A closed-form non-stationary solution of linear systems with fractional order 1/2 subjected to stochastic excitation

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ABSTRACT: This paper develops a novel method for determining a closed-form non-stationary stochastic response of linear systems with fractional derivative order 1/2 and subjected to stationary stochastic excitation. This is achieved by relying on the Laplace transform-based method for the linear fractional system, where the closed-form solution of the pulse response function is obtained by the eigenvector expansion of the state-space equation of the linear system with fractional derivative order 1/2. Pertinent Monte Carlo simulations demonstrate the applicability and accuracy of the proposed method.

1. INTRODUCTION

Fractional order derivative can be regarded as an extension of the traditional integer-order derivatives by allowing derivative operations of arbitrary real or complex order [1]. The fractional order derivative concept has a history of over 300 years, and it can be traced back to the idea during the correspondence of Leibniz with L'Hospital in 1695 [2]. During the first century after its introduction, the fractional derivative was considered only an interesting topic without rigorous theoretical development before its first mathematical definition by Liouville in 1832. It was not until the middle of the twentieth century when Gemant [3] and Bosworth [4] first proposed using the fractional derivative on modeling viscoelastic materials the fractional derivative began to be widely used by scientists and engineers. In recent decades, the fractional derivative has been applied in many science and engineering fields, such as the modeling of materials [5], physics [6], civil engineering [7], applied mathematics [8], bio-engineering [9] and so forth.

By allowing the derivative of arbitrary order, fractional models exhibit distinct advantages in modeling many natural and man-made phenomena [10]. Consequently, the difficulty of the response determination is increased due to the complicated definition and system memorability [1]. Therefore, developing analytical/numerical methods for fractional systems under various excitation conditions has become an increasingly important and active research field [11]. Since fractional linear systems satisfy the superposition principle, a natural idea is the Duhamel integral to compute the deterministic/stochastic response, which requires the impulse response function (IRF) of the system. Therefore, a popular strategy has been attempting to obtain the fractional oscillator's IRF first [12]. Specifically, using the inverse Fourier transform of the frequency response function, Gaul et al. [13] gave the IRF of a linear system with fractional order $\alpha = 1/2$. Suarez and Shokooh [14] applied the Laplace domain method to obtain an analytical IRF of a fractional oscillator with fractional order $\alpha = 1/2$ and

$\alpha = 1/3$. By applying the generalized Mittag-Leffler functions, Achar et al. [15] expressed the IRF as a summation of infinity terms. Most recently, Cao et al. [16] proposed the pole-residue form solution for fractional systems with arbitrary order and computed the system deterministic response under arbitrary excitation. From the preceding literature review, it may be argued that, unlike the integer-order linear dynamic systems with a simple exact analytic-form solution, most of the IRFs of fractional systems obtained so far are in the forms of numerical, semi-numerical, or infinite series. Note for a particular case with fractional order 1/2, a simple analytical solution for the IRF was developed based on the eigenvector expansion of the state-space equation of the linear fractional systems[17].

Furthermore, considering the uncertainties associated with external excitations leads to the problem of determining the stochastic response of fractional linear/nonlinear systems under stochastic loads [18]. In the past few years, a great amount of work has been focused on this topic and dedicated to treating complicated engineering problems such as the stationary response of a hysteretic system with fractional derivative element[19], the non-stationary response of a single-degree-of-freedom (SDOF) [18, 20] and multi-degree-of-freedom (MDOF) nonlinear systems[21], and fractional order nonlinear systems subjected combined deterministic and stochastic excitations [11]. Note that most of these works are numerical or semi-analytical methods and rely on special assumptions to obtain approximate solutions. The analytical method for the close-form response is rare even for a very simple fractional system (a SDOF linear fractional system for example) [1, 22]. Recently, Cao el al. [23] proposed a Laplace domain method for the non-stationary response of fractional oscillators to evolutionary stochastic excitation, which can be regarded as an extension of the method proposed by Hu et al. [24] to fractional systems. This approach requires the IRF of the fractional order system, in which the inverse Fourier transform of the frequency response function and Prony's method are utilized for calculating the pole-residue form of

IRF. Therefore, the stochastic response obtained is approximate, and the accuracy of the method depends on the numerical method.

