lie group and is one of the basic objects in the theory of Poisson-Lie groups [4, 5]. Only (1+1)- and (2+1)-dimensional Yang-Mills models were considered in the paper. In the present paper a generalization of the consideration to the Yang-Mills theory in any space-time dimension is found.

The plan of the paper is as follows. In the second section we remind some simple results from the theory of the Heisenberg double and introduce the notations used in the paper. In the third section we formulate a classical mechanics lattice gauge model with first-class constraints on arbitrary lattice or graph, the phase space of the model being the direct product of the Heisenberg double over all links: $\prod_{links} D_+^\gamma$. Then we show that in the case of the regular hyper-cubic space lattice in d-dimensions the model constructed is lattice-regularized (d+1)-dimensional Yang-Mills theory. In Conclusion we discuss unsolved problems and perspectives.

2 Heisenberg double

In this section we remind some simple results from the theory of the Heisenberg double and fix notations. More detailed discussion of the subject can be found in refs.[4, 5, 6, 7, 8, 9].

Let G be a matrix algebraic group and $D = G \times G$. For definiteness we consider the case of the $SL(N)$ group. Almost all elements $(x, y) \in D$ can be presented in two equivalent forms as follows

$$
\begin{aligned}
(x, y) &= (U, U)^{-1}(L_+, L_-) = (U^{-1}L_+, U^{-1}L_-) \\
&= (\widetilde{L}_+, \widetilde{L}_-)^{-1}(\widetilde{U}, \widetilde{U}) = (\widetilde{L}_+^{-1}\widetilde{U}, \widetilde{L}_-^{-1}\widetilde{U})
\end{aligned}
\tag{2.1}
$$

where $U, \widetilde{U} \in G$, the matrices $L_+, \widetilde{L}_+$ and $L_-, \widetilde{L}_-$ are upper- and lower-triangular, their diagonal parts $l_+, \widetilde{l}_+$ and $l_-, \widetilde{l}_-$ being inverse to each other: $l_+ l_- = \widetilde{l}_+ \widetilde{l}_- = 1$.

Let all of the matrices be in the fundamental representation V of the group G ($N \times N$ matrices for the $SL(N)$ group). Then the algebra of functions on the group D is generated by the matrix elements x_{ij} and y_{ij}. The matrices $L_\pm$ and U or $\widetilde{L}_\pm$ and $\widetilde{U}$ can be considered as almost everywhere regular functions of x and y. Therefore, the matrix elements $L_{\pm ij}$ and U_{ij} (or $\widetilde{L}_{\pm ij}$ and $\widetilde{U}_{ij}$) define another system of generators of the algebra $FunD$. We define the Poisson structure on the group D in terms of the generators $L_\pm$ and U as

follows [7, 8]

$$\{U^1, U^2\} = \gamma[r_\pm, U^1 U^2] \tag{2.2}$$

$$\{L_+^1, L_+^2\} = \gamma[r_\pm, L_+^1 L_+^2]$$
$$\{L_-^1, L_-^2\} = \gamma[r_\pm, L_-^1 L_-^2]$$
$$\{L_+^1, L_-^2\} = \gamma[r_+, L_+^1 L_-^2] \tag{2.3}$$

$$\{L_+^1, U^2\} = \gamma r_+ L_+^1 U^2$$
$$\{L_-^1, U^2\} = \gamma r_- L_-^1 U^2 \tag{2.4}$$

Here γ is an arbitrary complex parameter, $r_\pm$ are classical r-matrices which satisfy the classical Yang-Baxter equation and the following relations

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \tag{2.5}$$

$$r_- = -P r_+ P, \qquad r_+ - r_- = C \tag{2.6}$$

where P is a permutation in the tensor product $V \otimes V$ ($Pa \otimes b = b \otimes a$) and the matrix C is the tensor Casimir operator of the Lie algebra of the group G. For the $SL(N)$ group the solution of eqs.(2.5-2.6) looks as follows

