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Some Results on the Distribution of a Statistic based on the Pareto and Log-normal Distributions and the Power of an Associated Test*

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I

In the October 1975 edition of this *Review*, Harrison and Nolan examine the "residual wealth" assumption which Lyons (1972 and 1975) made when using the mortality multiplier method to estimate the distribution of personal wealth in Ireland. The assumption was that all of the unexamined estates of adults who died in 1966 had a net value of zero. Scrutiny of such assumptions is valuable because, as Atkinson (1974) has shown, wealth distribution estimates seem to be highly sensitive to the values assumed for residual wealth.

Amongst other things, Harrison and Nolan devise a statistical procedure for testing the validity of an arbitrarily small assumed value of average residual wealth. On the maintained hypothesis that wealth is log-normally distributed, and using a result due to Cramer (1971), they argue that if the null hypothesis is true —that is, if the assumed value of average residual wealth is correct in the sense that it produces a variance for the estimated distribution of the natural logarithm of wealth that is in accord with the log-normal variance suggested by the Cramer result—then their test statistic is F distributed. However, Chesher and McMahon (1976), while not demonstrating that the statistic is not F distributed, have recently drawn attention to certain flaws in the Harrison-Nolan argument. Consequently the precise nature of the statistic, and the status of the conclusion reached by Harrison and Nolan on the basis of their use of the statistic, are in doubt. For this reason Harrison (1976), in his reply to Chesher and McMahon, undertook to investigate the statistic further using the method of simulation.

It is the purpose of this paper to briefly describe the study that was carried out and to report and comment on the results. Section II is concerned with the

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empirical approximation to the distribution of the statistic and the adequacy of the fit of an appropriate F distribution to it; Section III is concerned with the power of the associated test; and Section IV contains some concluding remarks.

II

Underlying the test procedure is the assumption that the departure from log-normality of the estimated wealth distribution is due to the deficiencies of the mortality multiplier method and inevitable random factors associated with the estimation procedure.[1] Of the former influences, the fact that there is usually a sizeable number of unexamined estates and hence a residual part of the population about whose wealth an assumption has to be made, is considered to be of prime significance. Thus the relation $fe_i = fn_i - d_i + v_i$, $i = 1, 2, \ldots, m$, may be written, where fe_i is the relative frequency of persons in the ith class of the estimated distribution of the logarithm of wealth, fn_i is the corresponding normal relative frequency, d_i is the size of the frequency component which, on the maintained hypothesis, is "missing" from the ith wealth class and actually constitutes part of the residual, v_i denotes all other influences, and m is the total number of wealth categories. The sum of the d_i's is the total residual relative frequency, f_r. It will be recalled that in the case of Lyons's wealth distribution estimates, the magnitude of f_r was about 0·63.

Since the unexamined estates overwhelmingly belong to the category of small estates, the d_i's are considered to be relatively large for the lower wealth categories and small or zero for the higher wealth categories. It follows that the fe_i's for the wealth categories above a certain level of wealth w^* will probably be quite accurate in the sense that, if the v_i's are not large, their values will probably be close to those of the corresponding fn_i's. It is these fe_i's which are used to estimate the Pareto coefficient, b. According to Cramer's result, the log-normal standard deviation "corresponding" to this Pareto coefficient may be obtained from the equation $\sigma_p = 1{\cdot}525/b$.

The test as it was originally formulated involves comparison of this "predicted" standard deviation with the log-normal standard deviation, σ_c, calculated using the fe_i's of the wealth categories below w^*, and f_r and the assumed average value of residual wealth, w_r. In response to the points made by Chesher and McMahon, however, Harrison (1976) has suggested that it may be better to base the calculation of σ_c on all of the fe_i's, although as Harrison and Nolan (1975, p. 69, footnote 7) have pointed out, this modification makes no appreciable difference to the numerical value obtained for σ_c when using Lyons's figures. The null hypothesis H_0 is that $w_r = \bar{w}_r$, the value of average residual wealth which yields a standard

1. For further details on the test procedure see Harrison and Nolan (1975, pp. 67–69). The deficiencies of the mortality multiplier method of estimating wealth distributions have been well documented; see, for example, Lyons (1972).

deviation equal to σ_p. The alternative hypothesis is that $w_r < \bar{w}_r$. It is suggested that a one-tail test of H_0 may be carried out using the statistic $c = \sigma^2_c / \sigma^2_p$, the value of which is larger, the larger is σ^2_c, that is the further w_r departs from $\bar{w}_r$. It is the nature of the distribution of c and the goodness of fit of an F distribution to it that is of immediate concern.

