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Gauge-invariant Hamiltonian formulation of lattice Yang-Mills theory and the Heisenberg double

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Abstract

It is known that to get the usual Hamiltonian formulation of lattice Yang-Mills theory in the temporal gauge $A_0 = 0$ one should place on every link the cotangent bundle of a Lie group. The cotangent bundle may be considered as a limiting case of a so called Heisenberg double of a Lie group which is one of the basic objects in the theory of Lie-Poisson and quantum groups. It is shown in the paper that there is a generalization of the usual Hamiltonian formulation to the case of the Heisenberg double .

1 Introduction

Lattice regularization of gauge theories is at present the only known nonperturbative regularization. There are two possible ways of discretization of the theories. In the approach of Wilson [1] one considers Euclidean formulation of gauge theories and discretizes all space-time, thus replacing the theory by some statistical mechanics model. In the approach of Kogut and Susskind [2] one considers gauge theories in the Minkowskian space-time and in the first-order (Hamiltonian) formulation and discretizes only the space

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directions remaining the time variable continuous. In this case the Yang-Mills theory is replaced by some classical mechanics model with constraints. Passing from the continuous Yang-Mills theory to a lattice model one replaces the infinite dimensional phase space of the Yang-Mills theory by the finite dimensional phase space of the lattice model. Thus although the first approach is more suitable for practical calculations, the second one seems to be more appropriate for the study of different nonequivalent representations of the infinite dimensional Heisenberg algebra of the canonical variables of the Yang-Mills theory.

In their paper [2] Kogut and Susskind considered from the very beginning quantum theory of lattice Yang-Mills model in the temporal gauge $A_0 = 0$. However it appears to be useful to begin with the gauge-invariant formulation of the classical lattice Yang-Mills theory and then to quantize it in some gauge (for example in the temporal gauge). In the present paper we consider such a formulation of the Yang-Mills theory and as an example of a useful application present a very simple solution of the (1+1)-dimensional Yang-Mills theory on a cylinder. To get this formulation, which seems to be known, one should place on every link the cotangent bundle of a Lie group. Thus the phase space of this lattice gauge theory is the direct product of the cotangent bundles over all links. The cotangent bundle can be considered as a limiting case of a so called Heisenberg double D_+^γ of a Lie group which is one of the basic objects in the theory of Lie-Poisson and quantum groups [3, 4, 5, 6, 7]. In the last section of this paper we show that there is a generalization of the usual Hamiltonian formulation of lattice gauge theory to the case of the Heisenberg double. We consider the (1+1)- and (2+1)-dimensional $SL(N, C)$ Yang-Mills theory and show that one can single out the real forms $SU(N)$ and $SL(N, R)$ by imposing the corresponding involutions of the Heisenberg double. The lattice theory obtained has two parameters a and γ and coincides with the continuous Yang-Mills theory in the continuum limit $a \rightarrow 0, \gamma/a \rightarrow const$. At the end of the paper we make some remarks on quantization of this lattice gauge theory.

2 Lattice Yang-Mills theory and cotangent bundle

In this section we remind some known results concerning the Hamiltonian formulation of the lattice (d+1)-dimensional Yang-Mills theory described by the following action

$$S = \frac{1}{8e^2} \int dx \operatorname{tr} F_{\mu\nu}^2 \quad (2.1)$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $A_\mu = A_\mu^a \lambda^a$ is a gauge field with values in the Lie algebra \mathfrak{g} of a simple Lie group G , λ^a are generators of the Lie algebra with the following relations

$$\operatorname{tr} \lambda^a \lambda^b = -2\eta^{ab}, \quad [\lambda^a, \lambda^b] = f^{abc} \lambda^c \quad (2.2)$$

where η^{ab} is the Killing tensor of the group G .

