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# Selling Online Display Advertising via Guaranteed Contracts and the Real-time Bidding Auctions

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of the requirements for the degree of
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# Abstract

Online display advertising has become a principal revenue stream for a multitude of online publishers and content providers. This form of advertising involves selling 'impressions' or views of display advertisements to advertisers, which are then shown to internet users visiting the publisher's website or platform. One primary selling mechanism for online display ads is guaranteed contracts. These contracts specify the quantity, timing, and characteristics of impressions to be delivered to advertisers, agreed upon before the impressions are actually realised. However, the development of Real-Time Bidding (RTB) has introduced a novel and dynamic way of selling impressions. RTB allows advertisers to place bids for each individual impression in real time as a user visits a webpage hosted by the publisher. This mechanism introduces competition among advertisers and allows for the price of each impression to be determined by their bids at the moment a webpage is loaded. Both guaranteed contracts and RTB have their unique strengths and trade-offs. Guaranteed contracts provide certainty and planning capabilities for advertisers, allowing them to secure a certain amount of impressions in advance. On the other hand, RTB allows for more granular targeting and pricing based on real-time user data, offering the potential for improved advertising effectiveness. However, relying on a single selling channel for ad impressions, either guaranteed contracts or RTB, is insufficient due to the vast supply and diverse preferences of both publishers and advertisers. To address this, this thesis analyses publishers' optimal decisions and advertisers' strategic behaviour in different scenarios involving both guaranteed contracts and RTB.

The thesis comprises three research problems. First, it investigates the optimal pricing of guaranteed contracts in the presence of RTB to maximise publisher revenue. Second, it explores the recruiting strategy for additional advertisers and their impact on publishers' decisions and original advertisers' behaviour under different information settings. Finally, it examines the allocation strategy for impressions among dual channels, considering impression quality and different types of guaranteed contracts. Overall, the thesis makes several significant contributions to the literature on online advertising markets by developing novel models and providing comprehensive analyses of interactions between guaranteed contracts and RTB

under various scenarios, information settings, and impression quality levels. The results provide valuable insights for publishers and advertisers in optimising their strategies within dual-channel online advertising markets.

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# 1  Introduction

In this chapter, we commence by providing a succinct overview of the online advertising industry, followed by a focused discussion of online display advertising markets. Before delving into the three principal research questions embedded in online display advertising markets, we outline the key participants and primary selling channels in these markets. Subsequently, we introduce the main contents of the three studies and the connections between them, summarising their collective contributions to both academia and practice. Finally, we present the structural framework of this thesis.

## 1.1  Background of Online Advertising Markets

The emergence of online advertising markets can be traced back to the early days of the Internet in the mid-1990s. The growth of the World Wide Web led to new opportunities for businesses to reach consumers, giving birth to the online advertising industry. One of the earliest and most well-known examples of online advertising is the banner ad, which was introduced by HotWired (now Wired.com) in 1994. AT&T was the first company to purchase a banner ad on HotWired, which was essentially a clickable image linking to the advertiser's website (Evans 2009). This marked the beginning of display advertising as a prominent form of online advertising.

The global spending on online advertising has experienced rapid and stable growth since its emergence. According to the report from Insider Intelligence (2023), global online advertising spending amounted to approximately \$80 billion in 2011, with the majority of spending allocated to search and display advertising. By 2015, the market had nearly doubled in size, reaching \$160 billion, driven primarily by the rapid growth of mobile advertising and the expanding reach of social media platforms. This report also indicates that the online advertising market has continued to experience robust growth in recent years, with global spending surpassing \$325 billion in 2019 and accounting for more than half of all advertising expenditures worldwide for the first time. Despite the negative influence of the Covid-19 pandemic on the global economy, the rising trend of this market persisted through

2021, with global online advertising spending reaching over \$500 billion, further emphasising the importance of digital marketing in contemporary business strategies.

Search engine advertising, such as Google Ads, emerged in the early 2000s as a way for businesses to target specific keywords and reach users who were actively searching for information related to their products or services. This form of advertising, also known as pay-per-click (PPC), allows advertisers to bid on keywords and only pay when a user clicks on their ad. Another significant development in online advertising was the rise of social media platforms like Facebook and Twitter. Social media advertising enables businesses to target specific demographics and interests, allowing for more personalised and engaging ad experiences (Laroche et al. 2013, Tuten and Solomon 2017). The combination of display, search, and social media advertising has resulted in a highly competitive and diverse online advertising market. As technology advanced, programmatic advertising and real-time bidding (RTB) systems emerged, automating the buying and selling of online ad space and making the process more efficient (Busch 2016). Additionally, native advertising, video advertising, and influencer marketing have also gained traction, further expanding the range of opportunities for businesses to reach their target audience.

Online advertising offers several advantages over traditional advertising, which have also contributed to its rapid growth and adoption by marketers worldwide. Firstly, online advertising provides businesses with unprecedented targeting capabilities, enabling them to reach specific demographics and geographic regions with greater precision (Goldfarb and Tucker 2011). This targeting efficiency, facilitated by the wealth of user data available on digital platforms, allows advertisers to deliver highly relevant and personalised content to their target audience, resulting in higher engagement and conversion rates (Li and Kannan 2014). Secondly, online advertising is highly cost-effective compared to traditional advertising mediums, such as television, radio, and print (Busch 2016). The pay-per-click (PPC) and pay-per-impression (PPI) models employed by digital advertising platforms ensure that advertisers only incur costs when users interact with their ads, leading to more efficient allocation of marketing budgets (Liu and Chao 2020). Lastly, online advertising enables both publishers and advertisers to access comprehensive analytics and performance data, which is not readily available with traditional advertising channels. This data-driven approach allows marketers to optimise their campaigns, measure the effectiveness of their strategies, and make informed decisions about future marketing investments (Busch 2016). Consequently, the advantages of online advertising over traditional advertising have driven its exponential growth and integration into contemporary marketing strategies.
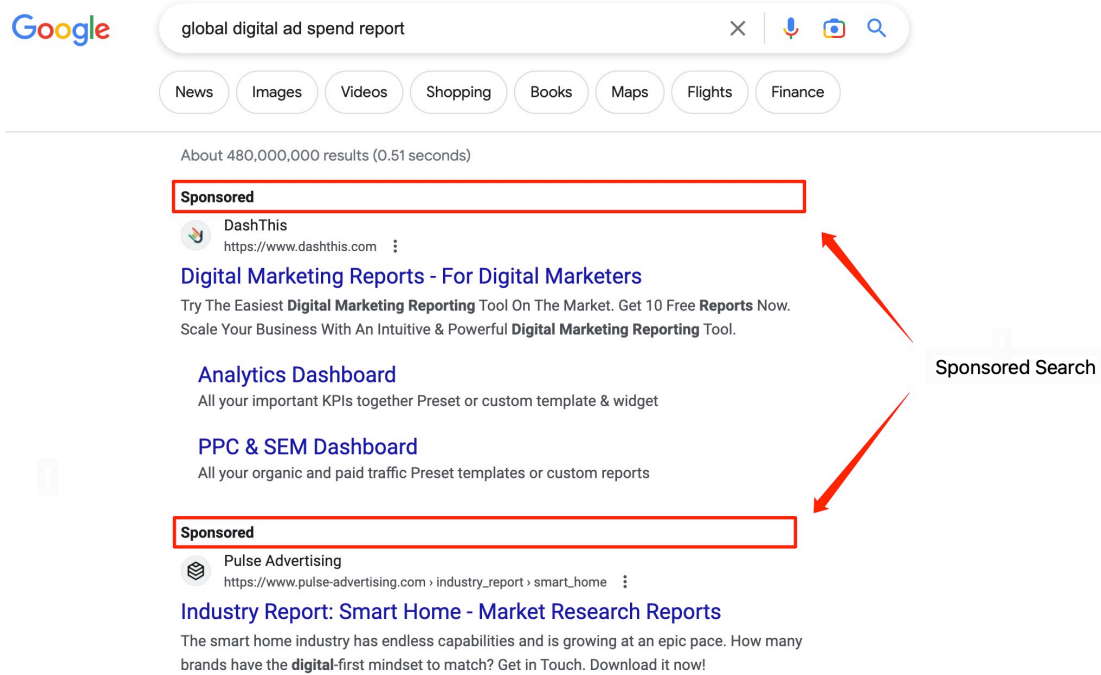
**Figure 1.1:** An example of sponsored search: In this example, the keywords are associated with "Digital", "Ads", and "Report" according to the search input. Therefore, at the top of the list of search outcomes are advertisements related to these keywords (Accessed: 9 May 2023).

## 1.2 Major Types of Online Advertising

There are numerous ways to classify the diverse range of online advertising formats. Focusing on the mechanisms employed, two primary categories emerge: sponsored search and display advertising (Zhang 2016). In 2022, sponsored search constituted 40.2% of the total online advertising revenue in the United States, while display advertising accounted for 30.0% (IAB 2022).

Sponsored search, also known as keywords search advertising, is a method where advertisers bid on specific keywords relevant to their products or services, to have their ads appear alongside search engine results when users search for those keywords (Ghose and Yang 2009). This form of advertising enables advertisers to target their audience based on search queries, leading to highly relevant and timely ad placements. Sponsored search has been popularised by platforms such as Google Ads and Microsoft Advertising, which dominate the search engine market (Jansen and Mullen 2008). Figure 1.1 illustrates an example of a sponsored search on Google.

Display advertising, on the other hand, encompasses a variety of ad formats, including banner ads, rich media, and video ads, which are typically placed on websites, mobile apps, and social media platforms (Li et al. 2002). These ads aim to generate brand awareness and engagement through visually appealing content,
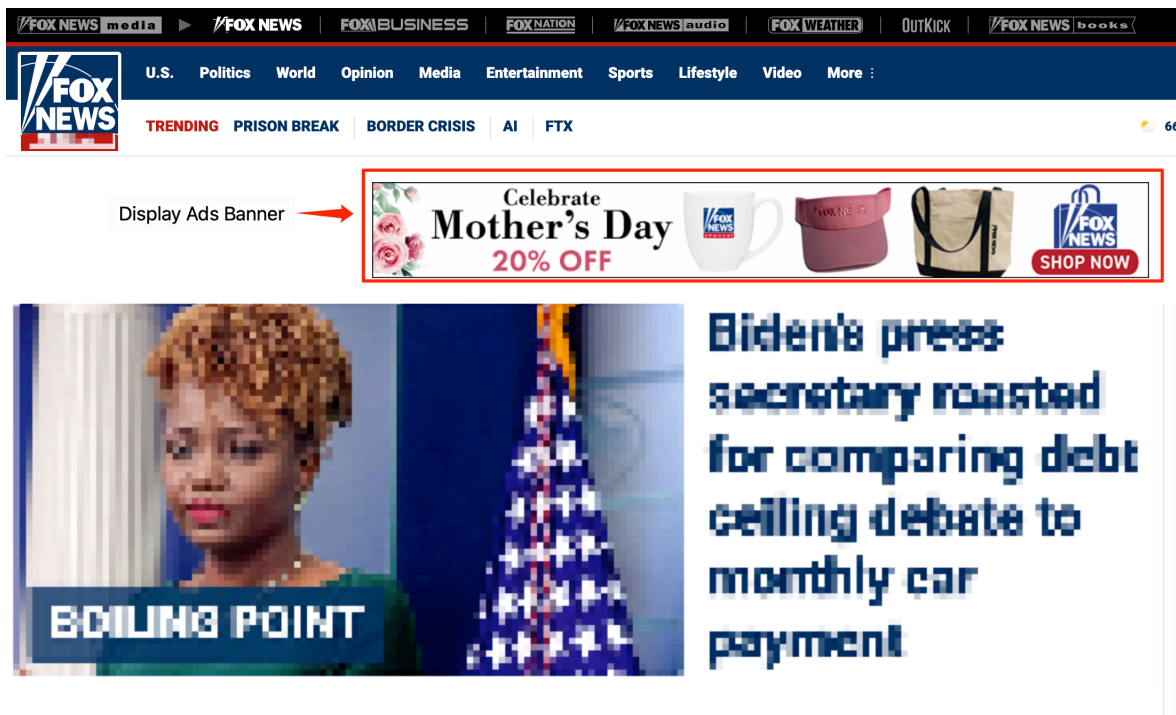
**Figure 1.2:** An example of display advertising: This is a banner advertisement on the website of Fox News. This advertisement is about promoting gifts for the coming Mother's Day, which is not related to the content on the current page, but is targeted by the time (Accessed: 9 May 2023).

capturing users' attention as they browse the Internet (Drèze and Hussherr 2003). Display advertising leverages targeting techniques, such as contextual targeting and behavioural targeting, to ensure that ads are shown to relevant audiences, enhancing the overall effectiveness of campaigns (Yan et al. 2009). An example of a display advertisement on Fox News is provided in Figure 1.2.

We choose online display advertising as the subject of this research. Focusing on online display advertising is essential for several reasons. Firstly, display advertising not only offers a wide range of creative formats, including banner ads, rich media, and video ads, but also can be deployed across different types of digital devices, such as computers, phones and tablets. This excellent flexibility can effectively convey brand messages and engage users through visually appealing content (Drèze and Hussherr 2003). Secondly, display advertising provides sophisticated targeting options, such as contextual and behavioural targeting, that allow advertisers to reach their desired audience with greater precision, thereby enhancing the overall effectiveness of campaigns (Yan et al. 2009). Lastly, the continuous growth and popularity of social media platforms, which serve as primary channels for display advertising, ensure that marketers can achieve extensive reach and visibility among diverse audiences (Hilde A. M. Voorveld and Bronner 2018).

## 1.3 Online Display Advertising Markets

### 1.3.1 Participants

The key participants in the online display advertising market can be categorised into three main groups - advertisers, publishers, and intermediaries. Advertisers are companies or individuals that create and pay for ads to promote their products or services. Publishers are the owners of websites or mobile applications where ads are displayed to reach their audience. Intermediaries facilitate the buying and selling of online display advertising inventory, connecting advertisers and publishers (Evans 2009).

Among intermediaries, there are several types of entities, including ad networks, demand-side platforms (DSPs), supply-side platforms (SSPs), ad exchanges, and data management platforms (DMPs) (Alaimo and Kallinikos 2018). Ad networks aggregate ad inventory from multiple publishers and sell it to advertisers, often targeting specific audiences or content categories. DSPs enable advertisers to purchase display ad inventory across multiple ad exchanges through a single interface, while SSPs help publishers manage and sell their ad inventory to the highest bidder, typically using RTB mechanisms. Ad exchanges act as marketplaces for buying and selling display ad inventory, also employing RTB technology. Lastly, DMPs collect, analyse, and manage data from various sources to support the targeting and optimisation of advertising campaigns.

In this thesis, we mainly focus on the interplay of decisions between sellers and buyers in this market, neglecting the support functions of intermediaries. Therefore, we take publishers and advertisers as the main research objects in our studies.

### 1.3.2 Selling Channels for Online Display Advertising

The selling unit of online display advertising is usually *impressions*, which are the page views of users on the Internet. Publishers make profits by selling impressions to advertisers. There are two primary selling channels in the markets: guaranteed and non-guaranteed channels. A guaranteed channel often refers to advanced selling at a fixed price through guaranteed contracts, while the main mechanism in the non-guaranteed channel is RTB auctions (Choi et al. 2020).

**Guaranteed Channel**

The history of guaranteed advertising contracts dates back to the early days of online advertising, when publishers and advertisers would negotiate fixed-price

agreements for specific ad placements (Liu and Chao 2020). In these early days, the online advertising market was relatively small and fragmented, and it was difficult for advertisers to measure the effectiveness of their campaigns. Guaranteed advertising contracts provided a way for advertisers to get a guaranteed number of impressions or clicks, which helped them measure the effectiveness of their campaigns. As the industry evolved, advertisers sought more efficient and automated methods to purchase ad inventory, giving rise to the development of programmatic guaranteed contracts. Programmatic guaranteed, also known as automated guaranteed or programmatic direct, combines the advantages of traditional guaranteed contracts with the automation and efficiency of programmatic advertising (Chen et al. 2020).

In a programmatic guaranteed contract, advertisers and publishers agree on a fixed price and volume of ad impressions, with the actual purchasing and delivery of the ad inventory being automated through a demand-side platform (DSP) and supply-side platform (SSP). This streamlined process minimises manual intervention and allows for more efficient and precise ad delivery, while still offering the premium placements and brand safety associated with traditional guaranteed contracts (Chen 2016).

The advantages of guaranteed advertising channels include greater control over ad placements, brand safety, and campaign visibility, as well as the opportunity to negotiate customised packages and exclusive partnerships with high-quality publishers (Dukes and Gal–Or 2003). However, these benefits come with some disadvantages, such as potentially higher costs compared to non-guaranteed channels, and a less flexible approach to ad inventory purchasing and targeting (Chen 2015).

**Non-guaranteed channel**

The development of RTB can be traced back to the mid-2000s, with the advent of ad exchanges and the growing need for more efficient and scalable methods to buy and sell online ad inventory. RTB is an auction-based system that automates the buying and selling of ad inventory on a per-impression basis. Advertisers submit bids for individual ad impressions with the highest bidder winning the opportunity to display their ad to a specific user (Yuan et al. 2013). The payment of the winner depends on the format of RTB. The most widely implemented one is the second price auction, which has the most potential for incentive compatibility compared with other formats of auctions (Wang et al. 2017). Therefore, we also only consider the second price auction in RTB in our three studies. Under the second price auction, the payment of the winner is the bidding price of the second highest bidder.

Usually, the conduct of RTB is technically supported by ad intermediaries, for example, ad exchange. Advertisers involved in an ad exchange first set their bids for impressions under specific targeting criteria. In practice, advertisers may only need to set their daily/weekly/monthly budgets and goal-oriented bidding strategies. For example, Google Ads proposes a series of strategies with different goals to advertisers (Google Ads Help 2023). An RTB auction is triggered by a visit from a user to the websites held by a publisher. Once the user clicks the URL, an ad request is sent to cooperated ad exchanges. Ad exchange automatically pulls this request to matched advertisers. Then, advertisers' preset bidding prices are returned and consequently, the winner is determined by the rule of auctions. Finally, the winner's advertisement is delivered to the website and exposed to the user. This whole process is completed in less than milliseconds (Sayedi 2018).

RTB enables advertisers to leverage advanced targeting techniques, such as behavioural and demographic targeting, as well as granular performance data to optimise their campaigns and maximise return on investment (Ghose and Yang 2009). The advantages of non-guaranteed advertising channels include increased scalability, efficiency, and cost-effectiveness, as well as the ability to continually refine targeting strategies, creative assets, and bidding strategies. However, there are also disadvantages to non-guaranteed channels, such as potential concerns regarding brand safety and ad view ability due to the automated nature of ad placements. Additionally, the auction-based system can lead to fluctuations in ad inventory availability and pricing, making it more challenging for advertisers to predict and manage their advertising budgets (Choi et al. 2020).

## 1.4   Research Questions and Contributions

The reliance on a single selling channel for ad impressions is insufficient to accommodate the vast supply generated by the ever-growing population of Internet users. Furthermore, it may not adequately address the diverse preferences and requirements of both publishers and advertisers in different scenarios. The advantages of guaranteed advertising channels, such as greater control over ad placements, brand safety, and campaign visibility, cater to the needs of advertisers seeking premium placements and exclusive partnerships with high-quality publishers. On the other hand, non-guaranteed channels, namely RTB, offer increased scalability, efficiency, and cost-effectiveness, making them suitable for advertisers seeking a more flexible and data-driven approach to purchasing ad inventory. Therefore, a selling mechanism that incorporates both guaranteed and non-guaranteed selling channels can help address these challenges and leverage their positive aspects effectively.

### 1.4.1 Three Research Questions and Their Connections

In this thesis, we focus on the analysis of both publishers' optimal decisions and advertisers' strategic behaviour in different scenarios, with the two selling channels activated exclusively. Specifically, guaranteed contracts for future impressions are available in advance, and RTB is triggered when these impressions are generated by user visits.

Figure 1.3 illustrates a general process of a publisher selling impressions to advertising through dual channels over the selling horizon. Before the opening of the guaranteed channel, the publisher first sets prices for guaranteed contracts and then announces them to advertisers. While guaranteed contracts are available, advertisers decide whether to buy them or wait until the opening of RTB. After the close of guaranteed contracts and before the start of RTB, the publisher considers the recruitment of extra advertisers to the RTB. During the RTB, each auction is triggered by a user visit. The publisher needs to allocate each impression to different buyers strategically. The figure also illustrates where our research problems are located and the connections between them. With the combination of the three studies, we are able to construct a comprehensive answer to our research problems.
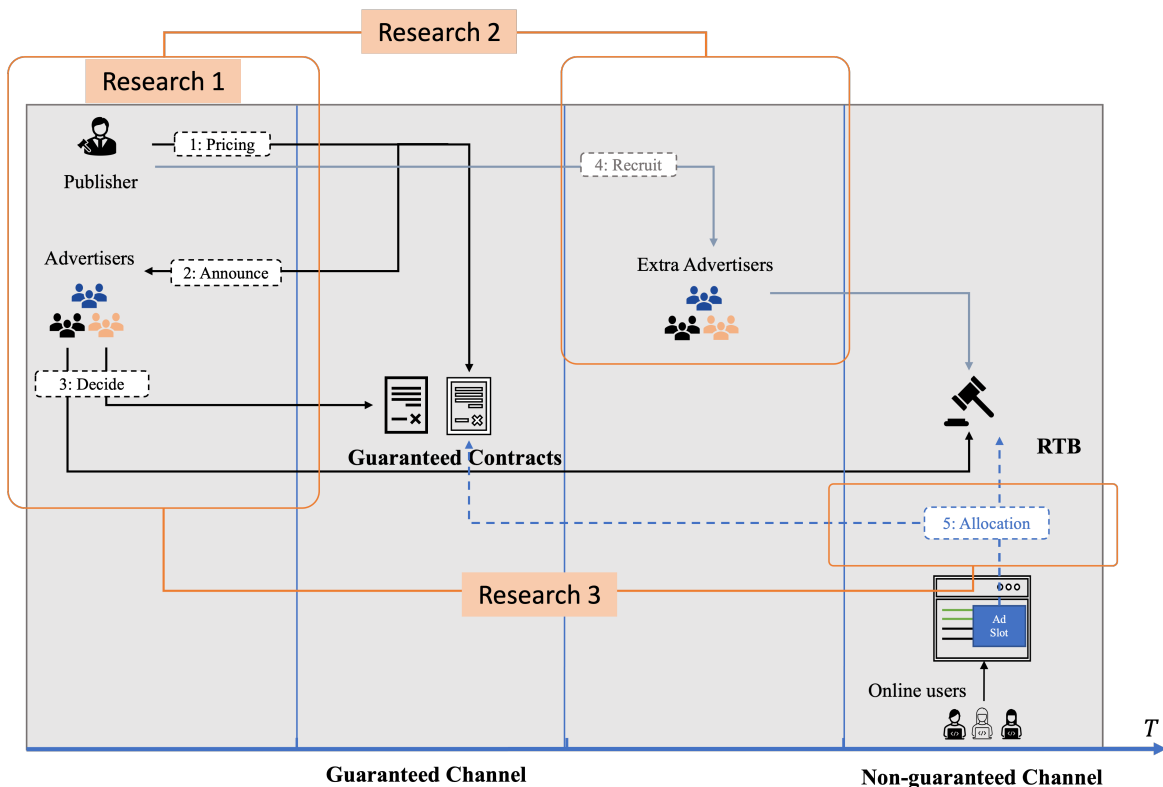


**Figure 1.3:** The overview of research problems in this thesis and their links: Research 1: Optimal pricing of guaranteed contracts under dual selling channels with the RTB. Research 2: Whether and how the publishers should advertise for their advertising slots? Research 3: Allocation of impressions among dual selling channels in online display advertising markets

In this thesis, the first research problem lays the groundwork for the subsequent inquiries, focusing on a publisher's optimal pricing strategy for guaranteed contracts in the presence of real-time bidding (RTB). We model a scenario where a publisher aims to sell a fixed number of homogenous impressions to multiple advertisers across two distinct periods. The first period is exclusively for guaranteed contracts, while the second period is reserved for RTB. Advertisers must decide between these two channels to maximise their utility, and their choices are influenced by the publisher's pricing of the guaranteed contracts. We approach this problem as a sequential game and explore equilibrium outcomes under varying parameters. Note that we can simplify the allocation of impressions between these two channels because of the assumption of homogenous impressions.

Building on this foundation, the second research problem delves into a more complex environment where additional advertisers may enter the campaign between the guaranteed and non-guaranteed channels. These advertisers could either join spontaneously or be actively recruited by the publisher at a certain unit cost. This addition introduces a new layer of complexity, affecting both the publisher's decisions and the behaviour of the original advertisers if they are aware of the potential for new entrants. We examine this scenario across a three-period selling horizon, considering different information structures to understand how the original advertisers might react to the publisher's plans for the middle period.

Finally, our third research problem extends the initial model by incorporating the allocation of impressions among different buyers. Here, the focus is on how a publisher can optimally allocate heterogenous impressions between guaranteed contracts and RTB, while also considering the strategic behaviour of advertisers. We introduce a model that accounts for the quality of impressions, reflecting the varied user information behind different page views. Advertisers' valuations are modelled as a function of the quality of impressions and their compatibility with each impression. We also explore two types of guaranteed contracts: quantity-guaranteed and quality-guaranteed contracts, each with its implications for the publisher's allocation policy and revenue.

Through these interconnected research problems, We aim to offer a comprehensive understanding of the complexities involved in managing online advertising inventory within a dual-channel selling environment.

### 1.4.2  Overall Contributions

This thesis offers a comprehensive investigation into the intricacies of online display advertising, with a specific focus on the dual channels of guaranteed contracts and real-time bidding (RTB). One of the most vital contributions is in the inclusion of

advertisers' strategic behaviour in the three research questions, which is neglected in the previous literature (e.g. Balseiro et al. 2014, Chen 2017, Shen 2018, Rhuggenaath et al. 2019, Wu et al. 2021 and Wang et al. 2022). It demonstrates how publishers' and advertisers' decisions interact mutually through sequential game frameworks, explaining when and why advertisers prefer one channel over the other.

Another major theme is the influence of guaranteed contracts on both advertisers and publishers. This thesis shows that the presence of guaranteed contracts can disrupt the conventional truth-telling rule among rational advertisers. This insight extends the current discourse on bidding strategies in second-price RTB (Menezes and Monteiro 2004, Balseiro et al. 2015, Balseiro and Gur 2019). Additionally, based on this finding, we explore the effect on publishers' various decisions (including pricing of guaranteed contracts, investing in attracting additional advertisers, and allocation strategy of impressions).

Revenue optimisation for publishers is another critical area that my thesis addresses. Our research indicates that publishers can maximise their revenues by judiciously combining guaranteed contracts and RTB, particularly under specific market conditions. These results widen the understanding of the online display markets, in which dual channels are made available. To guide publishers in attracting more advertisers, a cost-benefit indicator was introduced to provide a measurable feature for publishers. Furthermore, we emphasise the importance of balancing the supply and demand for guaranteed contracts when making an allocation strategy between two channels, offering new managerial insights that could control the cost of under-delivery.

Lastly, this thesis has practical implications that extend beyond the academia. We offer actionable insights for publishers in terms of setting prices of guaranteed contracts, investing in attracting additional advertisers and determining the allocation strategy of impressions. We also highlight the importance of the application of dual selling channels, which stands to benefit both publishers and advertisers.

In summary, this thesis advances our understanding of online advertising markets by developing and analysing models that capture the interactions between guaranteed contracts and RTB within various environments, incorporating various factors such as scarcity of impressions, different information settings, and heterogeneous impressions quality.

## 1.5 Structure of the Thesis

The rest of this thesis is organised as follows. In Chapter 2, we study the publisher's optimal pricing of guaranteed contracts in a selling horizon with dual channels in-

volved. Advertisers' strategic behaviour is considered and equilibrium outcomes are solved. We also discussed some extensions. Chapter 3 extends the publisher's problem by introducing additional advertisers between the dual selling channels. We discuss cases under different market settings and information structures. Chapter 4 concentrates on the allocation strategy of impressions among the two channels. To capture the allocation detail, we model the heterogenous both for advertisers and the quality of impressions. We examine the performance of two clusters of allocation policies under two types of guaranteed contracts through algorithms based on backward induction. Chapter 5 summarises the conclusion of this thesis and proposes some future research directions.

# 2 Study 1: Optimal Pricing of Guaranteed Contracts for Online Display Advertising Facing Dual Channels

## Abstract

Online display advertising produces substantial revenue for thousands of online publishers and content providers by selling impressions of display ads to advertisers. The main selling channels are guaranteed contracts and real-time bidding (RTB). Guaranteed contracts regulate what, when and how impressions should be delivered to advertisers before these impressions are realised, while RTB allows advertisers to bid for an impression only at the time when a user comes to visit the webpage hosted by the publisher. This paper investigates a problem where a publisher sells impressions to advertisers with unit demand during two periods, with guaranteed contracts and RTB exclusively activated in the first period and the second period. Impressions are generated in the second period. We consider discount factors of yields from the second period for both the publisher and advertisers. Advertisers make decisions about whether to buy guaranteed contracts or to join RTB. The publisher sets an optimal price for contracts to obtain maximum revenue. We find the Mixed Truth-telling Strategy for advertisers in RTB and then obtain the optimal pricing of guaranteed contracts for the publisher. We show that the implementation of both guaranteed contracts and RTB is more flexible and profitable than only one of them exists. Under different cases of parameters (i.e., the scarcity of impressions, discount factors), the publisher's optimal pricing for guaranteed contracts can either make all impressions consumed by advertisers through guaranteed contracts or RTB or both coexist. Besides, we further consider the case of the uncertain number of impressions, customer segmentation decisions, and different valuation distributions in extensions.

**Keywords:** online display advertising, strategic behaviours, guaranteed contracts, real-time bidding, mixed truth-telling

## 2.1 Introduction

The value of online advertising markets and their rising trends have been widely witnessed in recent years. The PwC IAB Internet Advertising Revenue Report (2022) states that the overall full-year revenue of online advertising in the United States achieved a 10.8% increase. As a main sub-category of online advertising, online display advertising refers to formats including banners, rich media and sponsorship etc. Revenue generated from online display advertising consisted of 30.3% of the total revenue and reached 63.5 billion dollars (IAB 2022). In general, the selling unit of online display advertising is an *impression*, which means page views of online users to an advertisement.

Online display ad impressions are mainly sold through *guaranteed contracts* and the *real-time bidding* (RTB) (Balseiro et al. 2014, Chen 2017, Chen et al. 2020, Liu and Chao 2020). Selling online display advertising has traditionally been conducted through guaranteed contracts after negotiations between the supply side (e.g., online media websites) and the demand side (i.e., advertisers). However, with the rapid growth of the internet and the consequent increase in demand for online advertising, the traditional negotiation-based approach became inefficient. To address this, a standardised automatic selling system was developed. This system allows publishers to post standardised contracts with pre-set rules (e.g., pricing, realisation periods), enabling advertisers to purchase contracts without personal negotiations with the publisher. In this paper, we refer to guaranteed contracts as automatic guaranteed contracts. RTB is another mechanism designed specifically for online advertising markets. An RTB auction completes in milliseconds, triggered when a user clicks a publisher's webpage. Advertisers' bidding prices are pre-set (as per advertisers' private valuations to the auctioned impression), and the winner is determined according to an auction rule, typically a second-price auction. The winning advertisement is then displayed on the clicked webpage.

Guaranteed contracts allow advertisers to secure future impressions, thereby providing publishers with a stable cash flow. However, such stability inevitably compromises flexibility. High-value impressions, for instance, might be undersold, whereas they could have been sold at premium prices to advertisers seeking to cherry-pick impressions in RTB (Choi et al. 2020). Furthermore, the unpredictable outcomes from RTB pose risks for both publishers and advertisers. The combination of these two channels could capitalise on their respective benefits while effectively mitigating their shortcomings (Balseiro et al. 2014, Chen et al. 2020). However, this approach gives rise to two novel challenges: a) why and under what conditions advertisers will opt for one channel over the other, and b) how publishers should price the guaranteed contracts to maximise their revenue while taking into account

advertisers' strategic choices between the two channels. In practice, these two selling channels are typically managed independently by distinct roles (Balseiro et al. 2014). Moreover, existing literature primarily focuses on either guaranteed contracts or RTB exclusively (will be discussed in the next section in detail). This research intends to fill these gaps by providing valuable insights into the two challenges.

In this paper, we propose a new selling mechanism that combines both guaranteed contracts and RTB, and allow advertisers to choose between these two channels strategically. More specifically, we consider a publisher selling a limited number of homogeneous impressions during two periods. The advertisers can choose to pay a fixed price to secure an impression in the first period before impressions are realised, or they can wait to participate in real-time second-price auctions in the second period. The setting of homogeneity among impressions enhances the fixed-price assumption for contracts. In reality, when the publisher groups advertisers according to different combinations of attributes, then impressions within a group can be treated as homogeneous approximately.

Our model is unique, because advertisers' rational expectation equilibrium decisions on whether to choose guaranteed contracts in period 1 have to account for the outcome from RTB in period 2, which is also an equilibrium emerging from all other advertisers' bids in RTB. Despite the complexity due to the advertisers' strategic behaviours and the auctions, we are able to obtain the closed-form solution and show that combining guaranteed contracts and RTB in selling online display advertisements can generate more revenue for the publishers in some specific scenarios, compared to the case when only RTB or guaranteed contracts is used. We further extend our model to examine the following questions to obtain more insights and check the robustness of the results. 1) The impact of uncertainty on the impression supply. When facing supply uncertainty, the publisher is more willing to combine both guaranteed contracts and RTB to sell the impressions. 2) Advertiser segmentation. Segmenting advertisers based on their valuations can improve the publisher's revenue when the supply of impressions is less scarce. The competition level of the selling campaign is intense enough when the impression scarcity is high, under which case the segmentation strategy is not necessary. 3) Different distributions for advertisers' valuation. Our main results hold when advertisers' valuation follows different types of distributions. We then utilise an RTB dataset from a UK SSP to conduct the counterfactual analysis and validate the proposed model and its extensions.

This paper contributes to the limited literature that studies selling impressions to strategic advertisers via both guaranteed contracts and RTB (which will be shown in Literature Review §2.2). Firstly, this research offers a novel perspective on advertisers' decision-making processes, revealing a threshold-type strategy that enriches

the current understanding of advertiser behaviour in dual channels. This strategy implies that advertisers with higher valuations would prefer buying guaranteed contracts in the initial period, while those with lower valuations would participate in RTB. Secondly, our research delves into the interplay between guaranteed contracts and RTB in the context of second-price auctions. Contrary to widely accepted conclusions drawn from existing literature on second-price auctions (Menezes and Monteiro 2004, Narahari 2014), we demonstrate that the presence of guaranteed contracts could potentially disrupt the conventional truth-telling rule among rational advertisers. We observe a truncation phenomenon, where an advertiser's bid may be determined not solely by its valuation, but rather limited by the price of the guaranteed contract. This revelation extends the current discourse on bidding strategy in second-price RTB with the presence of a fixed price channel. Lastly, our research posits that publishers could achieve higher revenues by judiciously combining guaranteed contracts and RTB. This strategy proves especially advantageous under conditions of lower scarcity in impressions (i.e. when the supply-to-demand ratio is high) and lower discount factors for both the publisher and advertisers (i.e., when the publisher's focus is on net present value and the advertisers show a greater propensity to secure impressions to circumvent RTB). By unearthing these conditions for optimal revenue, this study not only refines the academic understanding of revenue management in online advertising but also provides practical guidance for publishers aiming to optimise their revenue streams.

The rest of this paper is organised as follows. §2.2 reviews the related literature. §2.3-§2.4 demonstrates our basic model along with the closed-form equilibrium results under different cases. §2.5-§2.7 discuss extensions of the basic model and also experimental results from practical data. §2.8 concludes this paper and provides some future directions. Proofs are available in Appendix A1.

## 2.2   Literature Review

This research is mainly related to two research streams and the combination of them. The first stream is about operations management in the field of online display advertising. In this stream, we investigated them according to the different selling channels they considered. We first focused on literature about guaranteed contracts and RTB, respectively. Then we reviewed papers that covered these two channels together. The second stream is revenue management with strategic consumers facing multiple buying channels. At last, we especially discussed two papers that were closely related to this study.

Before the popularity of auctions in online advertising markets, guaranteed contracts were the main approach to sell impressions of online display advertis-

ing to advertisers (Liu and Chao 2020). Among various pricing schemes, i.e. CPM, CPC, CPA (cost per action) etc. (e.g. Mangani 2004, Fjell 2009, Asdemir et al. 2012 and Najafi-Asadolahi and Fridgeirsdottir 2014), of guaranteed contracts, the most popular one is CPM (Choi et al. 2020), which is also the scheme we assumed in this research. There are mainly two ways when impressions are sold through guaranteed contracts under the CPM scheme. First, advertisers proposed their willingness-to-pay and demand for impressions and publishers decided whether to accept them or not. Besides, the cancellation of promised contracts was allowed (Babaioff et al. 2009, Constantin et al. 2009, Roels and Fridgeirsdottir 2009). Second, the price of guaranteed contracts was determined by publishers according to advertisers' demands (Fridgeirsdottir and Najafi-Asadolahi 2018) or users' attributes (Bharadwaj et al. 2010). In our paper, we take the latter approach and assume a fixed price for guaranteed contracts to explore the effect of the combination of it with RTB.

In contrast to the scarce research on guaranteed contracts, there are plenty of studies focusing on RTB in online display advertising markets both from the sides of publishers and advertisers. From advertisers' perspective, the optimal bidding strategy for online display advertising is the most relevant topic to our study. Although truth-telling is commonly known as the weakly dominant strategy in the private-value second-price auction of a single object in theory (Menezes and Monteiro 2004), this may not always be the case due to various practical factors in the online display advertising markets. For example, the budget constraint of advertisers makes them intend to shade their true valuations in repeated bidding campaigns (Balseiro et al. 2015, Balseiro and Gur 2019). In other cases, advertisers may not know other advertisers' valuations, even their own valuations. Advertisers can post different bids in repeated auctions to learn others and their valuations through analysis of outcomes over time (Pin and Key 2011, Perlich et al. 2012, Iyer et al. 2014, Zhang et al. 2014, Cai et al. 2017). In terms of publishers, there are mainly three aspects of problems with their optimal pricing policies. Similar to the channel of guaranteed contracts, the first is the choice among different pricing schemes. Hu (2004), Dellarocas (2012) and Hu et al. (2016) studied the impact of different factors, including information asymmetry and double marginalisation, on the setting of CPM, CPC or CPA auctions. Their studies implied that different schemes were optimal under different conditions. Another popular direction is the optimisation (Balseiro et al. 2015, Paes Leme et al. 2016, Choi and Mela 2018) or learning (Amin et al. 2013, Cesa-Bianchi et al. 2014, Mohri and Medina 2014) of reserve price in auctions. Unlike previous studies that consider only one available channel, either guaranteed contracts or RTB, our study focuses on advertisers' bidding strategy in the RTB period, considering the impact of the presence of guaranteed contracts.

There are few research discussing selling display advertising impressions through dual channels, and most of them merely focused on the problem of impression

allocation between guaranteed contracts and real-time bidding from the perspective of publishers. Roels and Fridgeirsdottir (2009), Bharadwaj et al. (2010), Salomatin et al. (2012), Wu et al. (2021), Li et al. (2016) and Rhuggenaath et al. (2019) discussed the allocation solutions or policies through designed algorithms. Balseiro et al. (2014) and Chen (2017) constructed stochastic control models to solve the optimal allocation problem by the dynamic programming approach. Chen et al. (2014), Chen (2016) and Chen et al. (2020) focused on the pricing problem of guaranteed contracts under the scenario of selling display advertising through the dual channel. These papers only concentrated on one channel and treated another as input for their models, and they also did not take advertisers' strategic behaviours into consideration. Therefore, they fail to explain when and why one selling channel should be chosen over the other from the perspective of advertisers.

Since we take the advertisers' strategic decisions between guaranteed contracts and RTB into consideration, our paper is related to the literature on revenue management problems with strategic consumers involved. One stream of these literature investigated the timing of purchase decisions for strategic consumers when facing publisher's dynamic pricing policies in a multi-period selling horizon (Su 2007, Aviv and Pazgal 2008, Levin et al. 2009, Yin et al. 2009, Osadchiy and Vulcano 2010, Mersereau and Zhang 2012, Correa et al. 2016, Kremer et al. 2017). Another stream focused on problems about the strategic choices of consumers between different channels, i.e. online and offline channels or omnichannel, simultaneously (Ofek et al. 2011, Gao and Su 2017, 2019, Nageswaran et al. 2020, Gao et al. 2022). Regardless of the focus, existing studies assumed that the payments are set by the seller. In our study, advertisers' payments in RTB are decided by their opponents' bidding prices, which distinguishes our paper from the existing studies.

Furthermore, Sayedi (2018) and Cohen et al. (2023) share similar settings with ours. We all consider fixed-price guaranteed contracts and RTB simultaneously. However, both studies simplify RTB differently, which makes advertisers' bidding strategy unclear under their scenario. Sayedi (2018) simplified the analysis of the outcomes of the auctions in three aspects. a) There are only two advertisers in the campaign. b) With the assumptions of consumers located on a Hotelling line interval and the two advertisers located at the two endpoints of that interval, the two advertisers' valuations to an impression are not only known to each other but also complementary. This means that the outcomes of auctions are determinate to each other once the valuations of impressions are realised. c) His results promise a high reserve price in auctions such that the payments from auctions are always the value of that reserve price. Cohen et al. (2023) proposed two assumptions so that the payment from each auctioned unit is made the same. First, they assume that the distribution of $v_{I+1}$ is known to all buyers, in which $I$ is the number of homogeneous goods and $v_{I+1}$ is the valuation of the $(I+1)$th highest value. Second, they

demonstrate that the payment of each unit from the auction is $v_{I+1}$ under the VCG mechanism in a large market. Furthermore, the value of $v_{I+1}$ is known to all buyers right before the start of auctions. However, in our settings, there are a large number of advertisers involved and their valuations to an impression is private information, which is more allied with reality. These settings make our study more applicable in practice than theirs. Our paper also demonstrates that the presence of guaranteed contracts truncates advertisers' bidding price from their true valuation for impressions by the price of guaranteed contracts, which is contrary to the well-known truth-telling strategy when only the second-price auction exists.

## 2.3 Model Setup

Consider a market with one publisher and $N$ advertisers (assume that $N$ is sufficiently large). The publisher owns a website with a display advertising slot that can generate $Q$ impressions. We assume that all impressions are homogeneous (Caldentey and Vulcano 2007, Cohen et al. 2023) and $N > Q$. Advertisers have unit demand for the impressions and their valuations on an impression follow a uniform distribution $U(0, \overline{v})$. The publisher can sell these impressions to advertisers through two channels. It can either sell the impressions in advance via a guaranteed contract with a fixed price $p$ (referred to as selling period 1), or via RTB when these impressions are triggered by website users' visits (selling period 2). The auction for each impression in period 2 is organised as a second-price auction.

Facing the two channels, each advertiser needs to decide from which channel to buy the impression to maximise its utility. If an advertiser $i$ buys an impression in advance via a guaranteed contract in period 1, the utility is $u_{i,1} = v_i - p$, and if it waits for period 2 to attend the real-time bidding and wins the impression, the utility is $u_{i,2} = \delta_a(v_i - p_i^{(r)})$. Here, $p_i^{(r)}$ represents advertiser $i$'s payment in the real-time bidding, which is an equilibrium outcome depending on the bids from other advertisers who participate in the second-price auction. And $\delta_a$ is the discount factor for advertisers' utilities in period 2, which captures the disutility of the uncertainty and hassle of attending RTB[1]. For advertisers who do not buy in period 1 nor win in the auction in period 2, their utility is zero. We also assume a discount factor, $\delta_p$, for the publisher to represent the disutility of receiving the payment later in period 2.

