Complete decoupled adaptive robust design optimization

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ABSTRACT: Robust design optimization (RDO) can provide an optimum design solution that is relatively insensitive to input uncertainties, thus possesses significant value in the design of engineering structures. Generally, the RDO problem is a nested double-loop optimization process which requires huge computational costs especially for the case with complicated finite element model simulation. In view of this point, a completed decoupled adaptive RDO method based on the Kriging surrogate model is proposed, which can transform the nested double-loop uncertainty optimization process into the traditional deterministic optimization procedure. At the same time, closed-form expressions of the mean and standard deviation of performance functions under different design parameters are deduced and employed in the uncertainty propagation during the design optimization so to alleviate the post processing computational error. Moreover, the metamodeling uncertainty is considered in the established uncertainty propagation technique and an adaptive framework is introduced to improve the computational accuracy of uncertainty propagation so to further guarantee the estimation accuracy of RDO problems. An engineering application is introduced to illustrate the effectiveness of the established complete decoupled adaptive RDO method.

Engineering structures inevitably involve various uncertainties, such as deviation of material properties and geometric dimensions, which may severely degrade the performance of engineering structures (Chen and Li 2015). Considering these uncertainties, it is urgent to conduct design optimization of engineering structures. Moreover, with the increasing demand for high-quality engineering structures, the solution of design optimization should be robust under these uncertainties. Robust design optimization (RDO) is a theory that aims at providing a design optimization solution that is both optimum and relatively insensitive to input uncertainties.

Generally, the RDO problem requires making a tradeoff between the mean and standard deviation of the objective function, and this is usually completed by aggregating the mean and standard deviation in a single function with a weighted sum method. Directly solving the RDO problem is a nested double-loop optimization process, in which the outer loop searches for the optimal design parameter results and the inner loop estimates the mean and standard deviation of performance functions. This estimation process needs huge computational costs, especially for the case with time-consuming finite element model simulation. One possible way of addressing this challenge is to employ the surrogate model to replace the time-consuming computer simulation (Moustapha et al. 2016; Shi et al. 2020). Moreover, there always exists a discrepancy between the computer model and the surrogate model at unsampled points, introducing the so-called "metamodeling uncertainty" (Arendt et al. 2012; Zhang et al. 2013). The metamodeling uncertainty possesses an important influence on the accuracy of the surrogate model and further affects the solution of the RDO, thus it's necessary to take the metamodeling uncertainty into account in the surrogate model-assisted RDO, such as the methodology with a Bayesian framework (Apley et al. 2006), the approach with a multi-point objective-oriented sequential sampling strategy (Zhu et al. 2015) and the method with the Gaussian process model (Zhang et al. 2016). Generally, after the surrogate model of the performance function is well constructed by using the above surrogate model-assisted methods, several numerical integration techniques are usually used to estimate the mean and standard deviation of the performance function during the RDO procedure. However, although it is cheap to perform the uncertainty propagation based on the surrogate model, this process may introduce post-processing computational error (Shi et al. 2018).

In this work, an adaptive decoupled RDO method based on the Kriging surrogate model is proposed, in which a new description of metamodeling uncertainty that can simultaneously reflect the uncertainties in the prediction and the mean of performance function in estimating the standard deviation of the performance function is established. Furthermore, the closed-form expressions of the mean and standard deviation of performance function under different design parameters are deduced and employed in the uncertainty propagation during the design optimization, to alleviate the post-processing computational error, and the nested double-loop uncertainty optimization process is transformed into the traditional deterministic optimization procedure. Furthermore, a novel uncertainty propagation technique based on the Kriging surrogate model is established in this work, in which the design parameter is separated from inputs and the closed-form expressions of output moments under different design parameters are obtained based on only one surrogate model for each performance function. At the same time, a new description of metamodeling uncertainty is considered in the established uncertainty propagation technique and an adaptive framework is introduced to improve the computational accuracy of uncertainty propagation to further guarantee the estimation accuracy of RDO problems.

1. REVIEW OF THE ROBUST DESIGN OPTIMIZATION

RDO aims at providing a design optimization solution that is both optimum and relatively insensitive to input uncertainties of structures. Generally, the mathematical model of the RDO problem can be expressed as follows (Zhang et al. 2013):

$$\begin{aligned}
&\text{Minimize: } O=\mu\left[f\left(\mathbf{X},\mathbf{P},\mathbf{d}\right)\right]+c\sigma\left[f\left(\mathbf{X},\mathbf{P},\mathbf{d}\right)\right]\\
&\text{Subject to: } C_i=\mu\left[g_i\left(\mathbf{X},\mathbf{P},\mathbf{d}\right)\right]+c\sigma\left[g_i\left(\mathbf{X},\mathbf{P},\mathbf{d}\right)\right]\le 0\\
&\qquad (i=1,2,\ldots,n_g)\\
&\qquad \mathbf{d}^L\le\mathbf{d}\le\mathbf{d}^U,\boldsymbol{\mu}_{\mathbf{X}}^L\le\boldsymbol{\mu}_{\mathbf{X}}\le\boldsymbol{\mu}_{\mathbf{X}}^U
\end{aligned}\tag{1}$$