This research proposed a novel method to compute the closed-form non-stationary solution for determining the stochastic response of linear systems endowed with fractional derivative order 1/2 and subjected to stationary stochastic excitation. Compared to the work of Cao et al. [23], an analytical expression of IRF with closed-form Laplace transform obtained by the eigenvector expansion method [17] is introduced. By doing so, the stochastic non-stationary response of the considered system can be derived in a closed form. Pertinent Monte Carlo simulations demonstrate the applicability and accuracy of the proposed method.

2. MATHEMATICAL FORMULATION

The governing equation of motion of an SDOF linear system endowed with fractional derivative elements and subject to stochastic excitation is given by

$$m\ddot{x}(t) + c\mathrm{D}^{\alpha}x(t) + kx(t) = F(t), \qquad (1)$$

where m, c and k are mass, damping and stiffness coefficient, respectively; D^{α} represents the fractional derivative based on the Riemann-Liouville definition:

$$\mathrm{D}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}\frac{x(t-\tau)}{\tau^{\alpha}}\mathrm{d}\tau \quad 0 < \alpha < 1, \tag{2}$$

where α is the factional order; $\Gamma(\cdot)$ is the Gamma function; $F(t)$ is a zero mean stationary Gaussian stochastic excitation with the power spectrum density (PSD) $S_F(\omega)$ and correlation function (CF) $R_F(\tau)$. The following Fourier transform pair relate the PSD and CF

$$S_F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_F(\tau)\mathrm{e}^{-i\omega\tau}\mathrm{d}\tau, \qquad (3)$$

$$R_F(\tau) = \int_{-\infty}^{\infty} S_F(\omega)\mathrm{e}^{-i\omega\tau}\mathrm{d}\omega. \qquad (4)$$

The random response $x(t)$ is a zero mean stochastic process, since the excitation of the linear system is zero mean. Assume the system is initially

2

at rest, that is, $\dot{x}(0) = 0$, $x(0) = 0$, the correlation function of the response can be written as [24]

$$
\begin{aligned}
R_x(t_1, t_2) &= \mathrm{E}[X(t_1)X(t_2)] \\
&= \int_0^{t_1} \int_0^{t_2} h(t_1 - \tau_1) h(t_2 - \tau_2) R_F(\tau_1, \tau_2) \, d\tau_1 d\tau_2,
\end{aligned}
$$

(5)

where $h(t)$ is the IRF of the fractional system. Eq. (5) illustrates the IRF plays a crucial role in the stochastic response determination. However, the closed-form solution of $h(t)$ can not determined as a simple analytical function for a linear system with arbitrary fractional derivative order. Therefore, it is unlikely to obtain an explicit analytical solution for the response. It is worthy noting that for some special cases, for example, when the fractional order α is a rational number, the closed-from solution of the IRF can be determined using eigenvector expansion method [17]. For simplicity and without loss of generality, a special fractional order $\alpha = 1/2$ is considered in this study for illustration.

3. CLOSED-FORM SOLUTION OF IRF WHEN $\alpha = 1/2$

The IRF is the response of the system subjected to impulse excitation. That is

$$
\ddot{x}(t) + 2\xi \omega_n^{3/2} \mathrm{D}^{1/2} x(t) + \omega_n^2 x(t) = \delta(t), \quad (6)
$$

where

$$
2\xi \omega_n^{3/2} = \frac{c}{m}, \quad \omega_n^2 = \frac{k}{m}. \quad (7)
$$

Let

$$
\begin{aligned}
\mathbf{z}(t) &= [z_1(t), z_2(t), z_3(t), z_4(t)]^{\mathrm{T}} \\
&= \left[\mathrm{D}^{3/2} x(t), \dot{x}(t), \mathrm{D}^{1/2} x(t), x(t) \right]^{\mathrm{T}},
\end{aligned}
$$