Figure 2.1 shows the sequence of events. In period 1, the publisher decides

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[1]Existing literature also considers the uncertainty and hassle of attending RTB by assuming that advertisers are risk averse, thus captures disutility of the uncertainty as a subtracted term in advertises' expected utility expression (Chen et al. 2020, Cohen et al. 2023). In our paper, for ease of exposition, we use the advertisers' discount factor to capture such disutility in the form of a multiplier.

the price of a guaranteed contract for one impression. Observing the price of a guaranteed contract, advertisers decide whether to buy a guaranteed contract or join RTB in period 2. In period 2, advertisers who participate in RTB, decide and submit their bidding price, and then impressions are distributed to the winners based on the rule of second-price auction.
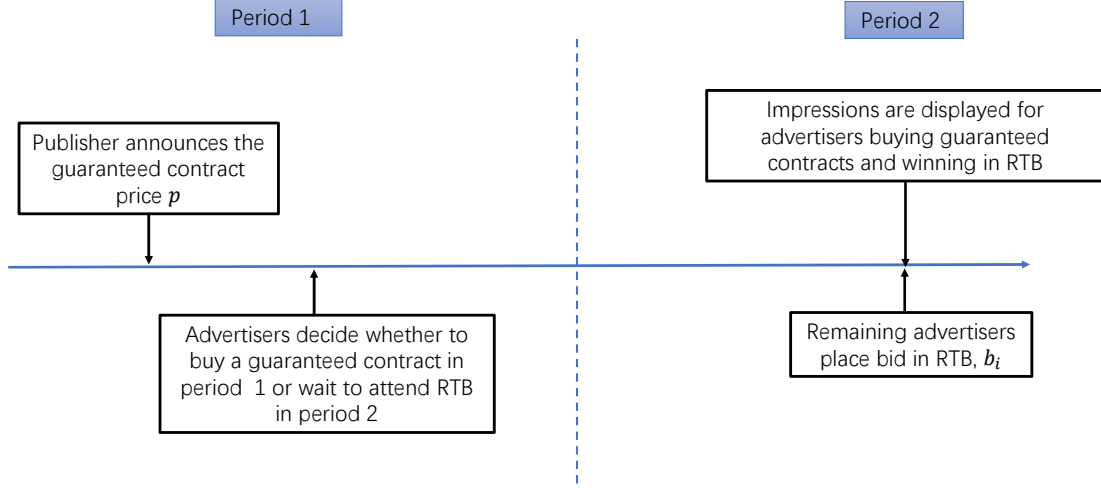


**Figure 2.1:** The sequence of events.

Our unique model setup captures the intricate dynamics between the publisher's pricing on the guaranteed contracts and advertisers' strategic deliberation over buying the guaranteed contracts or attending RTB. Such dynamics are complex because advertisers' rational expectation equilibrium decisions on whether to choose guaranteed contracts in period 1 have to account for the outcome from RTB in period 2. This outcome in RTB is also an equilibrium emerging from all other advertisers' decisions on buying the guaranteed contracts and their bids in auctions. In the following sections, we will show how to obtain the closed-form solution in equilibrium using backward induction.

## 2.4 Equilibrium Analysis and Results

### 2.4.1 Analysis on Advertisers' Decisions

Advertisers make decisions by comparing their expected utility from each channel. Namely, advertiser $i$ will buy a guaranteed contract in period 1 if

$$u_{i,1} > \max\{u_{i,2}, 0\}. \tag{2.1}$$

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Otherwise, advertiser $i$ will join RTB in period 2 if

$$u_{i,2} \geq \max\{u_{i,1}, 0\}. \tag{2.2}$$

Before the analysis of advertisers' behaviours under the coexistence of dual channels, we first consider how advertisers bid when only RTB exists, and the auctions are organised under the second-price rule.

**Lemma 2.1.** *In RTB, if every advertiser only has a unit demand of these homogeneous impressions, and all these impressions are merely sold through the multiple second-price auctions, then truth-telling is a dominant strategy for each advertiser when they are bidding.*

Lemma 2.1 indicates that advertisers should bid by their willingness-to-pay once they decide to join RTB. However, the presence of guaranteed contracts has different effects on different advertisers about their willingness-to-pay in RTB. The following analysis will explain these effects in detail.

We categorise advertisers into two groups based on their valuations relative to the price of a guaranteed contract $p$. The first group consists of advertisers whose valuations $v_i$ are less than or equal to $p$. These advertisers will opt for RTB and bid truthfully according to their valuations. This is because their payback will not exceed zero from guaranteed contracts, while in RTB they can obtain a positive return or at least zero if they bid by their valuations according to Lemma 2.1.

The second group includes advertisers with valuations $v_i > p$. Their motivation to participate in RTB rather than buying a contract is that they can win an impression in RTB. Their payments should not exceed an upper bound shown in the following inequation:

$$p_i^{(r)} < \frac{p - (1 - \delta_a)v_i}{\delta_a}, \tag{2.3}$$

such that $u_{i,2} > u_{i,1} > 0$. Inequation (2.3) also naturally defines a new willingness-to-pay (i.e. $\frac{p - (1 - \delta_a)v_i}{\delta_a}$) for these advertisers, instead of $v_i$ when they decide to attend RTB. Furthermore, according to Lemma 2.1, their bidding prices will also be $\frac{p - (1 - \delta_a)v_i}{\delta_a}$, which is subject to their altered willingness-to-pay.

This phenomenon is interesting. Because compared with their original valuations $v_i$, these advertisers seem to bid untruthfully. Essentially, the reason for this untruthful bidding behaviour is that the presence of guaranteed contracts truncates their valuations in RTB, and they bid "truthfully" according to their truncated valuations in these auctions.

From the above, we summarise a Mixed Truth-telling bidding Strategy in terms of their original valuations for advertisers joining RTB.

**Proposition 2.2** (Mixed Truth-telling Strategy)**.** *Advertisers with lower valuations bid truthfully in RTB, while those with higher valuations bid based on a truncated valuation. The strategy can be uniformly formulated as:*

$$b(v) = \frac{(v \wedge p) - (1 - \delta_a)v}{\delta_a} \tag{2.4}$$

*where* $(v \wedge p) := \min\{v, p\}$.

The bidding function $b(v)$ captures all advertisers' mixed truth-telling bidding strategy for $b(v) = v$ when $v \leq p$, and $b(v) = \frac{p - (1 - \delta_a)v}{\delta_a}$ when $v > p$. In the latter case, $b(v)$ is also decreasing as $v$ increases. Therefore, the upper bound of advertisers' bids in RTB is always $p$. This property is important to induce equilibrium outcomes.

Note that this proposition of mixed truth-telling is implemented by advertisers only after they decide to join RTB in period 2. Their motivation to devote themselves to RTB is that they can win an impression following this bidding strategy from the perspective of expectation. However, whether they can win an impression following this bidding strategy is dependent on the equilibrium outcome of all advertisers' decisions.

**Proposition 2.3.** *In the equilibrium states, if advertisers decide to buy impressions both through guaranteed contracts and RTB, then there must exist a threshold of valuation, denoted by $v'$, such that advertisers with $v > v'$ purchase guaranteed contracts in the first period, while those with $v \leq v'$ choose to attend RTB in the second period.*

Proposition 2.3 reveals a threshold-type pattern for advertisers' behaviours, although we do not need to know the specific value of $v'$ in the current stage. Similar threshold-like behaviours patterns are also obtained in other revenue management research with strategic consumers involved (Caldentey and Vulcano 2007, Aviv and Pazgal 2008, Yin et al. 2009, Osadchiy and Vulcano 2010, Papanastasiou and Savva 2017). This purchase pattern indicates that advertisers with higher valuations more intend to secure impressions through guaranteed contracts than advertisers with lower valuations. The latter are more likely to take risks for a greater surplus in RTB.

### 2.4.2 Analysis on the Publisher's Decisions

The publisher's goal is to determine the price of guaranteed contracts to maximise its revenue based on advertisers' strategic behaviours. We denote the revenue of the publisher as $\Pi$. The feasible strategy space for the publisher is that $p \in [0, +\infty)$.

**Proposition 2.4.** *When the publisher chooses prices of guaranteed contracts in different pricing segments, advertisers' behaviours will show different modes.*

1. *If the publisher sets the price of guaranteed contracts less than $\frac{N-Q}{N}\overline{v}$, all impressions will be sold to advertisers buying guaranteed contracts in period 1.*

2. *If the publisher chooses the price between $\frac{N-Q}{N}\overline{v}$ and $\frac{N-\delta_a Q}{N}\overline{v}$, consumers of these impressions consist of three groups: advertisers with $v \in (v', \overline{v}]$ will buy guaranteed contracts in period 1; advertisers with $v \in (p, v']$ will join RTB in period 2 and bid untruthfully; advertisers with $v \in [0, p]$ will also join RTB in period 2 and bid truthfully. Advertisers who join RTB will follow the bidding strategy $b(v)$ summarised in equation (2.4). The threshold value*

$$v' = \frac{Np - \delta_a(N-Q)\overline{v}}{(1-\delta_a)N}. \tag{2.5}$$

3. *If the publisher picks a price above $\frac{N-\delta_a Q}{N}\overline{v}$, all advertisers will join the real-time bidding and also follow the bidding strategy $b(v)$ in equation (2.4).*

To obtain the publisher's revenue when the price of guaranteed contracts varies, we need to understand how the $Q$ impressions are consumed by advertisers. Proposition 2.4 demonstrates three different cases that how impressions are delivered to advertisers through the two channels. The next is to clarify the payment from each unit impression.

The yield from a unit selling in the guaranteed channel is fixed and easy to capture. While in the non-guaranteed channel, each auctioned impression is paid by the winner with the second highest bid, which varies across different auctions. To address this complexity, we assume that there is a large number of advertisers. The detail is demonstrated in Lemma 2.5.

**Lemma 2.5.** *When the number of advertisers is very large, mathematically $N \to +\infty$, bidders' payments under the multiple second-price auctions approach those under a first-price mechanism, i.e., winners pay as the number of their own bids.*

From Proposition 2.4 and Lemma 2.5, we can obtain the revenue of the publisher when it chooses different prices, which is shown in equation (2.6):

$$\Pi(p) = \begin{cases} Qp, & p \in [0, \frac{N-Q}{N}\overline{v}) \\ \frac{\overline{v}-v'}{\overline{v}}Np + \frac{\delta_p}{2}\psi(v'), & p \in [\frac{N-Q}{N}\overline{v}, \frac{N-\delta_a Q}{N}\overline{v}) \\ \frac{\delta_p}{2}\psi(\overline{v}), & p \in [\frac{N-\delta_a Q}{N}\overline{v}, \overline{v}) \\ \frac{\delta_p}{2} \cdot \frac{(2N-Q)Q}{N}\overline{v}, & p \in [\overline{v}, +\infty) \end{cases}, \tag{2.6}$$

in which $\psi(x) = (p + b(x))\frac{(x-p)N}{\overline{v}} + \left(p + \frac{N-Q}{N}\overline{v}\right)\frac{Np - (N-Q)\overline{v}}{\overline{v}}$. Also note that $\Pi(p)$ is continuous on the domain of $[0, +\infty)$ although it is piece-wise on $\{0\} \cup \mathbf{R}^+$. From the top to the bottom piece in equation (2.6), advertisers move gradually from the guaranteed channel to the RTB channel. When $p \leq \frac{N-Q}{N}\overline{v}$, $\Pi(p)$ is linearly increasing as $p$ increases. While $p > \overline{v}$, $\Pi(p)$ is constant because all advertisers join RTB with truthful bids. Cases under $p \in (\frac{N-Q}{N}\overline{v}, \overline{v}]$ are quadratic, which will be illustrated in the following Figure 2.2.

For convenience, we let $s = \frac{N}{Q}$ to denote the scarcity of impressions. Figure 2.2 shows the tendency of the publisher's revenue under different cases of discount factors and the scarcity of impressions. Different colours on the different parts of a line represent different segments in the revenue function (2.6). This figure intuitively shows the concavity of the revenue function when $p \in (\frac{N-Q}{N}\overline{v}, \frac{N-\delta_a Q}{N}\overline{v}]$ and $p \in (\frac{N-\delta_a Q}{N}\overline{v}, \overline{v}]$ and also its continuity on the whole domain. As discount factors and the scarcity of impressions vary, the optimal price can be located on any piece of the function, which implies that the two channels have their own advantage under different cases. In theory 2.6, the detail of the optimal solution is demonstrated in closed-form.
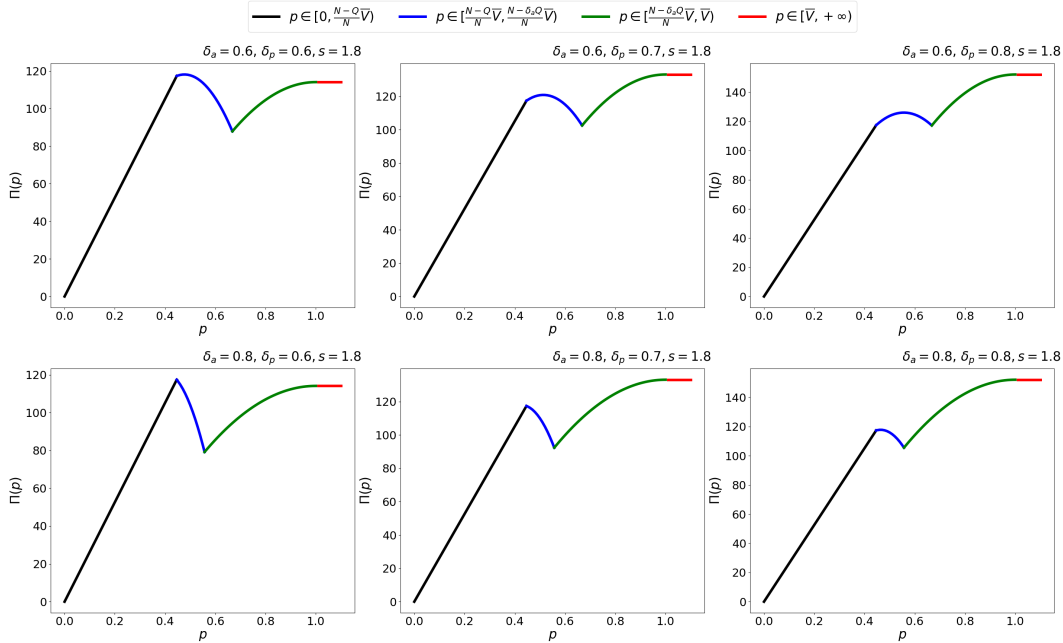


**Figure 2.2:** The publisher's revenue function of guaranteed contracts price

**Theorem 2.6.** *The optimality of the publisher's revenue depends on its discount factor, the advertisers' discount factor, the number of advertisers and the scarcity of impressions.*

1. *When $1 < s < 2$, if $\left(\delta_a \in (2-s, \frac{s}{2s-1}] \,\&\, \delta_p \in [0, \frac{\delta_a+s-2}{s-1})\right) | \left(\delta_a \in (\frac{s}{2s-1}, 1] \,\&\, \delta_p \in [0, \frac{2(s-1)}{2s-1}]\right)$; when $s \geq 2$, if $\left(\delta_a \in (0, \frac{s}{2s-1}] \,\&\, \delta_p \in [0, \frac{\delta_a+s-2}{s-1})\right) | \left(\delta_a \in (\frac{s}{2s-1}, 1] \,\&\,$*

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$\delta_p \in [0, \frac{2(s-1)}{2s-1}])$, *the optimal solution is*

$$p^* = \frac{N - Q}{N}\overline{v};$$

*the corresponding maximum revenue is*

$$\Pi(p^*) = \frac{(N - Q)Q}{N}\overline{v}.$$

*In this case, all impressions are consumed by contract buyers in period 1.*

2. *When* $s = 1$, *if* $\delta_p \in [0, 1 - \sqrt{\delta_a}]$; *when* $1 < s < 2$, *if* $\left(\delta_a \in [0, 2 - s] \,\&\, \delta_p \in [0, 1 - \sqrt{\frac{\delta_a(1-\delta_a)}{s^2 - 2\delta_a s + \delta_a}}]\right) \mid \left(\delta_a \in (2 - s, \frac{s}{2s-1}] \,\&\, \delta_p \in (\frac{\delta_a + s - 2}{s - 1}, 1 - \sqrt{\frac{\delta_a(1-\delta_a)}{s^2 - 2\delta_a s + \delta_a}}]\right)$; *when* $s \geq 2$, *if* $\left(\delta_a \in [0, \frac{s}{2s-1}] \,\&\, \delta_p \in (\frac{\delta_a + s - 2}{s-1}, 1 - \sqrt{\frac{\delta_a(1-\delta_a)}{s^2 - 2\delta_a s + \delta_a}}]\right)$, *the optimal solution is*

$$p = \frac{N - \delta_a Q}{(2 - \delta_p)N}\overline{v};$$

*the corresponding maximum revenue is*

$$\Pi(p^*) = \frac{(1 - \delta_p)^2 N^2 - (2\delta_p^2 - 4\delta_p + 2\delta_a)NQ + (\delta_a^2 + \delta_p^2 - 2\delta_p)Q^2}{2(1 - \delta_a)(2 - \delta_p)N}\overline{v}.$$

*In this case, there exists a threshold value*

$$v' = \frac{(1 + \delta_a\delta_p - 2\delta_a)N + \delta_a(1 - \delta_p)Q}{(1 - \delta_a)(2 - \delta_p)N}\overline{v} \in (p, \overline{v}),$$

*advertisers with* $v_i \in (v', \overline{v}]$ *will buy guaranteed contracts in period 1. Advertisers with* $v_i \in [0, v']$ *will follow the mixed truth-telling bidding strategy in period 2.*

3. *When* $s = 1$, *if* $\delta_p \in (1 - \sqrt{\delta_a}, 1]$; *when* $s > 1$, *if* $\left(\delta_a \in (0, \frac{s}{2s-1}] \,\&\, \delta_p \in (1 - \sqrt{\frac{\delta_a(1-\delta_a)}{s^2 - 2\delta_a s + \delta_a}}, 1]\right) \mid \left(\delta_a \in (\frac{s}{2s-1}, 1] \,\&\, \delta_p \in (\frac{2(s-1)}{2s-1}, 1]\right)$, *the optimal solution is*

$$p^* = \overline{v};$$

*the corresponding maximum revenue is*

$$\Pi(p^*) = \frac{\delta_p}{2} \cdot \frac{(2N - Q)Q}{N}\overline{v}.$$

*In this case, all advertisers participate in RTB and bid truthfully.*

Theorem 2.6 indicates three different types of equilibrium, with impressions consumed by advertisers through different channels. Each of them depends on the
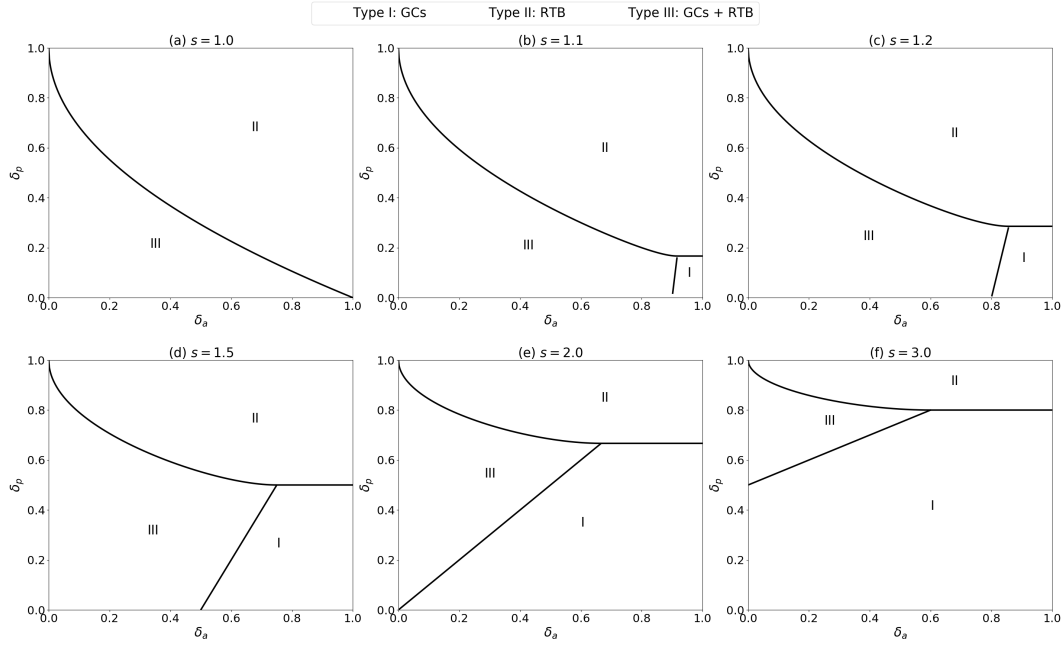
**Figure 2.3:** The composition of consumers for impressions in equilibrium under different cases

scarcity of impressions and the relationships between the scarcity and discount factors of both advertisers and the publisher. This theorem implies that the publisher needs to make a pricing strategy after a careful evaluation of key parameters mentioned above. The effectiveness of different channels is maximised under different parameter settings. According to results from Theorem 2.6, the publisher obtains a guideline for the optimal pricing scheme when both guaranteed contracts and RTB are available in the markets.

Figure 2.3 provides an intuitive understanding of results from Theorem 2.6, by illustrating the varying composition for consumers of impressions under the publisher's optimal pricing strategy, and taking into account different values of $\delta_a$ and $\delta_p$ in specific cases of $s$. The figure is segmented into three distinct zones, each representing a unique type of equilibrium outcome in Theorem 2.6:

- Type I: If the pair $(\delta_a, \delta_p)$ resides in this field, the publisher's most advantageous decision is to establish a guaranteed contract price that induces advertisers to acquire all impressions through guaranteed contracts in the initial period. In this region, RTB is not engaged, and the entire inventory is sold through guaranteed contracts.

- Type II: In this region, the combination of discount factors leads the publisher to set a price for guaranteed contracts that compels all advertisers to join the real-time bidding (RTB) process. This means that under these specific combinations of $\delta_a$ and $\delta_p$, the publisher sets a high price such that no impressions are sold through guaranteed contracts, and the entire inventory is directed to-

wards RTB.

- Type III: When parameters fall within this area, the publisher's optimal guaranteed contract price results in mixed channels. Some impressions are delivered through guaranteed contracts, while the remaining impressions are sold in RTB. This zone represents scenarios where both channels are utilised, reflecting a more nuanced strategy that takes advantage of both guaranteed contracts and RTB.

The figure provides a comprehensive visualisation of how the publisher's optimal pricing strategy is influenced by the interplay of the discount factors $\delta_a$ and $\delta_p$. Along with Theorem 2.6, this figure reveals some interesting insights: First, neither single channel, guaranteed contracts nor RTB, can dominate the other one in all scenarios. The price of guaranteed contracts acts as an adaptor for the publisher to control the choices of advertisers between the two channels. Therefore, the publisher needs to carefully evaluate the parameters of discount factors and the scarcity of impressions to set a proper price to get optimal revenue. Second, the channel of guaranteed contracts is more profitable than RTB when impressions are abundant, and especially when the publisher prefers instant revenue in the first period rather than waiting for RTB (i.e. $\delta_p$ is not that large). When the scarcity of impressions is small and $\delta_a$ and $\delta_p$ are located in the Type II in Figure 2.3, the usage of both channels yields the most revenue. At last, when RTB is dominant over guaranteed contracts, the publisher's optimal decision is to set a relatively higher price for guaranteed contracts, such that all advertisers bid truthfully in RTB. There is no case in equilibrium outcomes where all advertisers participate in RTB but some of them bid untruthfully.

## 2.5 Extension 1: Uncertain Supply of Impressions

In the basic model, we assume that the number of impressions is constant. However, in practice, publishers can only estimate the number of impressions based on historical visiting data or other methods, and they cannot predict with 100% certainty how many visitors will come to their website when they begin to sell guaranteed contracts for future impressions. Therefore, it is meaningful to consider the supply uncertainty of impressions in our model.

In this section, we extend our model to the case where both the risk-neutral publisher and advertisers face an uncertain number of impressions. More specifically, we hypothesise that the number of impressions can be either $Q_L$ or $Q_H$ ($0 < Q_L <$

$Q_H < N$), each with probability $\frac{1}{2}$. For computational convenience, we define:

$$\begin{cases} Q_L &= \alpha_1 N, \\ Q_H &= \alpha_2 N, \end{cases}$$

where $0 < \alpha_1 < \alpha_2 < 1$.

In scenarios where the supply is limited, if the publisher fails to deliver an impression to guaranteed contract buyers, the payment for that impression in period 1 would be returned to the buyers. In other words, we assume that the penalty cost for an undelivered impression equals the price of a guaranteed contract. Additionally, we assume that when the supply of impressions is less than the number required by contract buyers, impressions are allocated in descending order of advertisers' valuations, from higher to lower.

Since both the publisher and advertisers are risk-neutral, they make decisions to maximise their expected utilities, taking into account the uncertainty in the supply of impressions.

### 2.5.1 Analysis on Advertisers' Incentives

Advertisers aim to maximise their expected utilities either from period 1 or from period 2. In period 1, different from what we know in the basic model, advertisers are not guaranteed an impression after buying a contract due to supply uncertainty. By denoting the probability for advertiser $i$ of successfully receiving an impression after buying a guaranteed contract in period 1 as $\mathbb{P}^{i,1}_{win}$, advertiser $i$'s expected utility in period 1 can be given as

$$\mathbb{E}(u_{i,1}) = \mathbb{P}^{i,1}_{win} \cdot (v_i - p). \tag{2.7}$$

a) For advertisers with valuations in $((1 - \alpha_1)\overline{v}, \overline{v}]$, if they buy guaranteed contracts in period 1, they can get an impression no matter the supply of impressions is $Q_L$ or $Q_H$. Thus, we have $\mathbb{P}^{i,1}_{win} = \mathbb{P}\left(q > \frac{\overline{v} - v_i}{\overline{v}} N\right) = 1$. Then their expected utilities in period 1 are $\mathbb{E}(u_{i,1}) = v_i - p$.

b) For advertisers with valuations in $[(1 - \alpha_2)\overline{v}, (1 - \alpha_1)\overline{v}]$, only when the number of impressions is $Q_H$ can they get an impression through guaranteed contracts. Thus, for them, $\mathbb{P}^{i,1}_{win} = \mathbb{P}\left(q > \frac{\overline{v} - v_i}{\overline{v}} N\right) = \frac{1}{2}$. And their expected utilities in period 1 are $\mathbb{E}(u_{i,1}) = \frac{1}{2}(v_i - p)$.

c) For advertisers with valuations in $[0, (1 - \alpha_2)\overline{v})$, they have no chance to get an impression through guaranteed contracts in both cases. Thus, their winning probability in period 1 is $\mathbb{P}^{i,1}_{win} = \mathbb{P}\left(q > \frac{\overline{v} - v_i}{\overline{v}}\right) = 0$. Their expected utilities in period 1 are

$\mathbb{E}(u_{i,1}) = 0.$

Similar to the analysis in the basic model, we take $b_n(v_i)$ as advertiser $i$'s bidding price in period 2. According to Lemma 2.5, advertiser $i$'s payment in period 2 will be very close to its bidding price. So if advertiser $i$ can win an impression in period 2 by bidding with $b_n(v_i)$, its utility is

$$\mathbb{E}(u_{i,2}) = \delta_a(v_i - b_n(v_i)). \tag{2.8}$$

As demonstrated in the basic model, an advertiser's incentive to participate in RTB rather than purchasing a guaranteed contract is driven by the possibility of winning an impression in RTB at a price that yields a profit not less than buying a contract at price $p$. Consequently, advertisers' bidding strategy is dependent on the price of guaranteed contracts.

At the commencement of the selling horizon, advertisers are aware of their valuations for an impression and observe the price $p$. For those advertisers with valuations less than $p$, their utilities would be negative if they were to purchase a contract in period 1. As a result, they will opt for RTB and bid truthfully, as outlined in Lemma 2.1.

For the remaining advertisers, their decisions are contingent on the pricing of guaranteed contracts. Therefore, we explore their behavioural patterns under various pricing segments, which are encapsulated in the following proposition.

**Proposition 2.7.** *Let* $v_1' = \frac{p - \delta_a(1 - \alpha_2)\overline{v}}{1 - \delta_a}$, $v_2' = \frac{2p - \delta_a(1 - \alpha_2)\overline{v}}{2 - \delta_a}$, $v_3' = \frac{p - \delta_a(1 - \alpha_1)\overline{v}}{1 - \delta_a}$, *for advertisers with* $v_i > p$, *their behavioural modes contingent on the pricing of guaranteed contracts are:*

1. *If* $p \in \left[0, (1 - \alpha_2)\overline{v}\right]$:

   (a) *When* $v_i \in \left(p, \overline{v}\right]$, *advertiser i will buy guaranteed contracts.*

2. *If* $p \in \left((1 - \alpha_2)\overline{v}, [1 - \alpha_1 - \frac{\delta_a}{2}(\alpha_2 - \alpha_1)]\overline{v}\right]$:

   (a) *When* $v_i \in \left(p, v_1' \vee (1 - \alpha_1)\overline{v}\right]$, *advertiser i will join RTB and* $b_n(v_i) = \frac{p - (1 - \delta_a)v_i}{\delta_a}$.

   (b) *When* $v_i \in \left(v_1' \vee (1 - \alpha_1)\overline{v}, \overline{v}\right]$, *advertiser i will buy guaranteed contracts.*

3. *If* $p \in \left([1 - \alpha_1 - \frac{\delta_a}{2}(\alpha_2 - \alpha_1)]\overline{v}, (1 - \delta_a\alpha_1)\overline{v}\right]$:

   (a) *If* $\alpha_1 \leq \frac{\alpha_2}{2}$,

      i. *When* $v_i \in \left(p, v_3' \vee (1 - \alpha_1)\overline{v}\right]$, *advertiser i will join RTB and* $b_n(v_i) = \frac{p - (1 - \delta_a)v_i}{\delta_a}$.

ii. When $v_i \in \left( v_3' \vee (1 - \alpha_1)\overline{v}, v_2' \wedge \overline{v} \right]$, advertiser $i$ will join RTB and $b_n(v_i) = \frac{2p - (2 - \delta_a)v_i}{\delta_a}$.

iii. When $v_i \in \left( v_2', \overline{v} \right]$ if $v_2' \wedge \overline{v} = v_2'$, advertiser $i$ will buy guaranteed contracts.

(b) If $\alpha_1 > \frac{\alpha_2}{2}$,

i. If $p \in \left( [1 - \alpha_1 - \frac{\delta_a}{2}(\alpha_2 - \alpha_1)]\overline{v}, [1 - \alpha_1 + (1 - \delta_a)(\alpha_2 - \alpha_1)]\overline{v} \right]$:

A. When $v_i \in \left( p, v_3' \vee (1 - \alpha_1)\overline{v} \right]$, advertiser $i$ will join RTB and $b_n(v_i) = \frac{p - (1 - \delta_a)v_i}{\delta_a}$.

B. When $v_i \in \left( v_3' \vee (1 - \alpha_1)\overline{v}, v_2' \right]$, advertiser $i$ will join RTB and $b_n(v_i) = \frac{2p - (2 - \delta_a)v_i}{\delta_a}$.

C. When $v_i \in \left( v_2', \overline{v} \right]$, advertiser $i$ will buy guaranteed contracts.

ii. If $p \in \left( [1 - \alpha_1 + (1 - \delta_a)(\alpha_2 - \alpha_1)]\overline{v}, (1 - \delta_a\alpha_1)\overline{v} \right]$:

A. When $v_i \in \left( p, v_3' \right]$, advertiser $i$ will join RTB and $b_n(v_i) = \frac{p - (1 - \delta_a)v_i}{\delta_a}$.

B. When $v_i \in \left( v_3', \overline{v} \right]$, advertiser $i$ will join RTB and $b_n(v_i) = \frac{2p - (2 - \delta_a)v_i}{\delta_a}$.

4. If $p \in \left( (1 - \delta_a\alpha_1)\overline{v}, \overline{v} \right]$:

(a) When $v_i \in \left( p, \overline{v} \right]$, advertiser $i$ will join RTB and $b_n(v_i) = \frac{p - (1 - \delta_a)v_i}{\delta_a}$.

Proposition 2.7 reveals that the uncertain supply of impressions increases the complexity of advertisers' behaviours compared with Proposition 2.4 in the basic model. The pricing segments also depend on the variance between $Q_L$ and $Q_H$. Different advertisers follow different truncated bidding functions ($\frac{2p - (2 - \delta_a)v_i}{\delta_a}$ and $\frac{p - (1 - \delta_a)v_i}{\delta_a}$) when the price of contracts is locating in specific segments. Advertisers with higher valuations tend to explore a lower bidding price $\frac{2p - (2 - \delta_a)v_i}{\delta_a}(< \frac{p - (1 - \delta_a)v_i}{\delta_a})$ to seek a higher expected payback than that from guaranteed contracts under some specific scenarios. However, a small part of top-valued advertisers still want to secure impressions through guaranteed contracts when the price of guaranteed contracts is not too expensive.

### 2.5.2 The Publisher's Optimal Decision Analysis

In theory, it is possible to derive the publisher's optimal decision regarding the pricing of guaranteed contracts, as well as its maximum revenue, in accordance with

Proposition 2.7. However, the formulation of this solution would be overly cumbersome to present explicitly. Instead, we provide a graphical illustration that depicts the varying composition of consumers for impressions under the publisher's optimal pricing strategy for guaranteed contracts. This illustration takes into account variations in the parameters, namely $\alpha_1, \alpha_2, \delta_1, \delta_2$.
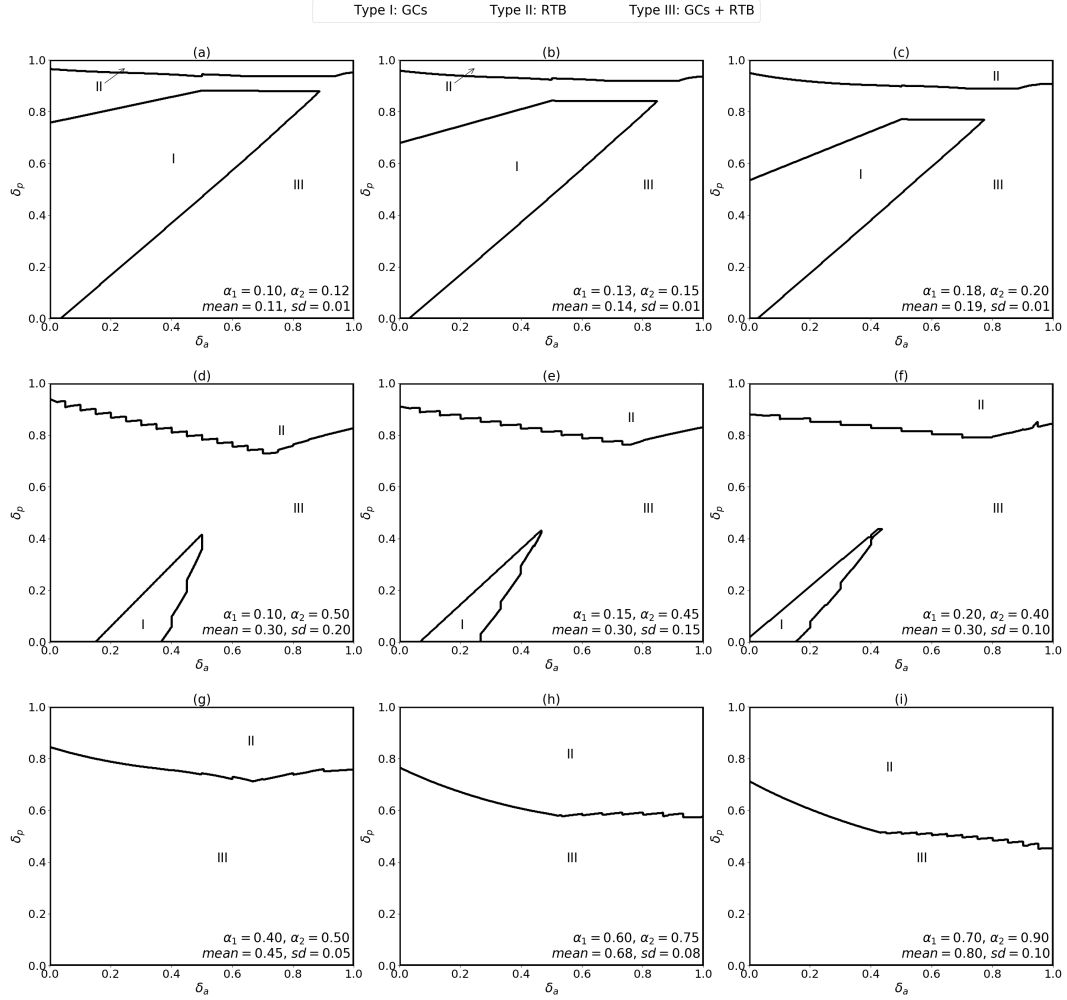


**Figure 2.4:** Composition for consumers of impressions in different cases under uncertain supply

In Figure 2.4, the subplots are organised to represent different scenarios. The first row of subplots shares the same standard deviation of $\alpha_1, \alpha_2$, with increasing means. The second row maintains the same mean for $\alpha_1, \alpha_2$, but with decreasing standard deviations. In the last row, both the means and standard deviations are increasing across the subplots. The figure is segmented into three distinct zones:

- Type I: This region represents scenarios where all impressions are sold to advertisers through guaranteed contracts.

- Type II: This zone signifies that some impressions are consumed through guaranteed contracts, while the remaining impressions are sold in RTB.

- Type III: In this area, all advertisers join the real-time bidding (RTB), and no impressions are sold through guaranteed contracts.

The figure reveals interesting dynamics in the publisher's optimal pricing strategy. When both $\alpha_1, \alpha_2$ are small, a wide range of $\delta_a$ and $\delta_p$ leads to all impressions being sold through guaranteed contracts (Type I). As the mean of $\alpha_1, \alpha_2$ increases, the Type I zone shrinks, and the Type II zone expands, revealing that the publisher leverages the RTB when the mean scarcity of impressions decreases. If the mean is held constant and the standard deviation decreases, the Type I zone shifts to the left, becoming longer and narrower. This implies that advertisers prefer guaranteed contracts more when the variance of the number of impressions increases.

This visualisation provides valuable insights into how the publisher's pricing strategy adapts to different levels of uncertainty and market conditions. When compared with Figure 2.3, the purple zone diminishes when the mean scarcity of impressions aligns with that in the basic model, while the green zone is more expansive in nearly all scenarios. This phenomenon indicates the efficacy of the dual-channel approach, as it enables the publisher to mitigate the negative effects of supply uncertainty. This strategy dominates scenarios where only one single channel is occupied, reflecting a more resilient and adaptive response to market fluctuations.

## 2.6 Extension 2: Advertiser Segmentation

Publishers who sell impressions via dual channels seek more revenue by implementing various strategies. One such strategy is customer segmentation, which is a strategic approach that involves dividing a business's customer base into distinct groups based on their specific characteristics, preferences, or behaviours. The primary goal of customer segmentation is to optimise revenue by tailoring pricing, marketing, and customer service strategies to better target and serve each segment (Sari et al. 2016).

This extension aims to investigate the effects of customer segmentation on a publisher's decision-making process and subsequent revenue generation. By examining the effect of the division of advertisers and impressions into two or more segments evenly, we aim to provide insights into the optimal pricing strategies and the conditions under which customer segmentation is most beneficial. Additionally, the impact of the scarcity of impressions and discount factors on the publisher's optimal revenue will be explored. The results derived from this extension serve as a valuable resource for publishers seeking to maximise their revenue through the effective implementation of customer segmentation strategies in the online advertising landscape.

### 2.6.1  Two Segments

At first, we explore the effect of dividing advertisers and impressions into two segments. In the first segment, there are $\frac{N}{2}$ advertisers with valuations in $[0, \frac{\bar{v}}{2}]$. The left $\frac{N}{2}$ advertisers are in the second segment. Their valuations are from $\frac{\bar{v}}{2}$ to $\bar{v}$. $Q$ impressions are assigned to these two segments evenly, i.e., $\frac{Q}{2}$ impressions to each segment. Other settings and parameters are kept the same as that in the basic model. In order to get the general form of the publisher's revenue in each segment, we first provide the publisher's revenue function when advertisers' valuations are uniformly distributed in $[v_l, v_h]$.

**Lemma 2.8.** *If there are $N$ advertisers with valuation uniformly distributed in $[v_l, v_h]$. There are $\hat{Q}$ impressions available. The discount factors for advertisers' utilities in period 2 is $\delta_a$, for the publisher's revenue from period 2 is $\delta_p$. Denote the price of guaranteed contracts as $p$. The publisher's revenue can be given as:*

$$
\Pi(p) = \begin{cases}
\hat{Q}p, & p \in [v_l, v_{t1}) \\[2mm]
\dfrac{(N - \delta_a \hat{Q})v_h + \delta_a \hat{Q} v_l - Np}{(1 - \delta_a)(v_h - v_l)}p + \dfrac{\delta_p}{2}\dfrac{p^2 - v_{t1}^2}{(1 - \delta_a)(v_h - v_l)}N, & p \in [v_{t1}, v_{t2}) \\[2mm]
\dfrac{\delta_p}{2}\left\{ \left( \dfrac{p - (1 - \delta_a)v_h}{\delta_a} + p \right) \dfrac{v_h - p}{v_h - v_l}N + \dfrac{p^2 - v_{t1}^2}{v_h - v_l}N \right\}, & p \in [v_{t2}, v_h) \\[2mm]
\dfrac{\delta_p}{2}\dfrac{v_h^2 - v_{t1}^2}{v_h - v_l}N, & p \in [v_h, +\infty)
\end{cases}
\quad (2.9)
$$

*in which* $v_{t1} = \dfrac{N - \hat{Q}}{N}v_h + \dfrac{\hat{Q}}{N}v_l$, $v_{t2} = \dfrac{N - \delta_a \hat{Q}}{N}v_h + \dfrac{\delta_a \hat{Q}}{N}v_l$.

By substituting $\hat{Q} = \frac{Q}{2}, v_l = 0$, $v_h = \frac{\bar{v}}{2}$ and $\hat{Q} = \frac{Q}{2}, v_l = \frac{\bar{v}}{2}$, $v_h = \bar{v}$ into (2.9), respectively, we can get the publisher's revenue function in both two segments. Consequently, the publisher's optimal pricing and maximum revenue in each segment can be obtained.

Accordingly, we can summarise the publisher's optimal revenue when dividing advertisers and impressions into two segments evenly.

**Proposition 2.9.** *When advertisers and impressions are evenly grouped into two subsets, the publisher's optimal decisions and revenue are shown in the following:*

1. *When $1 < s < 2$, if $\left( \delta_a \in (2 - s, \frac{s}{2s-1}] \ \& \ \delta_p \in [0, \frac{\delta_a + s - 2}{s-1}] \right) \mid \left( \delta_a \in (\frac{s}{2s-1}, 1] \ \& \ \delta_p \in [0, \frac{2(s-1)}{2s-1}] \right)$; when $s \geq 2$, if $\left( \delta_a \in (0, \frac{s}{2s-1}] \ \& \ \delta_p \in [0, \frac{\delta_a + s - 2}{s-1}] \right) \mid \left( \delta_a \in (\frac{s}{2s-1}, 1] \ \& \ \delta_p \in [0, \frac{2(s-1)}{2s-1}] \right)$, the optimal solution is*

$$
p_1^* = \frac{N - Q}{2N}\bar{v}, \ p_2^* = \frac{2N - Q}{2N}\bar{v};
$$

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*the corresponding maximum revenue is*

$$\Pi(p^*) = \frac{3NQ - 2Q^2}{4N}\overline{v}.$$

2. *When* $s = 1$, *if* $\delta_p \in [0, 1 - \sqrt{\delta_a}]$; *when* $1 < s < 2$, *if* $\left(\delta_a \in [0, 2 - s] \; \& \; \delta_p \in [0, 1 - \sqrt{\frac{\delta_a(1-\delta_a)}{s^2 - 2\delta_a s + \delta_a}}]\right) \mid \left(\delta_a \in (2 - s, \frac{s}{2s-1}] \; \& \; \delta_p \in (\frac{\delta_a + s - 2}{s - 1}, 1 - \sqrt{\frac{\delta_a(1-\delta_a)}{s^2 - 2\delta_a s + \delta_a}}]\right)$; *when* $s \geq 2$, *if* $\left(\delta_a \in [0, \frac{s}{2s-1}] \; \& \; \delta_p \in (\frac{\delta_a + s - 2}{s - 1}, 1 - \sqrt{\frac{\delta_a(1-\delta_a)}{s^2 - 2\delta_a s + \delta_a}}]\right)$, *the optimal solution is*

$$p_1^* = \frac{N - \delta_a Q}{2(2 - \delta_p)N}\overline{v}, \; p_2^* = \frac{2N - \delta_a Q}{2(2 - \delta_p)N}\overline{v};$$

*the corresponding maximum revenue is*

$$\Pi(p^*) = \frac{5(1 - \delta_a)^2 N^2 - 3(2\delta_p^2 - 4\delta_p + 2\delta_a)NQ + 2(\delta_a^2 + \delta_p^2 - 2\delta_p)Q^2}{8(1 - \delta_a)(2 - \delta_p)} \frac{\overline{v}}{N}.$$

3. *When* $s = 1$, *if* $\delta_p \in (1 - \sqrt{\delta_a}, 1]$; *when* $s > 1$, *if* $\left(\delta_a \in (0, \frac{s}{2s-1}] \; \& \; \delta_p \in (1 - \sqrt{\frac{\delta_a(1-\delta_a)}{s^2 - 2\delta_a s + \delta_a}}, 1]\right) \mid \left(\delta_a \in (\frac{s}{2s-1}, 1] \; \& \; \delta_p \in (\frac{2(s-1)}{2s-1}, 1]\right)$, *the optimal solution is*

$$p_1^* = \frac{\overline{v}}{2}, \; p_2^* = \overline{v};$$

*the corresponding maximum revenue is*

$$\Pi(p^*) = \frac{\delta_p}{2} \cdot \frac{(3N - Q)Q}{2N}\overline{v}.$$

There are three cases of equilibrium results in Proposition 2.9. The conditions of parameters in every case are the same in the two different segments. Moreover, under certain combinations of parameters, the equilibrium modes from both segments are the same. Specifically, when $p_1^* = \frac{N-Q}{2N}\overline{v}$ and $p_2^* = \frac{2N-Q}{2N}\overline{v}$, then impressions in both segments are consumed by guaranteed contracts buyers. When $p_1^* = \frac{N - \delta_a Q}{2(2 - \delta_p)N}\overline{v}$ and $p_2^* = \frac{2N - \delta_a Q}{2(2 - \delta_p)N}\overline{v}$, publisher allocate impressions not only to guaranteed contracts but also to RTB. When $p_1^* = \frac{\overline{v}}{2}$ and $p_2^* = \overline{v}$, all advertisers in these two segments join RTB and bid truthfully. Besides, the conditions of parameters in each case are the same as that in Theorem 2.6, which makes it convenient to compare the results to that in Theorem 2.6.