$$
\begin{aligned}
r_+ &= \sum_{i=1}^{N-1} h_i \otimes h_i + 2 \sum_{i<j}^{N} e_{ij} \otimes e_{ji} \\
&= -\frac{1}{N} I + \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + 2 \sum_{i<j}^{N} e_{ij} \otimes e_{ji}
\end{aligned}
\tag{2.7}
$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and h_i form an orthonomal basis of the Cartan subalgebra of the $SL(N)$ group: $\sqrt{i(i+1)} h_i = \sum_{k=1}^{i} e_{kk} - i e_{i+1,i+1}$.
In eqs.(2.2-2.6) we use the standard notations from the theory of quantum groups [10, 11]: for any matrix A acting in some space V one can construct two matrices $A^1 = A \otimes id$ and $A^2 = id \otimes A$ acting in the space $V \otimes V$, and for any matrix $r = \sum_a r_1(a) \otimes r_2(a)$ acting in the space $V \otimes V$ one can construct matrices $r^{12} = \sum_a r_1(a) \otimes r_2(a) \otimes id$, $r^{13} = \sum_a r_1(a) \otimes id \otimes r_2(a)$ and $r^{23} = \sum_a id \otimes r_1(a) \otimes r_2(a)$ acting in the space $V \otimes V \otimes V$.

The group D endowed with the Poisson structure (2.2-2.4) is called the Heisenberg double D_+^γ of the group G. It is not difficult to show that the matrices $\tilde{L}_\pm$ and $\tilde{U}$ have the same Poisson structure (2.2-2.4) and we shall need the Poisson brackets of $L_\pm$, U and $\tilde{L}_\pm$, $\tilde{U}$ [12]

$$
\begin{aligned}
\{L_\alpha^1, \tilde{L}_\beta^2\} &= 0 \quad for \quad any \quad \alpha, \beta = +, - \\
\{\tilde{L}_\pm^1, U^2\} &= -\gamma \tilde{L}_\pm^1 U^2 r_\pm \\
\{L_\pm^1, \tilde{U}^2\} &= -\gamma L_\pm^1 \tilde{U}^2 r_\pm \\
\{U^1, \tilde{U}^2\} &= 0
\end{aligned}
\tag{2.8}
$$

The cotangent bundle of the group G can be considered as a limiting case of the Heisenberg double . Namely, in the limit $\gamma \to 0$ and $L_\pm \to 1 + \gamma E_\pm$ the Poisson structure of the

Heisenberg double coincides with the canonical Poisson structure of the cotangent bundle T^*G.

Up to now we considered the Heisenberg double of a complex Lie group. One can single out some real forms by means of the following (anti)-involutions:

1. $SU(N)$ form for imaginary γ

$$U^* = U^{-1}, \quad L_+^* = L_-^{-1}, \quad L_-^* = L_+^{-1} \tag{2.9}$$

2. $SU(N)$ form for real γ [13, 12]

$$U^* = \widetilde{U}, \quad L_+^* = L_-, \quad L_-^* = L_+ \tag{2.10}$$

3. $SL(N)$ form for real γ

$$U^* = U, \quad L_+^* = L_+, \quad L_-^* = L_- \tag{2.11}$$

It can be easily checked that these (anti)-involutions are compatible with the Poisson structure (2.2-2.4).

3 Hamiltonian lattice Yang-Mills theory

In this section we consider Hamiltonian lattice gauge models based on the assignment of the Heisenberg double to each link of the lattice. We begin with the case of an arbitrary graph (regular hyper-cubic lattice, triangulation of a surface, the Bruhat-Tits tree, simplicial complexes and so on) which is described by a set of vertices and a set of links. Each link is thought of as either a path connecting two vertices v_1 and v_2 or a closed path with a marked vertex (tadpole). Two vertices can be connected by any finite number of links. Such a graph is certainly just an arbitrary connected Feynman diagram.

Let us now consider some vicinity of a vertex v which does not contain other vertices and closed paths. Let us denote the paths which go from the vertex v by $l_1(v),...,l_{N_v}(v)$. We call such a path as a vertex path. N_v is a common number of the paths and if there is no closed path for the vertex v then N_v coincides with the number of links going from v to some other vertices of the lattice. With each vertex path $l_i(v)$ one associates a field taking values in the Heisenberg double D_+^γ. This field is described by matrices $U(l_i(v))$, $L_+(l_i(v))$ and $L_-(l_i(v))$ with the Poisson structure (2.2-2.4) and fields corresponding to different paths have vanishing Poisson brackets.