in which $\mathbf{X}=\left[X_1,X_2,\ldots,X_{n_X}\right]^{\mathrm{T}}$ is the n_X-dimensional random design vector with mean value vector $\boldsymbol{\mu}_{\mathbf{X}}=\left[\mu_{X_1},\mu_{X_2},\ldots,\mu_{X_{n_X}}\right]^{\mathrm{T}}$, and lower and upper bounds being $\boldsymbol{\mu}_{\mathbf{X}}^L=\left[\mu_{X_1}^L,\mu_{X_2}^L,\ldots,\mu_{X_{n_X}}^L\right]^{\mathrm{T}}$ and $\boldsymbol{\mu}_{\mathbf{X}}^U=\left[\mu_{X_1}^U,\mu_{X_2}^U,\ldots,\mu_{X_{n_X}}^U\right]^{\mathrm{T}}$ respectively. $\mathbf{P}=\left[P_1,P_2,\ldots,P_{n_p}\right]^{\mathrm{T}}$ is the n_p-dimensional random parameter vector. The mean value vector $\boldsymbol{\mu}_{\mathbf{X}}$ of the random design vector and the random parameter vector $\mathbf{P}$ constitute the design parameter vector of the RDO problem. $\mathbf{d}=\left[d_1,d_2,\ldots,d_{n_d}\right]^{\mathrm{T}}$ is the n_d-dimensional deterministic design vector with $\mathbf{d}^L=\left[d_1^L,d_2^L,\ldots,d_{n_d}^L\right]^{\mathrm{T}}$ and $\mathbf{d}^U=\left[d_1^U,d_2^U,\ldots,d_{n_d}^U\right]^{\mathrm{T}}$ being its lower and upper bounds respectively. $\mu\left[f\left(\mathbf{X},\mathbf{P},\mathbf{d}\right)\right]$ and $\sigma\left[f\left(\mathbf{X},\mathbf{P},\mathbf{d}\right)\right]$

are the mean and standard deviation of the objective function $f(\mathbf{X},\mathbf{P},\mathbf{d})$ respectively. $\mu[g_i(\mathbf{X},\mathbf{P},\mathbf{d})]$ and $\sigma[g_i(\mathbf{X},\mathbf{P},\mathbf{d})]$ are the mean and standard deviation of the i-th constraint function $g_i(\mathbf{X},\mathbf{P},\mathbf{d})(i=1,2,...,n_g)$ respectively. c is a constant value that reflects the risk attitude of the designer.

2. COMPLETE DECOUPLED ADAPTIVE ROBUST DESIGN OPTIMIZATION

Before constructing the complete decoupled adaptive RDO method, the random design variable $X_i(i=1,2,...,n_X)$ and the random parameter variable $P_j(j=1,2,...,n_P)$ are first transformed into standard normal variables $U_{X_i}(i=1,2,...,n_X)$ and $U_{P_j}(j=1,2,...,n_P)$ respectively based on the equivalent probabilistic transformation technique (Rosenblatt 1952). Denote $\mathbf{U_X}=\left[U_{X_1},U_{X_2},...,U_{X_{n_X}}\right]^{\mathrm{T}}$ and $\mathbf{U_P}=\left[U_{P_1},U_{P_2},...,U_{P_{n_P}}\right]^{\mathrm{T}}$, then the objective function $f(\mathbf{X},\mathbf{P},\mathbf{d})$ and constraint function $g_i(\mathbf{X},\mathbf{P},\mathbf{d})(i=1,2,...,n_g)$ in the standard normal space can be described by $F(\mathbf{U_X},\mathbf{U_P},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})$ and $G_i(\mathbf{U_X},\mathbf{U_P},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})(i=1,2,...,n_g)$ respectively. Such a treatment is beneficial to the construction of the complete decoupled adaptive RDO method.

In each iterative optimization of RDO, the key is to calculate the mean and standard deviation of the performance function under the design parameter. Therefore, if closed-form expressions of the mean and standard deviation of the performance function under different design parameters can be obtained, the RDO problem will be transformed into a traditional deterministic optimization problem, which will provide great benefits to the robust design of complicated structures. Next, the closed-form expression of the mean and standard deviation under different design parameters will be established.

2.1. Closed-form expression of the mean

For the convenience of expression, denote $\mathbf{W}=[\mathbf{U_X},\mathbf{U_P},\boldsymbol{\mu}_\mathbf{X},\mathbf{d}]$ and $\mathbf{U}=[\mathbf{U_X},\mathbf{U_P}]$ respectively. Take the construction of the closed-form expression of the mean of objective function $F(\mathbf{U},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})$ for example, the similar way can be used for the constraint function $G_i(\mathbf{U},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})(i=1,2,...,n_g)$. It is supposed that there are two realizations of the input vector $\mathbf{W}$, i.e., $\mathbf{w}^{(i)}=\left[\mathbf{u}^{(i)},\boldsymbol{\mu}_\mathbf{X}^{(i)},\mathbf{d}^{(i)}\right]^{\mathrm{T}}$ and $\mathbf{w}^{(j)}=\left[\mathbf{u}^{(j)},\boldsymbol{\mu}_\mathbf{X}^{(j)},\mathbf{d}^{(j)}\right]^{\mathrm{T}}$, in which $\mathbf{u}^{(i)}=\left[\mathbf{u}_\mathbf{X}^{(i)},\mathbf{u}_\mathbf{P}^{(i)}\right]$ and $\mathbf{u}^{(j)}=\left[\mathbf{u}_\mathbf{X}^{(j)},\mathbf{u}_\mathbf{P}^{(j)}\right]$. Then, the Gaussian correlation function can be rewritten by the following expression when constructing the Kriging surrogate model (Lophaven et al. 2002) $\hat{F}(\mathbf{U},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})$ of the objective function $F(\mathbf{U},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})$:

$$R_\theta\left(\mathbf{w}^{(i)},\mathbf{w}^{(j)}\right)=\Upsilon_\mathbf{u}\left(\mathbf{u}^{(i)},\mathbf{u}^{(j)}\right)\Upsilon_{\boldsymbol{\mu}_\mathbf{X}}\left(\boldsymbol{\mu}_\mathbf{X}^{(i)},\boldsymbol{\mu}_\mathbf{X}^{(j)}\right)\Upsilon_\mathbf{d}\left(\mathbf{d}^{(i)},\mathbf{d}^{(j)}\right) \quad (2)$$

in which $\Upsilon_\mathbf{u}\left(\mathbf{u}^{(i)},\mathbf{u}^{(j)}\right)=\exp\left\{-\sum_{k=1}^{n_X+n_P}\theta_k\left(u_k^{(i)}-u_k^{(j)}\right)^2\right\}$, $\Upsilon_{\boldsymbol{\mu}_\mathbf{X}}\left(\boldsymbol{\mu}_\mathbf{X}^{(i)},\boldsymbol{\mu}_\mathbf{X}^{(j)}\right)=\exp\left\{-\sum_{k=1}^{n_X}\theta_{n_X+n_P+k}\left(\mu_{X_k}^{(i)}-\mu_{X_k}^{(j)}\right)^2\right\}$ and $\Upsilon_\mathbf{d}\left(\mathbf{d}^{(i)},\mathbf{d}^{(j)}\right)=\exp\left\{-\sum_{k=1}^{n_d}\theta_{2n_X+n_P+k}\left(d_k^{(i)}-d_k^{(j)}\right)^2\right\}$. $\theta_k(k=1,2,...,2n_X+n_P+n_d)$ is the correlation parameter in the Kriging surrogate model.

It is assumed that the training set and the corresponding output are $\mathbf{w}^*=\left\{\mathbf{w}^{*(1)},\mathbf{w}^{*(2)},...,\mathbf{w}^{*(m)}\right\}$ and $\mathbf{F}^*=\left[F(\mathbf{w}^{*(1)}),F(\mathbf{w}^{*(2)}),...,F(\mathbf{w}^{*(m)})\right]^{\mathrm{T}}$. Then, the regression parameter β and the Gaussian process variance σ_ε^2 can be obtained as follows:

$$\beta=\left(\mathbf{I}_{1\times m}\mathbf{R}^{-1}\mathbf{I}_{m\times 1}\right)^{-1}\mathbf{I}_{1\times m}\mathbf{R}^{-1}\mathbf{F}^* \quad (3)$$

$$\sigma_\varepsilon^2=\frac{1}{m}\left(\mathbf{F}^*-\mathbf{I}_{m\times 1}\beta\right)^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{F}^*-\mathbf{I}_{m\times 1}\beta\right) \quad (4)$$

where $\mathbf{R}$ is the correlation matrix with the element in i-th row and j-th column being $R_{ij}=R_\theta\left(\mathbf{w}^{*(i)},\mathbf{w}^{*(j)}\right)(i,j=1,2,...,m)$. The prediction mean $\mu_{\hat{F}}(\mathbf{u},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})$, the prediction variance $\sigma_{\hat{F}}^2(\mathbf{u},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})$, and the prediction covariance $\mathrm{COV}_{\hat{F}}\left([\mathbf{u},\boldsymbol{\mu}_\mathbf{X},\mathbf{d}],[\mathbf{u}',\boldsymbol{\mu}_\mathbf{X}',\mathbf{d}']\right)$ can be rewritten as follows:

$$\mu_{\hat{F}}(\mathbf{u},\boldsymbol{\mu}_\mathbf{X},\mathbf{d})=\beta+\sum_{i=1}^{m}\Upsilon_\mathbf{u}\left(\mathbf{u},\mathbf{u}^{*(i)}\right)\Upsilon_{\boldsymbol{\mu}_\mathbf{X}}\left(\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{X}^{*(i)}\right)\Upsilon_\mathbf{d}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)\mathrm{K}_i^{\mathrm{RFI}} \quad (5)$$

$$\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})=\sigma_{\varepsilon}^{2}\left\{1-\sum_{i=1}^{m}\sum_{j=1}^{m}\Upsilon_{\mathbf{u}}\left(\mathbf{u},\mathbf{u}^{*(i)}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)\mathrm{K}_{ij}^{\mathrm{R}}\Upsilon_{\mathbf{u}}\left(\mathbf{u},\mathbf{u}^{*(j)}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(j)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(j)}\right)+\mathrm{K}^{\mathrm{IRI}}\left[\sum_{i=1}^{m}\mathrm{K}_{i}^{\mathrm{IR}}\Upsilon_{\mathbf{u}}\left(\mathbf{u},\mathbf{u}^{*(i)}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)-1\right]^{2}\right\} \tag{6}$$