In the experiment to determine the approximate sampling distribution of c, and the modified form of the statistic, c', three sets of fn_i's were used, the value for m, the number of wealth classes, being 18 in each case. The values for the parameters of the underlying log-normal distributions were chosen so as to be in line with those suggested by Lyons's wealth distribution estimates. The mean μ was given the value 1·3 in each case, while the variance σ^2 was given the values 1·0, 1·5 and 2·0; the values represent the natural logarithm of wealth measured in units of thousands of pounds.

Four values for f_r were used, namely, 0·0, 0·4, 0·63 and 0·7, and in each case, except the first, the associated values for the d_i's were chosen to conform with what was said about their relative sizes above. The case of $f_r = 0{\cdot}63$ served as a benchmark in this respect in that the d_i's in this case were determined by comparing Lyons's estimated class frequencies, fe_i's, with the corresponding normal frequencies, fn_i's, for a variety of combinations of μ and σ^2 and then taking the mean differences. It is noteworthy that in this particular case the d_i's are indeed negligible for the higher wealth categories. The values of $\bar{w}_r$ for each combination of σ^2 and f_r were obtained in association with the appropriate set of d_i's, the method used being similar to that outlined in Appendix (ii) of Harrison and Nolan (1975).

The v_i's were assumed to be random normal variates each with zero mean and variances proportional to the corresponding fn_i's. Three factors of proportionality, s, were employed, namely, 0·2, 0·35 and 0·5. For each of the 36 combinations of σ^2, f_r and s, 1,000 sets of v_i's were generated. Together with the particular basic sets of fn_i's and d_i's, these were used to produce 1,000 sets of fe_i's, each set representing a different estimate of the relevant underlying log-normal distribution of wealth.

The Harrison-Nolan procedure was then applied to each of the 1,000 sets of fe_i's. The Pareto law was fitted by ordinary least squares regression to data on 10 wealth classes above the wealth value $w^* = £15{,}000$, and σ_c was calculated both as originally suggested by Harrison and Nolan, and also using all of the fe_i's. Thus 1,000 values of the statistic c and the modified statistic c' were computed for each combination of σ^2, f_r and s for the null hypothesis case of $w_r = \bar{w}_r$. These were grouped into 11 classes and the empirical approximation to the distributions of the two statistics thus formed.

Next, using probabilities obtained from Pearson and Hartley (1962), an F distribution was fitted to each empirical distribution. Although the moments of the empirical c and c' distributions could have been used to suggest the degrees of freedom of the respective F distributions, this was not in fact done. Since it is the test as used by Harrison and Nolan that Chesher and McMahon have called into question, the approach to the selection of the degrees of freedom used by Harrison and Nolan was retained, namely, that of choosing the degrees of freedom with regard to the number of wealth classes used, and the number of parameters estimated, in the calculation of the constituent variances σ^2_c and σ^2_p. Accordingly, an F^7_7 and an F^{17}_7 distribution were fitted to c and c' respectively. Finally, a χ^2 test of the goodness of fit of each F distribution was carried out.

Table 1: χ^2 *values for the goodness of fit of* F^7_7 *to* c

σ^2	f_r	$s = 0{\cdot}20$	$s = 0{\cdot}35$	$s = 0{\cdot}50$
1·0	0·70	3·67	2·63	2·43
	0·63	4·38	2·94	2·75
	0·40	11·72	6·58	4·27
	0·00	4·94	3·48	2·61
1·5	0·70	1·90	2·42	2·42
	0·63	3·77	2·79	2·77
	0·40	6·37	4·94	4·20
	0·00	5·67	4·22	2·43
2·0	0·70	8·13	7·96	6·73
	0·63	4·21	3·18	2·79
	0·40	4·00	3·93	3·64
	0·00	0·81	0·73	0·67

Table 2: χ^2 *values for the goodness of fit of* F^{17}_7 *to* c'

σ^2	f_r	$s = 0{\cdot}20$	$s = 0{\cdot}35$	$s = 0{\cdot}50$
1·0	0·70	7·90	6·24	5·85
	0·63	9·74	4·38	5·98
	0·40	25·01	15·83	9·55
	0·00	12·18	8·53	6·32
1·5	0·70	30·42	21·00	20·18
	0·63	29·57	19·49	17·21
	0·40	46·28	28·80	17·30
	0·00	12·85	10·75	7·46
2·0	0·70	69·60	48·39	45·19
	0·63	48·11	26·74	22·44
	0·40	44·93	27·23	17·92
	0·00	6·45	4·82	3·60