Let us now attach to the vertex v the following Gauss-law constraints [1]

$$G_\pm(v) = L_\pm(l_1(v))L_\pm(l_2(v))\cdots L_\pm(l_{N_v}(v)) = 1 \tag{3.12}$$

One can easily check that these constraints have the following Poisson brackets

$$\{G_+^1(v), G_+^2(v)\} = \gamma[r_\pm, G_+^1(v)G_+^2(v)]$$
$$\{G_-^1(v), G_-^2(v)\} = \gamma[r_\pm, G_-^1(v)G_-^2(v)]$$
$$\{G_+^1(v), G_-^2(v)\} = \gamma[r_+, G_+^1(v)G_-^2(v)] \tag{3.13}$$

We see that these Poisson brackets vanish on the constraints surface $G_\pm = 1$ and therefore they are first-class constraints. Thus one can consider gauge transformations which are

generated by these constraints. Namely, for any function X of the matrices $U(l_i(v))$, $L_\pm(l_i(v))$ an infinitesimal gauge transformation looks as follows

$$\delta X = \{X, \, \mathrm{tr}\,(G_+(v)\xi_+(v) + G_-(v)\xi_-(v))\} \qquad (3.14)$$

where $\xi_\pm$ are gauge parameters which do not depend on U and $L_\pm$.

Using the Poisson brackets (2.4) for U and $L_\pm$ one can easily verify that these constraints generate left gauge transformations of the field U

$$U(l_i(v)) \rightarrow g(l_i(v))U(l_i(v)) \qquad (3.15)$$

Let us note that $g(l_i(v))$ depends not only on the vertex v but on the vertex path $l_i(v)$ and the fields $L_\pm$ as well. But the gauge transformations for different vertex paths are certainly not independent.

A remarkable feature of these constraints is that they form the same quadratic Poisson algebra as the matrices $L_\pm$ do. In the limit $\gamma \rightarrow 0$, $L_\pm \rightarrow 1 + \gamma E_\pm$ one gets the following Gauss-law constraints

$$C_\pm(v) = \sum_{i=1}^{N_v} E_\pm(l_i(v)) = 0 \qquad (3.16)$$

which form the Lie algebra and were used by Kogut and Suskind [3] (more exactly they used the constraints $C(v) = C_+(v) - C_-(v)$). Thus the usual lattice gauge theory can be thought of as a particular case of the models under consideration.

Let us remark that the choice of the constraints $G_\pm(v)$ is not unique. One can use any constraint of the form

$$G_\pm(v, \sigma) = L_\pm(l_{\sigma_1}(v))L_\pm(l_{\sigma_2}(v)) \cdots L_\pm(l_{\sigma_{N_v}}(v)) = 1 \qquad (3.17)$$

where $\sigma \in Symm(N_v)$ is some permutation of 1,2,...,N_v.

These constraints form the same Poisson algebra (3.13) and in the limit $\gamma \rightarrow 0$ coincide with $C_\pm(v)$. However at finite γ only that constraints, which differ from each other by a cyclic permutation, are equivalent. Thus with each vertex one can in principle associate $(N_v - 1)!$ nonequivalent constraints.

Repeating the same procedure for all of the vertices one gets the phase space which is the direct product of the Heisenberg double over all of the vertex paths and a set of the Gauss-law constraints attached to the vertices. The Gauss-law constraints corresponding to different vertices have vanishing Poisson brackets. Taking into account that for each link there are two vertex paths one sees that one has placed on each link two different Heisenberg doubles. However one can impose on the fields attached to one link the following constraints

$$U^{-1}(1)L_\pm(1) = L_\pm^{-1}(2)U(2) \qquad (3.18)$$

Comparing eq.(3.18) with eq.(2.1) one concludes that the fields $U(1)$, $L_\pm(1)$ and $U(2)$, $L_\pm(2)$ are just different coordinates on the same Heisenberg double. Thus the phase space of the model is the direct product of the Heisenberg double over all links: $\prod_{links} D_+^\gamma$.