$$\begin{aligned}&\mathrm{COV}_{\hat{F}}\left([\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}],[\mathbf{u}',\boldsymbol{\mu}_{\mathbf{X}}',\mathbf{d}']\right)=\sigma_{\varepsilon}^{2}\Big\{\Upsilon_{\mathbf{u}}\left(\mathbf{u},\mathbf{u}'\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}'\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}'\right)-\\&\sum_{i=1}^{m}\sum_{j=1}^{m}\Upsilon_{\mathbf{u}}\left(\mathbf{u},\mathbf{u}^{*(i)}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)\mathrm{K}_{ij}^{\mathrm{R}}\Upsilon_{\mathbf{u}}\left(\mathbf{u}',\mathbf{u}^{*(j)}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}}',\boldsymbol{\mu}_{\mathbf{X}}^{*(j)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d}',\mathbf{d}^{*(j)}\right)+\\&\mathrm{K}^{\mathrm{IRI}}\left[\sum_{i=1}^{m}\mathrm{K}_{i}^{\mathrm{IR}}\Upsilon_{\mathbf{u}}\left(\mathbf{u},\mathbf{u}^{*(i)}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)-1\right]\left[\sum_{j=1}^{m}\mathrm{K}_{j}^{\mathrm{IR}}\Upsilon_{\mathbf{u}}\left(\mathbf{u}',\mathbf{u}^{*(j)}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}}',\boldsymbol{\mu}_{\mathbf{X}}^{*(j)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d}',\mathbf{d}^{*(j)}\right)-1\right]\Big\}\end{aligned} \tag{7}$$

in which $\mathbf{K}^{\mathrm{RFI}}=\mathbf{R}^{-1}\left(\mathbf{f}^{*}-\mathbf{I}_{m\times 1}\beta\right)$ and $\mathrm{K}_{i}^{\mathrm{RFI}}$ is the i-th element of the vector $\mathbf{K}^{\mathrm{RFI}}$. $\mathbf{K}^{\mathrm{R}}=\mathbf{R}^{-1}$ and $\mathrm{K}_{ij}^{\mathrm{R}}$ is the element in i-th row and j-th column of the matrix $\mathbf{K}^{\mathrm{R}}$. $\mathbf{K}^{\mathrm{IR}}=\mathbf{I}_{1\times m}\mathbf{R}^{-1}$ and $\mathrm{K}_{i}^{\mathrm{IR}}$ is the i-th element of the vector $\mathbf{K}^{\mathrm{IR}}$ and the constant $\mathrm{K}^{\mathrm{IRI}}=\left(\mathbf{I}_{1\times m}\mathbf{R}^{-1}\mathbf{I}_{m\times 1}\right)^{-1}$. According to the above expressions shown in Eqs. (12)-(14), the design parameter vector $[\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}]$ is derived from the inputs for the prediction mean, the prediction variance and the prediction covariance based on the Kriging surrogate model.

Based on the definition of the mean, the mean of the objective function under the design parameter vector $[\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}]$ based on the Kriging surrogate model can be expressed by:

$$\mathrm{E}_{\mathbf{U}}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]=\int_{\Omega_{\mathbf{U}}}\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\varphi_{\mathbf{U}}\left(\mathbf{u}\right)\mathrm{d}\mathbf{u} \tag{8}$$

where $\varphi_{\mathbf{U}}(\mathbf{u})$ is the joint probability density function of standard normal variable vector $\mathbf{U}=[\mathbf{U}_{\mathbf{X}},\mathbf{U}_{\mathbf{P}}]$, $\Omega_{\mathbf{U}}$ is the value domain of $\mathbf{U}$ and $\mathrm{E}_{\mathbf{U}}[\bullet]$ represents the expected operator concerning $\mathbf{U}$. Because $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$ follows the normal distribution with the mean $\mu_{\hat{F}}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$ and the variance $\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$, then $\mathrm{E}_{\mathbf{U}}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]$ is not a constant but a random variable under the design parameter vector $[\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}]$. Therefore, the expectation of concerning random variable $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$ can be used to be the mean of the objective function as follows:

$$\begin{aligned}\mu\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]&=\mathrm{E}_{\hat{F}}\left\{\mathrm{E}_{\mathbf{U}}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]\right\}\\&=\int_{\Omega_{\hat{F}}}\int_{\Omega_{\mathbf{U}}}\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\varphi_{\mathbf{U}}\left(\mathbf{u}\right)\mathrm{d}\mathbf{u}\,\phi_{\hat{F}}\left(\hat{f}\right)\mathrm{d}\hat{f}\end{aligned} \tag{9}$$

in which $\phi_{\hat{F}}\left(\hat{f}\right)$ is the probability density function of random variable $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$, $\Omega_{\hat{F}}$ is the value domain of $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$ and $\mathrm{E}_{\hat{F}}[\bullet]$ represents the expected operator with respect to $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$. The variance of $\mathrm{E}_{\mathbf{U}}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]$ with respect to random variable $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$ can be used to measure the variation of $\mu\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]$.

$$\begin{aligned}\mu^{\delta}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]&=\mathrm{V}_{\hat{F}}\left\{\mathrm{E}_{\mathbf{U}}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]\right\}\\&=\int_{\Omega_{\hat{F}}}\left\{\int_{\Omega_{\mathbf{U}}}\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\varphi_{\mathbf{U}}\left(\mathbf{u}\right)\mathrm{d}\mathbf{u}-\mathrm{E}_{\hat{F}}\left\{\mathrm{E}_{\mathbf{U}}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]\right\}\right\}^{2}\phi_{\hat{F}}\left(\hat{f}\right)\mathrm{d}\hat{f}\end{aligned} \tag{10}$$

where $\mathrm{V}_{\hat{F}}[\bullet]$ represents the variance operator with respect to $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$. According to Fubini's theorem (Briol et al. 2019), Eqs. (9) and (10) can be simplified as follows:

$$\mu\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]=\int_{\Omega_{\mathbf{U}}}\mu_{\hat{F}}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\varphi_{\mathbf{U}}\left(\mathbf{u}\right)\mathrm{d}\mathbf{u} \tag{11}$$

$$\mu^{\delta}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]=\int_{\Omega_{\mathbf{U}}}\int_{\Omega_{\mathbf{U}'}}\mathrm{COV}_{\hat{F}}\left([\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}],[\mathbf{u}',\boldsymbol{\mu}_{\mathbf{X}}',\mathbf{d}']\right)\varphi_{\mathbf{U}}\left(\mathbf{u}\right)\varphi_{\mathbf{U}'}\left(\mathbf{u}'\right)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{u}' \tag{12}$$