Without exception, as Table 1 shows, the resulting χ^2 values are such that the hypothesis that c is distributed as F^7_7 would not be rejected at the 5 per cent significance level for any of the combinations of parameters used, the 5 per cent critical value of χ^2_{10} being 18·3. There is some variation in goodness of fit with s, that is with the variances of the v_i's, the fit improving somewhat as the latter increases. There is also variation in fit with σ^2, and with f_r for any given σ^2. The same type of variations in goodness of fit are observable in the case of c', but as can be seen from Table 2, the fit of the F distribution is generally much worse than that in the case of c. Indeed, there are many combinations of parameters for which the χ^2 values are such that the hypothesis that c' is distributed as $F^{17}{}_7$ would have to be rejected at the 5 and 1 per cent levels of significance.

III

Given the positive results on the form of the distribution of the c statistic, it was decided to extend the study somewhat and investigate the power of c for a range of values of $w_r < \bar{w}_r$ with $\mu = 1{\cdot}3$, $s = 0{\cdot}35$ and combinations of the values $\sigma^2 = 1{\cdot}0$, $1{\cdot}5$, $2{\cdot}0$ and $f_r = 0{\cdot}70$, $0{\cdot}63$ and $0{\cdot}40$. Using essentially the same simulation procedure as before, the value of c was computed and the test of H_0 performed 500 times for each w_r and σ^2, f_r combination, the 5 per cent and 1 per cent critical values of $F^7{}_7$ being used. In each case the power of the test was estimated by the percentage of times the computed c statistic exceeded the critical F value, that is the percentage of times the false H_0 would have been rejected.

The resulting power function estimates, for the 5 per cent level of significance only, are given in Table 3, the percentages having been rounded to the nearest whole number. Because the value of $\bar{w}_r$ is different for each pair of σ^2 and f_r values, comparisons of the power of c for the various parameter combinations are difficult to make using the numbers in Table 3. The results have therefore been plotted in Figures 1, 2 and 3. To facilitate comparisons the horizontal axes of the graphs have been drawn to measure the ratio $w_r/\bar{w}_r$ which on the null hypothesis is unity for all σ^2 and f_r combinations.

As can be seen, and as is to be expected, the power of c increases as $w_r/\bar{w}_r$ decreases, being in excess of about 70 per cent for $w_r/\bar{w}_r \leqslant 0{\cdot}4$ for all combinations of σ^2 and f_r. There appear to be only slight increases in power for a given $w_r/\bar{w}_r$ value as f_r declines. For a given f_r there are no consistent differences in power for different σ^2 values; nevertheless, the general indications are that, for given f_r and $w_r/\bar{w}_r$, the power of c varies inversely with σ^2.

IV

In view of the remarks of Chesher and McMahon (1976), the main conclusion to emerge from the study is that for the range of combinations of parameter values used, the statistic c appears to be approximated very well by an F distribution with degrees of freedom chosen according to the procedure set out by Harrison and Nolan (1975). Contrary to expectations, the c' statistic does not appear to be well approximated by an F distribution, at least by an F distribution with the degrees of freedom that were used.

Table 3: *Power of c at the 5 per cent significance level* (%)

(i) $f_r = 0{\cdot}70$

w_r	1	5	6	7	8	9	10	20	30	40	50	75	100	150	200	250
σ^2																
1·0	100	100	100	100	100	100	100	100	100	100	100	94	80	41	13	4
1·5	100	100	100	100	100	100	100	100	100	74	49	13	5	3	—	—
2·0	100	98	76	62	35	15	8	1	—	—	—	—	—	—	—	—

(ii) $f_r = 0{\cdot}63$

w_r	1	10	20	30	40	50	75	100	150	200	250	275	300
σ^2													
1·0	100	100	100	100	100	100	99	98	84	38	11	7	8
1·5	100	100	100	100	97	92	42	26	5	—	—	—	—
2·0	100	99	31	9	2	1	—	—	—	—	—	—	—

(iii) $f_r = 0{\cdot}40$.

w_r	30	50	75	100	150	200	250	275	300	400	500	600	750
σ^2													
1·0	100	100	100	100	100	100	99	99	97	85	52	24	6
1·5	100	100	100	99	96	83	58	47	30	8	3	—	—
2·0	100	96	64	35	9	1	—	—	—	—	—	—	—

