

Coulomb Calligraphy

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The equilibrium configurations of N Coulomb charges confined by a harmonic potential in two dimensions are found to exhibit a wide variety of patterns that are pleasing to the eye, when adjacent points are joined up. We present a selection of these for $N = 50$. The patterns are reminiscent of early examples of algorithmic or computer-generated art, and might be of use for psycho-physical studies of the perception of structure and randomness. They could also find applications in the generation of *recognisable* symbols in an alphabet of infinite extent, easily generated and varied using the software that is provided here.

Key words: Algorithmic art, Ion crystals, Random point patterns

1. Introduction

The first exhibitions of computer generated graphics in art galleries date back to 1965 [1, 2]. The protagonists, such as Michael Noll, Georg Nees and Frieder Nake, had backgrounds in computer engineering, mathematics and physics. The algorithms that they designed made use of pseudo-random numbers to produce output that could be displayed using the very basic graphics plotters that were available at the time. Noll’s ‘Computer Composition with Lines’, for example, consists of randomly placed short vertical and horizontal lines, reminiscent of some of Piet Mondrian’s paintings, such as ‘Composition with Lines’ from 1917. Georg Nees’ work ‘Locken’ displays contours of randomly placed circles. In her series ‘Interruptions’ (1968–69) Vera Molnár, who has a classical fine art training, wrote computer code to randomly rotate and erase lines in an initially ordered grid [3].

Not surprisingly, the applications of computers in art has grown rapidly ever since. The results are variously called computer art, digital art, algorithmic art, generative art etc. [2, 4]. The expression “Künstliche Kunst” (artificial art) is also being used by some of the artists [5, 6]. As algorithms increasingly interfere in any aspects of life, it is argued that computer art is perfectly suited to engage with this development [7].

In the course of research on equilibrium states of ions in two dimensions we have come across patterns that appear to resonate with those early works of computer-generated art. We found distributions of points, obtained from an algorithm for the solution of a physical problem, which can be connected by lines following basic principles of human perception. Thousands of examples may be computed in a few minutes [8] for systems with about 10 ions (while a similar number of examples for systems containing 50

ions can be computed over the course of a few hours). All conform to and are generated by the same simple basic rule, a principle of physics. Each is defined by just a few parameters, and these can take many values.

An example is shown in Fig. 1a: it is a compact array of points, in which we can readily recognise lines, junctions and isolated points, a clear manifestation of what are called the grouping principles of proximity and good continuation in Gestalt psychology [9, 10]. We have chosen to process this primitive pattern by recognising the lines of points and joining them up smoothly, as in Fig. 1b–f, with a rule that draws the lines which is somewhat arbitrary. (All of the examples that are shown in later sections have been processed using the same rule, as defined below.)

Such modifications or interventions to an algorithmic outcome have been commonly employed in algorithmic art; they could take the form of Georg Nees simply turning off the plotter once he was satisfied with the produced image [1] or Manfred Mohr removing individual edges from his computer generated images of cubes, to disturb symmetries and create visual tension [2].

2. The Governing Rules

Each of the patterns described here begins as an equilibrium configuration of points in a plane, or equivalently charges, interacting with the repulsive Coulomb force ($1/r^2$) and confined by a harmonic potential. The scientific interest in such systems, which (to some extent) can be realised experimentally [11], is driven by their relevance to quantum computing [12].

The total energy \tilde{E} of a cluster of N charges Q , confined by a $2D$ isotropic harmonic potential, is given by

$$\tilde{E} = \frac{k}{2} \sum_i^N (X_i^2 + Y_i^2) + \frac{Q^2}{4\pi\epsilon_0} \sum_{i<j}^N [(X_i - X_j)^2 + (Y_i - Y_j)^2]^{-1/2}, \quad (1)$$