Substitute Eq. (5) into Eq. (11) and perform further derivation (Shi et al. 2018), the closed-form expression of the mean of the objective function under the design parameter vector $[\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}]$ can be obtained as follows:

$$\mu\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]=\beta+\sum_{i=1}^{m}\left|\mathbf{A}\right|^{\frac{1}{2}}\left|\mathbf{A}+\mathbf{I}\right|^{-\frac{1}{2}}\mathrm{K}_{i}^{\mathrm{RFI}}\exp\left\{-\frac{1}{2}\mathbf{u}^{*(i)\mathrm{T}}\left(\mathbf{A}+\mathbf{I}\right)^{-1}\mathbf{u}^{*(i)}\right\}\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right) \tag{13}$$

Similarly, the closed-form expression of $\mu^{\delta}\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ under the design parameter vector $[\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}]$ is shown below:

$$\begin{aligned}\mu^{\delta}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]=\sigma_{\varepsilon}^{2}\Bigg\{&\left|2\mathbf{A}^{-1}+\mathbf{I}\right|^{-\frac{1}{2}}-\sum_{i=1}^{m}\sum_{j=1}^{m}\Big[\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)\mathrm{K}_{ij}^{\mathrm{R}}\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(j)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(j)}\right)\times\\&\left|\mathbf{A}\right|\left|\mathbf{A}+\mathbf{I}\right|^{-1}\exp\left\{-\frac{1}{2}\mathbf{u}^{*(i)\mathrm{T}}\left(\mathbf{A}+\mathbf{I}\right)^{-1}\mathbf{u}^{*(i)}\right\}\exp\left\{-\frac{1}{2}\mathbf{u}^{*(j)\mathrm{T}}\left(\mathbf{A}+\mathbf{I}\right)^{-1}\mathbf{u}^{*(j)}\right\}\Big]+\mathrm{K}^{\mathrm{IRI}}\Bigg[\sum_{i=1}^{m}\mathrm{K}_{i}^{\mathrm{IR}}\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)\left|\mathbf{A}\right|^{\frac{1}{2}}\left|\mathbf{A}+\mathbf{I}\right|^{-\frac{1}{2}}\\&\exp\left\{-\frac{1}{2}\mathbf{u}^{*(i)\mathrm{T}}\left(\mathbf{A}+\mathbf{I}\right)^{-1}\mathbf{u}^{*(i)}\right\}-1\Bigg]\times\Bigg[\sum_{j=1}^{m}\mathrm{K}_{j}^{\mathrm{IR}}\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(j)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(j)}\right)\left|\mathbf{A}\right|^{\frac{1}{2}}\left|\mathbf{A}+\mathbf{I}\right|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\mathbf{u}^{*(j)\mathrm{T}}\left(\mathbf{A}+\mathbf{I}\right)^{-1}\mathbf{u}^{*(j)}\right\}-1\Bigg]\Bigg\}\end{aligned}\tag{14}$$

in which $\mathbf{A}=\mathrm{diag}\left(\frac{1}{2\theta_1},\frac{1}{2\theta_2},...,\frac{1}{2\theta_{n_X+n_P}}\right)$, $\mathbf{I}$ is the diagonal matrix with all diagonal elements being 1.

2.2. *Closed-form expression of the standard deviation*

Based on the definition of the variance, the variance of the objective function under the design parameter vector $[\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}]$ based on the Kriging surrogate model can be expressed by:

$$\mathrm{V}_{\mathbf{U}}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]=\int_{\Omega_{\mathbf{U}}}\left\{\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)-\mathrm{E}_{\mathbf{U}}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]\right\}^{2}\varphi_{\mathbf{U}}(\mathbf{u})d\mathbf{u}\tag{15}$$

in which $\mathrm{V}_{\mathbf{U}}[\bullet]$ represents the variance operator with respect to $\mathbf{U}$. Because $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$ follows the normal distribution with the mean $\mu_{\hat{F}}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$ and the variance $\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$, then $\mathrm{V}_{\mathbf{U}}\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ is not a constant but a random variable under the design parameter vector $[\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}]$. Therefore, the expectation of $\mathrm{V}_{\mathbf{U}}\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ with respect to random variable $\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})$ can be used to be the variance of the objective function as follows:

$$\begin{aligned}\sigma^{2}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]=\mathrm{E}_{\mathbf{U}}\left[\mu_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]+\mathrm{E}_{\mathbf{U}}\left[\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]-\\\mu^{2}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]-\mu^{\delta}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]\end{aligned}\tag{16}$$

where $\mathrm{E}_{\mathbf{U}}\left[\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ and $\mu^{\delta}\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ are introduced in the expression of the variance of the objective function to measure the metamodeling uncertainty. It should be noted that except $\mathrm{E}_{\mathbf{U}}\left[\mu_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ and $\mu^{2}\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$, the traditional description of the metamodeling uncertainty only considers $\mathrm{E}_{\mathbf{U}}\left[\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ in the expression of the variance of objective function. The new description of the metamodeling uncertainty can simultaneously reflect the uncertainties in the prediction and the mean of performance function, which plays a great role in improving the estimation accuracy of the RDO.