(8)

the equation of motion can be transform into the fractional-order state-space equation as

$$
\mathrm{D}^{1/2} \mathbf{z}(t) = \mathbf{G}\mathbf{z}(t) + \mathbf{Q}(t), \quad (9)
$$

where

$$
\mathbf{G} = \begin{bmatrix} 0 & 0 & -a & -b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta(t) \end{Bmatrix}, \quad (10)
$$

with $a = 2\xi \omega_n^{3/2}$, $b = \omega_n^2$. A solution based on the eigenvector expansion can be used for solving Eq. (9). In this regard, consider

$$
\mathbf{G}\{\Phi\}_j = \lambda_j \{\Phi\}_j, \quad (11)
$$

where, λ_j and $\{\Phi\}_j$, $j = 1, 2, 3, 4$, are the eigenvalues and eigenvectors of $\mathbf{G}$, respectively; See also Ref. [17] for details of determining λ_j and $\{\Phi\}_j$. Specifically,

$$
\begin{aligned}
\lambda_1 &= \lambda_2^* = p + iq, \\
\lambda_3 &= -p + is, \\
\lambda_4 &= -p - is.
\end{aligned}
$$

(12)

where $*$ means taking the complex conjugate, and

$$
\begin{aligned}
p &= \sqrt{\omega_n \kappa}, \\
q &= \sqrt{\omega_n \left(\kappa + \frac{\xi}{2\sqrt{\kappa}} \right)}, \quad s = \sqrt{\omega_n \left(\kappa - \frac{\xi}{2\sqrt{\kappa}} \right)}, \\
\kappa &= \frac{2^{1/3}}{4} \left[\left(\xi^2 + \sqrt{\xi^4 - \frac{16}{27}} \right)^{1/3} \right. \\
&\quad \left. + \left(\xi^2 - \sqrt{\xi^4 - \frac{16}{27}} \right)^{1/3} \right].
\end{aligned}
$$

(13)

$$
\{\Phi\}_j = \begin{bmatrix} \lambda_j^3 & \lambda_j^2 & \lambda_j & 1 \end{bmatrix}^{\mathrm{T}} \alpha_j, \quad (14)
$$

with $\alpha_j = 1/\sqrt{4\lambda_j^3 + a}$.

The eigenvectors satisfy the following orthogonality conditions

$$
\begin{cases} \{\Phi\}_i^{\mathrm{T}} \mathbf{G}\{\Phi\}_j = 0, & i \neq j, \\ \{\Phi\}_i^{\mathrm{T}} \mathbf{G}\{\Phi\}_j = \lambda_j, & i = j. \end{cases} \quad (15)
$$

Introducing a new variable $\mathbf{y} = [y_1, y_2, y_3, y_4]^{\mathrm{T}}$ and letting

$$
\mathbf{z} = [\Phi] \mathbf{y}, \quad (16)
$$

the system displacement $x(t)$ can be written as

$$
x(t) = \sum_{j=1}^{4} \Phi_{4j} y_j(t). \quad (17)
$$

Substituting Eq. (16) into Eq. (9), the state space equation becomes

$$
\mathrm{D}^{1/2} [\Phi] \mathbf{y} = \mathbf{G} [\Phi] \mathbf{y} + \mathbf{Q}(t). \quad (18)
$$