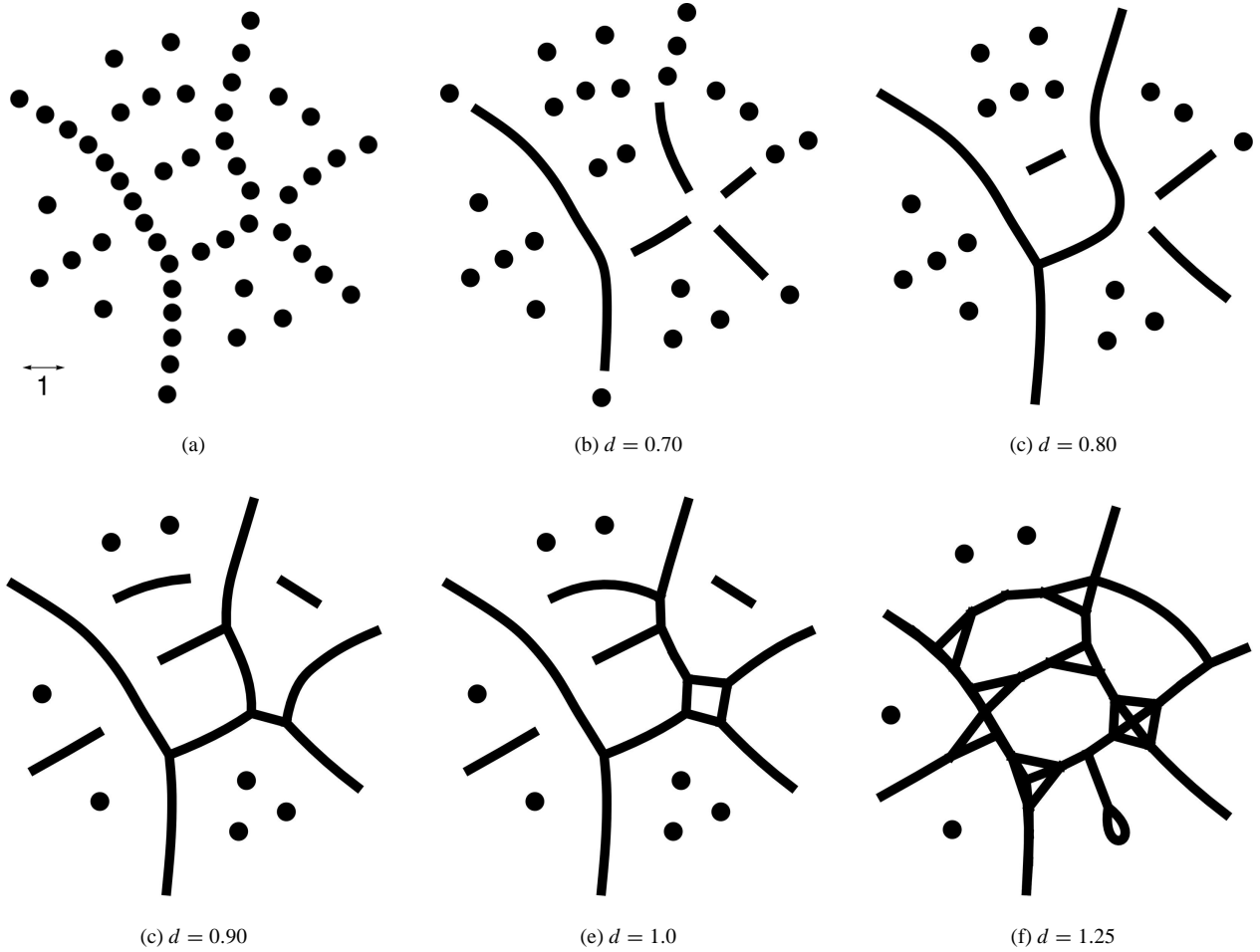


Fig. 1. (a) An example of the computed point pattern for $N = 50$ (energy $E = 651.545$, Eqn. (2)). The pattern can be modified by connecting points via (curved) lines, if they are less than a maximum (dimensionless) distance d apart from each other. The first image also shows a scale bar indicating a distance of 1. Figures (b–f) show the results of this procedure for several choices of d .