To the moment we have described the phase space of the model and imposed the Gauss-law constraints . The next problem is to find gauge-invariant functions on the phase space and to form from them a Hamiltonian. The simplest gauge-invariant functions can be obtained using a well-known theorem from the theory of Poisson-Lie groups [4, 5, 8]

5

which states that generators of the ring of the Casimir functions of the Poisson algebra (2.3) have the following form

$$h_k = \operatorname{tr}\left(L_+ L_-^{-1}\right)^k = \operatorname{tr} L^k \tag{3.19}$$

Due to the fact that the Gauss-law constraints (3.12) depend only on $L_\pm$ these functions are gauge-invariant. The functions (3.19) are not the only ones gauge-invariant. Just as in the case of the usual lattice gauge theory one can construct Wilson line observables. So let us consider a loop with is formed by the oriented links $l_1, l_2, ..., l_n$. Each oriented link which goes from a vertex v to a vertex u (it may be the same vertex) can be denoted by two vertex paths as follows

$$l(v, u) = l_i(v) l_j(u) \tag{3.20}$$

where i and j are the numbers of the vertex paths (we have numbered all vertex paths). Thus the loop is described by $2n$ vertex paths as follows

$$l_1, l_2, ..., l_n = l_{i_1}(v_1) l_{j_2}(v_2), l_{i_2}(v_2) l_{j_3}(v_3), ..., l_{i_n}(v_n) l_{j_1}(v_1) \tag{3.21}$$

i.e. we denote the link l_α going from v_α to $v_{\alpha+1}$ as $l_{i_\alpha}(v_\alpha) l_{j_{\alpha+1}}(v_{\alpha+1})$ and $i_\alpha, j_\alpha = 1, 2, ..., N_{v_\alpha}$.

In the usual lattice gauge theory the gauge-invariant Wilson line observable corresponding to the loop is of the form

$$W(l_1 \cdots l_n) = \operatorname{tr} U(l_{i_1}(v_1)) U(l_{i_2}(v_2)) \cdots U(l_{i_n}(v_n)) \tag{3.22}$$

We are looking for a similar expression for the Wilson line observable which coincides with (3.22) in the limit $\gamma \to 0$. It appears that to get a corresponding generalization one should attach to every vertex v_α some vertex operator $V(v_\alpha, j_\alpha, i_\alpha)$ which depends on the vertex paths $l_{i_\alpha}(v_\alpha)$ and $l_{j_\alpha}(v_\alpha)$. Then the gauge-invariant Wilson line observable for a loop without tadpoles looks as follows

$$W(l_1 \cdots l_n) = \operatorname{tr} U(i_1) V(j_2, i_2) U(i_2) V(j_3, i_3) \cdots U(i_n) V(j_1, i_1) \tag{3.23}$$

where we use short notations

$$U(l_{i_\alpha}(v_\alpha)) = U(i_\alpha)$$

$$V(v_\alpha, j_\alpha, i_\alpha) = V(j_\alpha, i_\alpha)$$

The choice of the vertex operator $V(j, i)$ is not unique. The simplest vertex operators have the form

$$V(j, i) = \begin{cases} L_{\epsilon_{j+1}}(j+1) \cdots L_{\epsilon_{i-1}}(i-1) & if \quad j < i-1 \\ 1 & if \quad j = i-1 \\ L_{\mu_j}^{-1}(j) \cdots L_{\mu_i}^{-1}(i) & if \quad j \geq i \end{cases} \tag{3.24}$$

where $\epsilon_k, \mu_l = \pm$.

To prove eq.(3.23) it is enough to show that the combination

$$U(l_k(u)) V(v, j, i) U(l_i(v)) = \tilde{U}(l_j(v)) V(v, j, i) U(l_i(v)) \tag{3.25}$$

is gauge-invariant under the transformations generated by the constraints $G_\pm(v)$. It is not difficult to do by using formulas (2.2-2.4, 2.8) for the Poisson brackets of U, $\tilde{U}$ and $L_\pm$. A

Hamiltonian of the model can be now written as some combination of the gauge-invariant functions (3.19) and (3.23).

Thus we have described the phase space, the constraints and gauge-invariant observables which can be used as Hamiltonians of the models and now we are passing to a particular model on the regular hyper-cubic space lattice. We shall show that this model is the lattice-regularized Yang-Mills theory.