Pre-multiplying Eq. (18) by $\{\Phi\}_j^T$ and considering Eq. (15) lead to four decoupled differential equations

$$D^{1/2} y_j(t) - \lambda_j y_j(t) = \Phi_{4j} \delta(t), \qquad (19)$$

where [17]

$$\Phi_{4j} = \frac{1}{\sqrt{4\lambda_j^3 + 2\xi \omega_n^{3/2}}}. \qquad (20)$$

The closed-form solution of Eq. (19) can be obtained by the Laplace transform method. This is achieved by relying on the Laplace transom property of the fractional derivative elements, i.e.,

$$L[D^\alpha x(t)] = s^\alpha X(s) - C, \qquad (21)$$

where $X(s)$ is the Laplace transform of $x(t)$, and C is a constant defined as

$$C = D^{\alpha-1} x(t)\big|_{t=0}. \qquad (22)$$

Furthermore, the Laplace transform of the impulse function is

$$L[\delta(t)] = 1. \qquad (23)$$

Taking Laplace transform of Eq. (19) and using Eq. (23) and Eqs. (21)-(22) for $\alpha = 1/2$, lead to

$$Y_j(s) = \frac{\Phi_{4j} + R_j}{\sqrt{s} - \lambda_j}, \qquad (24)$$

where $Y_j(s)$ is the Laplace transform of $y_j(t)$; R_j is a constant defined as

$$R_j = D^{-1/2} y_j(t)\big|_{t=0}. \qquad (25)$$

Taking inverse Laplace transform of Eq. (24) yields

$$y_j(t) = (\Phi_{4j} + R_j) L^{-1}\left(\frac{1}{\sqrt{s} - \lambda_j}\right). \qquad (26)$$

The inverse Laplace transform of $\left(1/\sqrt{s} - \lambda_j\right)$ can be determined using residue theory and contour integration:

$$L^{-1}\left[\frac{1}{\sqrt{s} - \lambda_j}\right] = \frac{1}{\sqrt{\pi t}} + \lambda_j e^{\lambda_j^2 t}\left[1 + \text{erf}\left(\lambda_j\sqrt{t}\right)\right], \qquad (27)$$

where $\text{erf}(\cdot)$ is the error function, defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \qquad (28)$$

Substituting Eqs. (26) - (28) into Eq. (17) leads to the closed-form response of the considered system under impulse excitation, i. e. the IRF

$$h(t) = x(t) = \frac{1}{\sqrt{\pi t}} \sum_{j=1}^4 \left(\Phi_{4j}^2 + \Phi_{4j} R_j\right)$$
$$+ \sum_{j=1}^4 \left(\lambda_j \Phi_{4j}^2 g_j(t) + \lambda_j \Phi_{4j} g_j(t) R_j\right), \qquad (29)$$

where

$$g_j(t) = e^{\lambda_j^2 t}\left[1 + \text{erf}\left(\lambda_j\sqrt{t}\right)\right]. \qquad (30)$$

If the system is initially at rest, the following additional conditions should be satisfied

$$\sum_{j=1}^4 \Phi_{4j}^2 = 0, \quad \sum_{j=1}^4 \lambda_j \Phi_{4j}^2 = 0, \quad \sum_{j=1}^4 \lambda_j \Phi_{4j}^2 R_j = 0. \qquad (31)$$

Finally, substituting Eq. (31) into Eq. (29) and replacing the eigenvector Φ_{4j} from Eq. (20), yields the closed-form solution for the IRF of the fractional system with fractional order $\alpha = 1/2$. That is

$$h(t) = \sum_{j=1}^4 \frac{\lambda_j/2}{2\lambda_j^3 + \xi \omega_n^{3/2}} g_j(t). \qquad (32)$$

The response in the Laplace domain can be written as

$$x(s) = h(s) f(s), \qquad (33)$$

where $x(s)$, $h(s)$ and $f(s)$ are the Laplace transforms of $x(t)$, $h(t)$ and $f(t)$, respectively. According to Eq. (32) and Eq. (30)

$$h(s) = \sum_{j=1}^4 a_j g_j(s), \qquad (34)$$

where

$$a_j = \frac{\lambda_j/2}{2\lambda_j^3 + \xi \omega_n^{3/2}}, \quad g_j(s) = \frac{1}{s - \lambda_j^2} + \frac{\lambda_j}{\sqrt{s}\left(s - \lambda_j^2\right)}. \qquad (35)$$