**Corollary 2.10.** *Comparing results from two segmentation solution from Proposition 2.9 and the no segmentation solution from Theorem 2.6 in the basic model, we have:*

1. *When* $s \geq 2$, *the publisher does not need to implement customer segmentation.*

2. *When $s < 2$:*

    (a) *If $2\delta_a + \delta_p \geq 2$:*

        i. *if $s > \dfrac{2 - \delta_p}{2(1 - \delta_p)}$, the publisher should take the two segmentation solution.*

        ii. *if $s \leq \dfrac{2 - \delta_p}{2(1 - \delta_p)}$, the publisher should not divide advertisers and impressions.*

    (b) *If $2\delta_a + \delta_p < 2$:*

        i. *if $s > \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}$, the publisher should take the two segmentation solution.*

        ii. *if $s \leq \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}$,*

            A. *if $\delta_a > (1 - \delta_p)^2$ and $\left(\delta_a + \dfrac{\sqrt{\delta_a \delta_p (1 - \delta_a)(2 - \delta_p)}}{1 - \delta_p}\right) < s \leq \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}$ or if $\delta_a \leq (1 - \delta_p)^2$, and if*

$$\frac{5(1 - \delta_a)^2 N^2 - 3(2\delta_p^2 - 4\delta_p + 2\delta_a)NQ + 2(\delta_a^2 + \delta_p^2 - 2\delta_p)Q^2}{8(1 - \delta_a)(2 - \delta_p)} \frac{\overline{v}}{N}$$
$$> \frac{(1 - \delta_a)^2 N^2 - (2\delta_p^2 - 4\delta_p + 2\delta_a)NQ + (\delta_a^2 + \delta_p^2 - 2\delta_p)Q^2}{2(1 - \delta_a)(2 - \delta_p)} \frac{\overline{v}}{N}$$

            *the publisher should take customer segmentation.*

            B. *if $\delta_a > (1 - \delta_p)^2$ and $s \leq \left(\delta_a + \dfrac{\sqrt{\delta_a \delta_p (1 - \delta_a)(2 - \delta_p)}}{1 - \delta_p}\right)$, the publisher should not divide advertisers and impressions.*

Figure 2.5 visualises the comparison results in this Corollary. The lower part represents scenarios where it is more optimal to group advertisers into two segments, depending on the location of $\delta_a$ and $\delta_p$. Conversely, the upper zone indicates that no segmentation is preferable under corresponding discount factors. As Figure 2.5 illustrates, when $s \geq 2$ (i.e., the demand for impressions is strong), the publisher does not need to implement a two-segmentation strategy. Only when $s < 2$ and the discount factor of the publisher does not roughly exceed a variable upper bound does the two-segmentation strategy outperform the basic one. Furthermore, this boundary shifts downward as the scarcity of impressions increases.
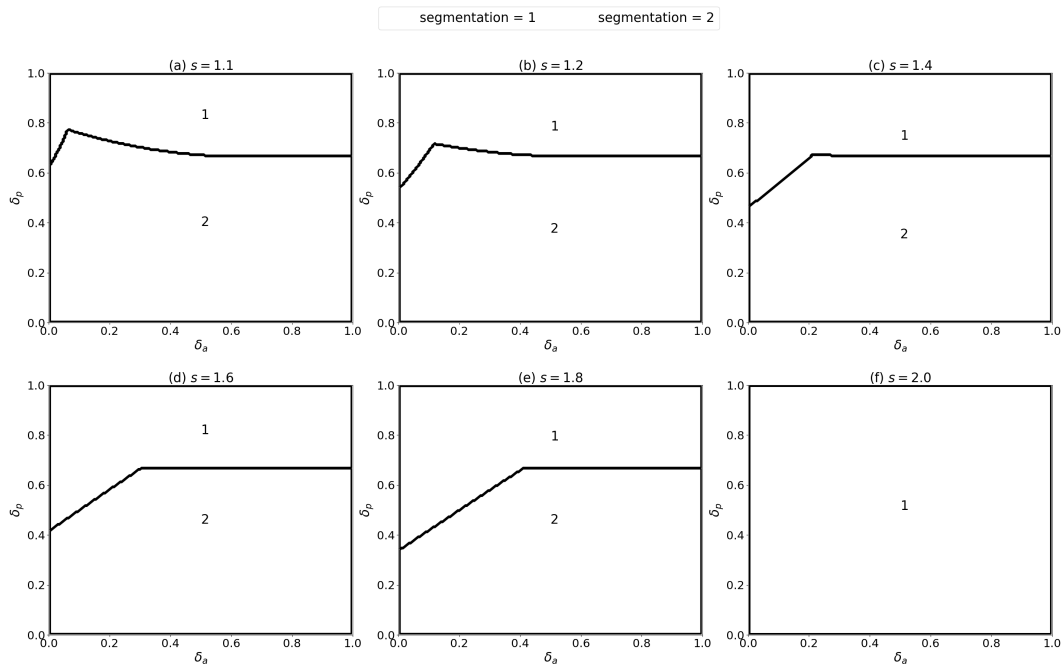
**Figure 2.5:** Two segmentation or no segmentation under different cases

## 2.6.2 More Segments

Due to the complexity of obtaining closed-form results under multiple segments, we numerically compare the basic model to models involving more segments. Figure 2.6 illustrates how the optimal revenue for the publisher changes as the scarcity of impressions increases. From this figure, it is evident that the maximum revenue obtained from customer segmentation surpasses that of the basic model when $s$ is sufficiently close to one. As $s$ increases, the solution that pools all advertisers and impressions together becomes more profitable than solutions of implementing a segmentation strategy.

Figure 2.7 demonstrates how the publisher's optimal choice of segmentation varies when $\delta_a$ and $\delta_p$ change. When $s$ is close to one, and $\delta_p$ is also close to one, the publisher should opt for the non-segmentation solution to maximise its revenue. For the cases in the leftmost region of $\delta_a$ and $\delta_p$, the solution involving the most segments generates the highest revenue, as evident from the plots in the first row. If $s$ is two or greater, the non-segmentation solution is applicable to all cases of $\delta_a$ and $\delta_p$. These observations are similar to what is revealed in Figure 2.5. By further looking into the differences between Figure 2.5 and Figure 2.7, there are two interesting findings:

1. In our experiments, the optimal strategies are either the no-segmentation or the most-segmentation scheme (i.e., the scheme where the number of segments is eight). This phenomenon may indicate that the publisher only needs to compare unified pricing (i.e., no segmentation) and totally personalised pricing
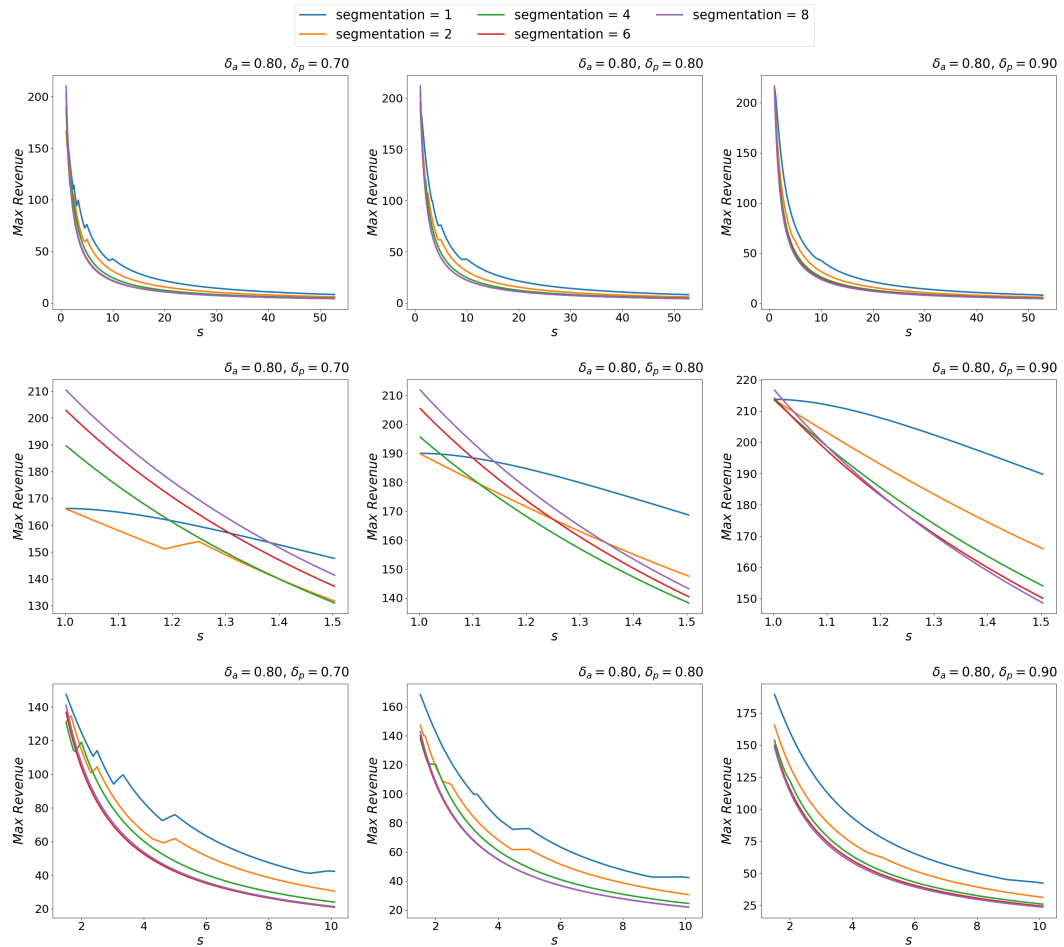
**Figure 2.6:** The effect of customer segmentation on the publisher's optimal revenue

when implementing the advertiser segmentation strategy. However, the limitation of this extension is that we cannot obtain the results of personalised pricing due to the calculation complexity.

2. The boundaries in Figure 2.7 that divide the two strategies in the former five subplots climb upward slightly compared with those in Figure 2.5. This means that the strategy containing eight segments dominates the no-segmentation strategy in some cases where the no-segmentation strategy was better than the strategy of two-segmentation.

This extension offers crucial insights for publishers when considering adopting customer segmentation. It reveals that this strategy can significantly enhance revenue when the number of impressions closely approaches the number of advertisers. In such scenarios, implementing a customer segmentation strategy can lead to a more effective allocation of impressions and optimised revenue generation. However, when the scarcity of impressions is high, the benefits of customer segmentation become less pronounced. In these cases, it may not be advantageous to implement segmentation, as the valuations of advertisers that can win an impres-
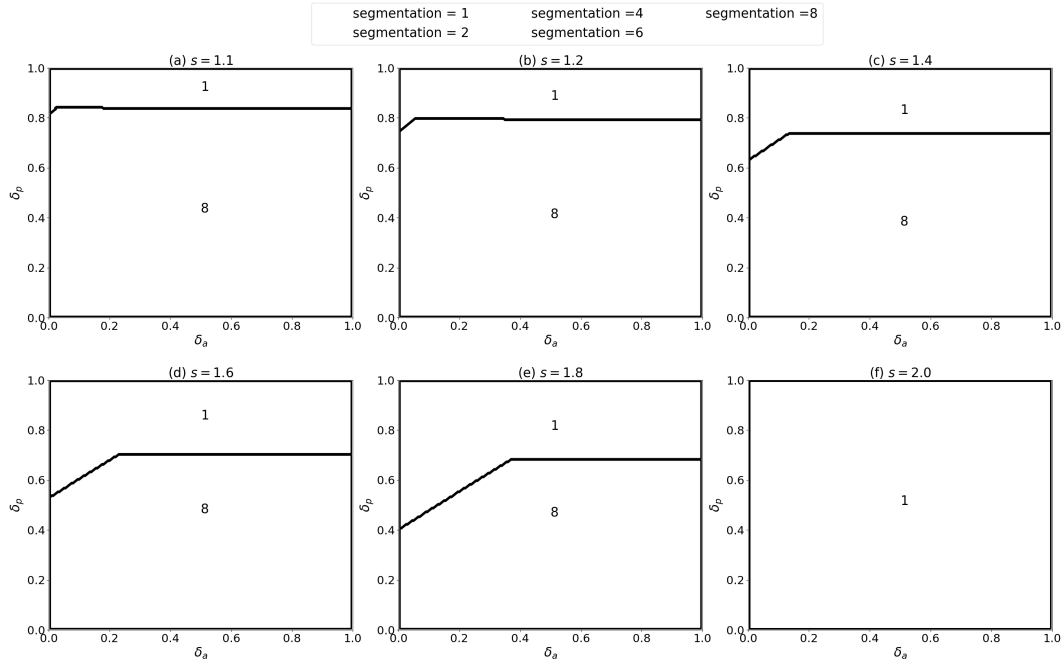
**Figure 2.7:** The optimal segmentation strategy under different cases

sion concentrate at the top. Moreover, the research underscores the importance of considering $\delta_a$ and $\delta_p$ values when determining optimal solutions, as they can influence the efficacy of different strategies. Besides, the limitation of this extension also inspires future research about exploring the personalised pricing of guaranteed contracts in online display advertising markets. In summary, these findings can guide publishers in making informed decisions about customer (advertiser) segmentation to maximise revenue potential.

## 2.7   Extension 3: Different Valuation Distributions

To derive the closed-form solution of the basic model, we simplify the distribution of advertisers' valuations to impressions as uniform. However, this assumption may not fully capture the complexities of real-world scenarios, where the distribution of advertisers' valuations is often diverse. Recognising this limitation, we extend our analysis to more general cases to ensure the robustness and applicability of our results. In this extension, we check the robustness of our findings derived from the basic model by considering general distributions of advertisers' valuations on both bounded and unbounded support. We then numerically demonstrate the composition for consumers of impressions under different valuation distributions. These numerical experiments further validate our theoretical findings and provide concrete examples of how our model can be applied in various market settings.

### 2.7.1 Models under General Distributions

In this section, we present the formulation of the publisher's revenue function without specifying the distribution of advertisers' valuations for homogeneous impressions. We use $f(\cdot)$ and $F(\cdot)$ to denote the probability density function ($p.d.f.$) and cumulative distribution function ($c.d.f.$) of the distribution, respectively. To capture the feature that there are always some advertisers who are not interested in a specific impression, especially when there is a large basis of advertisers, we set the support of the distribution starting from 0.

**Bounded support distribution**

In this case, the range of advertisers' potential valuation for an impression ranges from $[0, \overline{v}]$. Thus, we have

$$F(0) = 0, \; F(\overline{v}) = 1.$$

Similar to Proposition 2.4, we can give the publisher's revenue function as follows:

$$\Pi(p) = \begin{cases} Qp, & p \in [0, v_{(1)}) \\ \int_{v'}^{\overline{v}} Npf(v)\,\mathrm{d}v + \delta_p \psi(v'), & p \in [v_{(1)}, v_{(2)}) \\ \delta_p \psi(\overline{v}), & p \in [v_{(2)}, \overline{v}) \\ \delta_p \int_{v_{(1)}}^{\overline{v}} Nvf(v)\,\mathrm{d}v, & p \in [\overline{v}, +\infty) \end{cases}, \tag{2.10}$$

in which we define

$$\psi(\tau) = \int_{v_{(1)}}^{p} Nb(v)f(v)\,\mathrm{d}v + \int_{p}^{\tau} Nb(v)f(v)\,\mathrm{d}v, \; b(v) = \frac{(v \wedge p) - (1 - \delta_a)v}{\delta_a},$$

and

$$v' = \frac{p - \delta_a v_{(1)}}{1 - \delta_a}, \; v_{(1)} = F^{-1}\left(\frac{N - Q}{N}\right), \; v_{(2)} = (1 - \delta_a)\overline{v} + \delta_a v_{(1)}.$$

**Unbounded support distribution**

Because the distribution of advertisers' valuation to impressions is unbounded, for any large price $p_L$ of guaranteed contracts, there will be some advertisers with valuations greater than $\dfrac{p_L - \delta_a F^{-1}\left(\frac{N-Q}{N}\right)}{1 - \delta_a}$ will buy guaranteed contracts. Therefore, the cases in which all advertisers participate in RTB will not exist.

Consequently, if $v \in [0, +\infty)$, we have the publisher's revenue function below:

$$\Pi(p) = \begin{cases} Qp, & p \in [0, v_{(1)}) \\ \int_{v'}^{+\infty} N p f(v) \, \mathrm{d}v + \delta_p \psi(v'), & p \in [v_{(1)}, +\infty) \end{cases}. \qquad (2.11)$$

## 2.7.2  Results on Different Distributions

We test our results under different distributions, including truncated log-normal and practical distributions derived from industrial data. The industrial data for display advertising are obtained from an ad platform in the U.K. These data include various details such as the ID of advertisers, publisher's ID, the time of bidding, the bidding price, the final payment of the advertiser, and other related information. All the data were generated from real-time bidding under the second-price auction. Thus, we extract advertisers' bidding prices as their valuations for the auctioned impressions. This approximately aligns with our assumption of homogeneous impressions. To maintain consistency with this assumption, we filtered bidding prices to the same advertising slot within a half-hour time interval.

Figure 2.8 illustrates the histogram of advertisers' valuations extracted from a specified segment of this dataset. The histogram demonstrates a practical distribution of advertisers' valuations that cannot be well captured by elementary functions. However, this distribution still exhibits a widely observed approximate reverse U-shape with a slight right skew. This pattern illustrates that most advertisers participating in this campaign have high to medium valuations for impressions. Furthermore, we introduce a truncated log-normal distribution to our test, which is typically left-skewed, to compare equilibrium outcomes with those under uniform distribution in the basic model.

Figure 2.9 represents these comparisons, with each column in the subplots representing a type of distribution. The scarcity of impressions $s = 1.5$ in the upper row and $s = 2$ in the lower row of the subplots.

Figure 2.9 uncovers some interesting insights. On one hand, the pattern of equilibrium outcomes is similar across different distributions. Specifically:

- When both $\delta_a$ and $\delta_p$ are close to one, the optimal prices under different distributions are high, leading all advertisers to RTB.

- When $\delta_a$ is small and $\delta_p$ is high, especially when the scarcity of impressions is also high, the publisher should set prices to engage advertisers in both channels.
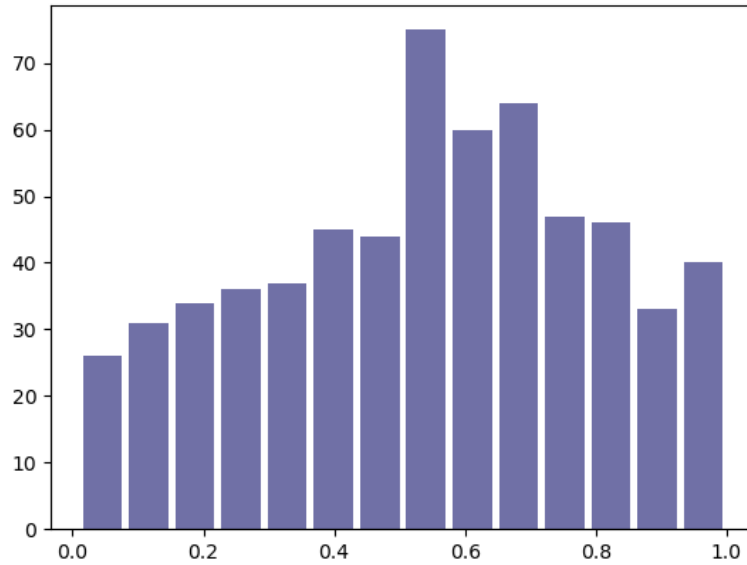
**Figure 2.8:** Histogram of advertisers' valuations from industrial data

- When $\delta_p$ is small and the scarcity is high, the publisher is more likely to set a lower guaranteed contract price to ensure that all impressions are consumed by contract buyers.

On the other hand, under the same scarcity level of impressions, equilibrium outcomes in both truncated log-normal and the practical distributions of valuations are more likely to lead to all impressions being consumed by guaranteed contracts than that in the uniform distribution.

These observations not only indicate the robustness of our results across different distributions of advertisers' valuations but also contribute to the understanding of the distribution of advertisers' valuations and the corresponding pricing strategies. Consequently, these insights provide valuable guidance for publishers seeking to optimise their revenue across various market conditions. By recognising the nuances in advertisers' valuation distributions and adapting pricing strategies accordingly, publishers can make more informed decisions that align with the specific characteristics of their market, thereby enhancing their ability to maximise revenue.

## 2.8 Conclusion

In this study, we investigate the impact of guaranteed contracts on advertisers' behaviours, advertisers' choices across dual channels, and their bidding strategy in the Real-Time Bidding (RTB) system. We discovered a threshold property of adver-
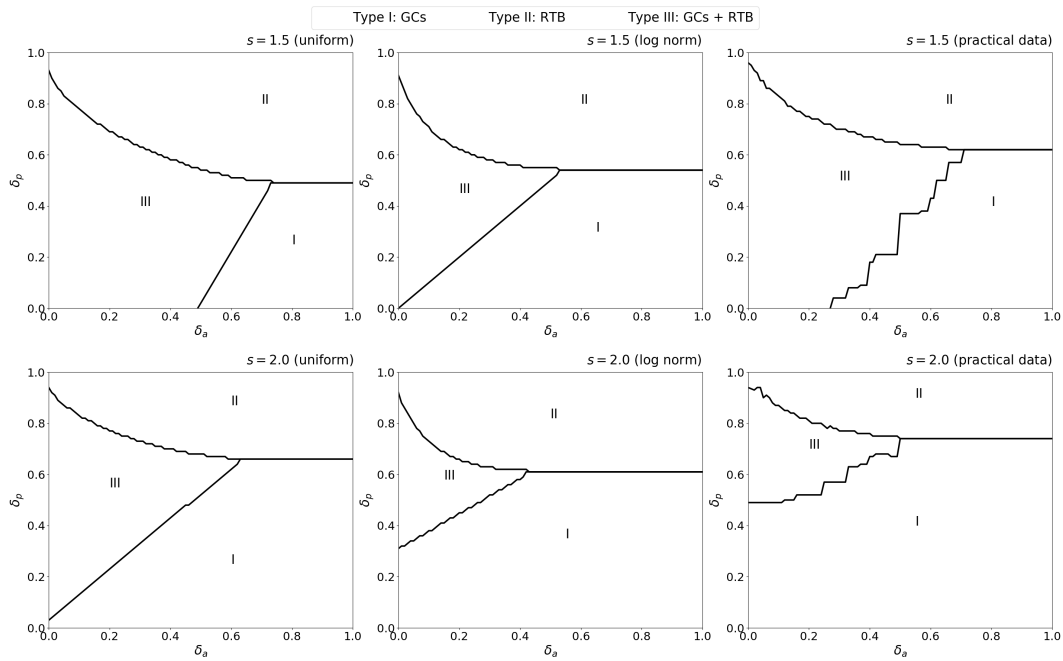
**Figure 2.9:** Composition for consumers of impressions in different cases

tisers' choice and found that the truth-telling strategy is not universally applicable to all advertisers in RTB. Instead, a mixed truth-telling strategy is more representative of advertisers' bidding behaviours. We also found that customer segmentation is not always beneficial and that the publisher tends to leverage the combination of dual channels to mitigate uncertainty in the supply of impressions. Finally, we show the robustness of our results under different distributions of advertisers' valuations.

This study contributes to the literature in three significant ways. First, it introduces a new perspective on advertisers' decision-making in dual channels, revealing a threshold-type strategy that deepens our understanding of strategic purchasing decisions in online display advertising (e.g. Chen et al. 2020 and Cohen et al. 2023). Second, it explores the interaction between guaranteed contracts and RTB in second-price auctions, showing that the presence of guaranteed contracts can influence an advertiser's bid limited to the price of the contract. This finding challenges the conventional wisdom on bidding strategy in second-price RTB with a fixed price channel (e.g. Sayedi 2018 and Cohen et al. 2023). Finally, the study suggests that publishers can increase their revenues by strategically combining guaranteed contracts and RTB, particularly in situations with a high supply-to-demand ratio and lower discount factors for both the publisher and advertisers. These insights not only enhance academic knowledge of revenue management in online advertising but also offer practical advice for publishers seeking to maximise their revenues.

There are still several limitations in this study and hence some valuable future topics. First, we assume that advertisers have a unit demand of impressions. Future

research can relax this constraint and then introduce budgets of advertisers. Specifically, it is worthwhile to investigate how the budget constraints affect advertisers' strategic behaviours across the dual channels. Second, we do not cover the allocation of specific impressions to advertisers because of the assumption of homogeneous impressions. When the differences in advertisers' preferences for impressions are captured, the allocation process becomes critical. Future work can also study the timing problem of deciding the interval between the dual channels, which may have an influence on the discount factors for both the publisher and advertisers.

# 3 Study 2: Impact of Additional Advertisers' Arrivals on Selling Impressions via Dual Channels

## Abstract

In the online display advertising market, publishers host and sell advertising impressions to advertisers for revenue. The primary selling channels are guaranteed contracts and real-time bidding (RTB), with the former providing a fixed revenue stream through pre-sold impressions, and the latter is triggered by the realisation of impressions. In this study, we consider a scenario in which publishers have an option to introduce more advertisers between the two selling channels. Specifically, the selling horizon consists of three periods. In the first and third periods, the guaranteed contracts and RTB channels are exclusively activated, respectively. In the second period, the publisher decides whether or not to recruit more advertisers to RTB. There is an original batch of advertisers who enter this campaign at the beginning who are unaware of the second period and must choose between the guaranteed and non-guaranteed channels because of unit demand. Next, we extend our basic model to an information transparency setting, under which the publisher's plan of attracting more advertisers becomes known to the original advertisers. This paper aims to understand the interaction and impact of the additional advertisers and dual-selling channels on the decisions of both the publisher and the original advertisers (under the information transparency setting) in online advertising markets. The study finds that the optimal strategy for recruiting additional advertisers is influenced by various market parameters such as impression scarcity, discount factors, and recruitment costs. When the original advertisers are aware of the publisher's plans, the applicability of dual selling channels expands to cover a broader range of scenarios, although other strategic considerations almost remain consistent. These findings offer publishers nuanced strategies for navigating the complex online advertising landscape.

## 3.1   Introduction

The online display advertising market has experienced exponential growth in recent years, becoming an indispensable component of the contemporary digital economy (IAB 2022). The ecosystem of this market comprises various entities, including publishers, advertisers, and ad intermediaries such as Supply Side Platforms (SSPs), ad networks, ad exchanges, and Demand Side Platforms (DSPs) (Liu and Chao 2020). These intermediaries play a pivotal role in bridging the gap between publishers and advertisers while providing them with essential technological services.

The emergence of intermediaries in the online display advertising market has significantly expanded the range of options available to publishers, enhancing their ability to manage and monetise ad inventory. This expansion manifests in two key aspects: diverse selling channels and access to a broader pool of advertisers. On one hand, publishers can strategically allocate their ad inventory across multiple channels, such as guaranteed contracts and Real-Time Bidding (RTB), to maximise revenue and optimise campaign performance. For instance, certain ad exchanges, like Google AdX, allow publishers to employ both guaranteed contracts and RTB simultaneously (Google Ad Manager Help 2023*a*). Guaranteed contracts enable publishers to secure a fixed revenue stream by selling a predetermined number of impressions at a fixed price before the impressions are realised. Conversely, RTB auctions facilitate dynamic pricing of impressions based on real-time demand, potentially yielding higher revenue for publishers' ad inventory. On another hand, the wide array of SSPs, ad networks, and ad exchanges available in the market provides publishers with the flexibility to reach a larger pool of advertisers, thereby enhancing the competitiveness of their selling campaigns. For example, SSPs like Rubicon Project or PubMatic offer automated tools that help publishers manage their ad inventory and maximise revenue by connecting them with a vast network of potential advertisers (Sharethrough 2015). Google Ad Manager promotes Open Bidding to help publishers facilitate with other third-party exchanges to compete for publishers' inventory (Google Ad Manager Help 2023*b*). By strategically investing in additional traffic through these platforms, publishers can increase the reach and effectiveness of their campaigns, leading to higher revenue (Choi et al. 2020). This ability to leverage the potential of advertising slots and adapt to the dynamic online advertising landscape marks a significant evolution in the industry.

However, the introduction of a broader advertiser base between the selling of guaranteed contracts and RTB introduces new complexities. Publishers must now

navigate the interplay between the two channels (i.e., guaranteed contracts and RTB) while considering the dynamic behaviours of both the original advertisers and the additional arrival advertisers. Such complexities not only pose operational challenges but also open up new avenues for academic inquiry. Therefore, this research is motivated by two pressing questions: a) Should publishers attract additional advertisers between the guaranteed and RTB channels, and if so, how? b) How does the influx of new advertisers influence the bidding and contract buying behaviours of the original advertisers, and whether such influence varies in scenarios when the original advertisers are aware or unaware of the new entrants? Addressing these questions is crucial for both industry practitioners aiming for more effective monetisation strategies and academics seeking to enrich the current understanding of online advertising ecosystems.

In this study, we develop models that capture the interplay of advertisers' decisions between guaranteed contracts and RTB, with the publisher's problem of introducing additional advertisers to RTB after the close of guaranteed contracts. Our model assumes that a publisher sells homogeneous impressions over a three-period selling horizon. In the first and third periods, the channels of guaranteed contracts and RTB are activated exclusively. In the second period, the publisher considers whether to invest in attracting more advertisers. Therefore, our model involves two batches of advertisers: the original batch of advertisers in period 1 and the extra batch of advertisers in period 2. For revenue and utility from RTB in the third period, we introduce discount factors for both the publisher and the original batch of advertisers. We assume that the additional advertisers enter this campaign closely before the start of RTB. Thus, there is no discount effect for the additional advertisers' utility from RTB. We also incorporate different information availability in this paper, namely whether the original advertisers know the plan of the publisher in period 2 or not. In the basic model, we consider information opacity where the original advertisers are unaware that the publisher will attract advertisers in period 2. The closed-form solution of equilibrium under these settings is obtained. According to the equilibrium solutions, we analyse how different parameters (e.g. scarcity of impressions and discount factors) affect the publisher's decisions of investing in more advertisers. Subsequently, we explore the case where the original advertisers know that the publisher may introduce more advertisers in period 2 with a fixed unit cost. This cost is known to both publishers and advertisers. However, they do not know the exact number of the additional advertisers that the publisher intends to hire. We show the impact of the additional advertisers on the original advertisers' behaviours and numerically demonstrate the equilibrium results under different parameters.

The results reveal that the optimal strategy for the recruitment of additional advertisers is highly contingent on several market parameters, including the scarcity

of impressions, both the publisher's and the original advertisers' discount factors of yields from RTB, and the unit cost of recruiting additional advertisers. When the original batch of advertisers is unaware of the publisher's recruitment plans for the second period, the findings suggest that if the scarcity of impressions is high, the publisher should set the price of guaranteed contracts such that all impressions are sold through guaranteed contracts, irrespective of other market conditions. However, if the publisher is contemplating recruiting additional advertisers, a careful cost-benefit analysis is imperative. Specifically, recruiting additional advertisers becomes profitable only when both the scarcity of impressions and the cost of attracting these advertisers are not prohibitively high. Moreover, the utility of employing dual selling channels—guaranteed contracts and RTB—becomes particularly significant when both discount factors are low. Interestingly, when the original batch of advertisers is aware of the publisher's recruitment plans, the applicability of dual selling channels expands to cover a broader range of scenarios compared to when they are unaware. Aside from this, the other strategic considerations almost remain consistent with the case where there is information opacity regarding the publisher's recruitment plans. These insights offer nuanced guidance for publishers navigating the complex landscape of online advertising. They highlight the need for strategic flexibility and the importance of considering multiple market parameters in decision-making.

Our research contributes to the existing literature by providing a comprehensive analysis of the seller's strategic investment in importing more demands during the interval of dual channels. Firstly, our models fill the gap in research on publishers' optimal decision of attracting more advertisers to existing selling campaigns with dual channels available (Sayedi 2018, Cohen et al. 2023). Theoretically, we extract a cost-benefit indicator to evaluate the value of one potential additional advertiser through the closed-form solution. Along with other parameters, this indicator can guide the publisher's decision of introducing additional advertisers under various scenarios. Secondly, we further complement the understanding of advertisers' strategic behaviour in online display markets by uncovering the impact of the additional arrival advertisers on their strategic choices between dual channels, and also their bids in RTB. As far as we know, no studies focus on former arrival advertisers taking the behaviour of future arrival advertisers into consideration. However, contrary to private marketplaces in which merely priority advertisers participate, advertisers not only need to consider existing opponent advertisers' actions, but also need to have fair expectations for the additional arrival advertisers. Lastly, our research indicates publishers need to consider not only the cost and revenue from new arrival advertisers, but also the alteration of the original advertisers' decisions. It provides publishers with new managerial insights when considering recruiting new advertisers between different selling periods.

The remainder of this paper is organised as follows. Section §3.2 provides a literature review related to this research. Section §3.3 discusses the model and closed-form solutions when the original advertisers do not know that the publisher will recruit extra advertisers in period 2. Section §3.4 demonstrates the model and numerical results under the case that the original advertisers know there will be extra advertisers in period 2. The conclusion and some future directions are stated in the last section. Proofs are available in Appendix A2.

## 3.2 Literature Review

This study intersects with three main research streams: dual-channel selling of online display advertising, strategic consumer (advertiser) behaviour, and the impact of marketing investment on a seller's revenue. While each stream has been extensively explored, their confluence presents blank that this research aims to navigate. To the best of our knowledge, no existing research has simultaneously considered these settings, presenting a unique opportunity for this study to contribute novel insights.

A growing body of research has emerged to investigate selling display advertising impressions through dual channels. This stream primarily focuses on the problem of impression allocation between guaranteed contracts (GC) and real-time bidding (RTB) from the publisher's perspective. Various allocation solutions and policies have been proposed through algorithm design (Li et al. 2016, Shen 2018, Rhuggenaath et al. 2019, Wu et al. 2021), and stochastic control models using dynamic programming approaches (Roels and Fridgeirsdottir 2009, Salomatin et al. 2012, Balseiro et al. 2014, Chen 2017). Other researchers have centred on the pricing problem of guaranteed contracts under the dual-channel selling scenario (Chen et al. 2014, Chen 2016, Chen et al. 2020). However, these studies have not fully addressed why or under what conditions one selling channel should be chosen over the other, as they did not consider advertisers' strategic behaviour.

There is extensive research on strategic consumer behaviour, especially in the field of revenue management. However, most of these studies focus on the interplay of decisions between sellers and consumers, neglecting the mutual effect of consumers' actions on each other. For example, Su (2007), Aviv and Pazgal (2008) and Papanastasiou and Savva (2017) modelled consumers' utilities in terms of the arrival time. Liu and Van Ryzin (2008), Prasad et al. (2011) and Du et al. (2015) considered risk preference in consumers' utilities. Under the setting of stochastic arrival consumers, few papers add the effect of future arrival consumers' behaviour to former arrival consumers' utilities in case of a scarce supply of selling objects. Elmaghraby et al. (2009) assumed one single object sold to Poisson arrival consumers

with the same valuation. The selling horizon consists of two periods, with an advance selling period and a spot selling period. Consumers who arrive earlier in the advance period evaluate whether they can get this object in the spot period. They must consider the behaviour of subsequent customers who may arrive later. Correa et al. (2016) and Zhang et al. (2021) also add a factor that reflects the advanced buyer's probability to get an object into their utility functions. Nevertheless, the significant difference between our research and these papers is that the arrival of the additional buyers is exogenous in their settings. Specifically, in this study, the seller needs to pay a unit cost to attract one additional buyer, which is related to papers about marketing efforts.

Literature on marketing efforts mainly focuses on the investments and performances of generic advertising and brand advertising in the duopoly and oligopoly competition markets. They proposed a differential game theory approach and statistical model to capture the dynamics and to address the problems under both information transparency and opacity settings (Bass et al. 2005, Isariyawongse et al. 2007, Qi et al. 2008, Tchumtchoua and Cotterill 2010, Ma et al. 2021). Mukhopadhyay et al. (2009) examined a contract design problem from the perspective of a manufacturer who relies on a sales agent for selling the product, to inspire the agent's motivation for marketing efforts. This type of advertising is called cooperative advertising (Doraiswamy et al. 1979, Bergen and John 1997). Ma et al. (2013) focused on a two-stage supply chain model and examined the players' different investments in cooperative advertising and the corresponding consequences on marketing performance. Further, they considered three different supply chain structures from the angle of game theory: manufacturer Stackelberg, retailer Stackelberg, and vertical Nash. Karray et al. (2022) investigated the manufacturer's problem of optimising the investments in cooperative advertising and own (brand) advertising. They characterised equilibrium solutions for four advertising scenarios for the manufacturer, ranging from no investment in any advertising activity to undertaking their own advertising and supporting cooperative advertising simultaneously. However, these studies often overlook the influence of marketing efforts on existing consumers' choices and the resultant effects on seller's revenue. Our research seeks to bridge this gap by examining how marketing investments to attract new advertisers influence the behaviour of existing advertisers and the overall revenue dynamics.

In this study, we aim to bridge this gap by providing a comprehensive analysis of selling online display advertising to strategic consumers (advertisers) through two selling channels respectively available in two periods, with the flexibility for the seller (publisher) to attract additional advertisers between the two selling channels. Unlike existing literature, we not only take future arrival advertisers' behaviour into account when considering the original advertisers' strategic decisions, but also the trade-off between revenue and cost for the publisher of attracting additional de-

mands, and its impact on the original advertisers' behaviours. These complex dynamics are not explored by existing studies.

## 3.3  Information Opacity with Additional Advertisers

### 3.3.1  Model Settings

Similar to study 1, we consider a scenario where a publisher sells $Q$ homogeneous impressions to advertisers via two channels: guaranteed contracts and RTB. The homogeneity of impressions can be approximated in reality by focusing on a specific group of advertisers with similar attributes. Guaranteed contracts are sold at a fixed price before impressions are realised, i.e., before visitors browse the publisher's webpage. In contrast, each RTB event is triggered when a visitor clicks on the website. With the aid of statistical learning, the publisher can accurately estimate the number of future impressions during a given horizon. Therefore, we take the number of $Q$ impressions as given in this study.

The selling horizon comprises three periods, named as period 1, 2, and 3, respectively. Period 1 and 2 are planned before the impressions are realised, and period 3 is the interval that these impressions are generated. Specifically, the selling of guaranteed contracts for future impressions occurs in period 1. At the beginning of this period, the publisher announces the price $p$ of a contract for one impression to $N$ advertisers. These advertisers decide whether to buy a guaranteed contract in this period or join RTB in period 3 to maximise their expected utilities. These advertisers are committed to the campaign and will not leave without securing an impression until the end of the selling horizon. For instance, advertisers who have previously achieved good advertising performance through this publisher's webpage may follow the release of impressions from this page. Alternatively, some advertisers may be attracted by the targeting attributes of potential visitors from these impressions. At the end of period 1, the guaranteed contracts channel closes. To simplify the problem, we avoid allocating impressions between contracts and RTB by randomly and firstly assigning impressions to guaranteed contract buyers, with the remaining impressions being released to the spot markets.

In period 2, the publisher can recruit more advertisers to join RTB by broadcasting its advertising opportunities to various intermediaries. For convenience, we refer to the $N$ advertisers from period 1 as the original batch of advertisers and the advertisers introduced in period 2 as the extra/additional advertisers. We assume that every $c$ cost will attract one extra advertiser. Thus, if the publisher decides to introduce $x$ extra advertisers in this period, it incurs $cx$ fee. Finally, advertisers from the original batch who did not buy a contract in period 1 and extra advertisers from

period 2 will join RTB to compete for the remaining impressions in period 3. RTBs in period 3 are organised under the widely used second-price rule.

We assume that for advertisers either from period 1 or period 2, their valuations for all homogeneous impressions are the same and are uniformly independently identically distributed on $[0, \overline{v}]$. All advertisers have a unit demand for impressions. Then, advertisers who joined in period 1 should decide whether to buy a guaranteed contract or to participate in RTB later by comparing the expected utilities from the two channels. Note that the publisher does not have to inform advertisers in the first period that they may introduce more advertisers in period 2. Then we maintain the information opacity that the original batch of advertisers have no knowledge about the publisher's advertising plan in period 2.

The publisher's revenue from guaranteed contracts is obtained instantly and can be controlled by the pricing. However, they need to wait until period 3 to get the revenue from RTB. And there also contains more uncertainty of revenue from auctions. Because the payments of advertisers in these second-price auctions are dependent on other advertisers' bidding prices, which is out of the publisher's control. Thus, we introduce a discount factor, $\delta_p$, for the revenue derived from RTB to address the effect of waiting time and uncertainty on it. Advertisers who arrive in period 1 also face a similar problem caused by the gap between period 1 and period 3. For example, if they buy a guaranteed contract in period 1, then they can leave this campaign to do other work. Otherwise, they need to wait until period 3 and bid in each auction unless they successfully get an impression. Besides, through a guaranteed contract, they can definitely get an impression while the outcomes of RTB are uncertain. Therefore, we similarly introduce $\delta_a$ to capture the discount of advertisers' expected utilities from RTB.

The information of $N$, $Q$, $\delta_a$, $\delta_p$, $v \sim U(0, \overline{v})$ are common knowledge for all parties in this model. Besides, the publisher also knows the cost of introducing an extra advertiser, $c$.

The sequence of events illustrated in Figure 3.1 is stated as follows:

1. Before the start of period 1, $N$ advertisers (the original batch) enter the campaign. The publisher announces the price of guaranteed contracts $p$ and the number of selling impressions $Q$ to them.

2. In period 1, the original advertisers must decide whether to buy a guaranteed contract in this period or participate in RTB when the impressions are realised. It's important to note that they are unaware that the publisher may attract more advertisers later.

3. In period 2, the guaranteed contracts selling channel closes, and RTB has not yet begun. The publisher decides whether to recruit more advertisers (the
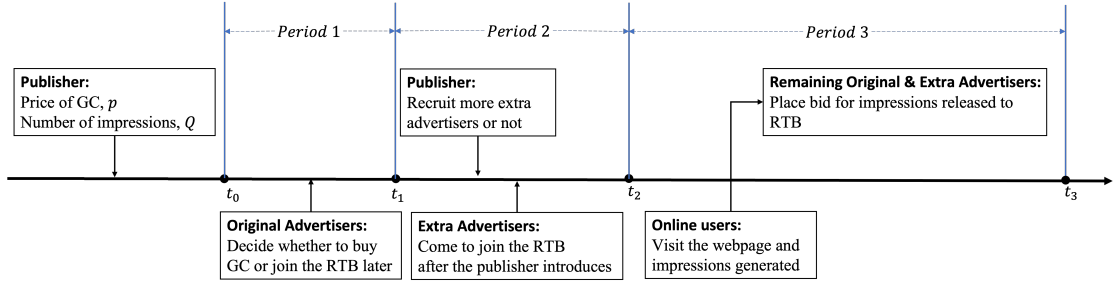
**Figure 3.1:** The sequence of events

extra batch) or not.

4. In period 3, advertisers from the original batch who decided to join RTB and advertisers who arrived in period 2 bid for impressions that are being released.

### 3.3.2 Analysis on Advertisers' Incentives

For advertiser $i$ from the original batch, we denote their valuation as $v_i$. As it does not make decisions in period 2, we discuss their utility in period 1, denoted by $u_i^1$, and in period 3, denoted by $u_i^3$.