So let us consider the regular hyper-cubic space lattice in d-dimensions. In this case an arbitrary vertex of the lattice can be denoted by a vector $\mathbf{n} = (n_1, ..., n_d)$ with integers n_i. We choose some orientation of the lattice and denote the orthonormal lattice vectors which define the orientation as $\mathbf{e}_i$, $i = 1, ..., d$ and introduce the notation $\mathbf{e}_{-i} = -\mathbf{e}_i$. With each link one can associate a positive link $(\mathbf{n}, \mathbf{e}_i)$ and a negative link $(\mathbf{n} + \mathbf{e}_i, -\mathbf{e}_i)$. The vertex paths of the vertex $\mathbf{n}$ are therefore oriented links $(\mathbf{n}, \mathbf{e}_\alpha)$, $\alpha = \pm 1, ..., \pm d$. We place on each vertex path $(\mathbf{n}, \mathbf{e}_\alpha)$ (or oriented link) the Heisenberg double which is described by the fields $U(\mathbf{n}, \alpha), L_\pm(\mathbf{n}, \alpha)$. Due to eq.(3.18) we have the following relations

$$
\begin{aligned}
U(\mathbf{n} + \mathbf{e}_i, -i) &= \tilde{U}(\mathbf{n}, i) \\
L_\pm(\mathbf{n} + \mathbf{e}_i, -i) &= \tilde{L}_\pm(\mathbf{n}, i)
\end{aligned}
\tag{3.26}
$$

Let us note that in the limit $\gamma \to 0$, $L_\pm \to 1 + \gamma E_\pm$ one gets the usual equations $U(l^{-1}) = U^{-1}(l)$ and $E(l^{-1}) = -U^{-1}(l)E(l)U(l)$ for any oriented link l.

There are $(2d-1)!$ nonequivalent choices of the constraints (3.12). In the paper we use the following Gauss-law constraints

$$
\begin{aligned}
G_\pm(\mathbf{n}) &= L_\pm(\mathbf{n}, -1)L_\pm(\mathbf{n}, 1)L_\pm(\mathbf{n}, -2)L_\pm(\mathbf{n}, 2) \cdots L_\pm(\mathbf{n}, -d)L_\pm(\mathbf{n}, d) \\
&= \tilde{L}_\pm(\mathbf{n} - \mathbf{e}_1, 1)L_\pm(\mathbf{n}, 1) \cdots L_\pm(\mathbf{n} - \mathbf{e}_d, d)L_\pm(\mathbf{n}, d) = 1
\end{aligned}
\tag{3.27}
$$

We see from this expression that it is natural to introduce the notation $G_\pm(\mathbf{n}, i) = L_\pm(\mathbf{n}, -i)L_\pm(\mathbf{n}, i)$. One can easily verify that the (anti)-involutions (2.9) and (2.11) are compatible with the constraints. A Hamiltonian of the model can be written in the following form (which is certainly not unique)

$$
H = \frac{e^2}{2\gamma^2}a^{2-d} \sum_{links} \mathrm{tr}\,(L^2(l) - 1) + \frac{a^{d-4}}{2e^2} \sum_{plaqettes} (W(\square) + W^*(\square))
\tag{3.28}
$$

where the summation is taken over all positive and negative links and over all plaqettes, e is the coupling constant and a is the lattice length. The Wilson term $W(\square)$ is determined by eqs.(3.22,3.24) and is equal to

$$
\begin{aligned}
W(\square_{ij}) = \;& \mathrm{tr}\,U(\mathbf{n}; i)V(\mathbf{n} + \mathbf{e}_i; -i, j)U(\mathbf{n} + \mathbf{e}_i; j)V(\mathbf{n} + \mathbf{e}_i + \mathbf{e}_j; -j, -i) \\
& U(\mathbf{n} + \mathbf{e}_i + \mathbf{e}_j; -i)V(\mathbf{n} + \mathbf{e}_j; i, -j)U(\mathbf{n} + \mathbf{e}_j; -j)V(\mathbf{n}; j, i)
\end{aligned}
\tag{3.29}
$$

This formula can be simplified if one uses the following equation expressing $U(\mathbf{n} + \mathbf{e}_i; -i)$ through $U(\mathbf{n}; i)$, $L_\pm(\mathbf{n}, i)$ and $\tilde{L}_\pm(\mathbf{n}, i)$

$$
U(\mathbf{n} + \mathbf{e}_i; -i) = \tilde{L}_\pm(\mathbf{n}, i)U^{-1}(\mathbf{n}, i)L_\pm(\mathbf{n}, i) = L_\pm(\mathbf{n} + \mathbf{e}_i; -i)U^{-1}(\mathbf{n}, i)L_\pm(\mathbf{n}, i)
\tag{3.30}
$$