where X_i and Y_i are the Cartesian coordinates of the i th charge, while k is the force constant for the harmonic confining potential. For convenience, we use $a = \frac{Q^2}{4\pi\epsilon_0}$ (ϵ_0 is the permittivity of free space) in what follows and consider the dimensionless energy E , defined as $E = \tilde{E}/(a^{2/3}k^{1/3})$. It is given by

$$E = \frac{1}{2} \sum_i^N (x_i^2 + y_i^2) + \sum_{i < j}^N [(x_i - x_j)^2 + (y_i - y_j)^2]^{-1/2}. \quad (2)$$

Here $x_i = X_i/l_0$ and $y_i = Y_i/l_0$ are dimensionless coordinates, with the length scale $l_0 = (a/k)^{1/3}$. Eqn. (2) is the equation whose equilibrium solutions for the N coordinates (x_i, y_i) we seek, with examples displayed in Figs. 1 and 2.

In previous work [8] we have also explored the effects of an anisotropic potential, i.e. the case where the force constant k in Eqn. (1) is replaced by constants k_x and k_y , acting in x and y directions, respectively. This allows the aspect ratio of the resulting pattern of ions to be varied, further adding to the richness of the results that could be generated. In the following, however, we will restrict ourselves to the isotropic case of Eqns. (1) and (2).

The algorithm that we use to find equilibrium solutions of Eq. (2) starts from a random initial arrangement of points in the $x - y$ plane (see [8] for details). We seek by iteration a configuration for which E is stationary, i.e. in terms of physics a solution for which the sum of forces on any given charge is zero. Most of these configurations produced are *unstable* equilibrium solutions, typically consisting of a compact set of points arranged in a fashion that is suggestive of lines, junctions and isolated points, as in Fig. 1a.

In the case of an unstable equilibrium solution even a slight perturbation may lead to the collapse of the configuration to a stable arrangement. Unstable solutions are to be contrasted with stable solutions, of the type reported by others (see for example [13]), which are (typically) crystalline in nature. Although both (stable and unstable solutions) can be found by numerical minimisation techniques, in the case of large systems (such as $N = 50$) the number of unstable solutions vastly outnumbers the stable solutions.

Having identified unstable configurations of charges, we proceed to introduce lines to some of the points as follows. In the first stage any points within a chosen dimensionless cut-off distance d of each other are considered to be connected. In the second stage, such ions or points are fitted by cubic *basis splines* (B-splines), resulting in smoothly bend-