Then, the standard deviation $\sigma\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ of the objective function can be expressed as follows:

$$\sigma\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]=\sqrt{\begin{aligned}&\mathrm{E}_{\mathbf{U}}\left[\mu_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]+\mathrm{E}_{\mathbf{U}}\left[\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]-\\&\mu^{2}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]-\mu^{\delta}\left[\hat{F}\left(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}\right)\right]\end{aligned}}\tag{17}$$

$\mu^{2}\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ and $\mu^{\delta}\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ can be estimated by Eqs. (13) and (14) respectively. By using the similar derivation process in estimation of mean, the closed-form expressions of $\mathrm{E}_{\mathbf{U}}\left[\mu_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ and $\mathrm{E}_{\mathbf{U}}\left[\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ can be expressed as follows:

$$\begin{aligned}\mathrm{E}_{\mathbf{U}}\left[\mu_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]=\beta^{2}+2\beta\sum_{i=1}^{m}\left|\mathbf{A}\right|^{\frac{1}{2}}\left|\mathbf{A}+\mathbf{I}\right|^{-\frac{1}{2}}\mathrm{K}_{i}^{\mathrm{RFI}}\exp\left\{-\frac{1}{2}\mathbf{u}^{*(i)\mathrm{T}}\left(\mathbf{A}+\mathbf{I}\right)^{-1}\mathbf{u}^{*(i)}\right\}\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)+\\\sum_{i=1}^{m}\sum_{j=1}^{m}\Big\{\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)\mathrm{K}_{i}^{\mathrm{RFI}}\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(j)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(j)}\right)\mathrm{K}_{j}^{\mathrm{RFI}}\times\left|\mathbf{A}\right|\left|2\mathbf{A}\right|^{-\frac{1}{2}}\left|\frac{1}{2}\mathbf{A}+\mathbf{I}\right|^{-\frac{1}{2}}\\\exp\left\{-\frac{1}{2}\left(\mathbf{u}^{*(i)}-\mathbf{u}^{*(j)}\right)^{\mathrm{T}}\left(2\mathbf{A}\right)^{-1}\left(\mathbf{u}^{*(i)}-\mathbf{u}^{*(j)}\right)\right\}\times\exp\left\{-\frac{1}{2}\left(\frac{1}{2}\mathbf{u}^{*(i)}+\frac{1}{2}\mathbf{u}^{*(j)}\right)^{\mathrm{T}}\left(\frac{1}{2}\mathbf{A}+\mathbf{I}\right)^{-1}\left(\frac{1}{2}\mathbf{u}^{*(i)}+\frac{1}{2}\mathbf{u}^{*(j)}\right)\right\}\Big\}\end{aligned}\tag{18}$$

$$\begin{aligned}
&\mathrm{E}_{\mathbf{U}}\left[\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]=\int_{\Omega_{\mathbf{U}}}\sigma_{\hat{F}}^{2}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\varphi_{\mathbf{U}}(\mathbf{u})\mathrm{d}\mathbf{u}\\
&=\sigma_{\varepsilon}^{2}\Bigg\{1-\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\mathrm{K}_{ij}^{\mathrm{R}}-\mathrm{K}^{\mathrm{IRI}}\mathrm{K}_{i}^{\mathrm{IR}}\mathrm{K}_{j}^{\mathrm{IR}}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(j)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(j)}\right)\times\\
&|\mathbf{A}||2\mathbf{A}|^{-\frac{1}{2}}\left|\frac{1}{2}\mathbf{A}+\mathbf{I}\right|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(\mathbf{u}^{*(i)}-\mathbf{u}^{*(j)}\right)^{\mathrm{T}}(2\mathbf{A})^{-1}\left(\mathbf{u}^{*(i)}-\mathbf{u}^{*(j)}\right)\right\}\times\exp\left\{-\frac{1}{2}\left(\frac{1}{2}\mathbf{u}^{*(i)}+\frac{1}{2}\mathbf{u}^{*(j)}\right)^{\mathrm{T}}\left(\frac{1}{2}\mathbf{A}+\mathbf{I}\right)^{-1}\left(\frac{1}{2}\mathbf{u}^{*(i)}+\frac{1}{2}\mathbf{u}^{*(j)}\right)\right\}-\\
&2\mathrm{K}^{\mathrm{IRI}}\sum_{i=1}^{m}\mathrm{K}_{i}^{\mathrm{IR}}\Upsilon_{\boldsymbol{\mu}_{\mathbf{X}}}\left(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{X}}^{*(i)}\right)\Upsilon_{\mathbf{d}}\left(\mathbf{d},\mathbf{d}^{*(i)}\right)|\mathbf{A}|^{\frac{1}{2}}|\mathbf{A}+\mathbf{I}|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\mathbf{u}^{*(i)\mathrm{T}}(\mathbf{A}+\mathbf{I})^{-1}\mathbf{u}^{*(i)}\right\}+\mathrm{K}^{\mathrm{IRI}}\Bigg\}
\end{aligned}\tag{19}$$

2.3. *Adaptive framework for robust design optimization*

Since the closed-form expressions of the mean and standard deviation of the objective function and the constraint function can be obtained by the above procedure, the RDO problem shown in Eq. (1) can be rewritten as follows:

$$\begin{aligned}
&\text{Minimize: } O=\mu\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]+c\sigma\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]\\
&\text{Subject to: } C_i=\mu\left[\hat{G}_i(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]+c\sigma\left[\hat{G}_i(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]\le 0\\
&(i=1,2,...,n_g)\\
&\mathbf{d}^{L}\le\mathbf{d}\le\mathbf{d}^{U},\boldsymbol{\mu}_{\mathbf{X}}^{L}\le\boldsymbol{\mu}_{\mathbf{X}}\le\boldsymbol{\mu}_{\mathbf{X}}^{U}
\end{aligned}\tag{20}$$

Because $\mu\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$, $\sigma\left[\hat{F}(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$, $\mu\left[\hat{G}_i(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right]$ $(i=1,2,...,n_g)$ and $\sigma\left[\hat{G}_i(\mathbf{u},\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d})\right](i=1,2,...,n_g)$ are analytical functions concerning the design parameter vector $[\boldsymbol{\mu}_{\mathbf{X}},\mathbf{d}]$, then the above expression shows a traditional deterministic optimization problem.