Using this equation and eq.(3.24) for the vertex operators one gets for $W(\square)$

$$
\begin{aligned}
W(\square_{ij}) = \;& \mathrm{tr}\,U(\mathbf{n}; i)V_{ij}(\mathbf{n} + \mathbf{e}_i)U(\mathbf{n} + \mathbf{e}_i; j)V_{ji}(\mathbf{n} + \mathbf{e}_i + \mathbf{e}_j) \\
& U^{-1}(\mathbf{n} + \mathbf{e}_j; i)V_{ij}(\mathbf{n} + \mathbf{e}_j)U^{-1}(\mathbf{n}; j)V_{ji}(\mathbf{n})
\end{aligned}
\tag{3.31}
$$

where

$$V_{ij}(\mathbf{n}) = \begin{cases} L_{\epsilon_i}(\mathbf{n}, i)G_{\epsilon_{i+1}}(\mathbf{n}, i+1)\cdots G_{\epsilon_{j-1}}(\mathbf{n}, j-1)L_{\epsilon_j}(\mathbf{n}, -j) & if \quad i < j \\ 1 & if \quad i = j \\ (L_{\mu_j}(\mathbf{n}, j)G_{\mu_{j+1}}(\mathbf{n}, j+1)\cdots G_{\mu_{i-1}}(\mathbf{n}, i-1)L_{\mu_i}(\mathbf{n}, -i))^{-1} & if \quad i > j \end{cases} \quad (3.32)$$

Let us mention that the Wilson line observable corresponding to an arbitrary loop formed by oriented links $l_1, l_2, ..., l_n$ has the same form

$$W(l_1 l_2...l_n) = \text{tr}\, U(l_1)V_{l_1 l_2}U(l_2)V_{l_2 l_3}\cdots U(l_n)V_{l_n l_1} \quad (3.33)$$

where one should take $U(l) = U(\mathbf{n}; i)$ for any positive link $l = (\mathbf{n}, \mathbf{e}_i)$ and $U(l^{-1}) = U^{-1}(\mathbf{n}, i)$ for any negative link $l^{-1} = (\mathbf{n} + \mathbf{e}_i, -\mathbf{e}_i)$ and the vertex operator $V_{l_k l_{k+1}}$ is given by eq.(3.32). Let us stress that in eq.(3.33) one has to use $U^{-1}(\mathbf{n}, i)$ for any negative link but not $\widetilde{U}(\mathbf{n}, i)$.

One can choose for example the positive sign for all of ϵ_k and negative sign for all of μ_l. Then the vertex operator $V_{ij}(\mathbf{n})$ has the following transformation law with respect to the anti-involution (2.9) which singles out the $SU(N)$ real form

$$V_{ij}^*(\mathbf{n}) = V_{ji}(\mathbf{n}) \quad (3.34)$$

The formula (3.34) ensures that the Wilson line observable $W(l_1 l_2...l_n)$ is complex- conjugated to $W(l_n l_{n-1}...l_1)$.

Now taking into account that in the limit $\gamma \to 0$, $L_\pm \to 1 + \gamma E_\pm$ the vertex operator V_{ij} goes to the unity one recovers the Gauss-law constraints and the Hamiltonian of the usual lattice gauge theory [3]. Thus we have shown that in the case of the regular hyper-cubic space lattice the model proposed is just a lattice-regularized Yang-Mills theory.

4   Conclusion

In this paper we considered the Hamiltonian formulation of classical gauge models on an arbitrary lattice. In this formulation we placed on each link the Heisenberg double of a Lie group and attached Gauss-law constraints to each vertex of the lattice. We have shown that the models on the regular hyper-cubic lattices correspond to lattice-regularized Yang-Mills theory. We discussed only the case of the pure Yang-Mills theory. It would be very interesting to include fermions to this construction. It seems to be plausible that a proper description of fermions would lead to a lattice version of the Faddeev-Shatashvili-Mickelsson 2-cocycle [14, 15, 16].

Another interesting problem is to classify all integrable lattice gauge models, i.e. to find all corresponding graphs and Hamiltonians.