4. LAPLACE DOMAIN METHOD FOR THE RESPONSE

With the fully understanding of the IRF, the stochastic response of the system can be further investigated. Substituting Eq. (4) into Eq. (5), yields

$$
R_x(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} h(t_1 - \tau_1) h(t_2 - \tau_2)
$$
$$
\times \left[\int_{-\infty}^{\infty} S_F(\omega) e^{i\omega(\tau_1 - \tau_2)} d\omega \right] d\tau_1 d\tau_2.
$$
(36)

Regrouping Eq. (36) to separate variables τ_1 and τ_2, leads to the following equation

$$
R_x(t_1, t_2) = \int_{-\infty}^{\infty} \left[\int_0^{t_1} h(t_1 - \tau_1) e^{i\omega\tau_1} d\tau_1 \right]
$$
$$
\times \left[\int_0^{t_2} h(t_2 - \tau_2) e^{-i\omega\tau_2} d\tau_2 \right] \cdot S_F(\omega) d\omega.
$$
(37)

When $t_1 = t_2$, Eq. (37) reduces to the equation of the mean square value (MSV) of the displacement as follows

$$
E\left[X^2(t)\right] = \int_{-\infty}^{\infty} \left[\int_0^t h(t - \tau_1) e^{i\omega\tau_1} d\tau_1 \right]
$$
$$
\times \left[\int_0^t h(t - \tau_2) e^{-i\omega\tau_2} d\tau_2 \right] \cdot S_F(\omega) d\omega.
$$
(38)

Introducing new variables

$$
p_1(t, \omega) = \int_0^t h(t - \tau) e^{i\omega\tau} d\tau,
$$
(39)

$$
p_2(t, \omega) = \int_0^t h(t - \tau) e^{-i\omega\tau} d\tau = p_1^*(t, \omega).
$$
(40)

Substituting Eq. (39) - (40) into Eq. (38), leads to

$$
E\left[X(t)^2\right] = \int_{-\infty}^{\infty} \Theta(t, \omega) d\omega,
$$
(41)

where

$$
\Theta(t, \omega) = |p_1(t, \omega)|^2 S_F(\omega),
$$
(42)

is the non-stationary power spectrum density of the response. In the ensuing analysis, $p_1(t, \omega)$ is derived in a closed form via the Laplace transform.

Taking Laplace transform of Eq. (39) and considering the closed-form solution of $h(s)$ shown in Eq. (34) lead to the Laplace transform of $p_1(t, \omega)$ in the form

$$
p_1(s, \omega) = \sum_{j=1}^{4} a_j \left[\frac{1}{s - \lambda_j^2} + \frac{\lambda_j}{\sqrt{s}\left(s - \lambda_j^2\right)} \right] \left(\frac{1}{s - i\omega} \right),
$$
(43)

Further, Eqs. (43) can be cast into

$$
p_1(s, \omega) = \sum_{j=1}^{4} a_j A_j \left[\frac{1}{s - \lambda_j^2} - \frac{1}{s - i\omega} \right.
$$
$$
\left. + \lambda_j \frac{1}{\sqrt{s}\left(s - \lambda_j^2\right)} - \lambda_j \frac{1}{\sqrt{s}(s - i\omega)} \right],
$$
(44)

where $A_j = 1/\left(-i\omega + \lambda_j^2\right)$.

Eq. (44) can be cast into a compact form as

$$
p_1(t, \omega) = \sum_{j=1}^{4} \sum_{k=1}^{4} D_{jk},
$$
(45)

where

$$
\begin{aligned}
D_{j1} &= a_j A_j e^{\lambda_j^2 t}, \\
D_{j2} &= -a_j A_j e^{i\omega t}, \\
D_{j3} &= a_j A_j e^{\lambda_j^2 t} \operatorname{erf}\left(\sqrt{\lambda_j^2 t}\right), \\
D_{j4} &= -a_j A_j \lambda_j \frac{1}{\sqrt{i\omega}} e^{i\omega t} \operatorname{erf}\left(\sqrt{i\omega t}\right).
\end{aligned}
$$
(46)

Finally, substituting Eq. (45) into Eq. (42) leads to the non-stationary power spectrum density of the response

$$
\Theta(t, \omega) = \left| \sum_{j=1}^{4} \sum_{k=1}^{4} D_{jk} \right|^2 \cdot S_F(\omega).
$$
(47)

Integrating Eq. (47) with respect to ω, yields the mean square value of the displacement of the considered system subjected to stationary stochastic excitation.