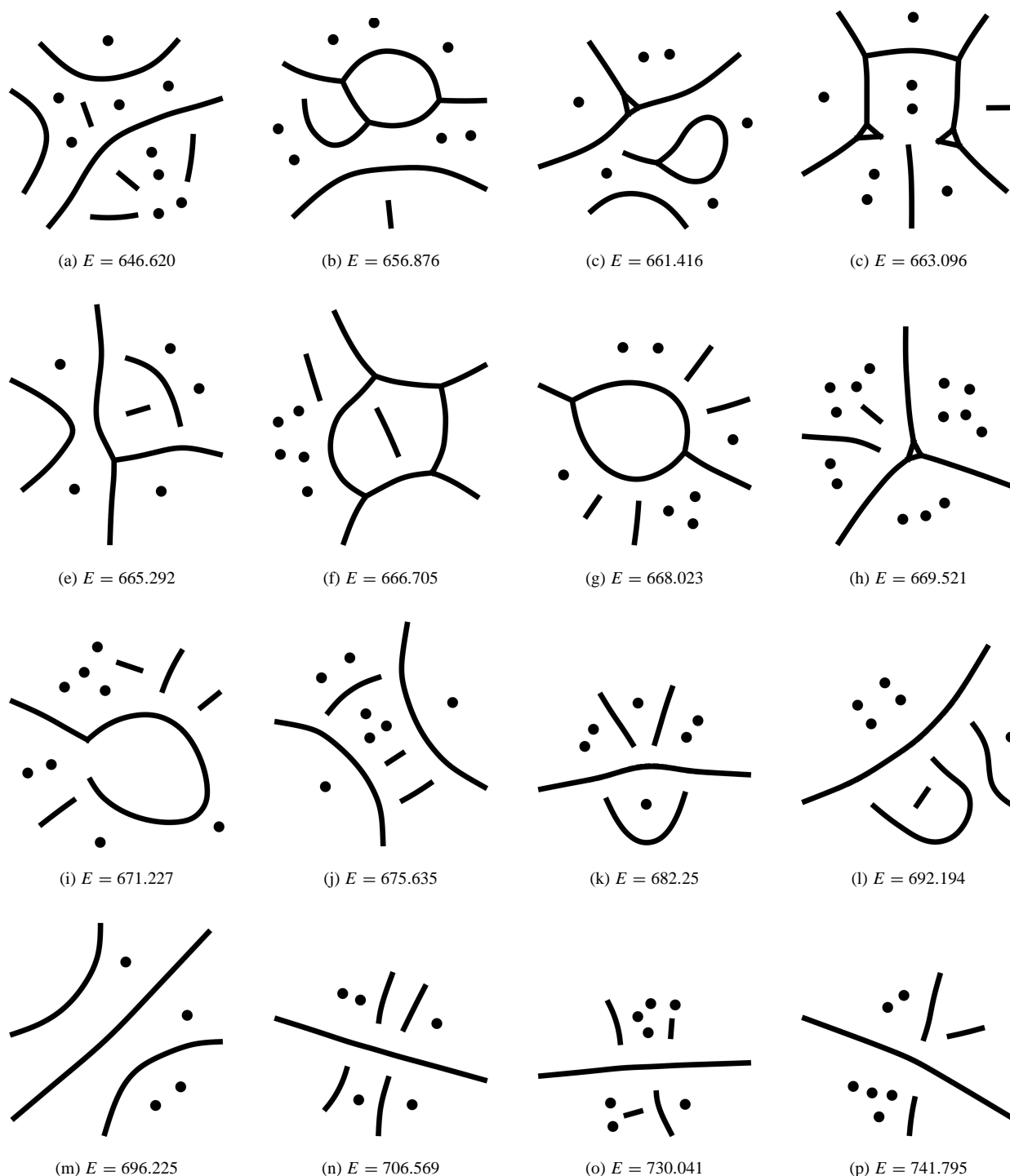


Fig. 2. Gallery of patterns for $N = 50$, showing a very small fraction of the total number of computed arrangements. Others may be sampled using the software specified in Appendix A. All patterns represent equilibrium solutions of Eqn. (2), with $d = 0.9$ for the maximum distance which determines which nearby points will be connected by a line.

ing curves that fit the sequence of points. Such splines are constrained to pass through the first and last point in the sequence, but not necessarily constrained to pass precisely through the intermediate points [14, 15].

Fig. 1b–f show some of the possibilities that can arise as this cut-off distance is varied. For the rest of the images shown below we chose a value of $d = 0.9$ on purely subjective aesthetic grounds (for comparison Fig. 1a shows a scale bar indicating a distance of 1).

3. Representative Patterns for $N = 50$

For the purposes of this presentation, we will confine our attention to the case $N = 50$, for which we have computed around six hundred patterns (by no means an exhaustive catalogue), of which a sample is shown in Fig. 2. We have placed a complete catalogue of our results, online at [16], together with the Mathematica script that generates them. See Appendix A for details.

Each such pattern may be uniquely identified in terms of

its energy E (Eqn. (2)). In practice, we never find any two equilibrium configurations with values of E close enough to make this designation ambiguous (at least this is the case for the approximately six-hundred arrangements that we have catalogued - see [16], for each of which the energy is stated to three-decimal places). Note however that each configuration may be reflected and rotated to create others of the same energy.