Observing the price of guaranteed contracts, $p$, its utility of buying a guaranteed contract in period 1 is expressed as $u_i^1 = v_i - p$. Otherwise, if it decides to join RTB in period 3, considering the discount factor, its utility of winning an impression is $u_i^3 = \delta_a(v_i - p_i^r)$, in which, $p_i^r$ denotes advertiser $i$'s payment in the auctions, which is dependent on other advertiser's bidding price. If it fails to get an impression at last, then $u_i^3 = 0$.

For these $N$ advertisers, their utilities and behavioural modes are the same as that in the basic model in Chapter 2 since they do not know the publisher will recruit more advertisers in period 2. According to Proposition 2.2 in Chapter 2, if they decide to join RTB, their bidding strategy following bidding function shown in equation (2.4), i.e. $b(v) = \frac{(v \wedge p) - (1 - \delta_a)v}{\delta_a}$, are subject to a *mixed truth-telling* scheme to maximise their utilities.

For extra advertisers from period 2, their choices are only to participate in RTB. Referring to Proposition 2.1, they bid under the truth-telling strategy in period 3.

### 3.3.3 Analysis on the Publisher's Revenue Function

The publisher should price the guaranteed contracts $p$ and announce it to $N$ original advertisers at the beginning of period 1. Then advertisers who decide to buy guaranteed contracts will pay $p$ to the publisher. Other advertisers wait for RTB

in period 3. In period 2, the publisher will decide whether to recruit $x$ additional advertisers to RTB with the unit cost of $c$.

For a given price $p$, we can obtain the behavioural pattern of these $N$ advertisers from Proposition 2.4 in Chapter 2. Since all extra advertisers go to RTB and bid truthfully according to their valuations, we need to evaluate the bids of these two groups of advertisers together, and to determine the outcomes of repeated auctions in RTB. Also, we should consider the remaining impressions in RTB after the selling of guaranteed contracts in period 1.

Specifically, if advertisers from the original batch buy all impressions through guaranteed contracts in period 1, then the publisher will not hire any extra advertisers. If some impressions are left after period 1, then according to the bidding strategy shown in equation (2.4), we know that the upper bound of the original advertisers' bidding price is $p$. Therefore, if the publisher hires the extra advertisers such that the number of them with valuations larger than $p$ exceeds the number of left impressions, then advertisers from the original batch cannot win a single impression in period 3. Consequently, with the increasing number of extra advertisers, the source of revenue will transfer from the original advertisers to the extra advertisers. On the other hand, recruiting more extra advertisers will incur more costs. So the publisher's problem is to set a proper price $p$ to adjust the consumption of guaranteed contracts and then introduce an optimal number of $x$ extra advertisers in period 2.

The publisher's revenue function on the whole map of its domain of $p$ and $x$ can be expressed as the following:

$$
\Pi(p,x) = \begin{cases}
\Pi_1, & x \in [0,+\infty), & p \in \left(0, \frac{N-Q}{N}\overline{v}\right]; \\[2mm]
\Pi_2^1, & x \in \left[\frac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}, +\infty\right), & p \in \left(\frac{N-Q}{N}\overline{v}, \frac{N-\delta_a Q}{N}\right]; \\[2mm]
\Pi_2^2, & x \in \left[0, \frac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}\right), & \\[4mm]
\Pi_3^1, & x \in \left[\frac{Q\overline{v}}{\overline{v}-p}, +\infty\right), & \\[2mm]
\Pi_3^2, & x \in \left[\frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}, \frac{Q\overline{v}}{\overline{v}-p}\right), & p \in \left(\frac{N-\delta_a Q}{N}\overline{v}, \overline{v}\right]. \\[2mm]
\Pi_3^3, & x \in \left[0, \frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}\right), &
\end{cases}
\tag{3.1}
$$

in which $\Pi_i^j$ denotes the formulation of the $j$th segment of $x$ under the $i$th segment of $p$. And $\Pi_1 = Qp - cx$,

$\Pi_2^1 = N_2^1 p + \delta_p \int_{\frac{x-N_2^2}{x}\overline{v}}^{\overline{v}} xv\frac{1}{\overline{v}}\,dv - cx$,

$\Pi_2^2 = N_2^1 p + \delta_p \left\{ \int_p^{b^{-1}(v'')} Nb(v)\frac{1}{\overline{v}}\,dv + \int_{v''}^p Nv\frac{1}{\overline{v}}\,dv + \int_{v''}^{\overline{v}} xv\frac{1}{\overline{v}}\,dv \right\} - cx$,

$\Pi_3^1 = \delta_p \int_{\frac{x-Q}{x}\overline{v}}^{\overline{v}} x v \frac{1}{\overline{v}} \, dv - cx,$

$\Pi_3^2 = \delta_p \left\{ \int_p^{b^{-1}(v^{(3)})} Nb(v) \frac{1}{\overline{v}} \, dv + \int_{v^{(3)}}^p Nv \frac{1}{\overline{v}} \, dv + \int_{v^{(3)}}^{\overline{v}} xv \frac{1}{\overline{v}} \, dv \right\} - cx,$

$\Pi_3^3 = \delta_p \left\{ \int_p^{\overline{v}} Nb(v) \frac{1}{\overline{v}} \, dv + \int_{v^{(4)}}^p Nv \frac{1}{\overline{v}} \, dv + \int_{v^{(4)}}^{\overline{v}} xv \frac{1}{\overline{v}} \, dv \right\} - cx.$

For a given price of guaranteed contracts, $p$, the publisher's revenue have different formulations depending on the segment in which it lies. The sequence of decisions about $p$ and $x$ behind this problem guarantees the implementation of the backward induction method. Initially, we fix the price $p$ and identify the extremum points of $x$ in different segments. Then, we compare the corresponding extremum values in different $x$ segments to determine the global optimal $x$ under the current $p$. Next, we find the extremum point of $p$ in the current segment of $p$. Finally, we compare the extremum values in different segments to identify the global optimal $p$. This approach allows us to systematically explore the solution space and identify the optimal pricing strategy.

**Lemma 3.1.** *There exists a cost-benefit indicator (CBI, $\equiv \sqrt{\frac{2c}{\delta_p \overline{v}}}$) to describe the value of introducing an extra advertiser in period 2, we denote it as $q$. Only when $q \leq 1$ is it possible for the publisher to yield positive revenue from extra arrival advertisers. Otherwise, they should never recruit extra advertisers in period 2.*

Lemma 3.1 implies a profitable threshold of the introducing cost for extra advertisers. Although the value of CBI can be any positive number, we only discuss cases in which CBI is not larger than one, as this paper is to figure out when to invest in extra advertising. Interestingly, $q = 1$ means $c = \delta_p \frac{\overline{v}}{2}$, in which $\frac{\overline{v}}{2}$ equals to the expectation of extra arrival advertisers' valuations (since we assume that their valuations are subject to uniformly independent identically distribution on $[0, \overline{v}]$.) Thus, $\delta_p \frac{\overline{v}}{2}$ reveals an expected revenue from a unit advertiser's payment from RTB along with the discount factor of revenue from this period. If the unit cost $c$ is larger than the expected unit revenue, which means $q > 1$, then it's not wise to recruit any more advertisers.

Besides the CBI (denoted by $q$, and $q \in (0, 1]$), the equilibrium outcomes are also dependent on other indicators, $\frac{N}{Q}$ and $\delta_a, \delta_p$. The larger $\frac{N}{Q}$ is, the more intense the demand for impressions is. Thus, we call the indicator $\frac{N}{Q}$ as the scarcity of the supply of impressions, denoted by $s$. As for $\delta_a, \delta_p$, they affect the introducing decision through influencing the publisher's revenue and advertisers' utilities from period 2.

**Theorem 3.2.** *The possible optimal solutions are shown in the following (the definition of $D, \bar{q}, q_{h_1}, q_{h_2}, q_{h_3}$ and $S_1(q), S_2(q), S_3(q), S_4(q), S_5(q), S_6(q), S_7(q)$ are attached in the proof in Appendix A2):*

1. When $2\delta_a + \delta_p \geq 2$, if $\left(q \in (0, q_{h_1}] \ \& \ s \in (0, S_5(q)]\right) | \left(q \in (q_{h_1}, \frac{2(1-\delta_p)}{2-\delta_p}] \ \& \ s \in (0, S_6(q)]\right) | \left(q \in (\frac{2(1-\delta_p)}{2-\delta_p}, 1] \ \& \ s \in (0, S_1(q)]\right)$; when $2\delta_a + \delta_p < 2 \ \& \ \delta_a \geq (1 - \delta_p)^2$, if $\left(q \in (0, q_{h_2}] \ \& \ s \in (0, S_5(q)]\right) | \left(q \in (q_{h_2}, q_{h_3}] \ \& \ s \in (0, S_7(q)]\right) | \left(q \in (q_{h_3}, 1] \ \& \ s \in (0, S_1(q)]\right)$; when $2\delta_a + \delta_p < 2 \ \& \ \delta_a < (1 - \delta_p)^2$, if $\left(q \in (0, q_{h_2}] \ \& \ s \in (0, S_5(q)]\right) | \left(q \in (q_{h_2}, 1] \ \& \ s \in (0, S_7(q)]\right)$, the optimal solution is

$$
\begin{cases}
p^* = \ \overline{v} \\
x^* = \ \frac{Q}{q} - N
\end{cases} ;
$$

the corresponding maximum revenue is

$$
\Pi_{max} = \delta_p \bar{q} Q \overline{v} + \frac{\delta_p q^2}{2} N \overline{v}.
$$

2. When $2\delta_a + \delta_p \geq 2$, if $\left(q \in (0, q_{h_1}] \ \& \ s \in (S_5(q), S_3(q)]\right)$; when $2\delta_a + \delta_p < 2$, if $\left(q \in (0, \frac{1-\delta_p}{2-\delta_a-\delta_p}] \ \& \ s \in (S_5(q), S_3(q)]\right) | \left(q \in (\frac{1-\delta_p}{2-\delta_a-\delta_p}, q_{h_2}] \ \& \ s \in (S_5(q), S_4(q)]\right)$, the optimal solution is

$$
\begin{cases}
p^* = \ \frac{(1+\delta_p \bar{q})N - \delta_a Q}{2N} \overline{v} \\
x^* = \ \frac{(2-\delta_a)Q - (1-\delta_p \bar{q})N}{2(1-\delta_a)q}
\end{cases} ;
$$

the corresponding maximum revenue is

$$
\Pi_{max} = \frac{(1 - \delta_a \bar{q})^2 N^2 - 2(\delta_a + \delta_a \delta_p \bar{q} - 2\delta_p \bar{q})NQ + \delta_a^2 Q^2}{4(1 - \delta_a)N} \overline{v}.
$$

3. When $2\delta_a + \delta_p < 2 \ \& \ \delta_a \geq (1 - \delta_p)^2$, if $\left(q \in (\frac{1-\delta_p}{2-\delta_a-\delta_p}, q_{h_2}] \ \& \ s \in (S_4(q), S_1(q)]\right) | \left(q \in (q_{h_2}, q_{h_3}] \ \& \ s \in (S_7(q), S_1(q)]\right)$; when $2\delta_a + \delta_p < 2 \ \& \ \delta_a < (1 - \delta_p)^2$, if $\left(q \in (\frac{1-\delta_p}{2-\delta_a-\delta_p}, q_{h_2}] \ \& \ s \in (S_4(q), S_1(q)]\right) | \left(q \in (q_{h_2}, 1] \ \& \ s \in (S_7(q), S_1(q)]\right)$; the optimal solution is

$$
\begin{cases}
p^* = \ \frac{N - \delta_a Q}{(2-\delta_p)N} \overline{v} \\
x^* = \ \frac{Q - qN}{(1-\delta_a)q}
\end{cases} ;
$$

the corresponding maximum revenue is

$$
\Pi_{max} = \frac{[1 - \delta_p(2 - \delta_p)(1 - q^2)]N^2 - 2[\delta_a - 2\delta_p \bar{q} + \delta_p^2 \bar{q}]NQ + \delta_a^2 Q^2}{2(1 - \delta_a)(2 - \delta_p)N} \overline{v}.
$$

4. When $2\delta_a + \delta_p \geq 2$, if $q \in (\frac{2(1-\delta_p)}{2-\delta_p}, 1] \ \& \ s \in (S_1(q), \frac{2-\delta_p}{2(1-\delta_p)}]$ ; when $2\delta_a + \delta_p <$

$2$ & $\delta_a \geq (1 - \delta_p)^2$, if $q \in (q_{h_3}, 1]$ & $s \in (S_1(q), S_1(q_{h_3})]$, the optimal solution is

$$\begin{cases} p^* = & \overline{v} \\ x^* = & 0 \end{cases};$$

the corresponding maximum revenue is

$$\Pi_{max} = \frac{\delta_p}{2} \frac{(2N - Q)Q}{N} \overline{v}.$$

5. When $2\delta_a + \delta_p < 2$ & $\delta_a \geq (1 - \delta_p)^2$, if $\left( q \in (\frac{1-\delta_p}{2-\delta_a-\delta_p}, q_{h_3}] \ \& \ s \in (S_1(q), \frac{2-\delta_a-\delta_p}{1-\delta_p}] \right)$ $\left| \left( q \in (q_{h_3}, 1] \ \& \ s \in (S_1(q_{h_3}), \frac{2-\delta_a-\delta_p}{1-\delta_p}] \right) \right.$; when $2\delta_a + \delta_p < 2$ & $\delta_a < (1 - \delta_p)^2$, if $q \in (\frac{1-\delta_p}{2-\delta_a-\delta_p}, 1] \ \& \ s \in (S_1(q), \frac{2-\delta_a-\delta_p}{1-\delta_p}]$ the optimal solution is

$$\begin{cases} p^* = & \frac{N-\delta_a Q}{(2-\delta_p)N} \overline{v} \\ x^* = & 0 \end{cases};$$

the corresponding maximum revenue is

$$\Pi_{max} = \frac{(1 - \delta_p)^2 N^2 - 2(\delta_a - 2\delta_p + \delta_p^2)NQ + (\delta_a^2 + \delta_p^2 - 2\delta_p)Q^2}{2(1 - \delta_a)(2 - \delta_p)N} \overline{v}.$$

6. When $2\delta_a + \delta_p \geq 2$, if $\left( q \in (0, q_{h_1}] \ \& \ s \in (S_3(q), +\infty) \right) \left| \left( q \in (q_{h_1}, \frac{2(1-\delta_p)}{2-\delta_p}] \ \& \ s \in \right. \right.$ $(S_6(q), +\infty) \left. \right) \left| \left( q \in (\frac{2(1-\delta_p)}{2-\delta_p}, 1] \ \& \ s \in (\frac{2-\delta_p}{2(1-\delta_p)}, +\infty) \right) \right.$; when $2\delta_a + \delta_p < 2$, if $\left( q \in (0, \frac{1-\delta_p}{2-\delta_a-\delta_p}] \ \& \ s \in (S_3(q), +\infty) \right) \left| \left( q \in (\frac{1-\delta_p}{2-\delta_a-\delta_p}, 1] \ \& \ s \in (\frac{2-\delta_a-\delta_p}{1-\delta_p}, +\infty) \right) \right.$, the optimal solution is

$$\begin{cases} p^* = & \frac{N-Q}{N} \overline{v} \\ x^* = & 0 \end{cases};$$

the corresponding maximum revenue is

$$\Pi_{max} = \frac{(N - Q)Q}{N} \overline{v}.$$

This theorem outlines various scenarios for determining optimal solutions in Equation (3.1), depending on parameters including $\delta_a, \delta_p, q, s$, to define conditions under which different optimal solutions exist. There are six types of equilibrium when certain conditions are met. The theorem provides specific formulas for $p^*, x^*$, and $\Pi_{max}$ under each type of equilibria. These formulas offer a comprehensive guide for finding optimal solutions under various scenarios.

Theorem 3.2 demonstrates that there is no single optimal strategy for all situations. Publishers must adapt their strategies to the specific conditions they face
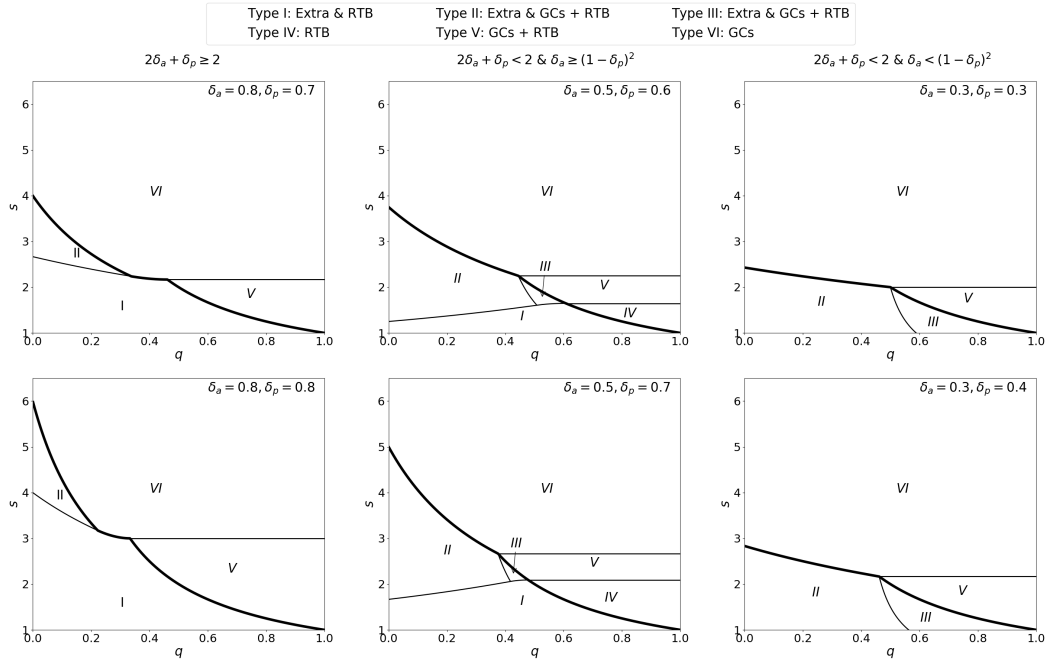
**Figure 3.2:** Different types of equilibria under different cases

in order to maximise their returns. Figure 3.2 illustrates the optimal strategies for different parameter combinations, with the horizontal and vertical axes of each subplot representing $q$ and $s$, respectively. The term 'Extra' in the legend indicates that in the corresponding area, the strategy involves recruiting extra advertisers. The number of types in the legend corresponds to the case order in Theorem 3.2, For example, $Type_{VI}$ in these subplots represents that the optimal solution is $p^* = \frac{N-Q}{N}\overline{v}$ and $x^* = 0$ when parameters are located in corresponding zones. This solution is demonstrated in the case of No.6 in Theorem 3.2.

Each subplot represents different combinations of $\delta_a$ and $\delta_p$, with subplots at each column representing a set of boundary conditions that $\delta_a$ and $\delta_p$ satisfy according to Theorem 3.2. The two subplots at the left column corresponds to $2\delta_a + \delta_p \geq 2$, subplots at the middle column to $2\delta_a + \delta_p < 2$ and $\delta_a \geq (1 - \delta_p)^2$, and subplots at the right column to $2\delta_a + \delta_p < 2$ and $\delta_a < (1 - \delta_p)^2$.

Within each subgraph, cases in which no additional advertisers are introduced are concentrated on the left upside, divided by the thick black line ($Type_{IV}$, $Type_V$ and $Type_{VI}$). This implies that the publisher does not have incentives to import additional advertisers, if both the scarcity of impressions $s$ and CBI $q$ are not small. Furthermore, when the scarcity of impressions $s$ is high enough, it is always optimal to sell all impressions via guaranteed contracts ($Type_{VI}$). For moderate $s$, a lower CBI encourages publishers to maximise revenue by selling through two channels and recruiting extra advertisers ($Type_{II}$ and $Type_{III}$). As $\delta_a$ and $\delta_p$ decrease (moving from left to right across the columns), this strategy becomes applicable to a wider range of $s$ and $q$ values.

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In the left column, $\delta_a$ and $\delta_p$ are large, publishers will set higher contract prices $p$ for low scarcity (i.e. when $s$ is close to one), and all impressions will be sold in RTB ($Type_I$ and $Type_{IV}$). Publishers will not recruit any additional advertisers if the CBI $q$ is too high. Subplots at the middle and right columns show that when $\delta_a$ and $\delta_p$ are small, publishers do not set high contract prices. Instead, they sell some impressions through contracts when scarcity is low and recruit additional advertisers for RTB if the CBI is low.

These observations uncover several managerial insights for publishers to make a proper investment strategy for introducing more advertisers between the activation of two selling channels:

- There is no one-size-fits-all strategy for publishers. The optimal approach is contingent on a variety of parameters, such as the scarcity of impressions and the CBI. Publishers must tailor their strategies to these specific conditions to maximise revenue.

- When impressions are scarce, selling them all via guaranteed contracts is the most profitable strategy. This is particularly true when both $\delta_a$ and $\delta_p$ are low. For moderate levels of impression scarcity, a lower CBI encourages publishers to diversify their selling channels by offering some guaranteed contracts and recruiting additional advertisers for RTB.

- The setting of contract prices and the decision to recruit additional advertisers for RTB are influenced by the interplay between impression scarcity, the two discount factors, and CBI. Two high discount factors favour higher contract prices and less recruitment for RTB, while two low discount factors make it advantageous to set moderate contract prices and actively recruit for RTB when scarcity is low and CBI does not exceed one.

These observations provide insights for publishers to wisely manipulate demands for impressions based on the specific market conditions they face. These findings also highlight the importance of continually reassessing and adapting strategies in response to changes in key parameters like impression scarcity, both publishers' and advertisers' discount factors, and the CBI.

## 3.4   Information Transparency with Additional Advertisers

In section 3.3, it is assumed that the advertiser is unaware of the publisher's plans to recruit additional advertisers in the second period. In practice, however, even if the publisher is not obliged to inform the advertiser of its plans, the original adver-

tisers can still learn their opponents' strategy through auction results and their payments after a series of campaigns (Balseiro and Gur 2019). Then, it's possible for the original advertisers to become aware of the entry of extra competitors by analysing historical data. Consequently, their awareness brings out behavioural adjustment, which leads to a new problem for the publisher: how to secure the maximum revenue under advertisers' new bidding behavioural modes? We thus, examine how the original advertisers will take action if they know that there will be additional advertisers coming in period 2, and how the publisher will adjust its strategy in this extension.

### 3.4.1   Voluntary Arrival of Additional Advertisers

In reality, extra advertisers may come from different sources. For example, Google has integrated many advertisers from different buying tools into its advertising sales platform as Figure 3.3 shows. Among these tools, Display&Video 360 and Google Ads are under the control of Google, while others are from third parties.
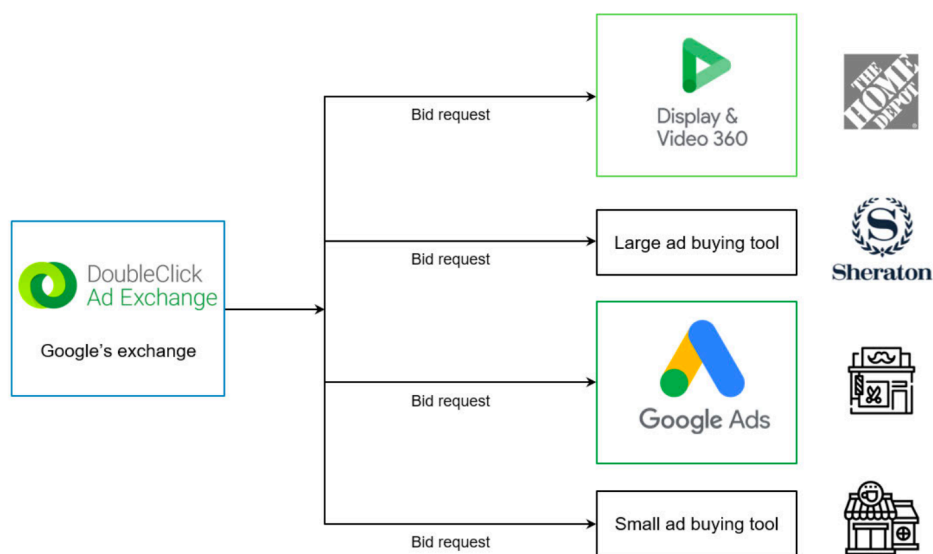


**Figure 3.3:** Demand Side Platforms (DSPs) in Google Ads ecosystem. Source: Antitrust Complaint against Google, Third Amended Edition. (2022)

In this case, we check the circumstance where extra advertisers are all from outside buying tools. This means that the extra advertisers are not attracted by the publisher with extra cost. We assume that the number of these advertisers is $M$, their valuations are also i.i.d. subject to $U(0, \overline{v})$. The publisher and the original advertisers know the information of these additional advertisers.

For the advertisers from the original batch, their incentives to choose between buying guaranteed contracts or joining RTB are still gaining better utilities by comparing $u_i^1 = v_i - p$ and $u_i^3 = \delta_a(v_i - p_i^r)$. Thus, if they decide to join RTB, their

bidding strategy $b(v)$ retains the same as shown in equation (2.4). Similarly, advertisers' behavioural modes depend on the price of guaranteed contracts.

**Proposition 3.3.** *The behavioural modes of advertisers from the original batch depend on the prices of guaranteed contracts:*

1. *$p \in [0, \frac{N-Q}{N}\overline{v}]$*

   *All advertisers with valuations $v > p$ will buy guaranteed contracts and the number of them exceeds the supply of impressions. Thus, no impressions were left in RTB.*

2. *$p \in (\frac{N-Q}{N}\overline{v}, \frac{N+M-Q}{N+M}\overline{v}]$*

   *Although all advertisers with valuations $v > p$ will also buy guaranteed contracts in this case, there are still $Q - \frac{\overline{v}-p}{\overline{v}}N$ impressions left in RTB. Because $p \leq \frac{N+M-Q}{N+M}\overline{v}$, then $Q - \frac{\overline{v}-p}{\overline{v}}N \leq \frac{\overline{v}-p}{\overline{v}}M$. The left impressions will be won by extra advertisers.*

3. *$p \in (\frac{N+M-Q}{N+M}\overline{v}, \frac{N+M-\delta_a Q}{N+M}\overline{v}]$*

   *In this case, there exists a threshold value $v'_M = \frac{(N+M)p - \delta_a(N+M-Q)\overline{v}}{(1-\delta_a)(N+M)}$ locates in $(p, \overline{v}]$. Therefore, if the valuations of advertisers from the original batch satisfy $v \in (v'_M, \overline{v}]$, they will buy guaranteed contracts in period 1. If $v \in (p, v'_M]$, these advertisers will join RTB and bid untruthfully. And for advertisers from the original batch with $v \in (0, p]$ and for all extra advertisers, they will join RTB with truthful bidding. Note that only advertisers with bids larger than $\frac{N+M-Q}{N+M}\overline{v}$ can win an impression in RTB.*

4. *$p \in (\frac{N+M-\delta_a Q}{N+M}\overline{v}, +\infty)$*

   *In this case, all advertisers, no matter which batch they are from, will join RTB. The original batch of advertisers follows the bidding strategy in equation (2.4). The additional batch of advertisers bid truthfully.*

Proposition 3.3 demonstrates how the original advertisers behave when there are extra *M* advertisers participating in period 2. Compared with results in Proposition 2.4, there is one more pricing segment. The threshold valuation is greater than the case where there is no additional advertiser involved. Based on Proposition 3.3, the publisher's revenue function can also be obtained segmentally.

$$\Pi(p) = \begin{cases} \Pi_1, & p \in [0, \frac{N-Q}{N}\overline{v}] \\ \Pi_2, & p \in (\frac{N-Q}{N}\overline{v}, \frac{N+M-Q}{N+M}\overline{v}] \\ \Pi_3, & p \in (\frac{N+M-Q}{N+M}\overline{v}, \frac{N+M-\delta_a Q}{N+M}\overline{v}] \\ \Pi_4, & p \in (\frac{N+M-\delta_a Q}{N+M}\overline{v}, \overline{v}] \\ \Pi_5, & p \in (\overline{v}, +\infty) \end{cases}, \tag{3.2}$$

in which $\Pi_i$ denote the publisher's revenue function under the $i_{\text{th}}$ pricing segment. And $\Pi_1 = Qp$,

$$\Pi_2 = \frac{\overline{v}-p}{\overline{v}}Np + \int_{M-(Q-\frac{\overline{v}-p}{\overline{v}}N)}^{\overline{v}} Mv\frac{1}{\overline{v}}\,\mathrm{d}v,$$

$$\Pi_3 = \frac{\overline{v}-v'_M}{\overline{v}}Np + \int_p^{v'_M} Nb(v)\frac{1}{\overline{v}}\,\mathrm{d}v + \int_{\frac{N+M-Q}{M}\overline{v}}^p Nv\frac{1}{\overline{v}}\,\mathrm{d}v + \int_{\frac{N+M-Q}{M}\overline{v}}^{\overline{v}} Mv\frac{1}{\overline{v}}\,\mathrm{d}v,$$

$$\Pi_4 = \int_p^{\overline{v}} Nb(v)\frac{1}{\overline{v}}\,\mathrm{d}v + \int_{\frac{N+M-Q}{M}\overline{v}}^p Nv\frac{1}{\overline{v}}\,\mathrm{d}v + \int_{\frac{N+M-Q}{M}\overline{v}}^{\overline{v}} Mv\frac{1}{\overline{v}}\,\mathrm{d}v,$$

$$\Pi_5 = \int_{\frac{N+M-Q}{M}\overline{v}}^{\overline{v}} Nv\frac{1}{\overline{v}}\,\mathrm{d}v + \int_{\frac{N+M-Q}{M}\overline{v}}^{\overline{v}} Mv\frac{1}{\overline{v}}\,\mathrm{d}v.$$

We can solve the publisher's optimal pricing by 1) finding the local extremum in each segment, and 2) comparing them to get the optimal price.

**Theorem 3.4.** *The effect of M extra advertisers on the publisher's revenue acts along with the size of the original batch. For convenience we let $s_M = \frac{N+M}{Q}(s_M > 1)$.*

1. *When $s_M < 2$, if $\left(\delta_a \in (2-s_M, \frac{s_M}{2s_M-1}] \,\&\, \delta_p \in [0, \frac{\delta_a+s_M-2}{s_M-1})\right) \mid \left(\delta_a \in (\frac{s_M}{2s_M-1}, 1] \,\&\, \delta_p \in [0, \frac{2(s_M-1)}{2s_M-1}]\right)$; when $s_M \geq 2$, if $\left(\delta_a \in (0, \frac{s_M}{2s_M-1}] \,\&\, \delta_p \in [0, \frac{\delta_a+s_M-2}{s_M-1}]\right) \mid \left(\delta_a \in (\frac{s_M}{2s_M-1}, 1] \,\&\, \delta_p \in [0, \frac{2(s_M-1)}{2s_M-1}]\right)$, the optimal price of guaranteed contracts is*

$$p^* = \frac{N+M-Q}{N+M}\overline{v},$$

*and the corresponding maximum revenue is*

$$\Pi(p^*) = \frac{2Q(N+M-Q)(N+\delta_p M)+\delta_p MQ^2}{2(N+M)^2}\overline{v}.$$

2. *When $s_M < 2$, if $\left(\delta_a \in [0, 2-s_M] \,\&\, \delta_p \in [0, 1-\sqrt{\frac{\delta_a(1-\delta_a)}{s_M^2-2\delta_a s_M+\delta_a}}]\right) \mid \left(\delta_a \in (2-s_M, \frac{s_M}{2s_M-1}] \,\&\, \delta_p \in (\frac{\delta_a+s_M-2}{s_M-1}, 1-\sqrt{\frac{\delta_a(1-\delta_a)}{s_M^2-2\delta_a s_M+\delta_a}}]\right)$; when $s_M \geq 2$, if $\left(\delta_a \in [0, \frac{s_M}{2s_M-1}] \,\&\, \delta_p \in (\frac{\delta_a+s_M-2}{s_M-1}, 1-\sqrt{\frac{\delta_a(1-\delta_a)}{s_M^2-2\delta_a s_M+\delta_a}}]\right)$, the optimal price of guaranteed contracts is*

$$p^* = \frac{N+M-\delta_a Q}{(2-\delta_p)(N+M)}\overline{v},$$

*and the corresponding maximum revenue is*

$$\Pi(p^*) = \frac{\begin{aligned}&(1-\delta_p)^2 N(N+M)^2 - (2\delta_p^2 - 4\delta_p + 2\delta_a)NQ(N+M)+\\ &2(1-\delta_a)\delta_p(2-\delta_p)MQ(N+M) - (1-\delta_a)\delta_p(2-\delta_p)Q^2(N+M)\\ &+ \delta_a[\delta_a - \delta_p(2-\delta_p)]NQ^2\end{aligned}}{2(1-\delta_a)(2-\delta_p)(N+M)^2}\overline{v}.$$

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*In this case, there exists a threshold*

$$v'_M = \frac{(1 + \delta_a \delta_p - 2\delta_a)(N + M) + \delta_a(1 - \delta_p)Q}{(1 - \delta_a)(2 - \delta_p)(N + M)}\overline{v} \in (p, \overline{v}].$$

*3. If* $\left(\delta_a \in (0, \frac{s_M}{2s_M - 1}] \ \& \ \delta_p \in (1 - \sqrt{\frac{\delta_a(1 - \delta_a)}{s_M^2 - 2\delta_a s_M + \delta_a}}, 1]\right) \mid \left(\delta_a \in (\frac{s_M}{2s_M - 1}, 1] \ \& \ \delta_p \in\right.$
$\left(\frac{2(s_M - 1)}{2s_M - 1}, 1]\right),$ *the optimal solution is*

$$p^* = \overline{v};$$

*the correspond maximum revenue is*

$$\Pi(p^*) = \frac{\delta_p}{2} \cdot \frac{[2(N + M) - Q]Q}{N + M}\overline{v}.$$

Theorem 3.4 reveals the publisher's optimal pricing strategy when there are $M$ extra advertisers entering in period 2. There are three types of equilibria depending on different levels of impression scarcity and both the publisher' and the original advertisers' discount factors. In the former two equilibria, impressions are sold through both guaranteed contracts and RTB. In the last equilibrium, all advertisers participate in RTB.

To compare the result with what we get in the basic model in the information opacity setting, we set $q \to 0$ and $M \to +\infty$. The reason is that we do not consider the cost of introducing $M$ advertisers in this extension. And if we set $q \to 0$ in the basic model, then the publisher would recruit a very large number, mathematically infinity, of advertisers. So we should then let $M \to +\infty$. By comparing the results under these two settings, we get the conclusion in Corollary 3.5.

**Corollary 3.5.** *By introducing adequate additional advertisers without cost, the publisher makes more revenue if the original advertisers are unaware of the existence of the additional arrival advertisers.*

It is intuitive to expect the results from Corollary 3.5 because the advantage of information benefits the publisher when there is no additional cost to attract more advertisers. What if the publisher can decide how many extra advertisers to hire with a fixed unit cost? We discuss it in the next section.

### 3.4.2 Attracting Additional Advertisers with Variable Cost

As we mentioned in the example of Google in the last section, the publisher may have the power to decide whether and how many advertisers it wants to include

in the campaign if the advertiser buying tools are under control. To explore how the extra advertisers affect the original advertisers' behavioural modes and the publisher's decision, we explore the case that original advertisers know the publisher will decide to hire extra advertisers in period 2. The cost of hiring a unit extra advertiser, denoted by $c$, is also common knowledge. The distribution of valuations of all advertisers involved in this campaign is i.i.d. on $U(0, \overline{v})$, which is also known to both of them. However, the original advertisers do not know how many extra advertisers the publisher will introduce in the future.

The sequence of events flows is similar to what is under the information opacity setting and is explained as follows. In period 1, the publisher announces the price of guaranteed contracts $p$ to these $N$ original advertisers. These advertisers know the publisher may introduce extra advertisers in period 2, which is a time window after the close of the guaranteed contracts selling channel and before the start of RTB. They also have the acknowledgement of the information about the cost $c$ for the publisher to addict one extra advertiser in period 2. After analysing, the original advertisers should make decisions of buying a guaranteed contract in period 1 or attending RTB in period 3. In period 2, the channel of guaranteed contracts closes. The publisher observes the number of sold contracts and the remaining impressions. If there still are impressions left, it decides the number of extra advertisers to hire by evaluating the cost and benefit of introducing extra advertisers. In period 3, RTB opens. The original advertisers that have not bought guaranteed contracts and extra advertisers imported in period 2 go to the competition of RTB together.

The bidding strategy of the original advertisers in RTB will remain the same as shown in equation (2.4) in Chapter 2. Because the motivation they participate in RTB is still $u_i^3 > u_i^1 > 0$. The critical differences are their decisions about whether to acquire a guaranteed contract or join RTB later as they know there will be additional advertisers entering this campaign in period 2.

After receiving the price information of guaranteed contracts in period 1, advertisers consider their strategy based on the import cost for the publisher. Roughly, if $c$ is very close to 0, the publisher can introduce a very large number of advertisers in period 2. If the number of extra advertisers is large enough, any original advertisers with valuations $v > p$ cannot win an impression in RTB by bidding $b(v)$. Thus, they will purchase a contract to secure their interests. On the other hand, if $c$ is too high (for example, higher than $\overline{v}$), the publisher will not introduce any extra advertiser. Then the original advertisers' behavioural modes are the same in the case of no extra advertisers existing. Namely, some advertisers with valuations $v > p$ will go to RTB to seek more utilities than that from guaranteed contracts by bidding $b(v)$. Thus, the original advertisers know that the higher cost leads to a lower number of extra advertisers.

Because the extra advertisers' bidding strategy is truth-telling, the expected payment of an extra advertiser is $\frac{\overline{v}}{2}$. Then, if $c > \delta_p \frac{\overline{v}}{2}$, the publisher's expected payoff from an extra advertiser would always be negative. Thus, like in the basic model, we also can introduce the CBI $q$ and only discuss cases that $q \leq 1$.

**Proposition 3.6.** *We denoted the number of extra advertisers as $x$. For a given price $p$, the equilibria $x^*(p)$ should satisfy a) $v = \frac{N+x-Q}{N+x}\overline{v}$ is the least valuation among the original advertisers that can get an impression in the campaign; b) the publisher's revenue under $x^*(p)$ is greater than any other $x$.*

According to condition a) in Proposition 3.6, we can get the behavioural modes of the original advertisers when condition a) holds. We can then build up the revenue function of the publisher by assuming current $x$ satisfying condition a). At last, we find a $x$ that can maximise the publisher's revenue under the current $p$, and condition b) is satisfied naturally. From condition a), the revenue function of the publisher is shown below:

$$
\Pi(p,x) = \begin{cases}
\Pi_1, & x \in [0,+\infty), & p \in \left(0, \frac{N-Q}{N}\overline{v}\right]; \\[2mm]
\begin{aligned}
\Pi_2^1, & \ x \in \left[\frac{Np-(N-Q)\overline{v}}{\overline{v}-p}, +\infty\right), \\
\Pi_2^2, & \ x \in \left[0, \frac{Np-(N-Q)\overline{v}}{\overline{v}-p}\right),
\end{aligned} & & p \in \left(\frac{N-Q}{N}\overline{v}, \frac{N-\delta_a Q}{N}\right]; \\[4mm]
\begin{aligned}
\Pi_3^1, & \ x \in \left[\frac{Np-(N-Q)\overline{v}}{\overline{v}-p}, +\infty\right), \\
\Pi_3^2, & \ x \in \left[\frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}, \frac{Np-(N-Q)\overline{v}}{\overline{v}-p}\right), & p \in \left(\frac{N-\delta_a Q}{N}\overline{v}, \overline{v}\right]; \\
\Pi_3^3, & \ x \in \left[0, \frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}\right),
\end{aligned} \\[4mm]
\Pi_4, & x \in [0,+\infty), & p \in (\overline{v}, +\infty).
\end{cases}
$$
(3.3)

in which $\Pi_i^j$ denotes the formulation of the $j$th segment of $x$ under the $i$th segment of $p$. And $\Pi_1 = Qp - cx$,

$\Pi_2^1 = \frac{\overline{v}-p}{\overline{v}}Np + \delta_p \int_{(N+x-Q)\overline{v}-Np}^{\overline{v}} xv\frac{1}{\overline{v}}\,dv - cx$,

$\Pi_2^2 = \frac{\overline{v}-v_x'}{\overline{v}}Np + \delta_p \left\{ \int_p^{v_x'} Nb(v)\frac{1}{\overline{v}}\,dv + \int_{b(v_x')}^p Nv\frac{1}{\overline{v}}\,dv + \int_{b(v_x')}^{\overline{v}} xv\frac{1}{\overline{v}}\,dv \right\} - cx$,

$\Pi_3^1 = \frac{\overline{v}-p}{\overline{v}}Np + \delta_p \int_{(N+x-Q)\overline{v}-Np}^{\overline{v}} xv\frac{1}{\overline{v}}\,dv - cx$,

$\Pi_3^2 = \frac{\overline{v}-v_x'}{\overline{v}}Np + \delta_p \left\{ \int_p^{v_x'} Nb(v)\frac{1}{\overline{v}}\,dv + \int_{b(v_x')}^p Nv\frac{1}{\overline{v}}\,dv + \int_{b(v_x')}^{\overline{v}} xv\frac{1}{\overline{v}}\,dv \right\} - cx$,

$\Pi_3^3 = \delta_p \left\{ \int_p^{\overline{v}} Nb(v)\frac{1}{\overline{v}}\,dv + \int_{\frac{N+x-Q}{N+x}\overline{v}}^p Nv\frac{1}{\overline{v}}\,dv + \int_{\frac{N+x-Q}{N+x}\overline{v}}^{\overline{v}} xv\frac{1}{\overline{v}}\,dv \right\} - cx$,

$\Pi_4 = \delta_p \left\{ \int_{\frac{N+x-Q}{N+x}\overline{v}}^{\overline{v}} Nv\frac{1}{\overline{v}}\,dv + \int_{\frac{N+x-Q}{N+x}\overline{v}}^{\overline{v}} xv\frac{1}{\overline{v}}\,dv \right\} - cx$.

As the decisions of the publisher of $p$ and $x$ are made in a sequence, we solve it

by the backward induction method. The first step is to treat $\Pi(p,x)$ as $\Pi_p(x)$, which is a function of $x$ with $p$ as a parameter. After getting the local extremum solutions of $x$, which would be dependent on $p$, we substitute solutions of $x$ into $\Pi_p(x)$ to reduce it as $\Pi(p)$. The next step is to find local extremums of $p$ and compare them to get the global optimal $p^*$. Finally, we find the corresponding extremum $x$ of current optimal $p^*$, then substitute $p^*$ to the $x$ to get the optimal $x^*$.
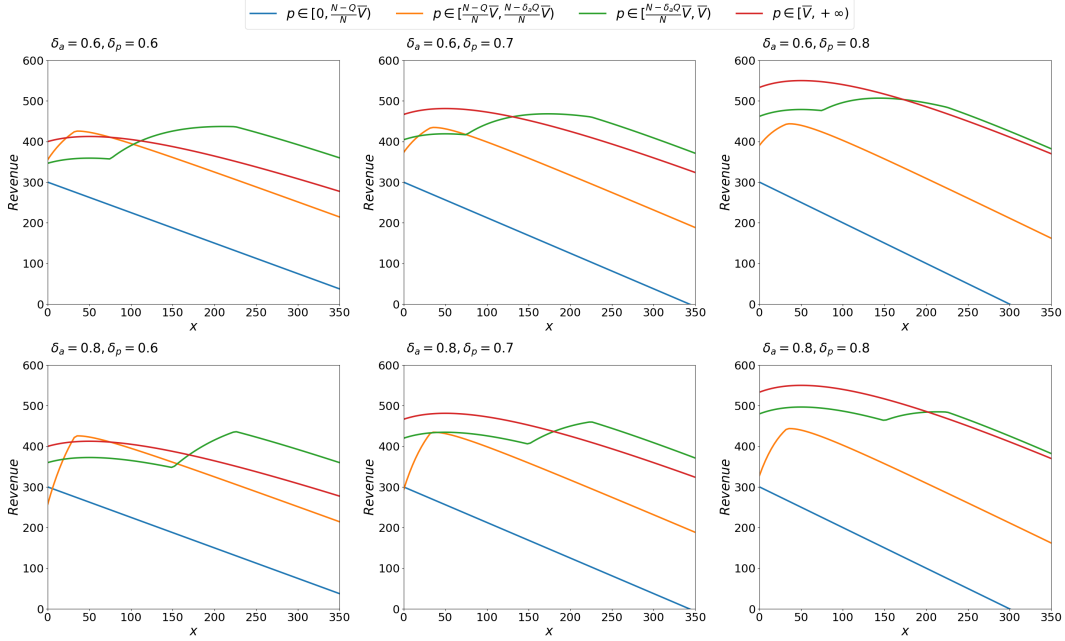


**Figure 3.4:** The publisher's revenue function of $x$ under different segments of $p$

Figure 3.4 shows the tendency of $\Pi_p(x)$ when $p$ is located in different segments. We set $Q = 100$, $s = 1.5$, $q = 0.5$, $\overline{v} = 10$. Different subplots reveal how the revenue functions vary under different cases of $\delta_a$ and $\delta_p$. When $p < \frac{N-Q}{N}\overline{v}$ and $p \geq \overline{v}$, $\Pi_p(x)$ is monotone decreasing. Furthermore, when $p \in [\frac{N-Q}{N}\overline{v}, \overline{v})$, we also can compute the closed-form of local optimal $x$ in $[0, \frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}]$ and $(\frac{Np-(N-Q)\overline{v}}{\overline{v}-p}, +\infty)$ by first-order condition. But for $\Pi_p(x)$ when $x \in (\frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}, \frac{Np-(N-Q)\overline{v}}{\overline{v}-p}]$, it's redundant to find the closed-form of local optimal solution. However, its concavity can be guaranteed, as shown in the following lemma.