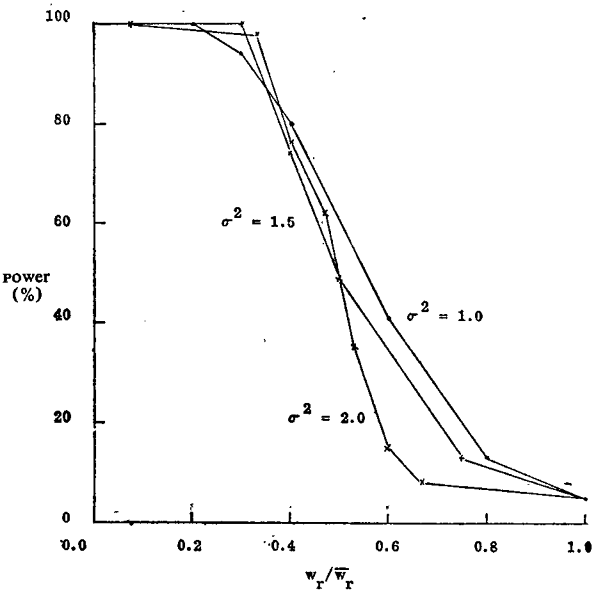

Figure 1: *Estimated power functions of c at the 5 per cent significance level for f_r = 0·70*

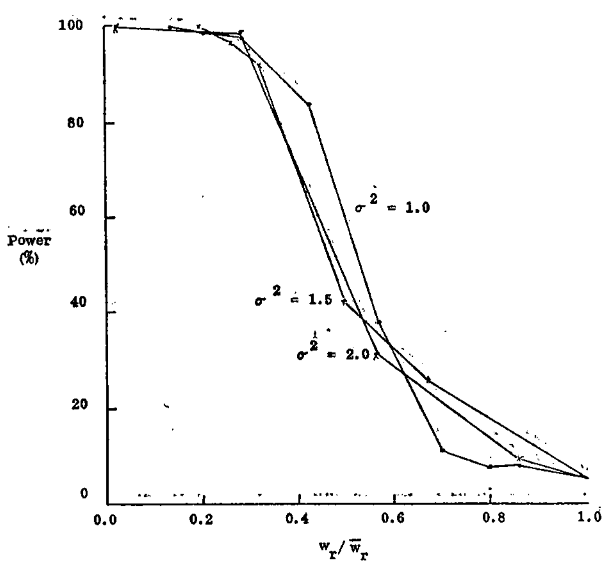

Figure 2: *Estimated power functions of c at the 5 per cent significance level for f_r = 0·63*

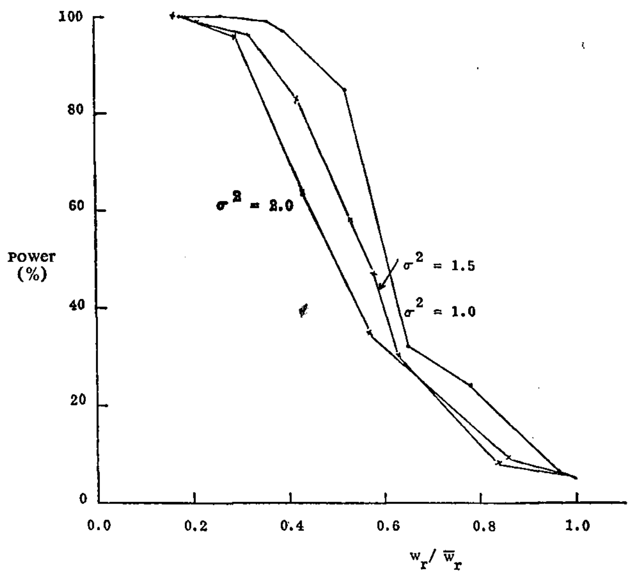

Figure 3: *Estimated power functions of c at the 5 per cent significance level for f_r = 0·40*

As the parameter values used were selected to simulate estimated wealth distributions which conformed closely to the distribution of Irish personal wealth as estimated by Lyons, the results also seem to support the use of the test procedure with Lyons's data, and suggest the retention of the conclusion so reached (Harrison and Nolan, 1975). It should be emphasised, however, that no claim of general applicability beyond the range of parameter values used in this study is made for the *c* test.

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