The action (2.1) can be rewritten in the first-order formulation as follows

$$S = -\frac{1}{2} \int dt d\mathbf{x} \operatorname{tr} \left(E_i \frac{\partial A_i}{\partial t} - \frac{e^2}{2} E_i^2 - \frac{1}{4e^2} F_{ij}^2 + A_0 (\partial_i E_i + [A_i, E_i]) \right), \quad (2.3)$$

where $i, j = 1, \dots, d$ are space indices.

It is obvious from eq.(2.3) that E_i and A_i are canonically-conjugated momenta and coordinates with the standard Poisson structure

$$\{A_i^a(\mathbf{x}), E_j^b(\mathbf{y})\} = \eta_{ab}\delta_{ij}\delta(\mathbf{x} - \mathbf{y}), \quad (2.4)$$

$$H = -\frac{1}{2} \int d\mathbf{x} \operatorname{tr} \left(\frac{1}{2} E_i^2 + \frac{1}{4} F_{ij}^2 \right) \quad (2.5)$$

is the Hamiltonian of the system, A_0 is a Lagrange multiplier and

$$G = \partial_i E_i + [A_i, E_i] = G^a \lambda^a \quad (2.6)$$

are the Gauss-law constraints forming the gauge algebra

$$\{G^a(\mathbf{x}), G^b(\mathbf{y})\} = f^{abc} G^c(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}), \quad (2.7)$$

Thus the classical Yang-Mills action describes a system with first-class constraints and an infinitely-dimensional phase space which can be presented as $\prod_{\mathbf{x}}(g \times g)_{\mathbf{x}} = \prod_{\mathbf{x}}(T^*g)_{\mathbf{x}}$.

Eqs.(2.4) and (2.7) can be rewritten in the following form which will be used in the paper

$$\{E_i^1(\mathbf{x}), A_j^2(\mathbf{y})\} = C \delta_{ij} \delta(\mathbf{x} - \mathbf{y}), \quad (2.8)$$

$$\{G^1(\mathbf{x}), G^2(\mathbf{y})\} = \frac{1}{2} [G^1(\mathbf{x}) - G^2(\mathbf{y}), C] \delta(\mathbf{x} - \mathbf{y}), \quad (2.9)$$

In eqs.(2.8) and (2.9) we use the standard notions from the theory of quantum groups [6]: for any matrix A acting in some space V one can construct two matrices $A^1 = A \otimes id$ and $A^2 = id \otimes A$ acting in the space $V \otimes V$, and matrix C is the tensor Casimir operator of the Lie algebra: $C = -\eta_{ab} \lambda^a \otimes \lambda^b$. For the case of the $SL(2)$ algebra $C = \sum_{a=1}^3 \sigma^a \otimes \sigma^a$, where σ^a are Pauli matrices.

Let us now introduce the regular hyper-cubic space lattice. An arbitrary vertex of the lattice is denoted by a vector $\mathbf{n} = (n_1, \dots, n_d)$ with integers n_i , the orthonormal lattice vectors are denoted \mathbf{e}_i , $i = 1, \dots, d$ and an arbitrary link is denoted by a vector \mathbf{n} and a lattice vector \mathbf{e}_i : $(\mathbf{n}, \mathbf{e}_i)$ or (\mathbf{n}, i) . To get a gauge-invariant lattice Yang-Mills theory, we place on each vertex an algebra-valued Lagrange multiplier $A_0(\mathbf{n})$ and on each link $(\mathbf{n}, \mathbf{e}_i)$ a group-valued field $U(\mathbf{n}; i)$ and an algebra-valued field $E(\mathbf{n}; i)$. Then the lattice Yang-Mills action which is invariant with respect to the following gauge transformations

$$\begin{aligned} U(\mathbf{n}; i) &\rightarrow g^{-1}(\mathbf{n}) U(\mathbf{n}; i) g(\mathbf{n} + \mathbf{e}_i) \\ E(\mathbf{n}; i) &\rightarrow g^{-1}(\mathbf{n}) E(\mathbf{n}; i) g(\mathbf{n}) \\ A_0(\mathbf{n}) &\rightarrow g^{-1}(\mathbf{n}) A_0(\mathbf{n}) g(\mathbf{n}) + g^{-1}(\mathbf{n}) \frac{dg(\mathbf{n})}{dt}, \end{aligned} \quad (2.10)$$