**Lemma 3.7.** *When $x \in (\frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}, \frac{Np-(N-Q)\overline{v}}{\overline{v}-p}]$, $\Pi_p(x)$ is concave on $x$.*

Next, we illustrate the publisher's revenue function of $p$ in Figure 3.5 after optimal $x$ is solved numerically with the settings of $Q = 100$, $\overline{v} = 10$. Different subplots reflect the variation of $\delta_a$ and $\delta_p$. From subplots in the left column to the right column, we witness the growth of the max revenue and also the increase in the corresponding optimal price when $\delta_p$ gets larger. Note that the segment where the optimal price is located also moves right, such that all contract buyers transfer to RTB. While comparing subplots from up and bottom in one column, the increase of $\delta_a$ does not have much effect on the value of max revenue.
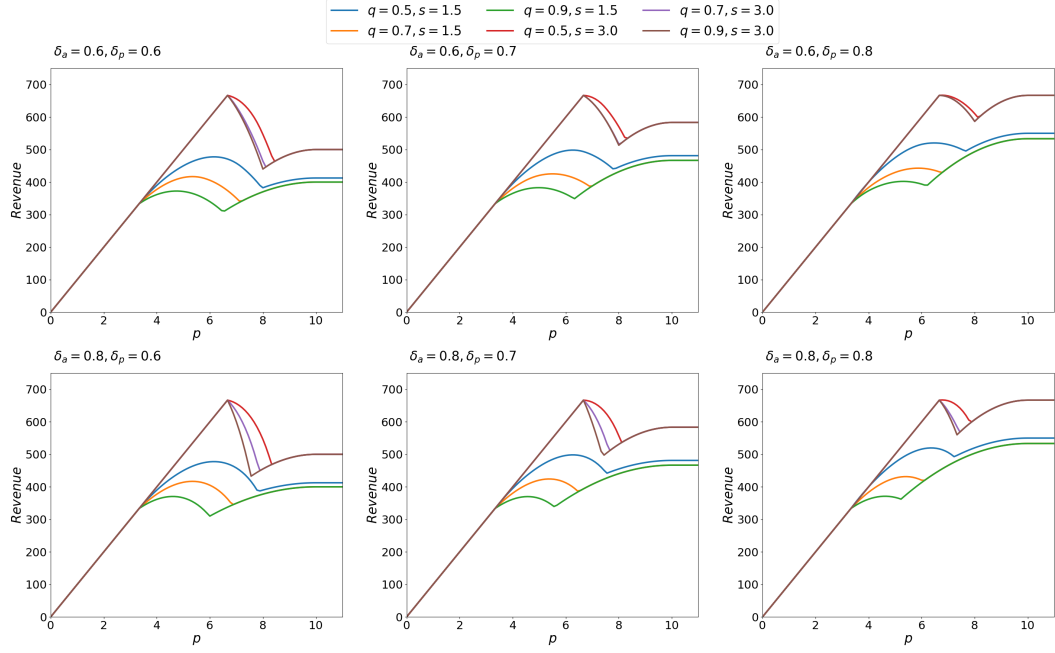
**Figure 3.5:** The publisher's revenue function of $p$ under different combinations of $q$ & $s$

In each subplot, different cases of the revenue function under $q : \{0.5, 0.7, 0.9\} \times s : \{1.5, 3\}$ are displayed. These lines are grouped into two clusters by different scarcity levels. When $s = 1.5$, the optimal price locates in $(\frac{N-Q}{N}\overline{v}, \frac{N-\delta_a Q}{N}\overline{v}]$ under the chosen CBIs. While $s$ increases into 3, the optimal price slides to $\frac{N-Q}{N}\overline{v}$, such that all impressions consumed by the original advertisers in period 1. When impression scarcity remains unchanged, a lower CBI leads to no less revenue than a higher CBI under the same price.

The equilibrium results of this problem are shown in Figure 3.6. The meaning of the legends in this figure is similar to that in Figure 3.2. The settings of $\delta_a$ and $\delta_p$ are allied with that in Figure 3.2. Therefore, comparing the results under the information opacity setting with that under the information transparency setting, there are more cases (of different combinations of $\delta_a$ and $\delta_p$) that the publisher tends to choose a moderate price of guaranteed contracts such that the impressions are sold to both contracts buyers and winners in auctions, with extra advertisers involved. The area of the region that represents all impressions sold to guaranteed contract buyers in Figure 3.6 reduces compared with that in Figure 3.2. The region that represents all advertisers participating in RTB also shrinks. The reason is that more original advertisers turn to guaranteed contracts instead of participating in RTB after knowing there may be more competitors, leading to their failure of winning an impression.

| Type I: Extra & RTB | Type II: Extra & GCs + RTB | Type III: Extra & GCs + RTB |
| Type IV: RTB | Type V: GCs + RTB | Type VI: GCs |
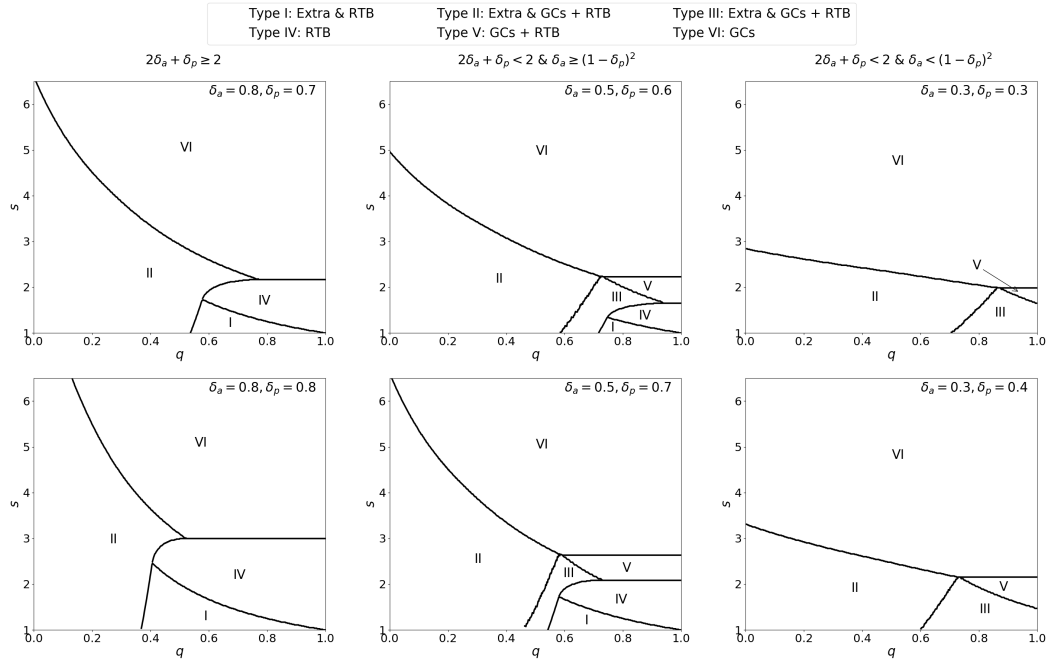
**Figure 3.6:** Different types of equilibria under different cases under info transparency

## 3.5 Conclusion

The prosperous online advertising markets provide publishers with abundant ways to reach advertisers by interfacing with various ad intermediaries. However, the strategic investment in attracting more advertisers, especially in the context of two different selling channels, has been largely unexplored. This paper addresses the question of whether a publisher should attract more advertisers to a selling campaign that includes both guaranteed contracts and real-time bidding (RTB), with an initial batch of advertisers participating from the start. Two scenarios are investigated, depending on whether the original advertisers are aware of the arrival of the additional advertisers or not.

This study serves as a first attempt to fill the blank of understanding the publishers' problem of attracting additional advertisers to the selling campaign between periods of guaranteed contracts and RTB. Existing literature on advertisers' strategic behaviour in dual-channel markets only considered the scenario that all advertisers enter the campaign at the beginning (e.g. Sayedi 2018 and Cohen et al. 2023). Our findings reveal several intriguing insights and offer practical guidance for publishers in attracting additional advertisers after the closure of the guaranteed contracts channel. First, we identify a cost-benefit indicator that serves as a threshold for the unit advertising cost of attracting more advertisers. If this indicator exceeds one, recruiting more advertisers is never beneficial. The decision also depends on the original scarcity of impressions and discount factors. When impressions are abun-

dant and discount factors are high, attracting more advertisers becomes more viable. Second, the awareness of the additional advertisers may prompt the original advertisers to purchase guaranteed contracts immediately rather than wait for RTB, unless there is an ample supply of impressions and a high cost-benefit indicator. This insight widens the understanding of advertisers' strategic behaviours from merely considering the publishers' decisions. Finally, publishers should be cautious about revealing their intention to recruit more advertisers after the guaranteed contracts close, as it may negatively affect their revenue.

There are still some valuable future directions contained in this problem. First, we assume that the extra advertisers are introduced in a single period, but the stochastic arrival of additional advertisers during the selling of guaranteed contracts is worth investigating in future research. Second, we make the hypothesis that both the original and additional advertisers' valuations for an impression are independent and identically distributed on the same uniform distribution due to the complexity. Future research can extend our study by considering the distribution of additional advertisers' valuations is different from that of the original advertisers. Besides, the heterogenous of impressions, the specific allocation strategy of impressions can also be included in future endeavours.

# 4 Study 3: Allocation of Impressions to Strategic Advertisers among Dual Selling Channels

## Abstract

Publishers making allocations among the dual channels–guaranteed contracts and real-time bidding (RTB)–face two challenges. First, the regulations in guaranteed contracts about whether under-delivery is allowed can influence the future allocation process. Second, publishers' allocation policy and advertisers' purchase decisions among the dual channels are mutually affected. To address the challenges, this paper introduces a sequential game theory model in which a publisher who sells heterogenous quality impressions to two advertisers across two periods. In the first period, only guaranteed contracts are available, while RTB is activated in the second period. There are two types of contracts: quantity-guaranteed contracts, where under-delivery is not permitted but quality is not assured, and quality-guaranteed contracts, which mandate specific quality levels and incur compensation costs for any under-delivery. The study delves into the subsequent decisions made by both the publisher and advertisers, depending on the chosen contract type. Due to the complexity of this problem, we propose two sets of threshold-type allocation policies in period 2: guaranteed contracts prior policy and RTB prior policy. The former allocates high-quality impressions preferentially to guaranteed contract holders, while the latter releases such impressions to auctions first. Utilising a numerical algorithm based on backward induction, the study evaluates the performance of these policies under both types of guaranteed contracts. The analysis reveals two salient points: firstly, quality-guaranteed contracts tend to generate more revenue for publishers, attributed to the compensatory incentives for advertisers; secondly, an RTB prior policy is generally more beneficial across both contract types, especially when maintaining long-term advertiser relationships is not a central objective for publishers.

## 4.1   Introduction

The online advertising sector remains the most dominant and fastest-growing segment within the digital marketing industry, amassing an impressive revenue of approximately \$476.46 billion in 2022 (Astute Analytica 2023). Online display advertising, a critical subset of this sector, accounts for approximately 32% of the online advertising market share in 2022. The revenue of online display advertising primarily comes from user visits to a multitude of websites managed by various publishers. Visits of online users to a webpage generate what is known as *'impressions'*, which provides advertisers an opportunity to show their ads to users. Publishers and advertisers can assess the quality of these impressions based on a range of user information, such as demographics, geographical location, device types, operating systems, past behaviour, and even click-through rates (CTR) through cookies (Balseiro et al. 2014). Advertisers can then purchase these impressions to display their ads to users.

Typically, publishers have two primary channels for selling these impressions to advertisers in online display advertising markets: the guaranteed channel of guaranteed contracts and the non-guaranteed channel of real-time bidding (RTB). Guaranteed contracts are pre-negotiated agreements between publishers and advertisers that delineate the quantity, price, and quality of impressions to be delivered. This guaranteed channel offers certainty and stability for both publishers and advertisers. However, since the selling of guaranteed contracts occurs before the realisation of impressions, both publishers and advertisers are uncertain about the quality of future impressions at this stage. Therefore, these prespecified contracts sacrifice the flexibility of both parties to dynamically adjust strategies according to real-time situations.

Unlike guaranteed contracts, RTB is an auction-based system triggered by a user's click on a webpage with an ad slot. Advertisers instantly receive user data, by which they can assess the quality of the impression, and submit bids based on preset strategies and the winner's ad is displayed. The entire process is completed within milliseconds while the loading of this webpage after the click. RTB offers greater flexibility and transparency, allowing advertisers to make real-time decisions on bidding for specific impressions. This also enables publishers to maximise revenue for high-quality impressions. However, the auction-based nature introduces uncertainty and volatility for both parties (Choi et al. 2020).

Previously, the channel of RTB was usually utilised by publishers to sell remnant impressions only after fulfilling all the demands of guaranteed contracts in the online display advertising markets. The advent of *header bidding* technology has altered this situation(Morrisroe 2023). Header bidding enables publishers to simultaneously offer their ad inventory to RTB campaigns before delivering them to contract buyers (Sayedi 2018). Therefore, this technology makes it possible for publishers to strategically allocate impressions between guaranteed contracts and RTB according to the quality of these impressions, to leverage advantages and mitigate disadvantages across these two channels. Nevertheless, these changes also give rise to two interesting research questions.

First, how to choose a proper type of guaranteed contract allied with an impression allocation strategy? Since publishers can release impressions to RTB before totally fulfilling guaranteed contracts after the implementation of header bidding, the problem of under-delivery comes to the surface. Existing literature handles this issue by modelling the types of contracts in two different ways. Balseiro et al. (2014), Sayedi (2018) and Cohen et al. (2023) assumed a strict contract rule that neither under-delivery nor over-delivery is allowed. Some other papers (e.g. Chen et al. 2014, Chen 2016, Chen 2017 and Chen et al. 2020) permitted the undelivered contracts with compensation to advertisers. It remains unclear which type of guaranteed contract is more effective.

Second, how to find the optimal impression allocation strategy across the dual channels while considering advertisers' strategic behaviour? Facing dual channels in the markets, advertisers also make strategic decisions about which channel to buy impressions from and how much they want to bid in RTB. Although the allocation of impressions across dual channels has attracted considerable interest, most of the extant literature neglected advertisers' strategic behaviour in this campaign (for details please refer to §3.2). In practice, advertisers' strategic decisions of purchasing guaranteed contracts and participation in RTB can be influenced by the publisher's allocation strategy, and in return affect the effectiveness of the allocation scheme subsequently. For instance, if publishers keep delivering high-quality impressions to RTB to seek more profit, advertisers' incentives to buy guaranteed contracts will decrease. However, guaranteed contracts can be beneficial for publishers who want to maintain long-term relationships with advertisers. Numerous studies also have explored optimal contract purchases (e.g. Ahmed et al. 2011, Pandey et al. 2011, Trusov et al. 2016 and Athey et al. 2018) and bidding strategies (e.g. Ghosh et al. 2009, Iyer et al. 2014, Balseiro et al. 2015 and Balseiro and Gur 2019) from the advertisers' perspective. This research highlights the importance of advertisers' strategic behaviour in the two channels. Therefore, it's necessary and significant for publishers to take advertisers' strategic behaviour into account when making the allocation strategy.

This study aims to answer these two questions by exploring optimal impression allocation strategies between different types of guaranteed contracts and RTB, and also taking advertisers' strategic behaviour into account. More specifically, we construct a model in which a publisher aims to sell $Q$ impressions to two distinct advertisers across two periods: the first for guaranteed contracts and the second for RTB. The model incorporates different quality of impressions. In practice, the quality of impressions varies and can be mapped from the distinct user information and historical behavioural data behind page views. Then, advertisers' valuations of impressions are modelled as functions of both impression quality and their compatibility with each impression. This method can both capture the relevance of advertisers' valuations to the quality of impressions and advertisers' different tastes to the same impression.

Next, we propose two alternative options for guaranteed contracts in period 1: a) quantity-guaranteed contracts, specifying only the compulsory number of impressions without quality constraints; and b) quality-guaranteed contracts, mandating a certain quality scope for a certain number of impressions delivered and requiring compensation for non-compliance. The publisher needs to decide the price, available number of guaranteed contracts in cases of both types and also the compensation cost for under-delivery contracts in quality-guaranteed contracts. Advertisers' decisions include their demands for guaranteed contracts and their bidding price in RTB. Because the sequences of events and allocation processes of impressions are different when choosing different types of contracts in the first period, we examine the publisher's allocation strategy under each type of them, respectively.

Furthermore, we focus on two sets of threshold-type allocation policies, i.e. guaranteed contract prior and RTB prior, when considering the allocation problem in period 2. The quality of impressions is the criterion in these allocation policies. Specifically, following a threshold-type guaranteed contracts prior policy, the publisher allocates impressions with quality exceeding a predetermined threshold to guaranteed contract buyers and releases others to RTB. While in a threshold-type RTB prior policy, impressions with quality over the threshold are released to RTB and the left are assigned to guaranteed contracts. Therefore, the problem of impression allocation is reduced to solving an optimal threshold in each set of allocation policies. The effectiveness of the two sets of policies is explored under the setting of two types of guaranteed contracts, respectively.

The complexity of our model arises from three key factors. First, the two periods in the model are heterogeneous, with different decision spaces and state spaces. Second, the decision space includes both continuous and discrete variables, which adds to the complexity of the analysis. Finally, the second period's dynamics add

complexity due to ongoing allocation decisions related to unfulfilled guaranteed contracts, leading to large state and action spaces and the potential for the curse of dimensionality. To overcome these difficulties, we develop an algorithm based on the backward induction approach and obtain several key findings. First, quality-guaranteed contracts generate higher revenue for publishers due to their flexibility and the incentivising effect of compensation clauses. Second, RTB prior policy dominates guaranteed contracts prior policy under two types of guaranteed contracts. Therefore, publishers are inclined to choose quality-guaranteed contracts and implement an RTB prior allocation strategy to maximise revenue, particularly when long-term relationships with advertisers are not a consideration. This result is contrary to the traditional way of allocating impressions with good quality to contract buyers first, which is widely implemented in the industry (Balseiro et al. 2014, Sayedi 2018, Morrisroe 2023). Furthermore, publishers should also carefully calibrate the supply of impressions allocated to the guaranteed channel to minimise the risk of under-delivery and associated penalties. Moreover, our results indicate that both channels are activated and most impressions are sold through RTB in equilibrium.

This study makes several contributions. From a methodological standpoint, this study contributes to existing literature by developing a sequential game theory framework that captures both the publisher's allocation process and advertisers' strategic decisions across dual channels (e.g. Balseiro et al. 2014, Li et al. 2016 and Chen 2017). This model is extensible and can be adapted to various scenarios, such as when impression quality distribution is empirically derived or when more complex allocation policies are considered. From a practical perspective, the research offers managerial insights for publishers. Specifically, our research provides publishers with valuable suggestions not only for designing the regulations for guaranteed contracts but also for making allocation policies of impressions across dual channels. Moreover, the balance of supply and demand for guaranteed contracts is emphasised in reducing the cost of under-delivery. Lastly, this study also highlights the importance of the application of dual selling channels, which benefits both publishers and advertisers.

The remainder of this study is structured as follows: Section §4.2 reviews the relevant literature; Section §4.3 introduces our model setup and the two types of guaranteed contracts we explore, and proposes two sets of allocation policies; Section §4.4 and Section §4.5 analyse the publisher's and advertisers decisions and solve their objectives under two types of guaranteed contracts, respectively; Section §4.6 concludes the paper, offering discussions and directions for future research. Proofs are available in Appendix A3.

## 4.2 Literature Review

While there is extensive literature on online display advertising, in this paper we focus on the literature discussing the allocation of online display advertising impressions between the guaranteed channel and RTB. There are primarily two streams, revenue management and data-driven perspective.

In the revenue management scope, researchers solved allocation problems by developing mathematical models to abstract practical scenarios. While handling the complexity of modelling revenue and cost (if applicable) from two distinct channels, the existing literature suggested three approaches. The first way was to develop a unified framework by modelling RTB as a contract without penalty cost (Roels and Fridgeirsdottir 2009, Rhuggenaath et al. 2019)). In Roels and Fridgeirsdottir (2009), revenue from RTB was fixed and set to be lower than payments from guaranteed contracts. The allocation process was oversimplified by merely setting RTB to consume remnant inventory after fulfilling guaranteed contracts in their model. While Rhuggenaath et al. (2019) accounted for uncertainty from spot markets by dividing Supply Side Platforms (platform providers for RTB) into two distinct groups, with uncertainty results (learnt from historical data) from the risky group and known results from the safe group. They further solved this problem by stochastic programming. However, their study failed to capture the different revenue of different auctions in RTB. The second way was to conversely set a bid price for each contract to participate in RTB (Jauvion and Grislain 2018) or set an opportunity cost of joining RTB instead of delivering current impression to contract buyers (Balseiro et al. 2014). They solved this problem by mixed integer programming or dynamic programming. Balseiro et al. (2014) also took the quality for advertisers from the guaranteed campaign into consideration. The highest and the second-highest bids from RTB were assumed to be known in their papers, neglecting advertisers' strategic bidding behaviours. But in our model, the bids are decided by advertisers' valuations of every impression. The last way was to handle guaranteed contracts and RTB separately and model the problem through bi-objective or multi-objective programming (Yang et al. 2012, Chen 2017, Shen 2018). Nevertheless, all of them regarded revenue from one or two channels as the input of their models. Yang et al. (2012) considered three objectives including revenue from non-guaranteed channels, brand awareness, and conversion rates for guaranteed contract buyers. Then they proposed several approaches to deal with the multi-objective programming problem depending on the data available in practice. Chen (2017) and Shen (2018) combined objectives through weighted sum methods. Different from these papers, we model the objectives of the publishers as revenue from both channels in this study. The revenue is obtained in detail by advertisers' payments and their bids in RTB, rather than merely assuming a parameterised revenue from any channel. This

makes our model closer to reality than theirs.

There are very limited papers studying the allocation of impressions with data-driven approaches. Li et al. (2016) and Zhang et al. (2022) developed efficient algorithms on practical data for programming models of the allocation problem between dual channels. Li et al. (2016) utilised the penalty cost of under-delivery guaranteed contracts and the reserve price in RTB as the decision variables to train their model. Zhang et al. (2022) proposed a unified guaranteed contracts allocation (UGA) framework, which consists of feature transform and differential sorting network modules, to solve a non-convex quadratically constrained quadratic programming(QCQP) problem. But both papers take the supply and demand of impressions as determinate, which is impossible in practice. In our research, the demands of impressions from different channels are determined by the equilibrium between the publisher and advertisers, which is closer to reality. Wu et al. (2021) and Wang et al. (2022) implemented machine learning techniques to solve this allocation problem. Wu et al. (2021) posted a unique bid for each guaranteed contract to RTB by solving a primary programming model, and then they implemented a multi-agent reinforcement learning (MARL) approach to enhance the cooperation between all contracts in the guaranteed campaign to increase the total revenue for the publisher. Wang et al. (2022) developed a cascade distillation-based framework called CONFLUX. The training of CONFLUX is first supervised by a linear programming model. Then a cumbersome network distils such paradigm by precisely modelling the competition at a request level and further transfers the generalisation ability to a lightweight student via knowledge distillation (Wang et al. 2022, p. 4070). Hence these studies are mainly driven by accessed data and solved by machine learning techniques, so it is hard to explain the optimality of their results.

In summary, almost all of these studies did not take advertisers' strategic decisions into account, particularly their purchase of guaranteed contracts and bidding behaviour in RTB as well as their influence on impressions allocation. Instead, these studies often assume related information, such as the demand for guaranteed contracts and the revenue from RTB, as the inputs of their models. Understanding advertisers' willingness to pay and their bidding strategies is crucial as it directly affects the pricing dynamics in both guaranteed and RTB channels, thereby affecting the publisher's revenue streams. Additionally, advertisers' choices between guaranteed contracts and RTB can significantly influence how publishers allocate their advertising slots between the two channels. To address this gap, this research focuses on the optimal allocation of impressions between guaranteed contracts and RTB while taking into consideration advertisers' strategic behaviour through dual channels.

## 4.3 Model Setup

In our model, a publisher sells $Q$ impressions to two advertisers, denoted as advertiser $i$ ($i \in \{1, 2\}$). In practice, publishers can get fairly good estimates of page views during a certain interval with the help of forecasting techniques. So we take the number of impressions, $Q$, as fixed and known in our model. For simplicity and without loss of generality, we assume that there is one advertising slot on the publisher's website, such that one user visit generates one impression. Therefore, the $Q$ impressions are generated by $Q$ visits to the webpage.

When a user visits the webpage, an advertising request is triggered and sent to the publisher along with the user's attributes (e.g. demographics, geographical location, device types, operating systems, past behaviour etc.). We assume that the publisher has the ability to infer the potential value from the user's information, i.e., impression quality, through historical data.

In practice, advertisers have heterogeneous valuations on the impression. More specifically, their valuations for an impression not only depend on the quality of this impression, but are also related to how this impression matches their interests (Balseiro et al. 2014, Sayedi 2018). For instance, an impression from a user with a high historical CTR in the age group of 25-34 may be identified as a good quality one by a publisher. However, advertisers selling eco-friendly products may evaluate this impression higher than advertisers selling luxury watches.

To capture this feature, we introduce a random parameter $\Theta$ to represent the quality of these impressions and another random parameter A to represent the matching degree between advertisers and an impression. While we do not specify a particular distribution for $\Theta$ and A at this stage, we assume that they have well-defined probability density function $f(\theta)$ and $g(\alpha)$, which are integrable and differentiable on their supports. This allows our model to be adapted to a variety of distributions, which can be determined based on empirical data or specific cases. For simplicity, we assume that advertiser $i$'s valuation for an impression $t$ is $v_i = \alpha_{i,t}\theta_t$, in which, $\theta_t$ is a realisation of $\Theta$, $\alpha_{i,t}$ is a realisation of A. We implement this formulation of their valuations to address these two advertisers' personal preferences and the relevance between their valuations and impression quality.

Two channels are activated exclusively in two periods of the entire selling horizon, i.e., guaranteed contracts in period 1 and RTB in period 2. In period 1, the publisher sells guaranteed contracts of future impressions to two advertisers in advance. Every contract is identical and contains only one impression. The supply of $Q$ impressions is produced in period 2. When one impression is generated, the publisher decides the allocation of it. If this impression is released to RTB, advertiser $i$

bids $b_i$ for it. The rule of RTB is the second-price auction. Besides, we assume that the distribution of A and $\Theta$ are both common knowledge. We also do not consider advertisers' budget constraints in this problem. Because our focus is the effect of heterogenous of impression quality on both advertisers' strategic choices and the publisher's allocation strategy between two channels. Considering the budget constraints will increase the complexity and distract the focus of this problem.

At the beginning, the publisher should decide the types of guaranteed contracts sold in period 1. Specifically, we consider two types of guaranteed contracts, namely quantity-guaranteed and quality-guaranteed contracts. In quantity-guaranteed contracts, publishers accomplish the agreed number of contracts to advertisers, without commitment to the quality of impressions delivered. While in quality-guaranteed contracts, publishers promise a specific number and quality of impressions to contract buyers. The quality of impressions assigned to advertisers must meet the requirements specified in contracts and the under-delivery incur penalty cost for publishers. Note that although the publisher does not promise the quality of impressions delivered to quantity-guaranteed contracts, the quality of impressions still serves as the criterion for the publisher when making the allocation scheme. Both the sequence of events and the allocation process are different after choosing different types of guaranteed contracts. To explore how these differences affect advertisers' decisions, the publisher's allocation process, and also the publisher's revenue, we discuss them in §4.4 and §4.5, respectively.

After choosing a type of guaranteed contract, the publisher needs to make an allocation scheme. However, finding a perfect allocation strategy is not a trivial task in practice for several reasons (Balseiro et al. 2014, Shamsi 2015, Wu et al. 2021): a) The severe time request. Once an impression is produced, the publisher should decide to assign it to contract buyers or release it to RTB in several milliseconds, which requires a high-efficiency decision method. b) The balance of revenue from dual channels. Revenue from guaranteed contracts and RTB are contradictory. The publisher should make decisions to maximise the total revenue from both channels, which makes it more complex than the traditional DP problem. c) Incentive compatibility. The publisher's revenue also depends on advertisers' decisions. Thus, it should design an allocation strategy aligned with advertisers' incentives.

The challenge of our model is three-fold. First, the heterogeneity between the two periods poses a significant challenge, as both the decision and state spaces for players differ in each period. The initial state in the second period is the outcome of the subgame in the first period. Second, the decision space types in our problem encompass both continuous (e.g. price of guaranteed contracts) and discrete (e.g. demands for guaranteed contracts) variables, which makes analysing potential solutions more difficult. Lastly, the dynamics of the problem in the second period

exacerbate its complexity, as the publisher must make allocation decisions as long as unfulfilled guaranteed contracts remain. Consequently, the action space scale is at least $O(2^U)$ ($U$ denotes the number of guaranteed contracts), which might invoke the curse of dimensionality (Powell 2007).

To address these challenges, we propose a policy-based allocation strategy inspired by the policy function approximation in approximate dynamic programming (ADP). A policy in ADP is usually parameterised. In our model, we focus on a series of policies structured by $\theta$, the quality of impressions. Specifically, for an impression with quality denoted by $\theta$. If $\theta \in \Theta_{GC}, (\Theta_{GC} \subseteq \Theta)$, the publisher assigns this impression to guaranteed contracts, otherwise releases it to RTB. The publisher's problem is to decide a proper $\Theta_{GC}$ to maximise its revenue. Since the publisher's allocation policy can be learned by advertisers as the selling campaign lasts and repeats in practice, we assume the policy the common knowledge among the publisher and advertisers. Based on this policy-based allocation scheme, we formulate our model under the two types of guaranteed contracts in §4.4.3 and §4.5.3, respectively.

When solving this problem, finding an optimal $\Theta_{GC}$ on $\Theta$ is still intricate because 1. $\Theta_{GC}$ is not necessary to be continuous, it may include several discrete intervals; 2. each interval contains two sides to be determined. Therefore, we only focus on two special sets of allocation policies: guaranteed contracts prior policy and RTB prior policy. Both policies are threshold-based and characterised by a threshold value, $\theta'$, which signifies the minimum acceptable quality level for an impression. In the guaranteed contracts prior policy, the publisher allocates an impression to a guaranteed contract if its quality $\theta$ surpasses the threshold value $\theta'$. If $\theta$ is less than or equal to $\theta'$, the impression is released to RTB for auction. Conversely, under RTB prior policy, the publisher releases an impression to RTB for auction if its quality $\theta$ is greater than or equal to the threshold value $\theta'$. If the quality of this impression falls below $\theta'$, the impression is allocated to the guaranteed contracts. Consequently, the decision of allocation policy is reduced to finding a single optimal threshold value to maximise its revenue. This type of policy is also understandable and acceptable, hence operable for the publisher in practice. We further examine the performances of the two sets of policies both under quantity-guaranteed contracts (§4.4.4) and quality-guaranteed contracts (§4.5.4), respectively. We summarise key notations in this game in Table A3.1 in Appendix A3.

## 4.4 Quantity-Guaranteed Contract

In quantity-guaranteed contracts, the publisher does not give a promise about the quality of delivered impressions. Instead, it makes sure that all contracts are fulfilled before impressions are sold out. Any under-delivery is not allowed in quantity-

guaranteed contracts. In this section, we first illustrate the sequence of events in the case that the publisher chooses this type of contract to sell. Then the allocation of impressions in period 2 is analysed. Following this analysis, we can obtain the publisher's expected revenue and advertisers' expected utilities. Finally, we develop a numerical algorithm to solve this problem.

## 4.4.1 Sequence of Events

As stated before, the impressions selling horizon consists of two periods. Figure 4.1 illustrates the sequence of events around the selling horizon under quantity-guaranteed contracts:

1. Before period 1 starts, the publisher announces the price for a one-impression contract, denoted by $p$, and the total available number of contracts, $U$. Note that $U$ should not exceed the total number of impressions $Q$.

2. When period 1 begins, these two advertisers observe the price for guaranteed contracts and also the available number of contracts. Then they make decisions about how many guaranteed contracts they claim to the publisher. Advertisers' demands for guaranteed contracts are denoted by $x_1$ and $x_2$, respectively.

3. The publisher receives $x_1, x_2$ and compares them with $U$. If $x_1 + x_2 \leq U$, advertisers are ensured to get $x_1$ and $x_2$ contracts. If $x_1 + x_2 > U$, advertisers' demand will be truncated by the proportion they post, i.e., advertiser $i$ ($i \in \{1, 2\}$) will only get $\frac{x_i}{x_1+x_2}U$ contracts. We denote the agreed number of guaranteed contracts as $U' = \min\{x_1 + x_2, U\}$.

4. Before period 2 starts, the publisher makes the allocation policy according to $U'$. This policy is also known to advertisers.

5. In period 2, there will be $Q$ impressions to sell in total. We use $t \in \{t | t = 1, 2, ..., Q\}$ to denote the $t^{th}$ user that visits the webpage, and $t = 0$ to mark the start of period 2. While the $t^{th}$ user comes, the publisher can acquire the quality of this impression, denoted by $\theta_t$. Then, the publisher decides whether to assign this impression to contract buyers or release it to RTB. When one impression is released to RTB, the publisher should also share the quality of this impression with advertisers such that they can accordingly analyse their matching degree to this impression and then obtain their valuations, which are expressed as follows

$$\begin{cases} v_{1,t} = \alpha_{1,t}\theta_t \\ v_{2,t} = \alpha_{2,t}\theta_t \end{cases}. \tag{4.1}$$

Subsequently, they post their bids $b_i$ and the winner gets this impression and

pays the other one's bidding price to the publisher. If one impression is assigned to guaranteed contracts, the publisher will deliver it to these two advertisers by the probability structured from their demands (i.e., advertiser $i$ will get a contract with the probability of $\frac{x_i}{x_1+x_2}$).
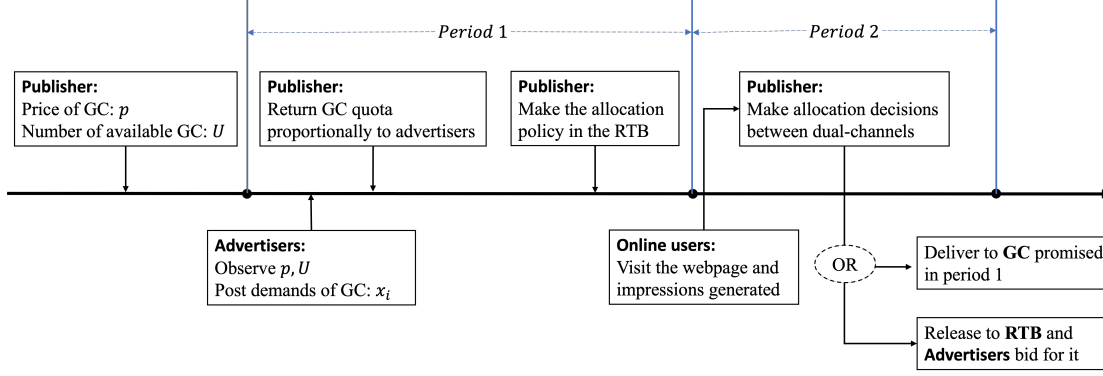


**Figure 4.1:** The sequence of events

## 4.4.2 Allocation Process

The allocation of quantity-guaranteed contracts needs continuous monitoring of the state of contract fulfilment (denote the number of unfulfilled contracts as $U_{\text{unmet}}$) and available impressions. To ensure that all guaranteed contracts are delivered, upon the arrival of the $t^{th}$ user, the publisher must assess whether the remaining number of impressions $(Q - t + 1)$ is sufficient to meet the unfulfilled demand from guaranteed contracts.

- If $U_{\text{unmet}} = 0$, all remaining impressions can be released to RTB.

- If $Q - t + 1 = U_{\text{unmet}} > 0$, all incoming impressions, including the $t^{th}$ one, must be allocated to fulfil guaranteed contracts.

- If $Q - t + 1 > U_{\text{unmet}} > 0$, the publisher has the discretion to allocate the incoming impression either to guaranteed contracts or to RTB. We denote the quality of the $t^{th}$ impression as $\theta_t$. If $\theta_t \in \Theta_{GC}, (\Theta_{GC} \subseteq \Theta)$, the publisher assigns this impression to guaranteed contracts, otherwise releases it to RTB.

The allocation process for the publisher under this $\theta$-based allocation policy can be summarised in Figure 4.2. Following this process, we can group all possible allocation outcomes into two categories, depending on whether all impressions that are allocated to guaranteed contracts satisfy $\theta \in \Theta_{GC}$ or not.

In the first category, not all impressions assigned to contracts satisfy the condition $\theta \in \Theta_{GC}$ because there exists a time point $t$ such that $Q - t + 1 = U_{\text{unmet}} > 0$ during the allocation process. Then all the left $Q - t + 1$ impressions need to be
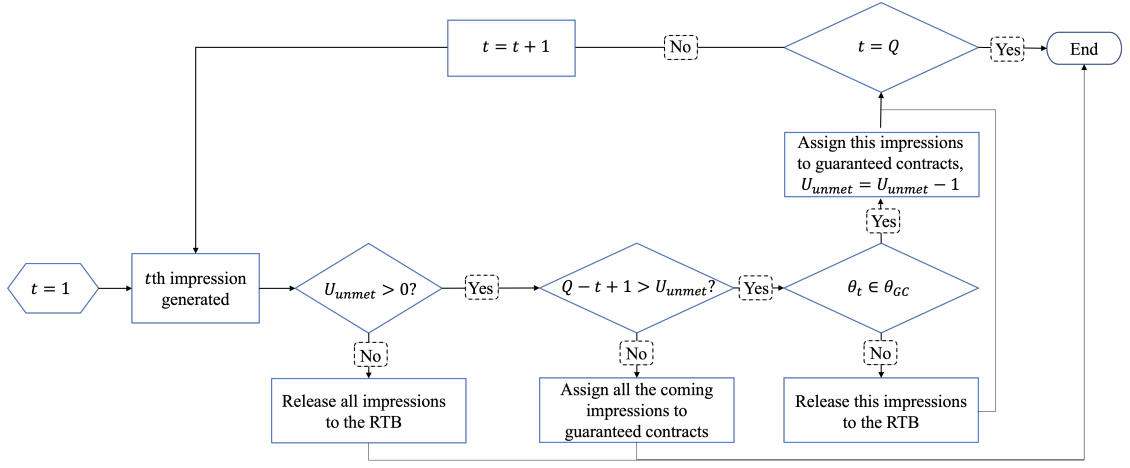
**Figure 4.2:** The publisher's allocation process in period 2 under quantity-guaranteed contracts

delivered to guaranteed contracts without checking their qualities. For ease of modelling, we introduce $y$ ($y \in [0, U' - 1]$) to represent $y$ guaranteed contracts are fulfilled when only $U' - y$ impressions remain before the arrival of the $t^{th}$ ($t = Q - U' + y + 1$) impression[1]. The quality of the last impression, $(Q - U' + y)^{th}$, should be in $\overline{\Theta}_{GC}$ ($\overline{\Theta}_{GC} = \{\theta \mid \theta \in \Theta \text{ and } \theta \notin \Theta_{GC}\}$) and be allocated to RTB[2]. Therefore, the probability of this case $y$ is

$$h(y) = \binom{Q-U'+y-1}{y} \left( \int_{\Theta_{GC}} f(\theta) \, d\theta \right)^y \left( \int_{\overline{\Theta}_{GC}} f(\theta) \, d\theta \right)^{Q-U'},$$

(4.2)

where $\binom{Q-U'+y-1}{y}$ means there are $y$ impressions among the former $Q - U' + y - 1$ impressions that meet the condition $\theta \in \Theta_{GC}$, $\int_{\Theta_{GC}} f(\theta) \, d\theta$ and $\int_{\overline{\Theta}_{GC}} f(\theta) \, d\theta$ are the probabilities that the quality of an impression is located in $\Theta_{GC}$ and $\overline{\Theta}_{GC}$, respectively. The schematic diagram for outcomes in the first category is shown in Figure 4.3.
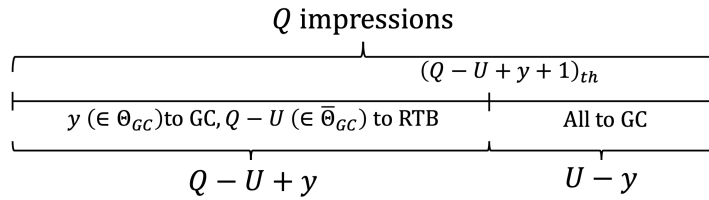


**Figure 4.3:** The schematic diagram for possible outcomes in the first category

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[1]$y = 0$ corresponds to the case that the quality of all the former $Q - U'$ impressions are not meet $\theta \in \Theta_{GC}$, then the remain $U'$ impressions need to be delivered to guaranteed contracts. $y = U' - 1$ corresponds to the case that there is still one guaranteed contract unfulfilled when only one last impression is left.

[2]The reason the quality of $(Q - U' + y)^{th}$ impression should be in $\overline{\Theta}_{GC}$ is that if this impression satisfies $\theta \in \Theta_{GC}$, then the remaining $U' - y$ impressions along with this impression should be delivered to guaranteed contracts buyers. Therefore, this scenario should be categorised to the case that $y - 1$ guaranteed contracts are fulfilled when only $U' - y + 1$ impressions are left.

In the second category, the quality of all impressions assigned to guaranteed contracts is located in $\Theta_{GC}$ because there exists a time point $t$ $(t < Q)$ such that $U_{\text{unmet}} = 0$. We implement $z$ $(z \in [U', Q-1])$ to denote that the $z^{th}$ impression comes with $\theta_z \in \Theta_{GC}$ and is assigned to the guaranteed contracts, then all guaranteed contracts are fulfilled before the arrival of the $t^{th}$ $(t = z + 1)$ impression[3]. This means $U' - 1$ guaranteed contracts have been fulfilled before the coming of $z^{th}$ impression. The probability of this case $z$ is

$$k(z) = \binom{z-1}{U'-1} \left( \int_{\Theta_{GC}} f(\theta) \, \mathrm{d}\theta \right)^{U'} \left( \int_{\overline{\Theta}_{GC}} f(\theta) \, \mathrm{d}\theta \right)^{z-U'}. \tag{4.3}$$

The schematic diagram for outcomes in the second category is shown in Figure 4.4.
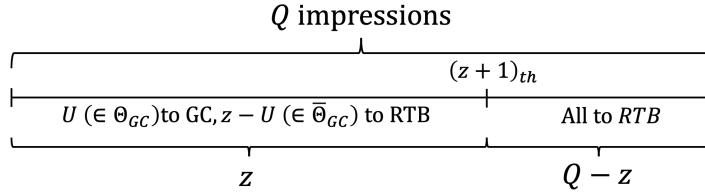


**Figure 4.4:** The schematic diagram for possible outcomes in the second category

### 4.4.3 Objectives of Advertisers and the Publisher

**Incentive of advertisers**  Advertisers decide how many guaranteed contracts they claim in period 1 and how much they bid for every impression released to RTB in period 2. For the channel of guaranteed contracts, advertiser $i$ has the probability of $\frac{x_i}{x_1+x_2}$ to get an impression. If an impression with quality $\theta_t$ is assigned to advertiser $i$, its utility is $\alpha_{i,t}\theta_t - p$.

In RTB, since we do not consider the budget constraints for advertisers, it's a typical second-price auction scenario for each auction. Then, following auction theory (Menezes and Monteiro 2004, Narahari 2014), we know that the truth-telling bidding is the weakly dominant strategy for both two advertisers. For a specific impression with quality $\theta_t$ released to RTB, if the matching degree for advertisers are $\alpha_{1,t}$, $\alpha_{2,t}$, respectively. We take advertiser 1 for an instance, its utility from bidding for this impression is $\max\{\alpha_{1,t} - \alpha_{2,t}, 0\}\theta_t$, i.e., getting $(\alpha_{1,t} - \alpha_{2,t})\theta_t$ if winning this impression otherwise 0.