5. NUMERICAL EXAMPLES
5.1. Deterministic response

Select the system parameters $m = 1, c = 0.1, k = 1, \alpha = 0.5$. Calculate the response of the system subjected to a sample of white noise with $S_F(\omega) =$

0.01 via the Duhamel integral method (DIM) and the Newmark-β method. The deterministic response of the fractional system is shown in Fig. 1. Note that the closed-form IRF shown in Eq. (34) is used for the Duhamel integral result. From Fig.
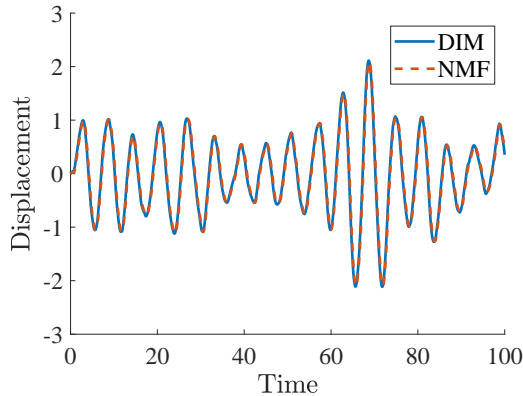

Figure 1: The response of the considered system subjected to a white noise

1, it seems that the response obtained by the DIM agrees with that obtained by the NMF quite well, proving the accuracy of the closed-form solution of the IRF.

5.2. Stochastic response

Investigate next the accuracy of the proposed method (PM) for determining the stochastic response of the considered system with typical parameters. The system parameters are the same as those used in Section 5.1. White noise with parameters $S_F(\omega) = 0.01, S_F(\omega) = 0.02, S_F(\omega) = 0.03$ are used for the excitation. The response MSVs of the system subjected to different excitation levels are shown in Fig. 2. The results obtained via the PM are compared with those estimated by the Monte Carlo simulation (MCS) over 1,000 samples. Fig. 2 shows that the responses obtained by the PM are in perfect agreement with those estimated by the MCS, both for the stationary and non-stationary phases of the response. Due to the limited number (1,000) of response samples, the MSC estimation exhibits obvious fluctuation, especially in the stationary phase. Besides, the fluctuation increases with increasing white noise strength. Obviously, the PM is computationally efficient for
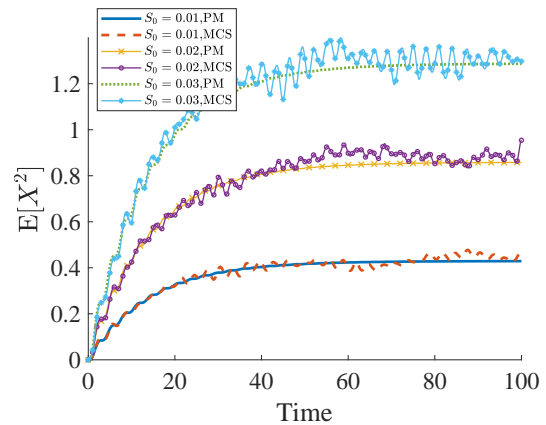

Figure 2: MSV of the system subjected to white noises with different PSD strength

only closed-form expressions are used in the calculation. Specifically, 870 seconds are needed for 10,000 times of simulation to obtain a satisfactory estimation, whereas only 3s for the PM.