can be written as follows

$$\begin{aligned} S_{lat} = & - \frac{1}{2} \operatorname{tr} \left[\sum_{\mathbf{n}, i} \left(E(\mathbf{n}; i) \frac{dU(\mathbf{n}; i)}{dt} U^{-1}(\mathbf{n}, i) - \frac{e^2}{2} a^{2-d} E^2(\mathbf{n}, i) \right) \right. \\ & + \sum_{\mathbf{n}} A_0(\mathbf{n}) \sum_i \left(E(\mathbf{n}; i) - U^{-1}(\mathbf{n} - \mathbf{e}_i; i) E(\mathbf{n} - \mathbf{e}_i; i) U(\mathbf{n} - \mathbf{e}_i; i) \right) \\ & \left. - \frac{a^{d-4}}{2e^2} \sum_{\text{plaquettes}} (W(\square) + W^*(\square)) \right] \end{aligned} \quad (2.11)$$

Here $W(\square)$ is the usual Wilson term

$$W(\square) = \text{tr} U(\mathbf{n}; i)U(\mathbf{n} + \mathbf{e}_i; j)U^{-1}(\mathbf{n} + \mathbf{e}_j; i)U^{-1}(\mathbf{n}; j) \quad (2.12)$$

In the continuum limit

$$\begin{aligned} a \rightarrow 0, \quad A_0(\mathbf{n}) &\rightarrow A_0(a\mathbf{n}) \\ E(\mathbf{n}; i) &\rightarrow a^{d-1}E_i(a\mathbf{n}) \\ U(\mathbf{n}; i) &\rightarrow 1 + aA_i(a\mathbf{n}), \end{aligned} \quad (2.13)$$

the action (2.11) reduces to the action (2.3).

The kinetic term in eq.(2.11) defines the Poisson structure for the fields $U(\mathbf{n}; i)$ and $E(\mathbf{n}; i)$

$$\begin{aligned} \{U^1(\mathbf{n}; i), U^2(\mathbf{m}; j)\} &= 0 \\ \{E^1(\mathbf{n}; i), E^2(\mathbf{m}; j)\} &= \frac{1}{2}[E^1(\mathbf{n}; i) - E^2(\mathbf{m}; j), C]\delta_{ij}\delta_{\mathbf{nm}} \\ \{E^1(\mathbf{n}; i), U^2(\mathbf{m}; j)\} &= CU^2(\mathbf{m}; j)\delta_{ij}\delta_{\mathbf{nm}} \end{aligned} \quad (2.14)$$

This Poisson structure being ultralocal coincides on every link with the canonical Poisson structure of the cotangent bundle of the group G : T^*G . Thus the phase space of the regularized model is the direct product of T^*G over all links: $\prod_{(\mathbf{n}, \mathbf{e}_i)}(T^*G)(\mathbf{n}, i)$.

Using eqs.(2.14) one can easily calculate the Poisson bracket of the Gauss-law constraints

$$\begin{aligned} G(\mathbf{n}) &= \sum_i (E(\mathbf{n}; i) - U^{-1}(\mathbf{n} - \mathbf{e}_i; i)E(\mathbf{n} - \mathbf{e}_i; i)U(\mathbf{n} - \mathbf{e}_i; i)) \\ \{G^1(\mathbf{n}), G^2(\mathbf{m})\} &= \frac{1}{2}[G^1(\mathbf{n}) - G^2(\mathbf{m}), C]\delta_{\mathbf{nm}} \end{aligned} \quad (2.15)$$

One can see from eqs.(2.14) and (2.15) that the field $E(\mathbf{n}; i)$ should be identified with the right-invariant momentum generating left gauge transformations of the field $U(\mathbf{n}; i)$ and the field $\tilde{E}(\mathbf{n}; i) = U^{-1}(\mathbf{n}; i)E(\mathbf{n}; i)U(\mathbf{n}; i)$ should be identified with the left-invariant momentum generating right gauge transformations of the field $U(\mathbf{n}; i)$. It is not difficult to check that $E(\mathbf{n}; i)$ and $\tilde{E}(\mathbf{n}; i)$ have the vanishing Poisson bracket. Let us note that in the continuum limit (2.13) the Poisson structure (2.14) reduces to eq.(2.8). Now imposing the temporal gauge $A_0 = 0$ and quantizing the Poisson structure (2.14) one gets the model considered by Kogut and Suskind [2].