**Incentive of the publisher**  The publisher's problem is to decide the price of guaranteed contracts $p$, the total number of available impressions $U$, and the range of

---
[3] $z = U'$ represents the former $U'$ impressions are eligible for the criteria of guaranteed contracts. $z = Q - 1$ represents that all guaranteed contracts are fulfilled when only one impression remains.

$\Theta_{GC}$. Then, the publisher's revenue from guaranteed contracts is always $U'p$. As for the revenue from RTB, the publisher gets $\min\{\alpha_{1,t}, \alpha_{2,t}\}\theta_t$ from the auction of an impression with quality $\theta_t$.

**Objectives**  Advertisers' total expected utilities and the publisher's total revenue are both generated from guaranteed contracts and RTB. How impressions are sold between the two channels depends on expected allocation outcomes. As we stated before, there are two categories of possible outcomes. Both the publisher's revenue and advertisers' utilities come from three segments of impressions in each category.

Let $\mathbb{E}(\alpha) = \int \alpha g(\alpha)\, d\alpha, \mathbb{E}(\theta) = \int \theta f(\theta)\, d\theta, \mathbb{E}_{GC}(\theta) = \int_{\Theta_{GC}} \theta f(\theta)|_{\Theta_{GC}}\, d\theta, \mathbb{E}_{RTB}(\theta) = \int_{\overline{\Theta}_{GC}} \theta f(\theta)|_{\overline{\Theta}_{GC}}\, d\theta$, in which, $f(\theta)|_{\Theta_{GC}}, f(\theta)|_{\overline{\Theta}_{GC}}$ are the conditional $p.d.f$ of $\theta$ on the interval $\Theta_{GC}$ and $\overline{\Theta}_{GC}$.

In the first category, impressions delivered to guaranteed contracts consist of two segments. One segment is the former $y$ impressions with quality distributed on $\Theta_{GC}$. Another segment is the remaining $U' - y$ impressions being assigned without checking their qualities. Thus, the quality of these impressions is expected to be distributed on $\Theta$. In RTB, all $Q - U'$ impressions are released after meet the condition $\theta \in \overline{\Theta}_{GC}$.

Therefore, advertisers' expected utility from these impressions under outcomes in the first category is

$$\Pi_1^{(i)}(x_i) = \underbrace{\frac{x_i}{x_1 + x_2} \left[ y\left(\mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p\right) + (U' - y)\left(\mathbb{E}(\alpha)\mathbb{E}(\theta) - p\right) \right]}_{\text{Expected utility from GC}} + \underbrace{(Q - U')\, \tau \mathbb{E}_{RTB}(\theta)}_{\text{Expected utility from RTB}},$$

(4.4)

in which $\tau = \mathbb{E}(\max\{\alpha_i - \alpha_j, 0\})$.

The publisher's expected revenue can also be obtained as

$$\Pi_1^{pub}(p, U', \Theta_{GC}) = U'p + \underbrace{\left[(Q - U')\xi \mathbb{E}_{RTB}(\theta)\right]}_{\text{Expected revenue from RTB}},$$

(4.5)

in which $\xi = \mathbb{E}(\min\{\alpha_1, \alpha_2\})$.

In the second category, impressions delivered to all $U'$ guaranteed contracts meet $\theta \in \Theta_{GC}$ with expected quality $\mathbb{E}_{GC}(\theta)$. While there are two segments of impressions released to RTB. The first segment is $z - U'$ impressions assigned before the $z^{th}$ impression comes. The quality of these impressions is distributed on $\overline{\Theta}_{GC}$. The other segment is the remaining $Q - z$ impressions that are released to RTB without checking their qualities.

Consequently, advertisers' expected utility from these impressions under outcomes in the second category is

$$\Pi_2^{(i)}(x_i) = \underbrace{\frac{x_i}{x_1 + x_2} \left[ U' \left( \mathbb{E}(\alpha) \mathbb{E}_{GC}(\theta) - p \right) \right]}_{\text{Expected utility from GC}} + \underbrace{(z - U') \tau \mathbb{E}_{RTB}(\theta) + (Q - z) \tau \mathbb{E}(\theta)}_{\text{Expected utility from RTB}}. \quad (4.6)$$

We also get the publisher's expected revenue as follows:

$$\Pi_2^{pub}(p, U', \Theta_{GC}) = U'p + \underbrace{\xi \left[ (z - U') \, \mathbb{E}_{RTB}(\theta) + (Q - z)\mathbb{E}(\theta) \right]}_{\text{Expected revenue from RTB}}. \quad (4.7)$$

Table 4.1 summarises the advertisers' utility and the publisher's revenue of a unit impression from each segment under the two categories.

**Table 4.1:** The unit utility and revenue of impressions from different parts under cases of the two categories

| Category 1 | Guaranteed Contracts | | Real-time Bidding |
|---|---|---|---|
| $h(y)$ | $y$ | $U' - y$ | $Q - U'$ |
| Advertisers' Utilities | $\mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p$ | $\mathbb{E}(\alpha)\mathbb{E}(\theta) - p$ | $\mathbb{E}(\max\{\alpha_i - \alpha_j, 0\})\mathbb{E}_{RTB}(\theta)$ |
| Publisher's Revenue | $p$ | $p$ | $\mathbb{E}(\min\{\alpha_1, \alpha_2\})\mathbb{E}_{RTB}(\theta)$ |
| Category 2 | Guaranteed Contracts | Real-time Bidding | |
| $k(z)$ | $U'$ | $z - U'$ | $Q - z$ |
| Advertisers' Utilities | $\mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p$ | $\mathbb{E}(\max\{\alpha_i - \alpha_j, 0\})\mathbb{E}_{RTB}(\theta)$ | $\mathbb{E}(\max\{\alpha_i - \alpha_j, 0\})\mathbb{E}(\theta)$ |
| Publisher's Revenue | $p$ | $\mathbb{E}(\min\{\alpha_1, \alpha_2\})\mathbb{E}_{RTB}(\theta)$ | $\mathbb{E}(\min\{\alpha_1, \alpha_2\})\mathbb{E}(\theta)$ |

In summary, advertiser $i$'s expected utility can be expressed as:

$$\Pi^{(i)}(x_i) = \sum_{y=0}^{U'-1} h(y)\Pi_1^{(i)}(x_i) + \sum_{z=U'}^{Q-1} k(z)\Pi_2^{(i)}(x_i). \quad (4.8)$$

The publisher's expected revenue is:

$$\Pi^{pub}(p, U', \Theta_{GC}) = \sum_{y=0}^{U'-1} h(y)\Pi_1^{pub}(p, U', \Theta_{GC}) + \sum_{z=U'}^{Q-1} k(z)\Pi_2^{pub}(p, U', \Theta_{GC}). \quad (4.9)$$

The decisions of both the publisher and advertisers have complex effects on their revenue or utilities. For example, if the publisher sets a low price for guaranteed contracts, then advertisers tend to post a high demand for it. However, the publisher does not want to sell too many contracts to them at a low price, which will reduce the potential revenue from RTB. So this publisher may only release a limited number of available contracts to advertisers. If the publisher sets a high

price for guaranteed contracts, then it would like to sell many contracts to gain stable revenue. However, advertisers would rather join RTB to take their chances than buy expensive contracts. Thus, the publisher needs to consider the advertisers' incentives when it sets the price and the available amount of guaranteed contracts at the start. For advertisers, when they decide their demands of guaranteed contracts, they should consider their expected utilities from RTB by analysing the publisher's allocation policy in the future.

**Proposition 4.1.** *At least one of these two advertisers' equilibrium demands for guaranteed contracts is the same, i.e., $x_1^* = x_2^* = x^*$.*

From Proposition 4.1, we can rewrite the agreed amount of guaranteed contracts as $U' = \min\{2x^*, U\}$. Furthermore, advertisers now can optimise $U'$ instead of $x_i$. And the probability that an impression assigned to guaranteed contracts can be delivered to one of the advertisers is $1/2$ when they post equilibrium demand. Therefore, we can also update advertisers' utility function by substituting $x_1 = x_2 = x^*$ into $\frac{x_i}{x_1+x_2}$:

$$
\Pi^{(i)}(U') = \sum_{y=0}^{U'-1} h(y) \left\{ \frac{1}{2} \left[ y \left( \mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p \right) + (U' - y) \left( \mathbb{E}(\alpha)\mathbb{E}(\theta) - p \right) \right] + (Q - U') \, \tau \mathbb{E}_{RTB}(\theta) \right\}
$$
$$
+ \sum_{z=U'}^{Q-1} k(z) \left\{ \frac{1}{2} \left[ U' \left( \mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p \right) \right] + (z - U') \, \tau \mathbb{E}_{RTB}(\theta) + (Q - z) \, \tau \mathbb{E}(\theta) \right\}
$$

(4.10)

### 4.4.4 Algorithm and Numerical Experiment

In this section, we design an algorithm to solve our model when both the quality of impressions and the matching degree between advertisers and impressions are uniformly distributed, i.e., $A \sim U(0, \bar{\alpha})$, $\Theta \sim U(0, \bar{\theta})$. Then, we compare the performance of the two sets of threshold-type policies through a numerical experiment. Specifically, we have $\Theta_{GC} = [\theta', \bar{\theta}]$ under guaranteed contracts prior policy and $\Theta_{GC} = [0, \theta')$ under RTB prior policy. The publisher's revenue (Equation (4.9)) and advertisers' utility function (Equation (4.10)) can be specified according to the allocation policy given the distribution of $A$ and $\Theta$. Table 4.2 summarised the details about both parties' objective functions in this case.

**Algorithm** We develop a numerical algorithm to solve this problem through the backward induction approach. Recall the sequences of events in Figure 4.1, the backward order of both parties' decisions in this game is: $b_i, \theta', U'(= \min\{2x^*, U\}), U, p$. Note that the advertisers' bidding strategy is truth-telling due to the absence of budget limitations. Therefore, we summarise three stages starting from solving optimal $\theta'$.

**Table 4.2:** The details of objectives under the two policies given uniform distributions of $A, \Theta$

| Guaranteed Contracts Prior Policy |
|---|

$$\mathbb{E}_{GC}(\theta) = \frac{\overline{\theta} + \theta'}{2} \qquad\qquad \mathbb{E}_{RTB}(\theta) = \frac{\theta'}{2}$$

$$h(y) = \binom{Q-U'+y-1}{y} \left(\frac{\overline{\theta} - \theta'}{\overline{\theta}}\right)^{y} \left(\frac{\theta'}{\overline{\theta}}\right)^{Q-U'} \qquad k(z) = \binom{z-1}{U'-1} \left(\frac{\overline{\theta} - \theta'}{\overline{\theta}}\right)^{U'} \left(\frac{\theta'}{\overline{\theta}}\right)^{z-U'}$$

$$\Pi^{pub}(p, U', \theta') = U'p + \frac{\overline{\alpha}\theta'}{6} \sum_{y=0}^{U'-1} h(y)(Q - U')$$

$$+ \frac{\overline{\alpha}}{6} \sum_{z=U'}^{Q-1} k(z) \left[(z - U')\theta' + (Q - z)\overline{\theta}\right]$$

$$\Pi^{(i)} = \sum_{y=0}^{U'-1} h(y) \left\{ \frac{1}{2} \left[ y \left( \frac{\overline{\alpha}(\overline{\theta} + \theta')}{4} - p \right) + (U' - y) \left( \frac{\overline{\alpha}\overline{\theta}}{4} - p \right) \right] + (Q - U') \frac{\overline{\alpha}\theta'}{12} \right\}$$

$$+ \sum_{z=U'}^{Q-1} k(z) \left\{ \frac{1}{2} \left[ U' \left( \frac{\overline{\alpha}(\overline{\theta} + \theta')}{4} - p \right) \right] + (z - U') \frac{\overline{\alpha}\theta'}{12} + (Q - z) \frac{\overline{\alpha}\overline{\theta}}{12} \right\}$$

| Real-time Bidding Prior Policy |
|---|

$$\mathbb{E}_{GC}(\theta) = \frac{\theta'}{2} \qquad\qquad \mathbb{E}_{RTB}(\theta) = \frac{\overline{\theta} + \theta'}{2}$$

$$h(y) = \binom{Q-U'+y-1}{y} \left(\frac{\theta'}{\overline{\theta}}\right)^{y} \left(\frac{\overline{\theta} - \theta'}{\overline{\theta}}\right)^{Q-U'} \qquad k(z) = \binom{z-1}{U'-1} \left(\frac{\theta'}{\overline{\theta}}\right)^{U'} \left(\frac{\overline{\theta} - \theta'}{\overline{\theta}}\right)^{z-U'}$$

$$\Pi^{pub}(p, U', \theta') = U'p + \frac{\overline{\alpha}(\overline{\theta} + \theta')}{6} \sum_{y=0}^{U'-1} h(y)(Q - U')$$

$$+ \frac{\overline{\alpha}}{6} \sum_{z=U'}^{Q-1} k(z) \left[(z - U')(\overline{\theta} + \theta') + (Q - z)\overline{\theta}\right]$$

$$\Pi^{(i)} = \sum_{y=0}^{U'-1} h(y) \left\{ \frac{1}{2} \left[ y \left( \frac{\overline{\alpha}\theta'}{4} - p \right) + (U' - y) \left( \frac{\overline{\alpha}\overline{\theta}}{4} - p \right) \right] + (Q - U') \frac{\overline{\alpha}(\overline{\theta} + \theta')}{12} \right\}$$

$$+ \sum_{z=U'}^{Q-1} k(z) \left\{ \frac{1}{2} \left[ U' \left( \frac{\overline{\alpha}\theta'}{4} - p \right) \right] + (z - U') \frac{\overline{\alpha}(\overline{\theta} + \theta')}{12} + (Q - z) \frac{\overline{\alpha}\overline{\theta}}{12} \right\}$$

**Stage 1** In this stage, the publisher determines the threshold $\theta'$ in an allocation policy given $p, U'$.

**Lemma 4.2.** *The publisher's revenue function $\Pi^{pub}(p, U', \theta')$ on $\theta'$ only contains infinite extrema on $[0, \overline{\theta}]$ under both guaranteed contract prior policy and RTB prior policy.*

Lemma 4.2 guarantees that we can get the local maximum in a quadratic convergence rate by implementing Newton's method if there is a good starting point. Theoretically, the optimal threshold can be expressed by $p, U'$ and other state parameters. Therefore, we denote it as $\theta'^{*}(p, U')$.

**Stage 2** The agreed number of guaranteed contracts is decided in this stage. Because the final knockdown amount is determined by advertisers after observing the total available number posted by the publisher. Hence, we first find the advertisers' best choice $U'^{*}$ under the range of $[0, U]$ to maximise their utilities by substituting $\theta'^{*}(p, U')$ in $\Pi^{(i)}$, in which $U$ is posted by the publisher. Then we get the optimal $U^{*}$ for the publisher to maximise its best revenue. Note that the complexity in this stage is $O(Q)$.

**Stage 3** Finally, the publisher decides an optimal price $p^{*}$ to maximise its revenue, $\Pi^{pub}(p, U', \theta')$. Unfortunately, we cannot find a good way to accelerate the computation at this stage because the closed-form of $U'$ is not accessible.

**Lemma 4.3.** *At least one optimal price of guaranteed contracts is no greater than $\overline{\alpha}\overline{\theta}$.*

However, according to Lemma 4.3, at least we can find a numerical solution of the price iteratively in $|\frac{\overline{\alpha}\overline{\theta}}{\sigma}|$ times by setting a step-size $\sigma$. And the gap between the optimal price between the numerical one will not excess $\frac{\sigma}{\overline{\alpha}\overline{\theta}}$. The procedure of this algorithm is shown below.

**Numerical experiments** We conducted numerical experiments to compare the performance of two distinct policies. Setting $Q = 50$, $A \sim U(0, 1)$, and $\Theta \sim U(0, 10)$, Figure 4.5 demonstrates the results of the guaranteed contracts prior policy, while Figure 4.6 presents those of RTB prior policy. In the left subplot, the blue dotted line represents the optimal $\theta'$ at the current price. In the right subplot, the blue and orange dotted lines depict the optimal number of guaranteed contracts for advertisers and the publisher as the price varies. The green line in both subplots illustrates the publisher's revenue.

Figure 4.5 reveals that the agreed number of guaranteed contracts between the publisher and advertisers is consistently zero under guaranteed contracts prior policy, resulting in all impressions being sold through RTB ($\Pi^{pub} \approx 83.3$). At low

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**Algorithm 1:** quantity-guaranteed Contracts
---
**Input:** $Q, \Theta, A, \sigma$

1   Initialisation;

2   **while** $p \leq \bar{\theta} * \bar{\alpha}$ **do**

3      **for** $U'$ *in* $[0, \ U]$ **do**

4         Use quasi-Newton's Method to get the optimal $\theta'^{*}(p, U')$ to maximise
         the publisher's revenue $\Pi^{pub}(p, U', \theta')$ under given $p, U'$;

5         Calculate $\Pi^{pub}_{\theta'^{*}}(p, U')$ and $\Pi^{(i)}_{\theta'^{*}}(p, U')$, respectively.

6         **if** $\Pi^{pub}_{\theta'^{*}}(p, U') > \Pi^{pub}_{\theta'^{*}}(p, U' - 1)$ *and* $\Pi^{(i)}_{\theta'^{*}}(p, U') > \Pi^{(i)}_{\theta'^{*}}(p, U' - 1)$
        **then**

7            **Update** $U'^{*} = U'$

8         **end**

9      **end**

10      Get the best choice of $U'^{*}$ to maximise $\Pi^{pub}_{\theta'^{*}, U'^{*}(p, U)}(p, U')$;

11      $p \leftarrow p + \sigma$.

12   **end**

13   Get the final results of $(p^{*}, U'^{*}, \theta')$
---
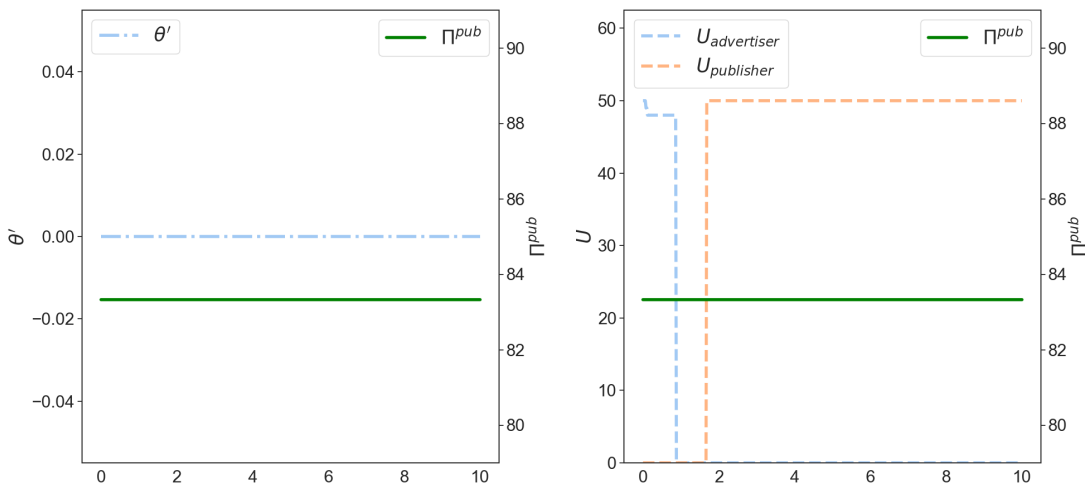


**Figure 4.5:** Numerical results in Guaranteed Contracts prior policy under quantity-guaranteed contracts
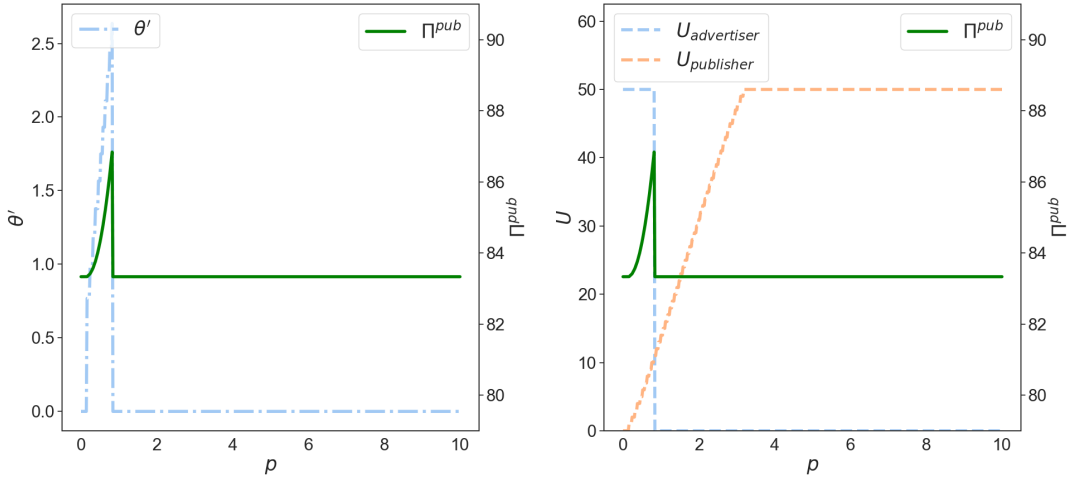
**Figure 4.6:** Numerical results in RTB prior policy under quantity-guaranteed contracts

prices, advertisers seek a high number of guaranteed contracts, while the publisher prefers selling all impressions via RTB. As the price increases, both advertisers' and the publisher's preferences shift entirely. In this scenario, the publisher's revenue solely originates from RTB. A potential explanation for this phenomenon is that both the publisher's revenue and advertisers' utilities are convex functions of the number of guaranteed contracts. The maximum is achieved either when $U = 0$ or when $U = Q$. For low prices, $\Pi_p^{(i)}(U = 0) > \Pi_p^{(i)}(U = Q)$ and $\Pi_p^{pub}(U = 0) < \Pi_p^{pub}(U = Q)$. Conversely, for high prices, $\Pi_p^{(i)}(U = 0) < \Pi_p^{(i)}(U = Q)$ and $\Pi_p^{pub}(U = 0) > \Pi_p^{pub}(U = Q)$. This leads to the result that the publisher and the two advertisers cannot arrive an agreed number of guaranteed contracts.

Figure 4.6 demonstrates that, in RTB priority policy, the optimal numerical solution is $p^* = 0.8$, $\theta'^* \approx 2.5$, $U'^* = 10$, and the corresponding revenue is approximate 86.5. The most notable difference compared to the guaranteed contracts priority policy is that the publisher's optimal choice of the number of guaranteed contracts gradually increases as the price of guaranteed contracts rises. This results in an intermediate intersection between the advertisers' demands and the publisher's offerings. As the price increases, the optimal threshold of impressions quality $\theta'$ also grows, signifying that a greater number of high-quality impressions will be allocated to guaranteed contracts. However, even though the quality of impressions assigned to guaranteed contracts improves as prices increase, advertisers will refrain from purchasing impressions if the price surpasses a certain value. Numerical results also indicate that the expected number of impressions satisfying $\Theta_{GC}$ is approximately 25, which is greater than the agreed number of guaranteed contracts. This observation suggests that when formulating an allocation policy, the publisher aims to establish a $\theta'$ such that the number of impressions allocated to guaranteed contracts aligns with the expected number of impressions with quality located on

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$\Theta_{GC}$.

In conclusion, RTB prior policy generates higher revenue for the publisher compared to the guaranteed contracts prior policy in the context of quantity-guaranteed contracts. This best revenue is achieved via selling impressions through dual channels.

## 4.5 Quality-Guaranteed Contracts

Unlike quantity-guaranteed contracts, each impression delivered to contract buyers under this setting must strictly meet prespecified quality requirements. We assume that the quality criteria ($\theta \in \Theta_{GC}$) contained in the requirements of guaranteed contracts align with what is in the allocation strategy. In other words, if the publisher promises that advertisers will receive impressions with quality satisfying $\theta \in \Theta_{GC}$, then they will deliver impressions to contract buyers as long as a) the quality of these impressions are in $\Theta_{GC}$ and b) there are still unmet contracts. Proper $\Theta_{GC}$ may depend on the price of guaranteed contracts. Therefore, we assume that the publisher first sets the price and then decides the scope of $\Theta_{GC}$.

In this section, we first illustrate the sequence of events and allocation process under the setting of quality-guaranteed contracts. Then we discuss advertisers and the publisher's objectives. Finally, we demonstrate a numerical algorithm to solve this model and a numerical experiment has been conducted.

### 4.5.1 Sequence of Events

The sequence of events under quality-guaranteed contracts is shown in Figure 4.7.

1. At first, the publisher announces the price for a one-impression contract $p$, the scope of quality of impressions delivered to contracts buyers $\Theta_{GC}$, the unit compensation for one unfulfilled contract $h$, and the total available number of contracts $U$.

2. When selling of guaranteed contracts starts in period 1, advertiser $i$ claims demand for guaranteed contracts $x_i$ to the publisher.

3. Receiving $x_1$ and $x_2$, the publisher returns quota of guaranteed contracts, i.e. $\min\{x_i, \frac{x_i}{x_1+x_2}U\}$, to advertiser $i$.

4. In period 2, these $Q$ impressions come one by one. We let $t = 0$ to mark the start of period 2 and $t \in \{t|t = 1, 2, ..., Q\}$ to denote the $t^{th}$ impression with quality $\theta_t$. Then, the publisher makes allocation decisions. If this impression is released to RTB, advertisers can get their valuations denoted as $\alpha_{i,t}\theta_t$ and bid

for it. Otherwise, if it is assigned to guaranteed contracts, advertiser $i$ will get it with the probability of $\frac{x_i}{x_1+x_2}$.
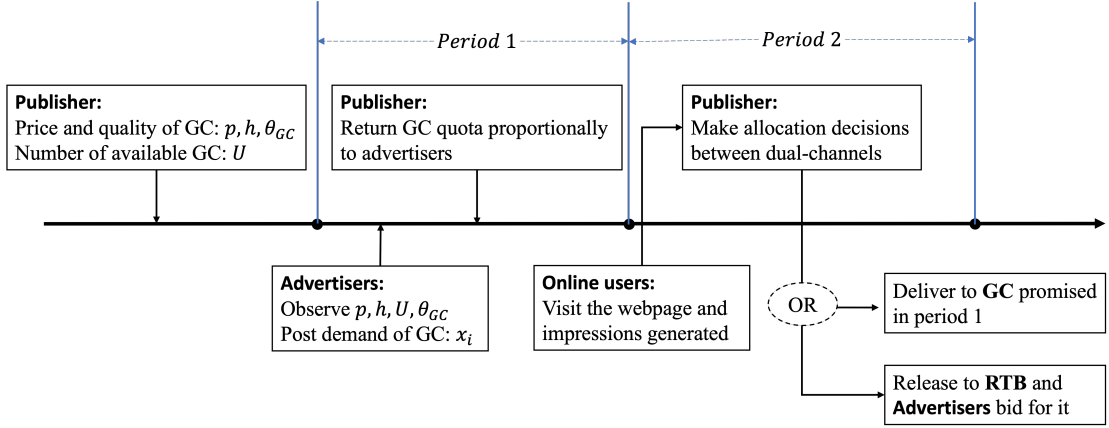


**Figure 4.7:** The sequence of events under the quality-guaranteed contracts setting

Compared with the sequence under quantity-guaranteed contracts, there are two main differences. First, the allocation strategy-making procedure is merged ahead into the quality requirements-making in guaranteed contracts. Second, the publisher needs to decide on one more variable, the compensation cost for unfulfilled contracts. Besides, the policy-based allocation process is also different from what is under the setting of quantity-guaranteed contracts, which is demonstrated next.

## 4.5.2  Allocation Process

Since under-delivery is permitted in this case, the publisher only needs to check whether there are unfulfilled contracts or not during the allocation process. Unlike in the quantity-guaranteed contract setting, the publisher does not need to monitor whether the remaining impressions are enough to fulfil contracts or not.

We implement $U_{unmet}$ to represent the number of unfulfilled contracts. At $t = 0$, we set $U_{unmet} = U'$. Upon the arrival of the $t^{th}$ impression, the publisher checks if $U_{unmet} > 0$. If so, they decide whether to assign this impression to guaranteed contracts or to RTB by comparing $\theta_t$ with $\Theta_{GC}$. Once $U_{unmet} = 0$ and the selling horizon has not yet ended, the publisher releases all remaining impressions to RTB. However, if there are still unfulfilled guaranteed contracts when all $Q$ impressions have been depleted, then the publisher must pay additional $hU_{unmet}$ to advertisers. The allocation process is illustrated in Figure 4.8. This process is simpler than that in Figure 4.2. Different possible allocation outcomes may occur because of various realisations of the quality of impressions. Depending on whether all contracts are successfully fulfilled or not, we can categorise these possible allocation outcomes

into two categories. For convenience, let $z$ denotes the number of impressions that satisfy $\theta_t \in \Theta_{GC}$.
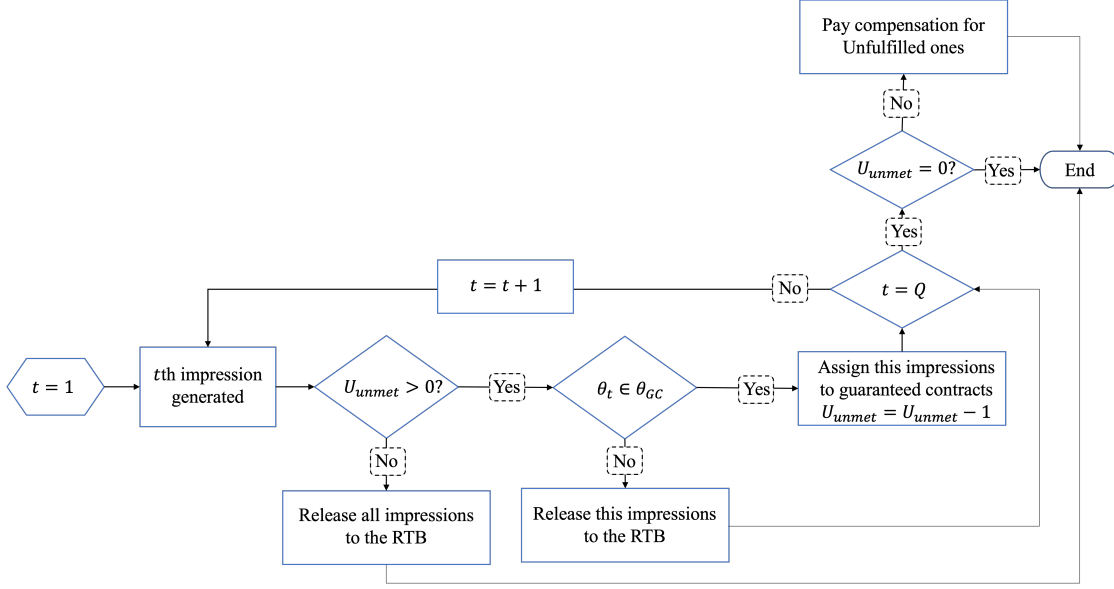


**Figure 4.8:** The publisher's allocation process in period 2 under quality-guaranteed contracts

In the first category, $z \in [0, U']$, the publisher delivers these eligible $z$ impressions to contract buyers. The other $Q - z$ impressions are released to RTB. Besides, there are $U' - z$ unfulfilled contracts after the end of allocation.

In the second category, $z \in (U', Q]$. Then among the $z$ eligible impressions, the former $U'$ impressions are delivered to fulfil all guaranteed contracts. The other $z - U'$ impressions with quality in $\Theta_{GC}$ together with the remaining $Q - z$ impressions with quality in $\Theta_{RTB}$ are released to RTB.

The probability that there are $z$ out of $Q$ impressions fulfilling the requirement of guaranteed contracts is given by:

$$l(z) = \binom{Q}{z} \left( \int_{\Theta_{GC}} f(\theta) \, d\theta \right)^z \left( \int_{\overline{\Theta}_{GC}} f(\theta) \, d\theta \right)^{Q-z} \tag{4.11}$$

### 4.5.3 Objectives of Advertisers and the Publisher

**Incentive of advertisers** Advertisers have two decisions to make: how many guaranteed contracts they want to buy in period 1 and how much they want to bid in RTB in period 2. In period 1, due to symmetry, both advertisers demand the same number of guaranteed contracts $x^*$ (Proposition 4.1). The agreed number of contracts is $U'(U' = \min\{2x^*, U\} \leq U)$. Hence the probability that advertiser $i$ gets an impression assigned to guaranteed contracts is $1/2$. If advertiser $i$ successfully gets an impression with quality $\theta_t$, its utility will be $\alpha_{i,t}\theta_t - p$. However, some contracts

may not be fulfilled by the end of the campaign. In that case, advertisers receive $h - p$ for each unfulfilled contract.

Their bidding incentives are the same as in the quantity-guaranteed setting: truth-telling is a weakly dominant strategy for each auction. Therefore, for an impression with quality $\theta_t$ that is released to RTB, advertiser $i$'s utility is $\max\{\alpha_{i,t} - \alpha_{3-i,t}, 0\}\theta_t$.

**Incentive of the publisher**   The publisher's revenue comes from two sources: advertisers' payments for guaranteed contracts in period 1 and their payments in auctions in period 2. Additionally, the publisher also needs to clear the compensation for unmet contracts at the end of this campaign. Once the number of contracts $U'$ is agreed upon, the publisher receives the payment of $U'p$ immediately. In period 2, for an impression with quality $\theta_t$ that is released to RTB, it obtains $\min\{\alpha_{1,t}, \alpha_{2,t}\}\theta_t$ after this auction. Finally, it pays $h$ back to advertisers for each unfulfilled contract.

**Objectives**   Similar to quantity-guaranteed contracts, advertisers' utilities and the publisher's revenue yield by different segments of impressions in each category we stated before.

In the first category, $z$ impressions are delivered to guaranteed contracts with their quality distributed on $\Theta_{GC}$. The other $Q - z$ impressions that are distributed on $\overline{\Theta}_{GC}$ are sent to RTB. Besides, for each unfulfilled contract, the publisher needs to pay $h$ and advertisers then can get $h - p$ back.

Therefore, advertiser $i$'s expected utility on possible comes in the first category can be expressed as:

$$\Pi_1^{(i)}(U') = \underbrace{\frac{1}{2}\left[z\left(\mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p\right) + (U' - z)(h - p)\right]}_{\text{Expected utility from GC}} + \underbrace{(Q - z)\tau\mathbb{E}_{RTB}(\theta)}_{\text{Expected utility from RTB}},$$

(4.12)

in which $\tau = \mathbb{E}(\max\{\alpha_i - \alpha_j, 0\})$.

Because the publisher makes revenue from advertisers' payments, we can also get its revenue as follows:

$$\Pi_1^{(pub)}(p, h, \Theta_{GC}, U') = \underbrace{zp + (U' - z)(p - h)}_{\text{Net expected revenue from GC}} + \underbrace{(Q - z)\xi\mathbb{E}_{RTB}(\theta)}_{\text{Expected revenue from RTB}}, \quad (4.13)$$

in which $\xi = \mathbb{E}(\min\{\alpha_1, \alpha_2\})$.

In the second category, the former $U'$ impression among $z$ impressions with

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quality in $\Theta_{GC}$ are delivered to guaranteed contracts. The other $z - U'$ impressions together with the remaining $Q - z$ impressions with quality in $\Theta_{RTB}$ are released to RTB.

Accordingly, we can get advertiser $i$'s expected utility as:

$$\Pi_2^{(i)}(U') = \underbrace{\frac{1}{2}U'\left(\mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p\right)}_{\text{Expected utility from GC}} + \underbrace{\tau\left[(Q - z)\,\mathbb{E}_{RTB}(\theta) + (z - U')\,\mathbb{E}_{GC}(\theta)\right]}_{\text{Expected utility from RTB}}.$$
(4.14)

The publisher's expected revenue is:

$$\Pi_2^{(pub)}(p, h, \Theta_{GC}, U') = U'p + \underbrace{\xi\left[(Q - z)\mathbb{E}_{RTB}(\theta) + (z - U')\mathbb{E}_{GC}(\theta)\right]}_{\text{Expected revenue from RTB}}$$
(4.15)

**Table 4.3:** The unit utility and revenue of impressions from different parts under cases of the two categories

| Category 1 | Guaranteed Contracts | | Real-time Bidding |
|---|---|---|---|
| $z \leq U'$ | $z$(Fulfilled) | $U' - z$(Unfulfilled) | $Q - z$ |
| Advertisers' Utilities | $\mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p$ | $h - p$ | $\mathbb{E}(\max\{\alpha_i - \alpha_j, 0\})\mathbb{E}_{RTB}(\theta)$ |
| Publisher's Revenue | $p$ | $p - h$ | $\mathbb{E}(\min\{\alpha_1, \alpha_2\})\mathbb{E}_{RTB}(\theta)$ |
| Category 2 | Guaranteed Contracts | Real-time Bidding | |
| $z > U'$ | $U'$ | $z - U'$ | $Q - z$ |
| Advertisers' Utilities | $\mathbb{E}(\alpha)\mathbb{E}_{GC}(\theta) - p$ | $\mathbb{E}(\max\{\alpha_i - \alpha_j, 0\})\mathbb{E}_{GC}(\theta)$ | $\mathbb{E}(\max\{\alpha_i - \alpha_j, 0\})\mathbb{E}_{RTB}(\theta)$ |
| Publisher's Revenue | $p$ | $\mathbb{E}(\min\{\alpha_1, \alpha_2\})\mathbb{E}_{GC}(\theta)$ | $\mathbb{E}(\min\{\alpha_1, \alpha_2\})\mathbb{E}_{RTB}(\theta)$ |

Table 4.3 summarises both the publisher's revenue and advertisers' expected utilities from a unit of impression from different segments under two categories. Accordingly, advertiser $i$'s expected utility is:

$$\Pi^{(i)}(U') = \sum_{z=0}^{U'} l(z)\Pi_1^{(i)}(U') + \sum_{z=U'+1}^{Q} l(z)\Pi_2^{(i)}(U').$$
(4.16)

The publisher's expected revenue is:

$$\Pi^{pub}(p, h, \Theta_{GC}, U') = \sum_{z=0}^{U'} l(z)\Pi_1^{pub}(p, h, \Theta_{GC}, U') + \sum_{z=U'+1}^{Q} l(z)\Pi_2^{pub}(p, h, \Theta_{GC}, U').$$
(4.17)

The additional decision $h$ for the publisher makes the analysis more complex than in the case of the quantity-guaranteed contract. Both the low price $p$ and the high compensation $h$ encourage advertisers to buy more guaranteed contracts.

However, their purchase willingness of guaranteed contracts is unclear when $p$ and $h$ are both small or large.

### 4.5.4 Algorithm and Numerical Experiment

Similar with what we did in §4.4.4, we can specialise the publisher's revenue (Equation (4.17)) and advertisers' utility function (Equation (4.16)) by assuming $A \sim U(0, \overline{\alpha})$, $\Theta \sim U(0, \overline{\theta})$ and implementing the two types of policies. The objective functions for advertisers and the publisher are shown in Table 4.4.

**Table 4.4:** The details of objectives under the two policies given uniform distributions of $A$, $\Theta$

| Guaranteed Contracts Prior Policy |
|---|
| $\mathbb{E}_{GC}(\theta) = \dfrac{\overline{\theta} + \theta'}{2}$ $\qquad\qquad$ $\mathbb{E}_{RTB}(\theta) = \dfrac{\theta'}{2}$ |
| $l(z) = \binom{Q}{z} \left( \dfrac{\overline{\theta} - \theta'}{\overline{\theta}} \right)^z \left( \dfrac{\theta'}{\overline{\theta}} \right)^{Q-z}$ |
| $\Pi^{pub}(p, h, U', \theta') = U'p + \displaystyle\sum_{z=0}^{U'} \left\{ l(z) \left[ -h(U' - z) + (Q - z)\dfrac{\overline{\alpha}\theta'}{6} \right] \right\}$ $\qquad + \displaystyle\sum_{z=U'+1}^{Q} \left\{ l(z) \left\{ \dfrac{\overline{\alpha}}{6}[(Q - z)\theta' + (z - U')(\overline{\theta} + \theta')] \right\} \right\}$ |
| $\Pi^{(i)} = \displaystyle\sum_{z=0}^{U'} l(z) \left\{ \dfrac{1}{2} \left[ z \left( \dfrac{\overline{\alpha}(\overline{\theta} + \theta')}{4} - p \right) + (U' - z)(h - p) \right] + \dfrac{\overline{\alpha}\theta'}{12}(Q - z) \right\}$ $\qquad + \displaystyle\sum_{z=U'+1}^{Q} l(z) \left\{ \dfrac{1}{2}U' \left( \dfrac{\overline{\alpha}(\overline{\theta} + \theta')}{4} - p \right) + \dfrac{\overline{\alpha}}{12}[(Q - z)\theta' + (z - U)(\overline{\theta} + \theta')] \right\}$ |
| Real-time Bidding Prior Policy |
| $\mathbb{E}_{GC}(\theta) = \dfrac{\theta'}{2}$ $\qquad\qquad$ $\mathbb{E}_{RTB}(\theta) = \dfrac{\overline{\theta} + \theta'}{2}$ |
| $l(z) = \binom{Q}{z} \left( \dfrac{\theta'}{\overline{\theta}} \right)^z \left( \dfrac{\overline{\theta} - \theta'}{\overline{\theta}} \right)^{Q-z}$ |
| $\Pi^{pub}(p, h, U', \theta') = U'p + \displaystyle\sum_{z=0}^{U'} \left\{ l(z) \left[ -h(U' - z) + (Q - z)\dfrac{\overline{\alpha}(\overline{\theta} + \theta')}{6} \right] \right\}$ $\qquad + \displaystyle\sum_{z=U'+1}^{Q} \left\{ l(z) \left\{ \dfrac{\overline{\alpha}}{6}[(Q - z)(\overline{\theta} + \theta') + (z - U')\theta'] \right\} \right\}$ |
| $\Pi^{(i)} = \displaystyle\sum_{z=0}^{U'} l(z) \left\{ \dfrac{1}{2} \left[ z \left( \dfrac{\overline{\alpha}\theta'}{4} - p \right) + (U' - z)(h - p) \right] + \dfrac{\overline{\alpha}(\overline{\theta} + \theta')}{12}(Q - z) \right\}$ $\qquad + \displaystyle\sum_{z=U'+1}^{Q} l(z) \left\{ \dfrac{1}{2}U' \left( \dfrac{\overline{\alpha}\theta'}{4} - p \right) + \dfrac{\overline{\alpha}}{12}[(Q - z)(\overline{\theta} + \theta') + (z - U)\theta'] \right\}$ |

**Algorithm**   In the current setting of quality-guaranteed contracts, the decision of $\theta'$ follows after the decision of the price for contracts. Therefore, the stages in the backward induction method for this case differ from those for quantity-guaranteed contracts. To simplify this problem, we sequentialise the publisher's decisions in the following order: $p, h, \theta', U'$.

**stage 1** In this stage, advertisers and the publisher achieve the agreed number of contracts. Advertisers observe the price $p$, compensation $h$ for a guaranteed contract, the criteria of impressions $\Theta_{GC}$, and the total available number $U$ posted by the publisher. They aim to find the desired number $U'^*$ ( $U' \in [0, U]$) that maximises their utilities $\Pi^{(i)}(U')$. Subsequently, the publisher identifies an optimal $U^*$ that maximises its revenue at $\Pi^{pub}(p, h, \theta', U'^*)$. The complexity of this stage is $O(Q)$.

**stage 2** In this stage, the publisher makes the allocation policy, which is also the criteria regulated in the guaranteed contracts. For given $p$ and $h$, the publisher determines a $\theta'$ to maximise $\Pi^{pub}_{U'^*}(p, h, \theta')$.

**stage 3** In the final stage, the publisher decides the penalty cost $h$ and then the price $p$ for a guaranteed contract to sequentially maximise the revenue function $\Pi^{pub}_{U'^*, \theta'^*}(p, h)$.

Since we cannot obtain the closed-form of the optimal guaranteed contracts, later decisions can only be assessed numerically. The algorithm is presented upside. The limitations of this algorithm are twofold: Firstly, we are unaware of the properties of the numerical results, such as the potential gap compared to the theoretical optimal results. Secondly, the algorithm's efficiency is low, restricting its application to small-scale problems.

**Numerical experiments**   The settings ($Q = 50$, $A \sim U(0, 1)$, $\Theta \sim U(0, 10)$) are identical to those in the context of quantity-guaranteed contracts, enabling a comparison between the results. Figure 4.9 and Figure 4.10 depict the numerical outcomes under the guaranteed contracts priority policy and the RTB priority policy, respectively. The relationship between these decisions and the final revenue is explored in various subplots. In the upper left subplot, the blue dash-dot line represents the optimal $\theta'$ under the current $p$. The upper right and lower right subplots feature blue and orange dashed lines, showcasing the number of guaranteed contracts posted by advertisers and the publisher, respectively. The red line in the lower subplots indicates the optimal compensation cost $h$ under the $p$. Green lines in all subplots represent the publisher's revenue under $p$.