We considered only classical theory and the next and most important problem is to quantize the models. There is no problem in quantizing the Poisson structure of the Heisenberg double. One just gets the quantized algebra of functions on the Heisenberg double [7, 8, 12] which includes as subalgebras the algebra of functions on the quantum group $Fun_q(G)$ and the quantized universal enveloping algebra $U_q(\mathcal{G})$, where $\mathcal{G}$ is the Lie algebra of the group G. The classical r-matrices $r_\pm$ are to be replaced by the R-matrices $R_\pm(q) = 1 + i\hbar\gamma r_\pm + \cdots$, where $q = e^{i\hbar\gamma}$. Thus a real γ corresponds to q lying on the unit circle of the complex plane and an imaginary γ corresponds to a real q. It is of

no problem to check that in quantum theory the Gauss-law constraints are first-class constraints and commute with the quantum Wilson line observables and therefore with the quantum Hamiltonian. Let us note that q has a nonpolynomial dependence on the Planck constant $\hbar$ and thus already "tree" correlation functions of the models will have a nonpolynomial dependence on $\hbar$ as well. It seems to be an indication that correlation functions of the models correspond to a summation over infinitely-many number of the usual Feynman diagrams. It is not excluded that the parameter γ plays the role of an infrared cut-off. Due to the fact that there is the additional parameter γ for the models one may expect that these models have more rich phase structure than the usual lattice gauge theory.

Let us finally notice that q-deformed lattice gauge theory was considered in refs.[17, 18, 19, 20] in connection with the Chern-Simons theory.

References

[1] S.A.Frolov, Gauge-invariant Hamiltonian formulation of lattice Yang-Mills theory and the Heisenberg double, hep-th/9501143.

[2] Wilson, *Phys.Rev.* **D10** (1974) 2445.

[3] Kogut and Susskind, *Phys. Rev.* **D11** (1975) 395.

[4] V.G.Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equation*,Sov.Math.Doklady **27** (1983),68–71.

[5] M.A.Semenov-Tian-Shansky, *Dressing transformations and Poisson-Lie group actions*, In: Publ. RIMS, Kyoto University 21, no.6, 1985, p.1237.

[6] N.Yu.Reshetikhin and M.A.Semenov-Tian-Shansky, *Lett.Math. Phys.* **19** (1990) 133-142.

[7] A.Yu.Alekseev and L.D.Faddeev, *Commun. Math. Phys.* **141**, 1991, 413-422.

[8] M.A.Semenov-Tian-Shansky, *Teor. Math. Phys.* **v93** (1992) 302 (in Russian).

[9] A.Yu.Alekseev and A.Z.Malkin, *Commun. Math. Phys.* **162** (1993) 147-173.

[10] V.Drinfeld, Quantum Groups, Proc. ICM-86, Berkeley, California, USA, 1986, 1987, pp.798-820.

[11] L.D.Faddeev, N.Yu.Reshetikhin and L.A.Takhtadjan, *Leningrad Math. J.* , **v1**, (1989), 178-206.

[12] A.Yu.Alekseev and L.D.Faddeev, An involution and dynamics for the q-deformed quantum top, hep-th/9406196.

[13] G. Mack and V. Schomerus, *Nucl. Phys.* **B370** (1992), 185.

[14] L.D.Faddeev, *Phys. Lett.* **145B** (1984) 1,2 81–86

[15] L.D.Faddeev, S.L.Shatashvili, *Theor. Math. Phys.* **60** (1985) 770 (*Teor. Mat. Fiz.* **60** (1984) 2 225)

[16] J.Mickelsson, *Commun. Math. Phys.* **97** (1985) 361.

[17] V.V. Fock and A.A. Rosly, *Poisson structures on moduli of flat connections on Riemann surfaces and r-matrices*, preprint ITEP 72-92, June 1992, Moscow.

[18] D.V. Boulatov, *Int. J. Mod. Phys.* **A8**, (1993), 3139.

[19] A. Yu. Alekseev, H. Grosse and V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern Simons theory I, II*, HUTMP 94-B336, HUTMP 94-B337.

[20] E. Buffenoir and Ph. Roche, *Two dimensional lattice gauge theory based on a quantum group*, preprint CPTH A 302-05/94.