As an application of this gauge-invariant formulation let us consider quantization of the (1+1)-dimensional Yang-Mills theory on a cylinder. In this case the lattice action (2.11) can be rewritten as follows

$$\begin{aligned} S_{lat} = & - \frac{1}{2} \text{tr} \left[\sum_{n=1}^N (E(n) \frac{dU(n)}{dt} U^{-1}(n) - \frac{e^2}{2} a E^2(n) \right. \\ & \left. + A_0(n)(E(n) - U^{-1}(n-1)E(n-1)U(n-1)) \right] \end{aligned} \quad (2.16)$$

Due to the gauge invariance of the action one can impose the gauge condition

$$U(n) = 1, \quad n \neq N; \quad U(N) \equiv U \quad (2.17)$$

Then one can easily solve the Gauss-law constraints $G(n) = 0$ for $n \geq 2$ and get the following solution

$$E(n) = E \quad \text{for any } n \quad (2.18)$$

Inserting this solution to the first constraint $G(1) = E(1) - U^{-1}E(N)U$ one gets a residual constraint $G = E - U^{-1}EU$ which generates the residual gauge transformations

$$U \rightarrow g^{-1}Ug, \quad E \rightarrow g^{-1}Eg \quad (2.19)$$

Finally the action (2.16) takes the form

$$S_{lat} = -\frac{1}{2} \text{tr} \left[E \frac{dU}{dt} U^{-1} - e^2 \pi R E^2 + A_0 (E - U^{-1}EU) \right] \quad (2.20)$$

where $R = aN/2\pi$ is the radius of the circle.

It is worthwhile to note that the resulting action (2.20), firstly obtained in ref.[8] by a different method, does not depend on the lattice length a and thus is the exact action describing partially-reduced Yang-Mills theory.

This gauge-invariant Hamiltonian formulation of the lattice Yang-Mills theory was based on the cotangent bundle of the group G . One could ask oneself whether it is possible to put on each link another phase space and on each vertex another first-class Gauss-law constraints and to get another lattice theory which in the continuum limit reduces to the continuous one. In the next section we show that such a possibility does exist and is based on a phase space which is called the Heisenberg double of a Lie group and is one of the basic objects in the theory of Poisson-Lie and quantum groups.

3 Heisenberg double and lattice Yang-Mills theory

In this section we present some simple results on the theory of the Heisenberg double which will be used later. More detailed discussion of the subject can be found in refs.[4, 7, 9].

Let G be a matrix algebraic group and $D = G \times G$. For definiteness we consider the case of the $SL(N)$ group. Almost all elements $(x, y) \in D$ can be presented as follows

$$(x, y) = (U, U)^{-1}(L_+, L_-), \quad (3.21)$$

where $U \in G$, the matrices L_+ and L_- are upper- and lower-triangular, their diagonal parts l_+ and l_- being inverse to each other: $l_+ l_- = 1$.