To guarantee the estimation accuracy of the RDO, an adaptive framework is established in this work. It is supposed that the current optimal design parameter is $\left[\tilde{\boldsymbol{\mu}}_{\mathbf{X}},\tilde{\mathbf{d}}\right]$ based on the current surrogate model, then the new sample will be identified to strengthen the surrogate model according to the current optimum. In this work, a modified version of learning function based on the one used for uncertainty propagation problems (Wei et al. 2020) is established as follows:

$$L(\mathbf{u})=\sigma_{\hat{F}}^{2}(\mathbf{u},\tilde{\boldsymbol{\mu}}_{\mathbf{X}},\tilde{\mathbf{d}})\varphi_{\mathbf{U}}(\mathbf{u})\int_{\Omega_{\mathbf{U}'}}\frac{\mathrm{COV}_{\hat{F}}\left(\left[\mathbf{u},\tilde{\boldsymbol{\mu}}_{\mathbf{X}},\tilde{\mathbf{d}}\right],\left[\mathbf{u}',\tilde{\boldsymbol{\mu}}_{\mathbf{X}},\tilde{\mathbf{d}}\right]\right)}{\varphi_{\mathbf{U}'}(\mathbf{u}')\mathrm{d}\mathbf{u}'}\tag{21}$$

Then, samples in the sample pool can be tested by the established learning function, and the new random sample $\tilde{\mathbf{u}}$ can be determined by:

$$\tilde{\mathbf{u}}=\max\left\{\mathbf{u}\,|\,L(\mathbf{u})\right\}\tag{22}$$

New training sample is $\left[\tilde{\mathbf{u}},\tilde{\boldsymbol{\mu}}_{\mathbf{X}},\tilde{\mathbf{d}}\right]$.

Before solving the RDO problem, initial surrogate models will be constructed for the objective function and constraint function respectively. Then, closed-form expressions of the mean and standard deviation of the objective function and constraint function will be calculated based on the current surrogate model, and the design optimization model shown in Eq. (20) will be solved to gain the current optimal design parameter $\left[\tilde{\boldsymbol{\mu}}_{\mathbf{X}},\tilde{\mathbf{d}}\right]$. After that, the established learning function will be employed to test the sample pool and new training samples will be deterministic for each surrogate model. Finally, these new training samples will be added to the original training sample set so to update these surrogate models respectively. This process will be carried out iteratively until the absolute value of the relative error of O at two consecutive iterations is less than the threshold.

$$\delta=\left|\frac{O^{(k)}-O^{(k-1)}}{O^{(k)}}\right|<\delta_0\tag{23}$$

in which $O^{(k-1)}$ and $O^{(k)}$ mean the optimization objective in the $(k-1)$ -th iteration and k -th iteration respectively. δ_0 is the threshold and it is suggested to be set $\delta_0=10^{-5}\sim10^{-3}$.

3. APPLICATIONS

A self-balancing vehicle (Shi et al. 2020) shown in Figure 1 is modified to demonstrate the effectiveness of the established method for solving the RDO problem. The chassis provides a platform for two driving wheels, one lighting module, and one control unit. The design optimization objective is to maximize the area of the chassis' upper surface so to make the self-balancing vehicle stable in the driving process. The deformation of the chassis under two cases of loads is used to design optimization constraints. The first case is that the deformation D_{Y_1} under the pressure Y_1 should be less than the allowable threshold to ensure the driving stability under the condition of turning. The other case is that the deformation D_{Y_2} under the pressure Y_2 should be less than the allowable threshold D_2^{allow} so to ensure the driving stability under the condition of acceleration. The RDO model of the self-balancing vehicle can be expressed as follows:

$$\begin{aligned}
&\text{Minimize: } O=\mu\left[f(\mathbf{X},\mathbf{P})\right]+c\sigma\left[f(\mathbf{X},\mathbf{P})\right]\\
&\text{Subject to: } C_i=\mu\left[g_i(\mathbf{X},\mathbf{P})\right]+c\sigma\left[g_i(\mathbf{X},\mathbf{P})\right]\le 0 (i=1,2)\\
&\qquad f(\mathbf{X},\mathbf{P})=-X_1X_2\\
&\qquad g_1(\mathbf{X},\mathbf{P})=D_{Y_1}(X_1,X_2,E,Y_1)-D_1^{\text{allow}}\\
&\qquad g_2(\mathbf{X},\mathbf{P})=D_{Y_2}(X_1,X_2,E,Y_2)-D_2^{\text{allow}}\\
&\qquad X_1\sim N\left(\mu_{X_1},4^2\right), X_2\sim N\left(\mu_{X_2},2^2\right)\\
&\qquad 400\text{mm}\le\mu_{X_1}\le 500\text{mm}, 200\text{mm}\le\mu_{X_2}\le 300\text{mm}
\end{aligned} \tag{24}$$

in which X_1 and X_2 are the width and depth of the chassis respectively, and E is the Young's Modulus. Random parameter variables E, Y_1 and Y_2 are normal variable, i.e., $E\sim N\left(72000,2000^2\right)$, $Y_1\sim N\left(1.5,0.1^2\right)$, $Y_2\sim N\left(1.5,0.1^2\right)$.