Investigate next the accuracy of the proposed method under different system/excitation parameters. For the convenience of comparison, define the time-averaged standard deviation (TASD) of the response as

$$\bar{\sigma} = \frac{1}{T} \int_0^T \sqrt{\mathrm{E}\left[X^2(t)\right]} \mathrm{d}t. \qquad (48)$$

The TASD of the response versus the strength of the stochastic excitation, the frequency, and the damping ratio of the system, are shown in Figs. 3-5, respectively. For each considered case, all the other parameters remain the same as those used in Fig. 2($S_0 = 0.01$), except for the varying parameter under investigation. The agreement of the relevant results obtained by the proposed method with the pertinent Monte Carlo data suggests the accuracy of this method for systems with different parameters.

6. CONCLUDING REMARKS

An analytical method has been proposed to derive a closed-form non-stationary solution for the stochastic response of linear systems endowed with 1/2-order fractional elements and subjected to stationary stochastic excitation. This has been achieved in two steps. First, the eigenvector expansion of the fractional state-space equation has been
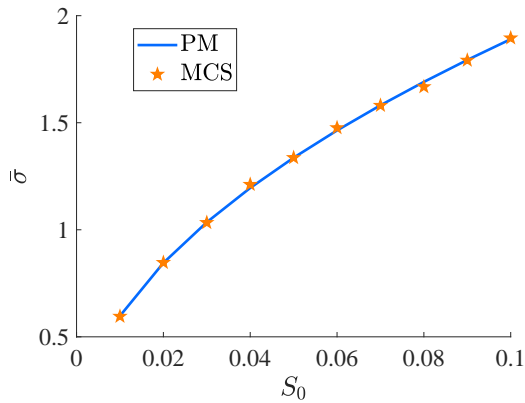
Figure 3: TASD of the considered system subjected to white noise stochastic excitation with various PSD strength
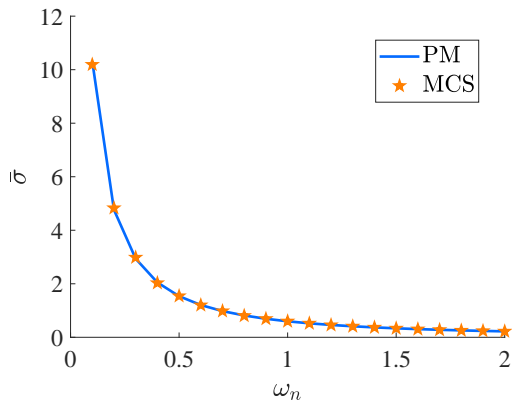

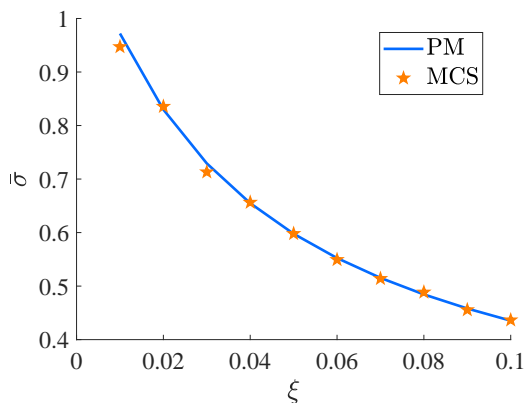*Figure 4: TASD of the considered system with different values of natural frequency ω*

Figure 5: TASD of the considered system with different values of damping ratio ξ

applied to obtain the system impulse response function in a closed form. Next, the derived closed-form

impulse response function and the Laplace transform have been utilized for determining the power spectral density of the non-stationary response. Numerical examples have shown the accuracy of the proposed method under different system/excitation parameters by comparing it to the Monte Carlo simulation. Besides, the proposed method has been proven to be computationally efficient due to the analytical solution. Therefore, the proposed analytical method may be helpful for the parameter optimization study of isolated structures with viscoelastic bearings. The proposed method can be further extended to multi-degree-of-freedom systems and/or subjected to stochastic excitation with evolutionary power spectrum density. Extensions to nonlinear fractional systems by applying the statistical linearization method also merit attention.

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