Let all of the matrices be in the exact matrix representation ρ , V of the group G . Then the algebra of functions on the group D is generated by the matrix elements x_{ij} and y_{ij} . The matrices L_{\pm} and U can be considered as almost everywhere regular functions of x and y . Therefore, the matrix elements $L_{\pm ij}$ and U_{ij} define another system of generators of the algebra $FunD$. We define the Poisson structure on the group D in terms of the generators L_{\pm} and U as follows [7]

$$\{U^1, U^2\} = \gamma[r_{\pm}, U^1 U^2] \quad (3.22)$$

$$\begin{aligned} \{L_+^1, L_+^2\} &= \gamma[r_{\pm}, L_+^1 L_+^2] \\ \{L_-^1, L_-^2\} &= \gamma[r_{\pm}, L_-^1 L_-^2] \\ \{L_+^1, L_-^2\} &= \gamma[r_+, L_+^1 L_-^2] \end{aligned} \quad (3.23)$$

$$\begin{aligned}\{L_+^1, U^2\} &= \gamma r_+ L_+^1 U^2 \\ \{L_-^1, U^2\} &= \gamma r_- L_-^1 U^2\end{aligned}\quad (3.24)$$

Here γ is an arbitrary complex parameter, r_\pm are classical r -matrices which satisfy the classical Yang-Baxter equation and the following relations

$$r_- = -Pr_+P \quad (3.25)$$

$$r_+ - r_- = C \quad (3.26)$$

where P is a permutation in the tensor product $V \otimes V$ ($Pa \otimes b = b \otimes a$). For example in the case of the $SL(2)$ group

$$r_+ = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 2 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad (3.27)$$

The group D endowed with the Poisson structure (3.22-3.24) is called the Heisenberg double D_+^γ of the group G . To understand the relation of the Heisenberg double D_+^γ to the cotangent bundle of the group G it is convenient to use the matrix $L = L_+ L_-^{-1}$ [10]. The Poisson brackets for L and U can be written as follows

$$\begin{aligned}\frac{1}{\gamma}\{L^1, L^2\} &= L^1 L^2 r_- + r_+ L^1 L^2 - L^1 r_- L^2 - L^2 r_+ L^1 \\ \frac{1}{\gamma}\{U^1, L^2\} &= r_- U^1 L^2 - L^2 r_+ U^1\end{aligned}\quad (3.28)$$

If one considers now the limit $\gamma \rightarrow 0$ and $L \rightarrow 1 + \gamma E$, $L_\pm \rightarrow 1 + \gamma E_\pm$ one gets the canonical Poisson structure of the cotangent bundle T^*G (see eq.(2.14)). Thus one can see that L is an analog of the right-invariant momentum and one can introduce the left-invariant momentum \tilde{L} by the following equation

$$\tilde{L} = U^{-1} L U \quad (3.29)$$

The matrix \tilde{L} can be decomposed into the product of upper- and lower-triangular matrices

$$\tilde{L} = \tilde{L}_+^{-1} \tilde{L}_- \quad (3.30)$$

where as before $\tilde{l}_+ \tilde{l}_- = 1$.

The matrices L_\pm and \tilde{L}_\pm are related to each other by means of the following equations

$$U^{-1} L_\pm = \tilde{L}_\pm^{-1} \tilde{U} \quad (3.31)$$

One can easily check that the matrices \tilde{L}_\pm, \tilde{U} have the Poisson brackets (3.22-3.24) and we shall need the Poisson brackets of L_\pm, U and \tilde{L}_\pm, \tilde{U}

$$\begin{aligned}\{L_\alpha, L_\beta\} &= 0 \quad \text{for any } \alpha, \beta = +, - \\ \{\tilde{L}_\pm^1, U^2\} &= -\gamma \tilde{L}_\pm^1 U^2 r_\pm \\ \{L_\pm^1, \tilde{U}^2\} &= -\gamma L_\pm^1 \tilde{U}^2 r_\pm \\ \{U^1, \tilde{U}^2\} &= 0\end{aligned}\quad (3.32)$$