Under the guaranteed contracts priority policy, the numerically optimal revenue is $\Pi^{pub} \approx 172.84$ and is attained at $p^* = 6.67$, $h^* = 12.2$, $U'^* = 24$, $\theta'^* = 5.05$.

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**Algorithm 2:** quality-guaranteed Contracts

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**Input:** $Q, \Theta, A, \{\sigma_i | i = p, h, \theta\}$

1 Initialisation: $p = 0$, $h = 0$, $\theta' = 0$;

2 **while** $p \leq \bar{\theta} * \bar{\alpha}$ **do**

3     **while** $\Pi^{pub}_{U'^*, \theta'^*}(p, h)$ *not converge* **do**

4        **while** $\theta' \leq \bar{\theta}$ **do**

5           **for** $U'$ *in* $[0, U]$ **do**

6              Calculate $\Pi^{pub}(p, h, \theta', U')$ and $\Pi^{(i)}(p, h, \theta', U')$ under given $(p, h, \theta')$, respectively.

7              **if** $\Pi^{pub}(p, h, \theta', U') > \Pi^{pub}(p, h, \theta', U' - 1)$ *and* $\Pi^{(i)}(p, h, \theta', U') > \Pi^{(i)}(p, h, \theta', U' - 1)$ **then**

8                 **Update** $U'^* = U'$

9              **end**

10           **end**

11           Get the $U'^*$;

12           $\theta' \leftarrow \theta' + \sigma_\theta$.

13        **end**

14        Get the current best $\theta'^*$ that max $\Pi^{pub}_{U'^*}(p, h, \theta')$;

15        $h \leftarrow h + \sigma_h$.

16     **end**

17     Get the current best $p^*$ that max $\Pi^{pub}_{U'^*, \theta'^*}(p, h)$;

18     $p \leftarrow p + \sigma_p$.

19 **end**

20 Get the final results of $(p^*, h^*, \theta', U'^*)$.

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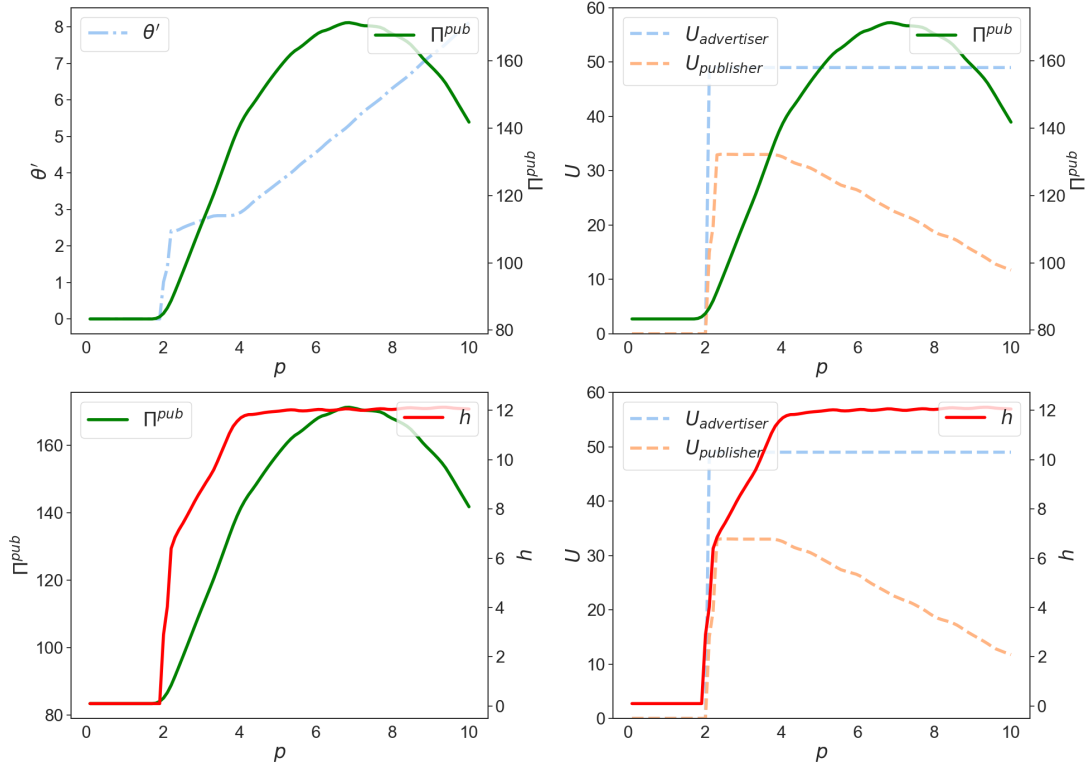
**Figure 4.9:** Numerical results in Guaranteed Contracts prior policy under quality-guaranteed contracts

From Figure 4.9, it is evident that neither advertisers nor the publisher wishes to trade via guaranteed contracts when $p < 2$, resulting in all impressions being sold in RTB. As the price exceeds 2, the publisher can entice advertisers to purchase guaranteed contracts by offering substantial compensation, which also generates increased revenue. Simultaneously, the publisher must provide impressions of higher quality as the price escalates. At the onset of price increases beyond 2, both the quality threshold $\theta'^*$ and compensation $h$ experience sharp growth. The agreed number of contracts diminishes due to the rising compensation cost.

Figure 4.10 displays results under RTB priority policy. The optimal revenue ($\Pi^{pub} \approx 189.45$) is achieved at $p^* = 5.96$, $h^* = 11$, $U'^* = 23$, $\theta'^* = 4.65$. Unlike the guaranteed contracts priority policy cases, all decisions and revenue change as the price of guaranteed contracts increases. The high compensation incentivises advertisers to buy contracts. Consequently, the publisher's strategy is to adjust the criteria in the policy to prevent excessive compensation payments, as evidenced by the similar trends of $\theta'^*$ and the number of contracts posted by the publisher. Another intriguing observation is that when the price of guaranteed contracts is sufficiently high, the publisher prefers to reduce the quota of guaranteed contracts. This is because higher prices demand higher quality, and the number of high-quality impressions is limited. If the publisher allocates a large quota to expensive guaran-
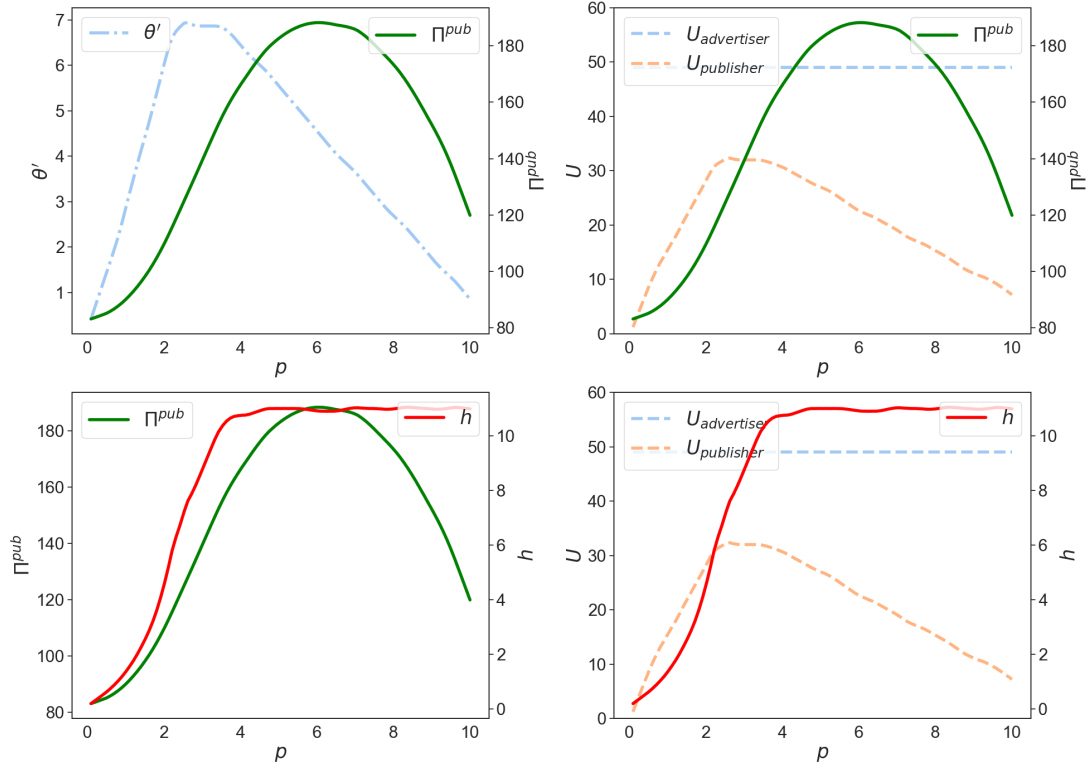
**Figure 4.10:** Numerical results in RTB prior policy under quality-guaranteed contracts

teed contracts, the probability of incurring prohibitive compensation payments also increases.

Comparing the performance of the two policies we find that, the RTB prior policy derives more expected revenue than guaranteed contracts prior policy. Since both the price and the final agreed number of guaranteed contracts are very close among the two policies, the contribution of the difference among the total revenue mainly comes from RTB. Impressions with better quality are allocated to RTB under the RTB prior policy, which generates more revenue. The results collectively reveal that the offer of compensation inspires advertisers' demands for guaranteed contracts.

## 4.5.5 Comparison of Results under Two Types of Guaranteed Contracts

**Benchmarks** For a comprehensive understanding of the publisher's revenue dynamics, we establish two benchmark scenarios: one where only guaranteed contracts are available, and another where only RTB is in operation.

In the first benchmark, we assume that there is no RTB channel in this market

and that the publisher sells all impressions through guaranteed contracts. Since advertisers' expected valuation of these impressions is $\mathbb{E}(\alpha)\mathbb{E}(\theta)$, the publisher can set the price of guaranteed contracts also as $\mathbb{E}(\alpha)\mathbb{E}(\theta)$ to extract the two advertisers' surplus. If we set $Q = 50$, $A \sim U(0,1)$, $\Theta \sim U(0,10)$, then the publisher's expected revenue is 125.

When the publisher only implements RTB when selling these $Q$ impressions, advertisers bid truthfully in each auction. Under the rule of second-price, the publisher gets the minimum of the two advertiser's bids. Therefore, the publisher's expected revenue from these impressions can be expressed as $Q\mathbb{E}(\min\{\alpha_1, \alpha_2\})\mathbb{E}(\theta)$. Under the setting of $Q = 50$, $A \sim U(0,1)$, $\Theta \sim U(0,10)$, its revenue is approximately 83.3.

**Comparison Discussion**  We compare the results under two types of guaranteed contracts - quantity-guaranteed contracts and quality-guaranteed contracts - in conjunction with two allocation policies: guaranteed contracts prior and RTB prior. The publisher's revenue under these four cases is illustrated in Figure 4.11. When the price of guaranteed contracts starts from zero, its revenue under the same policy remains very similar across both types of guaranteed contracts. However, as the prices exceed a certain threshold, its revenue under quality-guaranteed contracts continues to increase, while that under quantity-guaranteed contracts drops down. Furthermore, the adoption of the RTB prior policy with quantity-guaranteed contracts outperforms the exclusive RTB benchmark but falls short when compared to the guaranteed contracts-only benchmark. In contrast, the optimal revenue derived from quality-guaranteed contracts elevates the publisher's revenue potential, surpassing both benchmarks.

This finding suggests that quality-guaranteed contracts are more effective than quantity-guaranteed contracts from a revenue generation perspective. Compared to quantity-guaranteed contracts, quality-guaranteed contracts offer more flexibility to the publisher by incorporating a compensation mechanism. This feature allows the publisher to benefit more from RTB opportunities. RTB prior policy outperforms the guaranteed contracts prior policy as it can capitalise on the variable value inherent in high-quality impressions. This is due to the fact that the price of guaranteed contracts has already determined their value, allowing the RTB prior policy to focus on maximising the revenue from high-quality impressions.
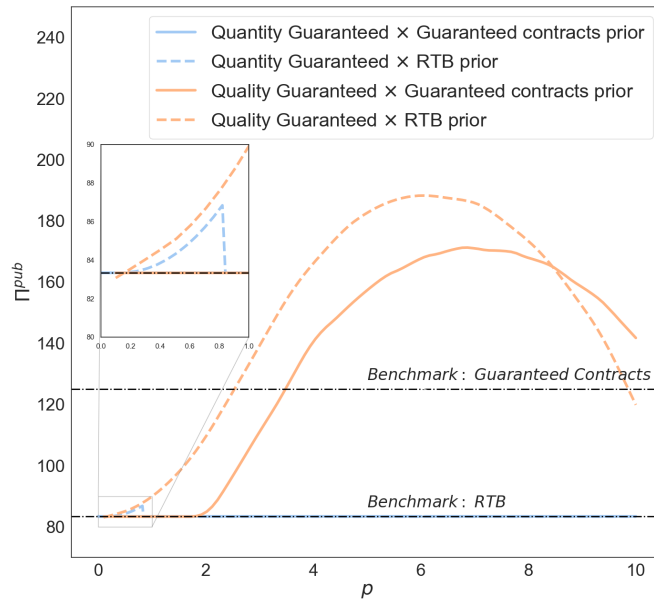
**Figure 4.11:** Publisher's revenue under different types of guaranteed contracts and allocation policies

## 4.6 Conclusion

Because guaranteed contracts and RTB are the main selling channels in the online display advertising markets, how to allocate impressions between them becomes a significant topic in the interests of publishers. This paper focuses on this problem while taking advertisers' strategic choice between these two channels into consideration. It contributes to the understanding of impressions allocation between two channels in online display advertising markets (e.g. Roels and Fridgeirsdottir 2009, Yang et al. 2012, Balseiro et al. 2014, Li et al. 2016, Chen 2017, Jauvion and Grislain 2018, Shen 2018, Rhuggenaath et al. 2019, Wu et al. 2021, Wang et al. 2022 and Zhang et al. 2022). By introducing the quality of every impression decided by distinct users' information behind, we are able to model the heterogeneous of different impressions. Furthermore, we check the performance of two different types of guaranteed contracts, namely quantity-guaranteed contracts and quality-guaranteed contracts, respectively. Under each type of guaranteed contracts, we also examine two kinds of threshold-like allocation policies based on the quality of impressions.

This research provides some interesting managerial insights. Firstly, from the perspective of generating more revenue for the publisher, quality-guaranteed contracts are better than quantity-guaranteed contracts, and the RTB prior policy dominates the guaranteed contracts prior policy. Consequently, the RTB prior policy under the quality-guaranteed contracts is the best choice for the publisher in this

setting. Secondly, the mechanism of compensation contained in quality-guaranteed contracts is significant for inspiring advertisers' demand for contracts, which creates a more flexible space for the publisher to make decisions. Due to the expensive penalty cost, the publisher should try to match the agreed number of guaranteed contracts with the number of impressions that are eligible for regulations in contracts, to decrease the probability of under-delivery. Lastly, the equilibrium results reveal the benefit of the combination of the dual channels.

This paper is among the first to understand the impact of advertisers' strategic behaviour on publishers' allocation strategy, which leaves several valuable future directions. Firstly, we only considered two advertisers in this campaign due to the complexity of this problem. This inevitably narrows the application of our model to oligopoly markets. Expanding to markets with a broader advertiser base could yield different allocation strategies. Second, we neglect the budget constraints for advertisers. In practice, advertisers usually have limited budgets when investing in the markets. Some research has explored how the budget limitation affects advertisers' bidding strategy in single RTB markets (e.g., Balseiro et al. 2015, Lu et al. 2015, Shin 2015 and Balseiro and Gur 2019). However, its effect in the dual-channel markets still remains unclear. As such, we recommend that future research to consider advertisers' strategic behaviours with limited budgets, when addressing the allocation of impressions between two channels. Finally, we only explored two types of threshold-like allocation policy. It's worthwhile for future research to find a global optimal allocation strategy, instead of optimising allocation strategy under limited types of policies.

# 5    Conclusion

The digital era has ushered in a complex and multi-faceted advertising landscape, presenting both challenges and opportunities for publishers and advertisers alike. This thesis has taken a deep dive into the world of online display advertising, with a particular focus on the strategic behaviours that both publishers and advertisers exhibit across dual selling channels, namely guaranteed contracts and RTB. Through three meticulously designed studies, this research sheds light on the optimal pricing of guaranteed contracts, the strategy for attracting additional advertisers between the two channels and the allocation policy of impressions among the guaranteed contracts and RTB.

## 5.1    Summary of Key Findings and Contributions

The first study in this series was instrumental in exploring the nuanced impact of guaranteed contracts on advertisers' behaviour, as well as their strategic choices across dual channels. It broke new ground by revealing that advertisers often employ a mixed truth-telling strategy in RTB, thereby challenging conventional wisdom. Moreover, the study found that publishers are increasingly leveraging the synergies between dual channels to mitigate the uncertainties associated with the supply of impressions. Following this, the second study turned the spotlight onto the publisher's strategic considerations, particularly in terms of attracting additional advertisers. It introduced a novel cost-benefit indicator that serves as a critical threshold for determining the unit advertising cost of attracting more advertisers. This study also enriched our understanding of how the arrival of additional advertisers can influence the behaviours and strategies of the original advertisers. The third study took the analysis a step further by examining optimal impression allocation strategies between guaranteed contracts and RTB, all while accounting for the strategic behaviours of advertisers. It emerged that quality guaranteed contracts, coupled with an RTB prior policy, offer the most lucrative revenue generation opportunities for publishers.

From a theoretical standpoint, this research has made several significant con-

tributions. The first study introduced a fresh perspective on advertisers' decision-making processes in dual channels, thereby deepening our understanding of strategic purchasing decisions in online display advertising. The second study filled an existing gap in the literature by offering a comprehensive analysis of the publisher's strategic investment in attracting more advertisers between the guaranteed contracts and RTB channels. The third study developed a sequential game framework that captures both the publisher's allocation process and advertisers' strategic decisions across dual channels, thereby adding a new layer of complexity and realism to existing models. On the managerial front, the findings from these studies offer actionable insights for publishers. For instance, the research suggests that publishers can maximise their revenues by strategically combining guaranteed contracts and RTB, particularly in situations characterised by a high supply-to-demand ratio and lower discount factors for both the publisher and advertisers. Moreover, publishers need to consider not just the cost and revenue implications of attracting new advertisers but also how such actions could alter the behaviours and strategies of the original advertisers. Specifically, publishers are advised to opt for quality guaranteed contracts and to implement an RTB prior allocation strategy to maximise revenue.

## 5.2    Limitations and Future Research

While this thesis has made significant strides in understanding the complexities of online display advertising, it is important to acknowledge its limitations, which in turn offer fertile ground for future research.

The first study assumed a unit demand for impressions from advertisers to highlight advertisers' choices between two channels. Similar literature that considers problems of a seller selling objects to consumers both through fixed-price channel (corresponding to guaranteed contracts) and auctions channel (corresponding to RTB in our study) also made unit demand assumption for consumers (e.g. Caldentey and Vulcano 2007, Chen et al. 2020, Cohen et al. 2023). This simplification may not capture the full range of advertisers' behaviours and strategies. Future research could relax this constraint to include variable demand, thereby providing a more realistic portrayal of market dynamics. The second study considered the arrival of additional advertisers in a single period. This static approach does not account for the stochastic nature of advertiser arrivals, which is more reflective of real-world scenarios. Future research could employ stochastic models to investigate how random arrivals of advertisers impact both publishers' and original advertisers' strategies. The third study was constrained by its focus on just two advertisers, primarily due to the computational complexity involved in solving the model.

This limitation narrows the applicability of the model to oligopoly markets. Future work could explore how the dynamics change with a larger pool of advertisers, potentially employing machine learning algorithms or other advanced computational methods to handle the increased complexity. The third study examined only two kinds of threshold-like allocation policies based on the quality of impressions. While these policies offer valuable insights, they may not represent the global optimum. Future research could explore a broader range of allocation strategies, potentially identifying more effective approaches through optimisation techniques.

Across all studies, the budget constraints that advertisers often face in real-world scenarios were not considered. Previous research has explored how budget limitations affect bidding strategies in single RTB markets (e.g., Balseiro et al. 2015, Lu et al. 2015, Shin 2015 and Balseiro and Gur 2019), but its impact in dual-channel markets remains an open question. Future studies could incorporate budget constraints to provide a more comprehensive view of advertisers' strategic choices. The studies primarily focused on single-period interactions between publishers and advertisers. In practice, these entities often engage in long-term relationships, which could significantly influence their strategies. Future research could extend the models to multi-period settings to capture the dynamics of long-term interactions. While the studies are grounded in robust theoretical models, empirical validation through real-world data could enhance their practical relevance. Future work could involve collaborating with industry partners to test the model's predictions against actual market behaviours. By addressing these limitations, future research can build upon the foundational insights provided by this thesis, offering a more comprehensive and nuanced understanding of online display advertising markets.

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# A1  Appendix for Study 1

## A1.1  Proof of Lemma 2.1.

*Proof.* Proof of Lemma 2.1

Since all impressions are homogenous and advertisers have unit demand for them, the repeated RTB can be regarded as a one-shot *"generalised second-price"* auction with $Q$ homogenous auctioned objects. Referring to the derivation process of truth-telling bidding strategy in single object second-price auction in *An Introduction to Auction Theory* (Menezes and Monteiro 2004), we start from the perspective of one of the advertisers, say Advertiser 1. Suppose this advertiser has a valuation $V = v$ and believes that other advertisers follow a bidding strategy $b(\cdot)$. $b(\cdot)$ is increasing in their valuations. Knowing its value, the distribution for valuations of other advertisers, the total number of advertisers $N$, and the total supply of impressions $Q$, Advertiser 1 has to figure out what is its best reply. If we denote Advertiser 1's reply is reporting a signal $x$ and its bid is $b(x)$, thus its expected profits are given by:

$$\pi(x) = \int_0^x (v - b(v'))f(v')(N - Q)F(v')^{N-Q-1}dv'.$$

Because there are Q homogeneous impressions, Advertiser 1 only needs to report a signal $x$ that makes it ranked from 1 to $Q$. Thus, when formulating the probability that it ranks from 1 to $Q$ among $N$ advertisers, we only need its reported signal $x$ to be larger than at least $N - Q$ advertisers' valuation. More specifically, first picking one advertiser from the $N - Q$ advertisers randomly, denoted by Advertiser 2, with valuation $v'$. Then the left $N - Q - 1$ advertisers' valuations are less than $v'$. Finally, Advertiser 2's valuation $v'$ should be less than Advertiser 1's signal $x$.

In a symmetric equilibrium, the expected profit is maximised at $x = v$ (Menezes and Monteiro, 2005). Thus, $\pi'(v) = 0$. Because

$$\pi'(x) = (v - b(x))f(x)(N - Q)F(x)^{N-Q-1},$$

$$\pi'(v) = (v - b(v))f(v)(N - Q)F(v)^{N-Q-1}.$$

From $\pi'(v) = 0$ we can obtain that $b(v) = v$ because of $f(v)(N - Q)F(v)^{N-Q-1}$ is always positive. That is to say, telling the truth valuation is the dominant strategy in this special multiple second-price auctions with homogeneous objects. $\square$

## A1.2 Proof of Proposition 2.3.

*Proof.* Proof of Proposition 2.3

We give proof by contradiction. Assume that there exist two advertisers denoted by $i, j$ with valuations $v_i > v_j > p$. In the equilibrium, advertiser $i$ decides to join the real-time bidding in period 2 while advertiser $j$ chooses to buy guaranteed contracts in period 1 to maximise their own utilities.

Advertiser $i$ chooses to join the real-time bidding in period 2 means that it can win an impression by bidding with $b(v_i)$, which makes it earn not less than buying a guaranteed contract in period 1. Because $v_i > v_j > p$, through applying equation (2.4), we have $b(v_i) < b(v_j) < p$. As advertiser $i$ can win an impression in period 2 by bidding with $b(v_i)$, advertiser $j$ also can win an impression in period 2 by bidding with $b(v_j)$, which yields not less utility for it than buying a guaranteed contract. Thus, advertiser $j$ will also join the real-time bidding in period 2, which contradicts our assumption at first. $\square$

## A1.3 Proof of Proposition 2.4.

*Proof.* Proof of Proposition 2.4

We prove this proposition by analysing advertisers' utilities.

1. When $p \in [0, \dfrac{N-Q}{N}\bar{v})$, for advertisers with $v \geq p$, if they buy guaranteed contracts, we have
$$u^1 = v - p > 0,$$

   while if they go to join the real-time bidding in period 2, their bidding prices are subject to $b(v)$. According to Proposition 2.3, if there exists a threshold valuation $v'(v' > p)$ such that advertisers with valuation $v < v'$ join the real-time bidding in period 2 and advertisers with $v \geq v'$ buy guaranteed contracts in period 1. For an advertiser $i$ with $v_i = v'$, its bidding price is

$$b(v_i) = \frac{p - (1 - \delta_a)v'}{\delta_a}.$$

We have $b(v_i) < b(v)$, in which $v \in (\dfrac{p-(1-\delta_a)v'}{\delta_a}, v')$. Advertiser $i$'s incentive to join the real-time bidding instead of buying a guaranteed contract is that it believes it can obtain an impression in period 2 by bidding with $b(v')$. However, in this case, there are advertisers with valuations $v \in (\dfrac{p-(1-\delta_a)v'}{\delta_a}, v')$ bid more than advertiser $i$, and advertisers with valuations $v \in (v', \overline{v})$ have bought guaranteed contracts in period 1. These advertisers are prior to advertiser $i$ in this market. The number of these advertisers are

$$\dfrac{v' - \dfrac{p-(1-\delta_a)v'}{\delta_a}}{\overline{v}}N + \dfrac{\overline{v}-v'}{\overline{v}}N. \tag{A1.1}$$

Because $p < \dfrac{N-Q}{N}\overline{v}$ and $v' > p$, we have (A1.1) is larger than $Q$ in this case, so $u_{i,2} = 0 < u_{i,1}$. Thus, advertiser $i$ won't join the real-time bidding and there doesn't exist a threshold value in $(p, \overline{v})$ such that (A1.1) is less than $Q$. Furthermore, because $\dfrac{\overline{v}-p}{\overline{v}}N > Q$, all impressions will be sold in the first period.

2. When $p \in [\dfrac{N-Q}{N}\overline{v}, \dfrac{N-\delta_a Q}{N}\overline{v})$, we also assume a threshold value $v'$ first. Advertisers with valuations $v < v'$ join the real-time bidding in period 2 and advertisers with $v \geq v'$ buy guaranteed contracts in period 1. If there is an advertiser $i$ with valuation $v_i = v'$, it will be the last one that can win an impression in period 2. Thus,

$$\dfrac{\overline{v}-b(v')}{\overline{v}}N = Q.$$

We can solve this equation and get

$$v' = \dfrac{p}{1-\delta_a} - \dfrac{\delta_a}{1-\delta_a}\dfrac{N-Q}{N}\overline{v} > p.$$

Because $\dfrac{N-Q}{N}\overline{v} \leq p < \dfrac{N-\delta_a Q}{N}\overline{v}$, we have $p \leq v' < \overline{v}$. Thus, this assumption of existence of $v'$ is reasonable in this case. It means that, for advertisers with $v \in (v', \overline{v}]$, $0 = u^2 < u^1$. For advertisers with $v \in (p, v']$, $0 < u^1 < u^2$. For advertiser with $v \in [0, p]$, $u^1 \leq 0 \leq u^2$. Accordingly, advertisers with $v \in (v', \overline{v}]$ will buy guaranteed contracts. Advertisers with $v \in (p, v']$ will join the real-time bidding by bidding by $\dfrac{p-(1-\delta_a)v}{\delta_a}$ and advertisers with $v \in [0, p]$ will go to the real-time bidding and bid truthfully. Note that only part of the advertisers can win an impression in the last group.

3. When $p \in [\dfrac{N - \delta_a Q}{N}\bar{v}, +\infty)$, we discuss two sub-cases as below.

   (a) When $p \in [\dfrac{N - \delta_a Q}{N}\bar{v}, \bar{v}]$, following the bidding strategy $b(v)$, even advertisers with $v = \bar{v}$ can win an impression in period 2. The reason is that

   $$\dfrac{\bar{v} - \dfrac{p - (1 - \delta_a)\bar{v}}{\delta_a}}{\bar{v}} N < Q.$$

   This indicates that for advertisers with $v \in (p, \bar{v}]$, $0 < u^1 < u^2$. For advertisers with $v \in [0, p]$, $u^1 \leq 0 \leq u^2$. So all advertisers will join the real-time bidding in period 2 and bid by $b(v)$.

   (b) When $p \in (\bar{v}, +\infty)$, for all advertisers, we have $u^1 < 0 \leq u^2$. Thus, all advertisers will join the real-time bidding and bid truthfully.

   $\square$

## A1.4   Proof of Lemma 2.5.

*Proof.* Proof of Lemma 2.5

If there are $N$ advertisers in total, and all advertisers' bids are subject to an i.i.d. described by a p.d.f $f(\cdot)$ and the corresponding cumulative distribution function is $F(\cdot)$. Focus on an advertiser called by advertiser 1. We implement $\varphi(b)$ as advertiser 1's position if it bids by $b$. Thus, we have

$$\mathbb{P}(\varphi(b) = k) = \binom{N-1}{k-1} F(b)^{N-k} (1 - F(b))^{k-1}.$$

We use $p(b)$ as advertiser 1's payment under bidding price $b$. Then,

$$\mathbb{P}[(p(b) = x) \cdot (\varphi(b) = k)] = \binom{N-1}{k-1}\binom{N-k}{1} f(x) F(x)^{N-k-1} (1 - F(b))^{k-1}.$$

Accordingly, we can get advertiser 1's expected payment under the condition that it ranks $k^{th}$ position by bidding $b$.

$$\mathbb{E}(p(b)|\varphi(b) = k) = \int_0^b x \cdot \mathbb{P}[(p(b) = x)|(\varphi(b) = k)]\, \mathrm{d}x$$

$$= b - \int_0^b \dfrac{F(x)^{N-k}}{F(b)^{N-k}}\,\mathrm{d}x.$$

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Thus, its expected payment when bidding by $b$ in this auction is

$$\mathbb{E}(p(b)) = \sum_{k=1}^{N} \left( b - \int_0^b \frac{F(x)^{N-k}}{F(b)^{N-k}} dx \right) \left[ \binom{N-1}{k-1} F(b)^{N-k} (1 - F(b))^{k-1} \right]$$

$$= b - \int_0^b [F(x) + (1 - F(b))]^{N-1} dx.$$

Because $F(x)$ is less than $F(b)$, $[F(x) + (1 - F(b))] < 1$. Thus, when the number of advertisers gets larger, advertiser 1's payment becomes closer to its bid. $\square$

## A1.5 Proof of Equation (2.6)

*Proof.* Proof of Equation (2.6)

We use $\Pi_i^j(p)$ to denote the revenue of the publisher obtained from the $j$th group of advertisers when pricing during the $i$th segment. $N_i^j$ denotes the number of advertisers from the $j$th group when the publisher sets the price in the $i$th segment.

1. If $p \in [0, \frac{N-Q}{N}\overline{v}]$, because all impressions are sold through period 1, we have the publisher's revenue in this interval is

$$\Pi_1(p) = Qp.$$

   Note that there are more than $Q$ advertisers who want to buy guaranteed contracts, but we don't consider the allocation of impressions among advertisers in our problem. Instead, we assume that the publisher randomly chooses $Q$ advertisers to sell these impressions when $p < \dfrac{N-Q}{N}\overline{v}$. Besides, we will show that the publisher will not choose a price less than $\dfrac{N-Q}{N}\overline{v}$ to maximise its revenue later.

2. If $p \in [\dfrac{N-Q}{N}\overline{v}, \dfrac{N-\delta_a Q}{N}\overline{v})$, the number of advertisers that buy guaranteed contracts in period 1 is

$$N_2^1 = \frac{\overline{v} - v'}{\overline{v}} N.$$

   Thus, the publisher's revenue for this part is

$$\Pi_2^1(p) = \int_{v'}^{\overline{v}} Np \frac{1}{\overline{v}} dv = \frac{\overline{v} - v'}{\overline{v}} Np.$$

   Advertisers with valuations $v \in (p, v')$ will join the real-time bidding in period

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2 and can win impressions. The number of them is

$$N_2^2 = \frac{v' - p}{\overline{v}} N.$$

According to Lemma 2.5, advertisers' payments in RTB can be approximately regarded as their own bids. Thus, we can get the revenue of the publisher from these advertisers is

$$\Pi_2^2(p) = \int_p^{v'} N \frac{p - (1 - \delta_a)v}{\delta_a} \frac{1}{\overline{v}} \, dv = \frac{1}{2} \left( p + b(v') \right) \frac{v' - p}{\overline{v}} N.$$

Advertisers with valuation $v \in (b(v'), p]$ also can win impressions in period 2 by bidding truthfully. The number of them is

$$N_2^3 = \frac{p - b(v')}{\overline{v}} N.$$

We also can get the publisher's revenue from them is

$$\Pi_2^3(p) = \int_{b(v')}^p N v \frac{1}{\overline{v}} \, dv = \frac{1}{2} \left( p + b(v') \right) \frac{p - b(v')}{\overline{v}} N.$$

Considering the discount factor of the publisher's revenue in period 2, we summarise its revenue in this price segment,

$$\Pi_2(p) = \frac{\overline{v} - v'}{\overline{v}} N p + \frac{\delta_p}{2} \left[ \left( p + b(v') \right) \frac{v' - p}{\overline{v}} N + \left( p + b(v') \right) \frac{p - b(v')}{\overline{v}} N \right].$$

3. If $p \in [\frac{N - \delta_a Q}{N} \overline{v}, \overline{v}]$, all advertisers will join the real-time bidding in period 2. Following the bidding strategy, advertisers with $v \in (p, \overline{v}]$ will bidding by $\frac{p - (1 - \delta_a)v}{\delta_a}$. The number of these advertisers is

$$N_3^1 = \frac{\overline{v} - p}{\overline{v}} N.$$

The publisher's revenue from this part is

$$\Pi_3^1(p) = \int_p^{\overline{v}} N \frac{p - (1 - \delta_a)x}{\delta_a} \frac{1}{\overline{v}} \, dv = \frac{1}{2} \cdot \left( p + b(\overline{v}) \right) \frac{\overline{v} - p}{\overline{v}} N.$$

Advertisers with $v \in [0, p]$ will also join the real-time bidding and bid truthfully. The number of advertisers and win impressions among advertisers with

$v \in [0, p]$ is

$$N_3^2 = \frac{p - \dfrac{N - Q}{N}\overline{v}}{\overline{v}} N.$$

The revenue that the publisher can get from this part is

$$\Pi_3^2(p) = \int_{\frac{N-Q}{N}\overline{v}}^{p} Nv \frac{1}{\overline{v}} dv = \frac{1}{2} \cdot \left(p + \frac{N-Q}{N}\overline{v}\right) \frac{p - \dfrac{N-Q}{N}\overline{v}}{\overline{v}} N.$$

From the above, we know the publisher's revenue in this pricing interval is

$$\Pi_3(p) = \frac{\delta_p}{2} \left[ (p + b(\overline{v})) \frac{\overline{v} - p}{\overline{v}} N + \left(p + \frac{N-Q}{N}\overline{v}\right) \frac{p - \dfrac{N-Q}{N}\overline{v}}{\overline{v}} N \right].$$

4. If $p \in (\overline{v}, +\infty)$, all advertisers will join real-time bidding in period 2 and bid truthfully. So the publisher's revenue is

$$\Pi_4(p) = \int_{\frac{N-Q}{N}\overline{v}}^{\overline{v}} Nv \frac{1}{\overline{v}} dv = \frac{\delta_p}{2} \cdot \frac{(2N-Q)Q}{N}\overline{v}.$$

Summarise the analysis above we can get the publisher's revenue function. $\square$

## A1.6   Proof of Theorem 2.6.

*Proof.* Proof of Theorem 2.6

The publisher's revenue is increasing at the first segment, i.e., $p \in [0, \frac{N-Q}{N}\overline{v})$. Thus, it is obvious that the maximum of

$$\max \Pi_1(p) = \frac{(N-Q)Q}{N}\overline{v}.$$

In the third segment, the publisher's revenue is increasing because the function is concave and we can see that $\Pi_3'(\overline{v}) = 0$, and its maximum at this segment is equal to its constant value at the last segment, i.e., $\frac{\delta_p}{2} \cdot \frac{(2N-Q)Q}{N}\overline{v}$.

For the second segment, the revenue function $\Pi_2(p)$ is also concave. Let $\Pi_2'(p) =$

0, we have

$$p^{(*)} = \frac{N - \delta_a Q}{(2 - \delta_p)N}\overline{v}.$$

However, the range of $p$ in this segment is $[\frac{N - Q}{N}\overline{v}, \frac{N - \delta_a Q}{N}\overline{v})$.

1. If $N < \frac{2 - \delta_a - \delta_p}{1 - \delta_p}Q$, $p^{(*)} \in [\frac{N - Q}{N}\overline{v}, \frac{N - \delta_a Q}{N}\overline{v})$. Note that $\Pi(p)$ is continuous on $[0, +\infty)$, so

$$\max \Pi_1(p) = \Pi_2(\frac{N - Q}{N}\overline{v}) < \Pi_2(p^{(*)}) = \max \Pi_2(p);$$

Then we explore the relationship between $\Pi_2(p^{(*)})$ and $\max \Pi_3(p)$ to decide the global maximum of $\Pi(p)$.

$$\Pi_2(p^{(*)}) - \max \Pi_3(p) = \frac{(1 - \delta_p)^2 N^2 - 2\delta_a(1 - \delta_p)^2 NQ + (\delta_a^2 - 2\delta_a\delta_p + \delta_a\delta_p^2)Q^2}{2(1 - \delta_a)(2 - \delta_p)}\frac{\overline{v}}{N}$$

Because $\delta_a \in (0, 1), \delta_p \in (0, 1)$, thus, $(1 - \delta_a)(2 - \delta_p) > 0$. Let

$$A = (1 - \delta_p)^2 N^2 - 2\delta_a(1 - \delta_p)^2 NQ + (\delta_a^2 - 2\delta_a\delta_p + \delta_a\delta_p^2)Q^2,$$

We have

$$\begin{aligned} A &= [(1 - \delta_p)N - \delta_a(1 - \delta_p)Q]^2 - \delta_a\delta_p(1 - \delta_a)(2 - \delta_p)Q^2 \\ &= \frac{1}{Q^2}\{[(1 - \delta_p)\frac{N}{Q} - \delta_a(1 - \delta_p)]^2 - \delta_a\delta_p(1 - \delta_a)(2 - \delta_p)\} \end{aligned} \tag{A1.2}$$

Let $\frac{N}{Q} = k, (k > 1)$, and

$$B = [(1 - \delta_p)k - \delta_a(1 - \delta_p)]^2 - \delta_a\delta_p(1 - \delta_a)(2 - \delta_p)$$

If $B = 0$, we can get the roots of this equation:

$$k_1 = \delta_a + \frac{\sqrt{\delta_a\delta_p(1 - \delta_a)(2 - \delta_p)}}{1 - \delta_p}$$

$$k_2 = \delta_a - \frac{\sqrt{\delta_a\delta_p(1 - \delta_a)(2 - \delta_p)}}{1 - \delta_p}$$

Because $k \in (1, +\infty)$, $k_2$ should be eliminated. Then the key is to compare $k_1$

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and 1.

$$k_1 - 1 = \frac{\sqrt{\delta_a \delta_p (1 - \delta_a)(2 - \delta_p)} - (1 - \delta_a)(1 - \delta_p)}{(1 - \delta_p)}$$

$$= \frac{(1 - \delta_a)[\delta_a - (1 - \delta_p)^2]}{(1 - \delta_p)(\sqrt{\delta_a \delta_p (1 - \delta_a)(2 - \delta_p)} + (1 - \delta_a)(1 - \delta_p))}$$

(A1.3)

Thus, if $\delta_a > (1 - \delta_p)^2$, then $k_1 > 1$. a. It means, when $k > k_1$, $B > 0$, i.e., when $\frac{N}{Q} > k_1$, $\Pi_2(p^{(*)}) > \max \Pi_3(p)$, and $\max \Pi(p) = \Pi_2(p^{(*)})$. b. When $1 < k \le k_1$, $B \le 0$, i.e., when $1 < \frac{N}{Q} \le k_1$, $\Pi_2(p^{(*)}) \le \max \Pi_3(p)$, and $\max \Pi(p) = \max \Pi_3(p)$.

If $\delta_a \le (1 - \delta_p)^2$, then $k_1 \le 1$. It means that, we always have $k > 1 > k_1$, then $B > 0$, then $\Pi_2(p^{(*)}) > \max \Pi_3(p)$, and $\max \Pi(p) = \Pi_2(p^{(*)})$.

2. If $N \ge \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p} Q$, $p^{(*)} \le \dfrac{N - Q}{N} \overline{v}$. In this case, we have

$$\max \Pi_1(p) = \max \Pi_2(p) = \Pi_2(\frac{N - Q}{N} \overline{v}).$$

Then we compare $\max \Pi_1(p)$ and $\max \Pi_3(p)$ to decide the global maximum of $\Pi(p)$.

$$\max \Pi_1(p) - \max \Pi_3(p) = \left[ 2(1 - \delta_p)NQ - (2 - \delta_p)Q^2 \right] \frac{\overline{v}}{2N}.$$

Thus, if $\frac{N}{Q} \ge \dfrac{2 - \delta_p}{2(1 - \delta_p)}$, $\max \Pi_1(p) \ge \max \Pi_3(p)$, else $\max \Pi_1(p) < \max \Pi_3(p)$.

From the analysis above, we know that there are three key threshold values of the $\frac{N}{Q}$, i.e., $\dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}$, $\delta_a + \dfrac{\sqrt{\delta_a \delta_p (1 - \delta_a)(2 - \delta_p)}}{1 - \delta_p}$, $\dfrac{2 - \delta_p}{2(1 - \delta_p)}$. For convenience, we let

$$\begin{cases} \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p} & = C_1 \\ \delta_a + \dfrac{\sqrt{\delta_a \delta_p (1 - \delta_a)(2 - \delta_p)}}{1 - \delta_p} & = C_2 \\ \dfrac{2 - \delta_p}{2(1 - \delta_p)} & = C_3 \end{cases}$$

$C_1$ is the threshold value to decide whether $p^{(*)}$ is in the second segment or not. If so, the $\max\{\Pi_1(p), \Pi_2(p)\}$ is $\Pi_2(p^{(*)})$, otherwise it will be $\max \Pi_1(p)$. $C_2$ is for comparing $\Pi_2(p^{(*)})$ and $\max \Pi_3(p)$. $C_3$ is to compare $\max \Pi_1(p)$ and $\max \Pi_3(p)$. Besides, $C_1 > 1$ and $C_3 > 1$, and we should care about the relationship between $C_2$

and 1.

When $2\delta_a + \delta_p \geq 2$, we have:

$$\begin{cases} C_1 & \leq & C_2 \\ C_1 & \leq & C_3 \end{cases}$$

and when $2\delta_a + \delta_p < 2$, we have:

$$\begin{cases} C_1 & > & C_2 \\ C_1 & > & C_3 \end{cases}$$

Thus,

1. if $2\delta_a + \delta_p \geq 2$,

   (a) when $\dfrac{N}{Q} \geq C_1$, we have $\max\{\Pi_1(p), \Pi_2(p)\} = \max\Pi_1(p)$, then the next is to compare $\max\Pi_1(p)$ and $\max\Pi_3(p)$. Because $C_1 \leq C_3$, we have

      i. if $\dfrac{N}{Q} \geq C_3$, $\max\Pi(p) = \max\Pi_1(p)$, and

      $$p^* = \frac{N-Q}{N}\overline{v}$$

      ii. if $C_1 \leq \dfrac{N}{Q} < C_3$, $\max\Pi(p) = \max\Pi_3(p)$, and

      $$p^* = \overline{v}$$

   (b) when $\dfrac{N}{Q} < C_1$, we have $\max\{\Pi_1(p), \Pi_2(p)\} = \Pi_2(p^{(*)})$, then the next is to compare $\Pi_2(p^{(*)})$ and $\max\Pi_3(p)$. Because $C_1 \leq C_2$ and $C_1 > 1$, so we have $1 < \dfrac{N}{Q} < C_2$ under this case. Thus,

      $$p^* = \overline{v}$$

2. if $2\delta_a + \delta_p < 2$,

   (a) when $\dfrac{N}{Q} \geq C_1$, we have $\max\{\Pi_1(p), \Pi_2(p)\} = \max\Pi_1(p)$, then the next is to compare $\max\Pi_1(p)$ and $\max\Pi_3(p)$. Because $C_1 > C_3$, $\dfrac{N}{Q} > C_3$.