The sequence of patterns shown in Fig. 2 is arranged in order of increasing energy. A crude interpretation of the general nature of the patterns is as follows. Configurations with low energies (Fig. 2a) tend to be comprised of points arranged in compact clusters centred about the origin. Configurations with higher energies are more frequently comprised of one or two long lines, with isolated points (and shorter lines) scattered throughout (Fig. 2p).

Viewed imaginatively these patterns may conjure up images of human settlements, archaeological sites, battlefields, or Eastern forms of calligraphy, in extraordinary variety. The mind will search for the underlying principle—we have already stated it, in all its simplicity. Truly countless possibilities (defined by N and E) are available for exploration.

4. Point Patterns

While we chose to turn the point patterns into calligraphy-like symbols by replacing nearby points with lines, it might be instructive to analyse the point patterns themselves in terms of existing symmetries. This could be achieved for example via the computation of (2d) bond order parameters, which are used to characterize the structures of glasses or jammed materials [17].

Random point patterns are commonly used in psychophysical studies of human perception [18, 19, 20]. While too much order might be perceived as boring, too much randomness may lead to confusion of the viewer. An exploration of the tension between these two sensations and the successful identification of a “midpoint” between them characterises many works of art [21].

5. Conclusion and Outlook

We have presented samples of an extensive catalogue of 2d patterns made up of points and lines. While the position of the points is determined from numerically obtained equilibrium solutions to an equation from electrostatics (Eqn. (2)), the introduction of lines is based on elementary principles of Gestalt psychology (proximity, good continuation), again implemented numerically.

While the patterns may be appreciated solely for their aesthetic value, or may have merit for the study of symmetry and human perception, they could also lead to practical applications, as follows.

Throughout our initial experimentation with the (line) patterns we were intrigued how readily the brain seeks to interpret them as calligraphic symbols of some imaginary language. It might be possible to exploit this for the creation of a visual code, similar to the QR code. The code is readily produced computationally, and via the vast numbers of patterns (noting that here we only explored $N = 50$ in an isotropic potential) could result in an “infinite alphabet”

of symbols. Its advantage lies in the fact that, unlike a QR code, the patterns suggest some sort of subjective content or meaning, much like the images used in a Rorschach test. They are therefore easily committed to memory.

While we have only employed crude methods to join together the points into patterns, it is clear that much more sophisticated methods could be used. One possibility would be to ask test participants to join together the dots in a manner that appeals to them. The resulting data set could be used to train a neural network which could in turn produce further images, which would reflect the artistic instincts of the humans used to train it.

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Appendix A. Catalogue and Code

The original images representing positions of the Coulomb charges in equilibrium, along with the processed images (using a cut-off of $d = 0.9$ to join the points) can be found here [16]. Each pair of images is identified by a unique *id* number and by its energy E .

We also provide the algorithm used to generate the equilibrium configurations. This is in the form of a Mathematica notebook (https://www.dropbox.com/s/6y4und9mgelmifl/generate_equilibrium_configurations_published_23-01-30.nb?dl=0 [Accessed 10th February 2023]) which requires the use of Mathematica [22] to run (tested on version 13). We also provide a PDF print out of this file for users without access to Mathematica (https://www.dropbox.com/s/rj11s9529m94oxo/generate_equilibrium_configurations_published_23-01-30.pdf?dl=0 [Accessed 20th March 2023]).

The output of the Mathematica notebook is in the form of a text file. Each equilibrium configuration, generated by the algorithm for a system of N ions, is given on a single line of the text file. The first number on each line is the energy of the configuration, this is followed by N numbers which are the x-coordinates of the ions, followed by a further N numbers which are the corresponding y-coordinates of the ions.

Currently, the algorithm is set to run 1000 times. However the actual number of configurations listed may be less than this. The algorithm only outputs equilibrium configurations. A configuration is considered to be in equilibrium if the value of the objective function (as defined in [8]) is less than a small tolerance value (i.e. 1×10^{-10} , as defined in the Mathematica script) and only results fulfilling this condition are written to the file (as a result for systems with $N = 50$ it may take some minutes before valid results are written to the file).

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