Up to now we considered the Heisenberg double of a complex Lie group. For physical applications one should single out some real form. If γ is imaginary ($\gamma^* = -\gamma$) one can single out the $SU(N)$ form by means of the standard anti-involution

$$U^* = U^{-1}, \quad L_+^* = L_-^{-1}, \quad L_-^* = L_+^{-1} \quad (3.33)$$

It is well-known that for real γ the $SU(N)$ anti-involution $U^* = U^{-1}$ is not compatible with the Poisson structure (3.22). However as was pointed out in ref.[11, 12] the matrix \tilde{U} can be used to define an anti-involution on the Heisenberg double. Namely, taking into account that $\tilde{U} \rightarrow U^{-1}$ in the limit $\gamma \rightarrow 0$ one defines the anti-involution as follows

$$U^* = \tilde{U}, \quad L_+^* = L_-, \quad L_-^* = L_+ \quad (3.34)$$

The Heisenberg double of the $SL(N)$ group with this anti-involution reduces in the limit $\gamma \rightarrow 0$ to $T^*SU(N)$. It can be easily checked that this anti-involution is compatible with the Poisson structures (3.22-3.24) and (3.32). Let us notice that for real γ the involution which singles out the $SL(N, R)$ form looks as usual

$$U^* = U, \quad L_+^* = L_+, \quad L_-^* = L_- \quad (3.35)$$

Now we are ready to discuss a lattice gauge theory based on the Heisenberg double. So we place on every link $(\mathbf{n}, \mathbf{e}_i)$ a field taking values in the Heisenberg double D_+^γ . Thus the phase space of the model is the direct product of D_+^γ over all links: $\prod_{(\mathbf{n}, \mathbf{e}_i)} D_+^\gamma(\mathbf{n}, i)$. We suppose that the parameter γ goes to zero in the continuum limit $a \rightarrow 0$. To define a lattice gauge theory one should find lattice Gauss-law constraints and a lattice Hamiltonian which coincide with the Gauss-law constraints and the Hamiltonian of the continuous theory in the continuum limit. As was mentioned above the cotangent bundle T^*G may be considered as a limiting case of the Heisenberg double when $\gamma \rightarrow 0$. By this reason one could look for such lattice Gauss-law constraints which can be reduced to the lattice constraints (2.15) in the limit $\gamma \rightarrow 0$, a being constant. The form of the constraints depends on the space dimension and, so we begin with the simplest case of (1+1)-dimensional Yang-Mills theory on a cylinder.

In this case the constraints (2.13) look as follows

$$G(n) = E(n) - \tilde{E}(n-1) = 0 \quad (3.36)$$

One could try to generalize the constraints $G(n)$ by replacing $E(n)$ on $L(n)$ and $\tilde{E}(n)$ on $\tilde{L}(n)$. This replacement does work in the classical theory, i.e. the corresponding constraints $G(n) = L(n) - \tilde{L}(n-1)$ are first-class constraints. However one can show that quantization violates this property of the constraints. To get a proper modification it seems to be necessary to decompose $G(n)$ as follows

$$G(n) = G_+(n) - G_-(n) \quad (3.37)$$

where $G_\pm(n) = E_\pm(n) - \tilde{E}_\pm(n)$ are upper- and lower-triangular matrices and their diagonal parts $g_\pm(n)$ satisfy the following equation: $g_+(n) + g_-(n) = 0$.

Then the required modification (which seems to be the only possible) is of the form

$$G_\pm(n) = \tilde{L}_\pm(n-1)L_\pm(n) = 1 \quad (3.38)$$

One can easily calculate the Poisson brackets of the constraints $G_{\pm}(n)$

$$\begin{aligned}\{G_+^1(n), G_+^2(m)\} &= \gamma[r_{\pm}, G_+^1(n)G_+^2(m)]\delta_{nm} \\ \{G_-^1(n), G_-^2(m)\} &= \gamma[r_{\pm}, G_-^1(n)G_-^2(m)]\delta_{nm} \\ \{G_+^1(n), G_-^2(m)\} &= \gamma[r_+, G_+^1(n)G_-^2(m)]\delta_{nm}\end{aligned}\quad (3.39)$$

We see that these Poisson brackets vanish on the constraints surface $G_{\pm}(n) = 1$ and therefore these constraints are first-class constraints. A remarkable feature of these constraints is that they form not a Lie-Poisson algebra but the same quadratic Poisson algebra as the matrices L_{\pm} do. In the limit $\gamma \rightarrow 0$, $L_{\pm} \rightarrow 1 + \gamma E_{\pm}$, $\tilde{L}_{\pm} \rightarrow 1 - \gamma \tilde{E}_{\pm}$ one recovers the old Gauss-law constraints (3.36).