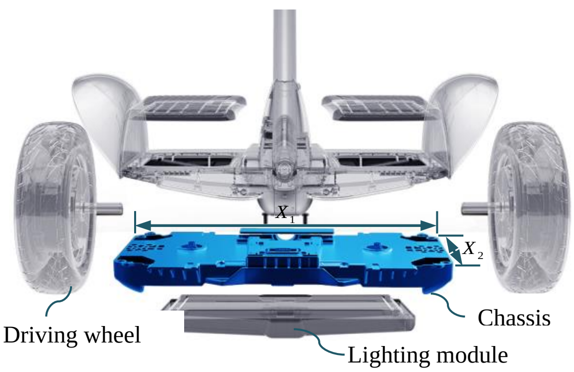


Figure 1: The self-balancing vehicle

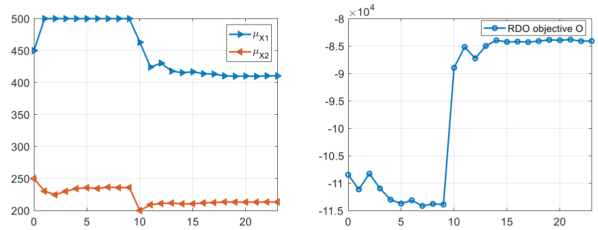
To test the accuracy of the proposed method (PM), the original Kriging surrogate model, the support vector machine (SVM), and the sparse polynomial chaos expansion (SPCE) are employed to approximate performance functions of these applications with the same computational cost of the PM. The design optimization result estimated by the general nested double-loop optimization process (GNDOP) based on real performance functions is used to be the true value of RDO. The final RDO solutions obtained with the Kriging, the SVM, the SPCE, and the PM based on real performance functions are also provided inside parentheses of the corresponding RDO solutions location in tables.

For this self-balancing vehicle robust design problem, since the objective function is a simple and explicit expression, then the surrogate models are employed to approximate constraint functions. For the PM, the total computational costs are 72 which contains 15 initial samples for each constraint function and 22 added samples for each constraint function. The GNDOP is used to be the reference, and 5×10^5 samples are used to estimate the mean and standard deviation of performance functions in each iteration and the number of iterations is 107. The RDO results with different methods are listed in Table 1. The evolution of RDO results with the number of iterations is shown in Figure 2. Evolution of output moments with the number of iterations is shown in Figure 3.

Table 1: RDO results of the self-balancing vehicle

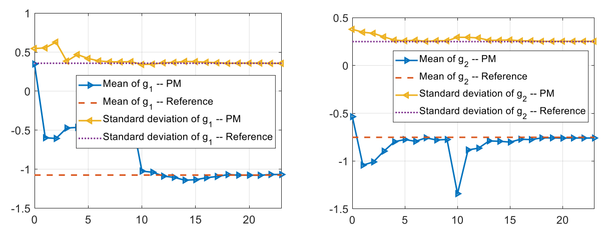
Methods	$\left[\mu_{X_1},\mu_{X_2}\right]$	$O/\times10^4$	C_1	C_2
Kriging	$[408.83, 213.08]$	-8.36	-0.0001 (-0.0165)	0.0002 (-0.0012)
SVM	$[405.53, 213.73]$	-8.31	-0.0005 (-0.0048)	0.0003 (0.0154)
SPCE	$[409.95, 213.58]$	-8.40	-0.0007 (0.0004)	0.0001 (0.0010)
PM	$[410.54, 213.43]$	-8.41	0.0000 (-0.0027)	0.0000 (-0.0023)
GNDOP	$[410.08, 213.58]$	-8.40	0.0004	-0.0002

From Table 1, it can be seen that the design parameter results estimated by the SPCE and the PM can match well with that of the GNDOP, which illustrates the high accuracy of the PM. Both the ordinary Kriging surrogate model and the SVM have a bit of error when compared with the GNDOP, and the error is mainly caused by the inaccurate approximations of constraint functions. Figure 2 shows the good convergence of the PM in estimating both design parameters and RDO objectives. Simultaneously, it can be seen from Figure 3 that the mean and standard deviation of constraint functions converge to the corresponding references, which demonstrates the high accuracy of established closed-form expressions of the mean and standard deviation for performing the uncertainty propagation during the estimation of the RDO.



(a) Design parameters *(b) Objective*

Figure 2 Evolution of RDO results



(a) Function g_1 *(b) Function* g_2

Figure 3 Evolution of output moments

4. CONCLUSIONS

An effective adaptive decoupled RDO method by considering a new description of metamodeling uncertainty of surrogate model is proposed for dealing with robust design problems. Based on the proposed method, the uncertainties in the prediction and the mean of performance function are employed to measure the metamodeling uncertainty. Compared with the traditional description of metamodeling uncertainty, the new expression plays an important role in providing accurate uncertainty propagation solution during design optimization. Simultaneously, an adaptive framework is introduced to improve the computational accuracy of uncertainty propagation so to further guarantee the estimation accuracy of RDO problems. The RDO results of the self-balancing vehicle illustrate that the proposed method is effective in solving robust design problems.

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