To complete the construction of the lattice theory one should find a lattice Hamiltonian. As is well-known from the theory of Poisson-Lie groups [3, 4, 7] generators of the ring of the Casimir functions of the Poisson algebra (3.23) have the following form

$$h_k = \text{tr } L^k \quad (3.40)$$

It is clear that these functions are invariant with respect to the gauge transformations which are generated by the constraints (3.38). In principle one can choose any combination of these functions as a Hamiltonian of the theory

$$H = \frac{a}{\gamma^2} \sum_{n=1}^N \sum_{k=-\infty}^{\infty} c_k (\text{tr } (L^k(n) - 1)) \quad (3.41)$$

Then in the limit $\gamma \rightarrow 0$ one gets the Hamiltonian

$$H = a \sum_{n=1}^N \text{tr } E^2(n) \sum_{k=-\infty}^{\infty} \frac{1}{2} c_k k^2 \quad (3.42)$$

which coincides up to a constant with the Hamiltonian used in the previous section.

However there is a special choice of the Hamiltonian which leads on the quantum level to a natural generalization of dynamics of the symmetric top

$$H \sim \text{tr } (\log L)^2 \quad (3.43)$$

This Hamiltonian was implicitly used in ref.[12].

The functions (3.40) are not the only ones gauge-invariant. Just as in the case of the usual lattice gauge theory one can construct Wilson line observables. For the (1+1)-dimensional model the simplest observables look as follows

$$\begin{aligned}W_{k_1 \dots k_N} &= \text{tr } (L^{k_1}(1)U(1)L^{k_2}(2)U(2) \dots L^{k_N}(N)U(N)) \\ W'_{k_1 \dots k_N} &= \text{tr } (U^{-1}(N)L^{k_N}(N)U^{-1}(N-1)L^{k_{N-1}}(N-1) \dots U^{-1}(1)L^{k_1}(1))\end{aligned}\quad (3.44)$$

where $L = L_+ L_-^{-1}$ and k_1, \dots, k_N are integers.

It is not difficult to show that the involutions (3.33-3.35) are compatible with the Gauss-law constraints (3.38).

Thus we have constructed the (1+1)-dimensional gauge-invariant lattice Yang-Mills theory based on the assignment of the Heisenberg double to every link and now we pass

to the (2+1)-dimensional case. The usual lattice Gauss-law constraints (2.15) look in this case as follows

$$G_{\pm}(n_1, n_2) = E_{\pm}(n_1, n_2; 1) + E_{\pm}(n_1, n_2; 2) - \tilde{E}_{\pm}(n_1 - 1, n_2; 1) - \tilde{E}_{\pm}(n_1, n_2 - 1; 2) \quad (3.45)$$

where just as for the two-dimensional case we decompose G into upper- and lower-triangular matrices.

There are (at least) six nonequivalent modifications of the constraints (3.45). In this paper we use the following constraints

$$G_{\pm}(n_1, n_2) = \tilde{L}_{\pm}(n_1 - 1, n_2; 1)\tilde{L}_{\pm}(n_1, n_2 - 1; 2)L_{\pm}(n_1, n_2; 1)L_{\pm}(n_1, n_2; 2) = 1 \quad (3.46)$$

These constraints form the same Poisson algebra as for the two-dimensional case with the only change $n \rightarrow \mathbf{n} = (n_1, n_2)$ in eq.(3.39). Let us notice that the product of matrices L_{\pm} , \tilde{L}_{\pm} is taken in the clock-wise order (starting from $\tilde{L}_{\pm}(n_1 - 1, n_2; 1)$). Namely due to this choice of the constraints (3.46) the following expressions generalizing the Wilson term (2.12)

$$\begin{aligned} W(n_1 + 1, n_2) &= \\ \text{tr}(G_{\pm}^{-1}(n_1 + 1, n_2)\tilde{U}(n_1, n_2; 1)U(n_1, n_2; 2)U(n_1, n_2 + 1; 1)\tilde{U}(n_1 + 1, n_2; 2)) & \\ W'(n_1 + 1, n_2) &= \\ \text{tr}(G_{\pm}(n_1 + 1, n_2)\tilde{U}^{-1}(n_1 + 1, n_2; 2)U^{-1}(n_1, n_2 + 1; 1)U^{-1}(n_1, n_2; 2)\tilde{U}^{-1}(n_1, n_2; 2)) & \end{aligned} \quad (3.47)$$

are gauge-invariant.

Let us mention that one can find a similar generalization of an arbitrary Wilson line observable but the corresponding formula depends on the local form of the Wilson line and will not be discussed in the paper.

These Wilson terms can be used to write down a gauge-invariant Hamiltonian

$$H = H(L) + \frac{1}{2a^2e^2} \text{tr} \sum_{n_1, n_2} (W(n_1, n_2) + W'(n_1, n_2)) \quad (3.48)$$

The first term in eq.(3.48) depends only on L_{\pm} and coincides with $\sum_{\mathbf{n}, i} \text{tr} E^2(\mathbf{n}, i)$ in the limit $\gamma \rightarrow 0$ (see the discussion of the two-dimensional model).

This completes the construction of the (2+1)-dimensional lattice Yang-Mills theory. For imaginary γ one can impose the anti-involution (3.33) to get the real form $SU(N)$. However for real γ the anti-involution (3.34) is not compatible with the Gauss-law constraints and thus we can get only the real form $SL(N, R)$. Let us notice that in these cases the Hamiltonian is real.

4 Discussion

In this paper we considered the gauge-invariant Hamiltonian formulation of classical lattice Yang-Mills theory in (1+1) and (2+1) dimensions. In this formulation we placed on every link the Heisenberg double of a Lie group. It is clear that the construction presented can be generalized to the case of the (d+1)-dimensional $SL(N)$ Yang-Mills model [13].

For the (1+1)-dimensional model it appears to be possible to make the limit $a \rightarrow 0$, γ being constant. The theory obtained in such a way seems to be related to Poisson- σ -models recently introduced in ref.[14] . It would be interesting to clarify this relation.

We have not written down the Lagrangians which correspond to these lattice gauge models. In principle one can do it by using some expressions for the symplectic form of the Heisenberg double which were found in ref.[9].

We considered only classical theory and it would be of great interest to quantize the models. There is no problem in quantizing the Poisson structure of the Heisenberg double. One just gets the quantized algebra of functions on the Heisenberg double which was introduced in ref.[15] . The classical r -matrices r_{\pm} are to be replaced by the R -matrices $R_{\pm}(q) = 1 + i\hbar\gamma r_{\pm} + \dots$, where $q = e^{i\hbar\gamma}$. Thus a real γ corresponds to q lying on the unit circle of the complex plane and an imaginary γ corresponds to a real q . It is not difficult to verify that in quantum theory the Gauss-law constraints [3.38] and [3.46] are first-class constraints and commute with the quantum Hamiltonians [3.41] and [3.48] if these Hamiltonians are finite polynomials of L and L^{-1} . It seems that the case of q being a root of unity, $q^N = 1$, is the most interesting one. However the present formulation permits to single out only $SL(N, R)$ real form for such a q . It is an interesting problem to find a similar formulation for the $SU(N)$ case. Let us finally notice that q -deformed lattice gauge theory was considered in refs.[16, 17, 18, 19] in connection with the Chern-Simons theory.

Due to the fact that there are two arbitrary parameters a and γ in the theory one may expect that the theory has more rich phase structure than the usual lattice gauge models. We hope to consider these problems in forthcoming publications.

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