Thus,
$$p^* = \frac{N-Q}{N}\overline{v}$$

(b) when $\frac{N}{Q} < C_1$, we have $\max\{\Pi_1(p), \Pi_2(p)\} = \Pi_2(p^{(*)})$, then the next is to compare $\Pi_2(p^{(*)})$ and $\max \Pi_3(p)$. Because $C_2 < C_1$, we should then discuss the relationship between $C_2$ and 1. So,

    i. when $\delta_a > (1-\delta_p)^2$, $C_2 > 1$,

        A. if $C_2 \leq \frac{N}{Q} < C_1$, we have

$$p^* = \frac{N-\delta_a Q}{(2-\delta_p)N}\overline{v}$$

        B. if $1 < \frac{N}{Q} < C_2$, we have
$$p^* = \overline{v}$$

    ii. when $\delta_a \leq (1-\delta_p)^2$, $C_2 \leq 1$. Thus we have $\frac{N}{Q} > C_2$. So,

$$p^* = \frac{N-\delta_a Q}{(2-\delta_p)N}\overline{v}$$

$\square$

# A2 Appendix for Study 2

## A2.1 Proof of Equation (3.1)

*Proof.* Proof of Equation (3.1)

We discuss the publisher's total revenue with different decisions of $x$ under a given $p$ in period 2 in the following. Because the revenue function is piece-wise liked, we use $\Pi_i^j$ to denote the formulation of the $j$th segment of $x$ under the $i$th segment of $p$.

1. If $p \in [0, \dfrac{N-Q}{N}\overline{v}]$, all impressions should be delivered to guaranteed contracts buyers, so there is no need to hire more advertisers in period 2. The publisher's revenue is

$$\Pi_1 = Qp - cx$$

2. If $p \in (\dfrac{N-Q}{N}\overline{v}, \dfrac{N-\delta_a Q}{N}\overline{v}]$, from Proposition 2.4, the number of advertisers that buy guaranteed contracts is

$$N_2^1 = \frac{\overline{v} - v'}{\overline{v}}N = \frac{(N - \delta_a Q)\overline{v} - Np}{(1 - \delta_a)\overline{v}}.$$

Then, the left impressions that are available in period 3 is

$$N_2^2 = Q - N_2^1 = \frac{Np - (N - Q)\overline{v}}{(1 - \delta_a)\overline{v}}.$$

According to $b(v)$, the upper bound of the original batch of advertisers' bidding price is $p$.

(a) When $\dfrac{\overline{v} - p}{\overline{v}}x \geq N_2^2$, i.e. $x \geq \dfrac{Np - (N - Q)\overline{v}}{(1 - \delta_a)(\overline{v} - p)}$, the expected number of extra arrival advertisers with $v \geq p$ is larger than the number of left impressions. Thus, all the left impressions will be won by extra arrival

advertisers from period 2. In this case, the publisher's revenue is

$$\Pi_2^1 = N_2^1 p + \delta_p \int_{\frac{x - N_2^2}{x}\bar{v}}^{\bar{v}} xv \frac{1}{\bar{v}} dv - cx$$

(b) When $\dfrac{\bar{v} - p}{\bar{v}} x < N_2^2$, i.e. $x < \dfrac{Np - (N - Q)\bar{v}}{(1 - \delta_a)(\bar{v} - p)}$, the left impressions will be shared both by advertisers from the period 1 and period 2. Let $v''$ be the lowest valuation of advertisers that can win an impression in the RTB, we have

$$\frac{b^{-1}(v'') - v''}{\bar{v}} N + \frac{\bar{v} - v''}{\bar{v}} x = N_2^2,$$

so, we get

$$v'' = (1 - \frac{Q}{N + (1 - \delta_a)x})\bar{v}$$

In this case, the publisher's revenue comes from four parts: payments of guaranteed contracts buyers, payments of advertisers from the original batch with $v \in (p, b^{-1}(v'')]$ bidding by $b(v)$, payments of advertisers from the original batch with $v \in (v'', p]$ bidding truthfully, payments of extra advertisers from period 2 with $v \in (v'', \bar{v}]$. So, the publisher's revenue is

$$\Pi_2^2 = N_2^1 p + \delta_p \left\{ \int_p^{b^{-1}(v'')} Nb(v)\frac{1}{\bar{v}} dv + \int_{v''}^p Nv\frac{1}{\bar{v}} dv + \int_{v''}^{\bar{v}} xv\frac{1}{\bar{v}} dv \right\} - cx$$

3. If $p \in (\dfrac{N - \delta_a Q}{N}\bar{v}, \bar{v}]$, from Proposition 2.4, all advertisers from the original batch will join the RTB in period 3. Specifically, the bids of advertisers with $v \in (p, \bar{v}]$ are located in $[\dfrac{p - (1 - \delta_a)\bar{v}}{\delta_a}, p)$; advertisers with $v \in [0, p]$ will bid truthfully.

(a) When $\dfrac{\bar{v} - p}{\bar{v}} x \geq Q$, i.e., $x \geq \dfrac{Q\bar{v}}{\bar{v} - p}$, all impressions will be won by extra arrival advertisers. The publisher's revenue is

$$\Pi_3^1 = \delta_p \int_{\frac{x - Q}{x}\bar{v}}^{\bar{v}} xv \frac{1}{\bar{v}} dv - cx$$

(b) When $\dfrac{\bar{v} - p}{\bar{v}} x < Q$ and $\dfrac{\bar{v} - b(\bar{v})}{\bar{v}}(N + x) \geq Q$, i.e., $\dfrac{Np - (N - \delta_a Q)\bar{v}}{\bar{v} - p} \leq x < \dfrac{Q\bar{v}}{\bar{v} - p}$, there exists a least valuation, denoted as $v^{(3)} \in [\dfrac{p - (1 - \delta_a)\bar{v}}{\delta_a}, p)$,

such that the $Q$ impressions are won by advertisers from the original batch with $v \in [v^{(3)}, b^{-1}(v^{(3)})]$ and extra advertisers with $v \in [v^{(3)}, \overline{v}]$. Thus, we have $v^{(3)}$ satisfy:

$$\frac{\overline{v} - v^{(3)}}{\overline{v}} x + \frac{b^{-1}(v^{(3)}) - v^{(3)}}{\overline{v}} N = Q,$$

by solving this we get

$$v^{(3)} = \overline{v} - \frac{(1 - \delta_a) Q \overline{v} + (\overline{v} - p) N}{N + (1 - \delta_a) x}.$$

Then the publisher's revenue is

$$\Pi_3^2 = \delta_p \left\{ \int_p^{b^{-1}(v^{(3)})} N b(v) \frac{1}{\overline{v}} dv + \int_{v^{(3)}}^p N v \frac{1}{\overline{v}} dv + \int_{v^{(3)}}^{\overline{v}} x v \frac{1}{\overline{v}} dv \right\} - cx$$

(c) When $\dfrac{\overline{v} - b(\overline{v})}{\overline{v}} (N + x) < Q$, i.e., $x < \dfrac{Np - (N - \delta_a Q)\overline{v}}{\overline{v} - p}$, let $v^{(4)} \Big( v^{(4)} \in$

$\Big( \dfrac{N - Q}{N} \overline{v}, b(\overline{v}) \Big) \Big)$ be the least valuation of the advertisers that can won an impression in the RTB, we have

$$\frac{\overline{v} - v^{(4)}}{\overline{v}} (N + x) = Q,$$

so we get

$$v^{(4)} = \frac{N + x - Q}{N + x} \overline{v}.$$

Thus, the publisher's revenue is

$$\Pi_3^3 = \delta_p \left\{ \int_p^{\overline{v}} N b(v) \frac{1}{\overline{v}} dv + \int_{v^{(4)}}^p N v \frac{1}{\overline{v}} dv + \int_{v^{(4)}}^{\overline{v}} x v \frac{1}{\overline{v}} dv \right\} - cx$$

In summary, we can obtain the publisher's revenue function on the whole map of its domain of $p$ and $x$.

$\square$

## A2.2 Proof of Lemma 3.1.

*Proof.* Proof of Lemma 3.1

Let $q \equiv \sqrt{\dfrac{2c}{\delta_p \overline{v}}} \geq 1$, we check the publisher's revenue function in terms of $x$ in each segment.

1. when $p \in (0, \frac{N-Q}{N}\overline{v}]$:

   $\Pi_1$: The optimal $x$ in this stage is always 0.

2. when $p \in (\frac{N-Q}{N}\overline{v}, \frac{N-\delta_a Q}{N}\overline{v}]$,

   (a) $\Pi_2^1$: when $(x_2^1)^* = \frac{Np-(N-Q)\overline{v}}{(1-\delta_a)q\overline{v}}$, we have $\Pi_2^{1\prime}((x_2^1)^*) = 0$. Because $q \geq 1$, then $p > 0 \geq (1-q)\overline{v}$. so $(x_2^1)^* < \frac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}$. Therefore, $\Pi_2^1$ is decreasing on $\left[\frac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}, +\infty\right)$;

   (b) $\Pi_2^2$: when $(x_2^2)^* = \frac{Q-qN}{(1-\delta_a)q}$, we have $\Pi_2^{2\prime}((x_2^2)^*) = 0$. Because $q \geq 1$ and $N > Q$, then $(x_2^2)^* < 0$. So $\Pi_2^2$ is decreasing on $\left[0, \frac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}\right)$.

   Note that $\Pi_2^1$ and $\Pi_2^2$ are continuous on $x$, so the global optimal point of $x$ on $[0, +\infty)$ is 0.

3. when $p \in \left(\frac{N-\delta_a Q}{N}\overline{v}, \overline{v}\right]$:

   Similarly, it's easy to show that $\Pi_3^1, \Pi_3^2, \Pi_3^3$ are continuous on $x$ and when $q \geq 1$ they are all decreasing on their domain. So, the global optimal point of $x$ on $[0, +\infty)$ is still 0.

Therefore, when the cost-benefit indicator $q \geq 1$, the publisher's optimal decision is not to introduce extra advertisers in period 2.

$\square$

# A2.3 Proof of Theorem 3.2.

*Proof.* Proof of Theorem 3.2

Let's first discuss this problem in different price segments. Then compare the local optimal revenue in different segments to get the global optimal results. Because the publisher first decides the price of guaranteed contracts, then the number of extra advertisers, from the backward induction, we first optimise $x$, then $p$ in each piecewise function.

**Finding Local extremum revenue under different price segments**

1. $p \in (0, \dfrac{N-Q}{N}\overline{v}]$

The optimal solutions are obviously $(p^*, x^*) = (\dfrac{N-Q}{N}\overline{v}, 0)$, and the correspond maximum revenue is $\dfrac{(N-Q)Q}{N}\overline{v}$.

2. $p \in (\dfrac{N-Q}{N}\overline{v}, \dfrac{N-\delta_a Q}{N}\overline{v}]$

The theoretical optimal $x$ of $\Pi_2^1$ and $\Pi_2^2$ are $(x_2^1)^* = \dfrac{Np - (N-Q)\overline{v}}{(1-\delta_a)q\overline{v}}$, $(x_2^2)^* = \dfrac{Q-qN}{(1-\delta_a)q}$, respectively.

If $p < \overline{q}\overline{v}$, $(x_2^1)^* \in \left[\dfrac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}, +\infty\right)$ and $(x_2^2)^* > \dfrac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}$, then $\Pi_{2max}^2(x) = \Pi_2^2(\dfrac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}) < \Pi_2^1((x_2^1)^*)$.

If $p \geq \overline{q}\overline{v}$, $(x_2^1)^* \leq \dfrac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}$ and $(x_2^2)^* \leq \dfrac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}$, then $\Pi_{2max}^1(x) = \Pi_2^1(\dfrac{Np-(N-Q)\overline{v}}{(1-\delta_a)(\overline{v}-p)}) < \Pi_2^2(x)$.

From the analysis above we can infer that the optimal $x$ depends on the comparison of $p$ and $\overline{q}\overline{v}$. Because $p \in (\dfrac{N-Q}{N}\overline{v}, \dfrac{N-\delta_a Q}{N}\overline{v}]$, we should discuss different cases of the interaction of $\overline{q}\overline{v}$ and the interval $(\dfrac{N-Q}{N}\overline{v}, \dfrac{N-\delta_a Q}{N}\overline{v}]$.

(a) If $\overline{q}\overline{v} \leq \dfrac{N-Q}{N}\overline{v}$, i.e., $s \geq \dfrac{1}{q}$

Then $p > \overline{q}\overline{v}$, so
$$\max_x \Pi_2(p, x) = \max_x \Pi_2^2(p, x)$$

And $s > \dfrac{1}{q}$,
$$(x_2^2)^* = \dfrac{Q-qN}{(1-\delta_a)q} < 0,$$
then
$$\max_x \Pi_2^2(p, x) = \Pi_2^2(p, 0)$$

As $d\Pi_2^2(\dfrac{N-\delta_a Q}{(2-\delta_p)N}\overline{v}, 0)/dp = 0$, $\dfrac{N-\delta_a Q}{(2-\delta_p)N}\overline{v}$ is naturally less than $\dfrac{N-\delta_a Q}{N}\overline{v}$.

i. if $s < \dfrac{2-\delta_a-\delta_p}{1-\delta_p}$

$$\dfrac{N-\delta_a Q}{(2-\delta_p)N}\overline{v} > \dfrac{N-Q}{N}\overline{v}$$

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The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - \delta_a Q}{(2 - \delta_p)N}\bar{v}, 0)$;

ii. if $s \geq \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}$

The function of $\Pi_2^2(p, 0)$ on $p$ is decreasing on $(\dfrac{N - Q}{N}\bar{v}, \dfrac{N - \delta_a Q}{N}\bar{v}]$, thus The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - Q}{N}\bar{v}, 0)$.

(b) If $\bar{q}\bar{v} \in (\dfrac{N - Q}{N}\bar{v}, \dfrac{N - \delta_a Q}{N}\bar{v}]$, i.e., $s \in [\dfrac{\delta_a}{q}, \dfrac{1}{q})$

i. When $p \in (\dfrac{N - Q}{N}\bar{v}, \bar{q}\bar{v}]$, $p \leq \bar{q}\bar{v}$.

$$\max_x \Pi_2(p, x) = \max_x \Pi_2^1(p, (x_2^1)^*)$$

By solving $d\Pi_2^1(p, (x_2^1)^*)/dp = 0$, we get

$$(p_2^1)^* = \dfrac{(1 + \delta_p \bar{q})N - \delta_a Q}{2N}\bar{v}.$$

Naturally, the next is to discuss whether $(p_2^1)^*$ is located in $(\dfrac{N - Q}{N}\bar{v}, \bar{q}\bar{v}]$ or not.

A. If $s > \dfrac{2 - \delta_a}{1 - \delta_p \bar{q}}$, i.e., $(p_2^1)^* < \dfrac{N - Q}{N}\bar{v}$,

The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - Q}{N}\bar{v}, (x_2^1)^*) = (\dfrac{N - Q}{N}\bar{v}, 0)$.

B. If $\left(s \leq \dfrac{2 - \delta_a}{1 - \delta_p \bar{q}}\right) \& \left((q \leq \dfrac{1 - \delta_p}{2 - \delta_p})|(q > \dfrac{1 - \delta_p}{2 - \delta_p} \& s \leq \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}})\right)$,

i.e., $\dfrac{N - Q}{N}\bar{v} \leq (p_2^1)^* \leq \bar{q}\bar{v}$,

The optimal solutions are $(p_2^*, x_2^*) = ((p_2^1)^*, (x_2^1)^*)$
$= (\dfrac{(1 + \delta_p \bar{q})N - \delta_a Q}{2N}\bar{v}, \dfrac{(2 - \delta_a)Q - (1 - \delta_p \bar{q})N}{2(1 - \delta_a)q})$.

C. If $q > \dfrac{1 - \delta_p}{2 - \delta_p} \& s > \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}$, i.e., $(p_2^1)^* > \bar{q}$,

The optimal solutions are $(p_2^*, x_2^*) = (\bar{q}\bar{v}, (x_2^1)^*) = (\bar{q}\bar{v}, \dfrac{Q - (1 - \bar{q})N}{(1 - \delta_a)q})$.

ii. When $p \in (\bar{q}\bar{v}, \dfrac{N - \delta_a Q}{N}\bar{v}]$, $p > \bar{q}\bar{v}$.

$$\max_x \Pi_2(p, x) = \max_x \Pi_2^2(p, (x_2^2)^*),$$

(Because $s < \dfrac{1}{q}$, $(x_2^2)^* = \dfrac{Q - qN}{(1 - \delta_a)q} > 0$) by solving $d\Pi_2^2(p, (x_2^2)^*)/dp = 0$, we get

$$(p_2^2)^* = \dfrac{N - \delta_a Q}{(2 - \delta_p)N}\bar{v} < \dfrac{N - \delta_a Q}{N}\bar{v}.$$

Thus, we only need to compare $(p_2^2)^*$ and $\bar{q}\bar{v}$.

A. If $\left(q \leq \dfrac{1 - \delta_p}{2 - \delta_p}\right) | \left(q > \dfrac{1 - \delta_p}{2 - \delta_p} \ \& \ s \leq \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}\right)$, i.e., $(p_2^2)^* \leq \bar{q}\bar{v}$,

The optimal solutions are $(p_2^*, x_2^*) = (\bar{q}\bar{v}, (x_2^2)^*) = (\bar{q}\bar{v}, \dfrac{Q - qN}{(1 - \delta_a)q})$.

B. If $q > \dfrac{1 - \delta_p}{2 - \delta_p} \ \& \ s > \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}$, i.e., $(p_2^2)^* > \bar{q}\bar{v}$,

The optimal solutions are $(p_2^*, x_2^*) = ((p_2^2)^*, (x_2^2)^*)$
$$= (\dfrac{N - \delta_a Q}{(2 - \delta_p)N}\bar{v}, \dfrac{Q - qN}{(1 - \delta_a)q}).$$

(c) If $\bar{q}\bar{v} > \dfrac{N - \delta_a Q}{N}\bar{v}$, i.e., $s < \dfrac{\delta_a}{q}$.

Then $p < \bar{q}\bar{v}$, so

$$\max_x \Pi_2(p, x) = \max_x \Pi_2^1(p, (x_2^1)^*)$$

We already know that

$$(p_2^1)^* = \dfrac{(1 + \delta_p \bar{q})N - \delta_a Q}{2N}\bar{v}.$$

Then we discuss whether $(p_2^1)^*$ in $(\dfrac{N - Q}{N}\bar{v}, \dfrac{N - \delta_a Q}{N}\bar{v}]$ or not.

i. If $s \geq \dfrac{2 - \delta_a}{1 - \bar{q}\delta_p}$, i.e., $(p_2^1)^* \leq \dfrac{N - Q}{N}\bar{v}$,

The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - Q}{N}\bar{v}, (x_2^1)^*) = (\dfrac{N - Q}{N}\bar{v}, 0)$.

ii. If $\dfrac{\delta_a}{1 - \bar{q}\delta_p} \leq s < \dfrac{2 - \delta_a}{1 - \bar{q}\delta_p}$, i.e., $(p_2^1)^* \in (\dfrac{N - Q}{N}\bar{v}, \dfrac{N - \delta_a Q}{N}\bar{v}]$,

The optimal solutions are $(p_2^*, x_2^*) = ((p_2^1)^*, (x_2^1)^*)$

$$= (\frac{(1+\delta_p\bar{q})N - \delta_a Q}{2N}\bar{v}, \frac{(2-\delta_a)Q - (1-\delta_p\bar{q})N}{2(1-\delta_a)q}).$$

iii. If $s < \dfrac{\delta_a}{1 - \bar{q}\delta_p}$, i.e., $(p_2^1)^* > \dfrac{N - \delta_a Q}{N}\bar{v}$,

The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - \delta_a Q}{N}\bar{v}, \dfrac{Q}{q})$.

Observing the analysis in session (2), we know that these cases are mainly dependent on the variation of $s$ and $q$. The key threshold points of $s$ are $\{\dfrac{\delta_a}{1 - \delta_p\bar{q}}, \dfrac{2 - \delta_a}{1 - \delta_p\bar{q}}, \dfrac{\delta_a}{q}, \dfrac{1}{q}, \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}, \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}\}$. So we need to discuss the relationship between these thresholds to sort the cases above. Because $\delta_a, \delta_p \in (0,1)$ and $q \in (0,1]$, it's easy to get $\dfrac{\delta_a}{1 - \delta_p\bar{q}} < \dfrac{2 - \delta_a}{1 - \delta_p\bar{q}}, \dfrac{\delta_a}{q} < \dfrac{1}{q}, \dfrac{\delta_a}{1 - \delta_p\bar{q}} < \dfrac{\delta_a}{q} < \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}$. And when $q < \dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p}, \dfrac{2 - \delta_a}{1 - \delta_p\bar{q}} < \dfrac{\delta_a}{q}$; when $q < \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}, \dfrac{2 - \delta_a}{1 - \delta_p\bar{q}} < \dfrac{1}{q} < \dfrac{\delta_a}{2 - (1 - \delta_p)\bar{q}}$; while $q \geq \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}, \dfrac{\delta_a}{2 - (1 - \delta_p)\bar{q}} \leq \dfrac{1}{q} \leq \dfrac{2 - \delta_a}{1 - \delta_p\bar{q}}$.

So, adding $\{\dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p}, \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}\}$ getting from sorting key points about $s$, the key threshold points of $q$ are $\{\dfrac{1 - \delta_p}{2 - \delta_p}, \dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p}, \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}\}$. Similarly, we have $\dfrac{1 - \delta_p}{2 - \delta_p} < \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}, \dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p} < \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}$.

When $s$ and $q$ are realised as specific values, some subcases in session (2) will be eliminated. And because the thresholds of $q$ are only related to $\delta_a, \delta_p$, while those of $s$ are also related to $q$, we first discuss the varies of $q$ and then come to $s$. For the discussion of $q$, it's easy to show that $\dfrac{1 - \delta_p}{2 - \delta_p}$ doesn't have affect on the subcases, so we only need to discuss the points of $\{\dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p}, \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}\}$.

- If $q \in (0, \dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p}]$.

For $s$ we have:

$$\frac{\delta_a}{1 - \delta_p\bar{q}} < \frac{2 - \delta_a}{1 - \delta_p\bar{q}} \leq \frac{\delta_a}{q} < \frac{1}{q} < \frac{\delta_a}{1 - (2 - \delta_p)\bar{q}},$$

and $\dfrac{1}{q} > \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}$.

Thus, sessions of (2(a)i), (2(b)iB), (2(b)iC), (2(b)iiB) are eliminated.

- If $q \in (\dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p}, \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}]$.

  For $s$ we have:

  $$\frac{\delta_a}{1 - \delta_p\bar{q}} < \frac{\delta_a}{q} < \frac{2 - \delta_a}{1 - \delta_p\bar{q}} \leq \frac{1}{q} \leq \frac{\delta_a}{1 - (2 - \delta_p)\bar{q}},$$

  and also $\dfrac{1}{q} \geq \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}$.

  In this scenario, sessions of (2(a)i), (2(b)iC), (2(b)iiB), (2(c)i) are eliminated.

- If $q \in (\dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}, 1]$ For $s$ we have:

  $$\frac{\delta_a}{1 - \delta_p\bar{q}} < \frac{\delta_a}{q} < \frac{\delta_a}{1 - (2 - \delta_p)\bar{q}} < \frac{1}{q} < \frac{2 - \delta_a}{1 - \delta_p\bar{q}},$$

  and also $\dfrac{1}{q} < \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}$.

  Then sessions of (2(b)iA), (2(c)i) are eliminated.

By looking into the details of optimal solutions in these valid cases under different segments of $s$ and $q$, we know that the results under $q \in (0, \dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p}]$ and $q \in (\dfrac{\delta_a(1 - \delta_p)}{2 - \delta_a - \delta_a\delta_p}, \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}]$ are the same, so we can combine the two intervals in the summary of session (2).

In summary,

- If $q \in (0, \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}]$

  (a) $s \in (0, \dfrac{\delta_a}{1 - \delta_p\bar{q}}]$

  The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - \delta_a Q}{N}\bar{v}, \dfrac{Q}{q})$, and the correspond max revenue is $\delta_p\bar{q}Q\bar{v}$.

  (b) $s \in (\dfrac{\delta_a}{1 - \delta_p\bar{q}}, \dfrac{2 - \delta_a}{1 - \delta_p\bar{q}}]$

The optimal solutions are $(p_2^*, x_2^*)$

$$= (\frac{(1 + \delta_p \bar{q})N - \delta_a Q}{2N}\bar{v}, \frac{(2 - \delta_a)Q - (1 - \delta_p \bar{q})N}{2(1 - \delta_a)q}), \text{ and the correspond}$$

max revenue is $\dfrac{(1 - \delta_p \bar{q})^2 N^2 - 2(\delta_a + \delta_a \delta_p \bar{q} - 2\delta_p \bar{q})NQ + \delta_a^2 Q^2}{4(1 - \delta_a)N}\bar{v}.$

(c) $s \in (\dfrac{2 - \delta_a}{1 - \delta_p \bar{q}}, +\infty)$

The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - Q}{N}\bar{v}, 0)$, and the correspond

max revenue is $\dfrac{(N - Q)Q}{N}\bar{v}.$

- If $q \in (\dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}, 1]$

(a) $s \in (0, \dfrac{\delta_a}{1 - \delta_p \bar{q}}]$

The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - \delta_a Q}{N}\bar{v}, \dfrac{Q}{q})$, and the corre-

spond max revenue is $\delta_p \bar{q} Q \bar{v}.$

(b) $s \in (\dfrac{\delta_a}{1 - \delta_p \bar{q}}, \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}]$

The optimal solutions are $(p_2^*, x_2^*)$

$$= (\frac{(1 + \delta_p \bar{q})N - \delta_a Q}{2N}\bar{v}, \frac{(2 - \delta_a)Q - (1 - \delta_p \bar{q})N}{2(1 - \delta_a)q}), \text{ and the correspond}$$

max revenue is $\dfrac{(1 - \delta_p \bar{q})^2 N^2 - 2(\delta_a + \delta_a \delta_p \bar{q} - 2\delta_p \bar{q})NQ + \delta_a^2 Q^2}{4(1 - \delta_a)N}\bar{v}.$

(c) $s \in (\dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}, \dfrac{1}{q}]$

The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - \delta_a Q}{(2 - \delta_p)N}\bar{v}, \dfrac{Q - qN}{(1 - \delta_a)q})$, and the

correspond max revenue is

$\dfrac{[1 - \delta_p(2 - \delta_p)(1 - q^2)]N^2 - 2[\delta_a - \delta_p(2 - \delta_p)\bar{q}]NQ + \delta_a^2 Q^2}{2(1 - \delta_a)(2 - \delta_p)N}\bar{v}.$

(d) $s \in (\dfrac{1}{q}, \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}]$

The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N - \delta_a Q}{(2 - \delta_p)N}\bar{v}, 0)$, and the corre-

spond max revenue is

$$\frac{(1-\delta_p)^2 N^2 - 2[\delta_a - \delta_p(2-\delta_p)]NQ + [\delta_a^2 - \delta_p(2-\delta_p)]Q^2}{2(1-\delta_a)(2-\delta_p)N}\bar{v}.$$

(e) $s \in (\dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}, +\infty)$

The optimal solutions are $(p_2^*, x_2^*) = (\dfrac{N-Q}{N}\bar{v}, 0)$, and the correspond

max revenue is $\dfrac{(N-Q)Q}{N}\bar{v}$.

3. $p \in (\dfrac{N - \delta_a Q}{N}\bar{v}, \bar{v}]$

The theoretical optimal $x$ of $\Pi_3^1$, $\Pi_3^2$, and $\Pi_3^3$ are

$$(x_3^1)^* = \frac{Q}{q}, (x_3^2)^* = \frac{(\bar{q}\bar{v} - p)N + (1 - \delta_a)\bar{v}Q}{(1 - \delta_a)q\bar{v}}, (x_3^3)^* = \frac{Q}{q} - N,$$

respectively.

- If $p \le \bar{q}\bar{v}$, then

$$(x_3^1)^* \in (\frac{Q\bar{v}}{\bar{v} - p}, +\infty), (x_3^2)^* > \frac{Q\bar{v}}{\bar{v} - p}, (x_3^3)^* > \frac{Np - (N - \delta_a Q)\bar{v}}{\bar{v} - p}.$$

Thus,

$$\max_x \Pi_3(x) = \Pi_3^1((x_3^1)^*)$$

- If $\bar{q}\bar{v} < p \le (1 - \delta_a q)\bar{v}$, then

$$(x_3^1)^* < \frac{Q\bar{v}}{\bar{v} - p}, (x_3^2)^* \in (\frac{Np - (N - \delta_a Q)\bar{v}}{\bar{v} - p}, \frac{Q\bar{v}}{\bar{v} - p}], (x_3^3)^* > \frac{Np - (N - \delta_a Q)\bar{v}}{\bar{v} - p}.$$

Thus,

$$\max_x \Pi_3(x) = \Pi_3^2((x_3^2)^*)$$

- If $p > (1 - \delta_a q)\bar{v}$, then then $(x_3^1)^* < \dfrac{Q\bar{v}}{\bar{v} - p}, (x_3^2)^* < \dfrac{Q\bar{v}}{\bar{v} - p}], (x_3^3)^* <$
  $\dfrac{Np - (N - \delta_a Q)\bar{v}}{\bar{v} - p}$. Thus,

$$\max_x \Pi_3(x) = \max_x \Pi_3^3(x),$$

Note that when $s \le \dfrac{1}{q}, (x_3^3)^* \ge 0$, then $\max_x \Pi_3^3(x) = \Pi_3^3((x_3^3)^*)$; other-
wise $\max_x \Pi_3^3(x) = \Pi_3^3(0)$.

Similar to the discussion in session (2), we need to check whether $\bar{q}\bar{v}$, $(1 - \delta_a q)\bar{v}$ located in the interval $(\dfrac{N - \delta_a Q}{N}\bar{v}, \bar{v}]$. Obviously, $\bar{q}\bar{v} < (1 - \delta_a q)\bar{v}$, both $\bar{q}\bar{v}$ and $(1 - \delta_a q)\bar{v}$ are less than $\bar{v}$, so we only need to check the left-side.

(a) If $\bar{q}\bar{v} \geq \dfrac{N - \delta_a Q}{N}\bar{v}$, i.e., $s \leq \dfrac{\delta_a}{q}$

   i. When $p \in (\dfrac{N - \delta_a Q}{N}\bar{v}, \bar{q}\bar{v}]$, $p \leq \bar{q}\bar{v}$.

$$\max_x \Pi_3(p, x) = \max_x \Pi_3^1(p, (x_3^1)^*)$$

But $\Pi_3(p, x)$ doesn't depend on $p$, which means $p$ can be any value on $(\dfrac{N - \delta_a Q}{N}\bar{v}, \bar{q}\bar{v}]$. The optimal solutions are $(p_3^*, x_3^*) = (p, \dfrac{Q}{q})$, the correspond maximum revenue is $\bar{q}\delta_p Q\bar{v}$.

   ii. When $p \in (\bar{q}\bar{v}, (1 - \delta_a q)\bar{v}]$,

$$\max_x \Pi_3(p, x) = \max_x \Pi_3^2(p, (x_3^2)^*)$$

By solving $\mathrm{d}\Pi_3^2(p, (x_3^2)^*)/\mathrm{d}p = 0$, we get $(p_3^2)' = \bar{q}\bar{v}$, also noted that $\mathrm{d}^2\Pi_3^2(p, (x_3^2)^*)/\mathrm{d}p^2 > 0$, then $\Pi_3^2(p, (x_3^2)^*)$ is increasing on $(\bar{q}\bar{v}, (1 - \delta_a q)\bar{v}]$. Thus,
$$(p_3^2)' = (1 - \delta_a q)\bar{v}.$$

   iii. When $p \in ((1 - \delta_a q)\bar{v}, \bar{v}]$,

$$\max_x \Pi_3(p, x) = \max_x \Pi_3^3(p, (x_3^3)^*)$$

By solving $\mathrm{d}\Pi_3^2(p, (x_3^2)^*)/\mathrm{d}p = 0$, we get

$$(p_3^3)^* = \bar{v}.$$

The optimal solutions are $(p_3^*, x_3^*) = (\bar{v}, \dfrac{Q}{q} - N)$, the correspond maximum revenue is $\bar{q}\delta_p Q\bar{v} + \dfrac{\delta_p q^2}{2}N\bar{v}$.

It's not hard to show this revenue function is continuous both on $p$ and $x$, so we can get that the optimal solutions of the case $s < \dfrac{\delta_a}{q}$ are obtained

from session 3(a)iii, i.e., $(p_3^*, x_3^*) = (\overline{v}, \frac{Q}{q} - N)$, the correspond maximum revenue is $\overline{q}\delta_p Q\overline{v} + \frac{\delta_p q^2}{2}N\overline{v}$.

(b) If $\overline{q}\overline{v} < \frac{N - \delta_a Q}{N}\overline{v} \leq (1 - \delta_a q)\overline{v}$, i.e., $s \in (\frac{\delta_a}{q}, \frac{1}{q}]$

Things are the same as subcases in sessions (3(a)ii) and (3(a)iii). Thus, the optimal solutions also remain the same.

(c) If $(1 - \delta_a q)\overline{v} < \frac{N - \delta_a Q}{N}\overline{v}$, i.e., $s > \frac{1}{q}$

Then $p > (1 - \delta_a q)\overline{v}$. Note that in this case, $(x_3^3)^* < 0$, so

$$\max_x \Pi_3(p, x) = \max_x \Pi_3^3(p, 0)$$

By solving $d\Pi_3^2(p, (x_3^2)^*)/dp = 0$, we get

$$(p_3^3)^* = \overline{v}.$$

The optimal solutions are $(p_3^*, x_3^*) = (\overline{v}, 0)$, the correspond maximum revenue is $\frac{\delta_p(2N - Q)Q}{2N}\overline{v}$.

By analysing the cases above, we can summarise these results as

- If $s \leq \frac{1}{q}$, The optimal solutions are $(p_3^*, x_3^*) = (\overline{v}, \frac{Q}{q} - N)$, and the correspond maximum revenue is $\overline{q}\delta_p Q\overline{v} + \frac{\delta_p q^2}{2}N\overline{v}$.

- If $s > \frac{1}{q}$, The optimal solutions are $(p_3^*, x_3^*) = (\overline{v}, 0)$, and the correspond maximum revenue is $\frac{\delta_p(2N - Q)Q}{2N}\overline{v}$.

As far as now, we have got the local optimal solutions under these three different segments of $p$. And we also know that the parameters $q$ and $s$ are the key indicators when discussing different cases. Thus, the next is to compare local extremum revenue under different cases of $q$ and $s$.

**Comparing different local results**

For $\Pi_1(p, x)$ and $\Pi_2(p, x)$, because

$$\Pi_1(\frac{N - Q}{N}\overline{v}, x) = \lim_{p \to \frac{N-Q}{N}\overline{v}^+} \Pi_2(p, x),$$

and $\Pi_1(p, x)$ on $p$ is increasing on $(0, \dfrac{N - Q}{N}\overline{v}]$, so the comparing of these two segments is naturally completed when exploring the second segment, that is when the optimal price is $\dfrac{N - Q}{N}\overline{v}$, then the optimal solution will be $\Pi_1(\dfrac{N - Q}{N}\overline{v}, x)$.

Hence the most complicated part is to compare results from the second and the last segment.

Because the threshold of $s$ when $p \in (\dfrac{N - \delta_a Q}{N}\overline{v}, \overline{v}]$ is only $\dfrac{1}{q}$, which is also contained in the threshold of $s$ when $p \in (\dfrac{N - Q}{N}\overline{v}, \dfrac{N - \delta_a Q}{N}\overline{v}]$, we take the thresholds of $s$ and $q$ from $p \in (\dfrac{N - Q}{N}\overline{v}, \dfrac{N - \delta_a Q}{N}\overline{v}]$ as the criteria. (For statement convenience, we refer the second price segment as $p \in (\dfrac{N - Q}{N}\overline{v}, \dfrac{N - \delta_a Q}{N}\overline{v}]$ and the last price segment as $p \in (\dfrac{N - \delta_a Q}{N}\overline{v}, \overline{v}]$ in the following proof.)

- If $q \in (0, \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}]$

  1. $s \in (0, \dfrac{\delta_a}{1 - \delta_p \overline{q}}]$

     From the analysis above we know that $s < \dfrac{1}{q}$, so the local extremum revenue from the last price segment is $\overline{q}\delta_p Q\overline{v} + \dfrac{\delta_p q^2}{2}N\overline{v}$.

     $\Pi_{2max} = \delta_p \overline{q} Q\overline{v}$, $\Pi_{3max} = \overline{q}\delta_p Q\overline{v} + \dfrac{\delta_p q^2}{2}N\overline{v}$. So the global optimal solutions are $(p^*, x^*) = (p_3^*, x_3^*) = (\overline{v}, \dfrac{Q}{q} - N)$, and the correspond maximum revenue is $\overline{q}\delta_p Q\overline{v} + \dfrac{\delta_p q^2}{2}N\overline{v}$.

  2. $s \in (\dfrac{\delta_a}{1 - \delta_p \overline{q}}, \dfrac{2 - \delta_a}{1 - \delta_p \overline{q}}]$

     Also $s < \dfrac{1}{q}$, we compare $\dfrac{(1 - \delta_p \overline{q})^2 N^2 - 2(\delta_a + \delta_a \delta_p \overline{q} - 2\delta_p \overline{q})NQ + \delta_a^2 Q^2}{4(1 - \delta_a)N}\overline{v}$ and $\overline{q}\delta_p Q\overline{v} + \dfrac{\delta_p q^2}{2}N\overline{v}$.

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Let

$$L_1 = \frac{(1 - \delta_p\bar{q})^2 N^2 - 2(\delta_a + \delta_a\delta_p\bar{q} - 2\delta_p\bar{q})NQ + \delta_a^2 Q^2}{4(1-\delta_a)N}\bar{v} - \left(\bar{q}\delta_p Q\bar{v} + \frac{\delta_p q^2}{2}N\bar{v}\right)$$

$$= \left\{\frac{\left[(1-\delta_p\bar{q})N - \delta_a Q\right]^2}{4(1-\delta_a)N} - \frac{\delta_p q^2}{2}N\right\}\bar{v}$$

$$= \frac{\left[\left(1 - \delta_p\bar{q} - \sqrt{2(1-\delta_a)\delta_p}q\right)N - \delta_a Q\right]\left[\left(1 - \delta_p\bar{q} + \sqrt{2(1-\delta_a)\delta_p}q\right)N - \delta_a Q\right]}{4(1-\delta_a)N}\bar{v}$$

Because $s > \dfrac{\delta_a}{1 - \delta_p\bar{q}}$, so $\left(1 - \delta_p\bar{q} + \sqrt{2(1-\delta_a)\delta_p}q\right)N - \delta_a Q > 0$. Then let

$$L_1' = \left(1 - \delta_p\bar{q} - \sqrt{2(1-\delta_a)\delta_p}q\right)N - \delta_a Q$$

$$= \left[(1 - \delta_p) + (\delta_p - \sqrt{2(1-\delta_a)\delta_p})q\right]N - \delta_a Q,$$

in which $\left[(1 - \delta_p) + (\delta_p - \sqrt{2(1-\delta_a)\delta_p})q\right] > 0$. This can be simply proved by contradiction:

$\left[(1 - \delta_p) + (\delta_p - \sqrt{2(1-\delta_a)\delta_p})q\right] \leq 0$ means $(\delta_p < 2(1-\delta_a)) \& (q \geq \dfrac{1 - \delta_p}{\sqrt{2(1-\delta_a)\delta_p} - \delta_p})$. But $(\delta_p < 2(1-\delta_a))$ induces

$$\frac{1 - \delta_p}{2 - \delta_a - \delta_p} < \frac{1 - \delta_p}{\sqrt{2(1-\delta_a)\delta_p} - \delta_p},$$

and we know that $q \leq \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}$, so it is impossible to meet $q \geq \dfrac{1 - \delta_p}{\sqrt{2(1-\delta_a)\delta_p} - \delta_p}$, then $\left[(1 - \delta_p) + (\delta_p - \sqrt{2(1-\delta_a)\delta_p})q\right] \leq 0$ is not real.

If $s > \dfrac{\delta_a}{1 - \delta_p\bar{q} - \sqrt{2(1-\delta_a)\delta_p}q}$, $\Pi_{2max} > \Pi_{3max}$; otherwise $\Pi_{2max} > \Pi_{3max}$. Because $s \in (\dfrac{\delta_a}{1 - \delta_p\bar{q}}, \dfrac{2 - \delta_a}{1 - \delta_p\bar{q}}]$, and we have

$$\frac{\delta_a}{1 - \delta_p\bar{q} - \sqrt{2(1-\delta_a)\delta_p}q} > \frac{\delta_a}{1 - \delta_p\bar{q}},$$ so we need to compare

$\dfrac{\delta_a}{1 - \delta_p\bar{q} - \sqrt{2(1-\delta_a)\delta_p}q}$ and $\dfrac{2 - \delta_a}{1 - \delta_p\bar{q}}$. We have that when

$$q \le \frac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p}-2(1-\delta_a)\delta_p},$$

$$\frac{\delta_a}{1-\delta_p\bar{q}-\sqrt{2(1-\delta_a)\delta_p q}} \le \frac{2-\delta_a}{1-\delta_p\bar{q}}.$$ Also we know $q$ must in $(0,\frac{1-\delta_p}{2-\delta_a-\delta_p}]$.

So we have when $2\delta_a+\delta_p \ge 2$, $\dfrac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p}-2(1-\delta_a)\delta_p} \le \dfrac{1-\delta_p}{2-\delta_a-\delta_p}$.

In this case, we have

(a) If $2\delta_a+\delta_p \ge 2$:

    i. if $q \in (0, \dfrac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p}-2(1-\delta_a)\delta_p}]$,

        – if $s \in (\dfrac{\delta_a}{1-\delta_p\bar{q}}, \dfrac{\delta_a}{1-\delta_p\bar{q}-\sqrt{2(1-\delta_a)\delta_p q}}]$:

        $\Pi_{2max} \le \Pi_{3max}$

        – if $s \in (\dfrac{\delta_a}{1-\delta_p\bar{q}-\sqrt{2(1-\delta_a)\delta_p q}}, \dfrac{2-\delta_a}{1-\delta_p\bar{q}}]$:

        $\Pi_{2max} > \Pi_{3max}$

    ii. if $q \in (\dfrac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p}-2(1-\delta_a)\delta_p}, \dfrac{1-\delta_p}{2-\delta_a-\delta_p}]$,

        $\Pi_{2max} \le \Pi_{3max}$

(b) if $2\delta_a+\delta_p < 2$:

    – if $s \in (\dfrac{\delta_a}{1-\delta_p\bar{q}}, \dfrac{\delta_a}{1-\delta_p\bar{q}-\sqrt{2(1-\delta_a)\delta_p q}}]$:

    $\Pi_{2max} \le \Pi_{3max}$

    – if $s \in (\dfrac{\delta_a}{1-\delta_p\bar{q}-\sqrt{2(1-\delta_a)\delta_p q}}, \dfrac{2-\delta_a}{1-\delta_p\bar{q}}]$:

    $\Pi_{2max} > \Pi_{3max}$

3. $s \in (\dfrac{2-\delta_a}{1-\delta_p\bar{q}}, \dfrac{1}{q}]$

Still $s \le \dfrac{1}{q}$, then we compare $\dfrac{(N-Q)Q}{N}\bar{v}$ and $\bar{q}\delta_p Q\bar{v} + \dfrac{\delta_p q^2}{2}N\bar{v}$.

Let

$$L_2 = \frac{(N-Q)Q}{N}\bar{v} - \left(\bar{q}\delta_p Q\bar{v} + \frac{\delta_p q^2}{2}N\bar{v}\right)$$

$$= \left[-\frac{\delta_p q^2}{2}s^2 + (1-\delta_p\bar{q})s - 1\right]\frac{Q^2}{N}\bar{v}$$

Solving $L_2 = 0$ as the equation of $s$, we get the roots

$$
\begin{cases}
s_1 & = \dfrac{2}{(1-\delta_p\bar{q}) - \sqrt{(1-\delta_p\bar{q})^2 - 2\delta_p q^2}} \\[4mm]
s_2 & = \dfrac{2}{(1-\delta_p\bar{q}) + \sqrt{(1-\delta_p\bar{q})^2 - 2\delta_p q^2}}
\end{cases}
$$

Because $\dfrac{1-\delta_p\bar{q}}{\delta_p q^2}/\dfrac{1}{q} > 1$ and also $s_1 = \dfrac{(1-\delta_p\bar{q}) + \sqrt{(1-\delta_p\bar{q})^2 - 2\delta_p q^2}}{\delta_p q^2} > \dfrac{1-\delta_p\bar{q}}{\delta_p q^2}$, so $s_1 > \dfrac{1}{q}$. Denote $L_1$ as $L_1(s)$, we have

$$L_2\left(\frac{1}{q}\right) = \frac{2(1-\delta_p) - (2-\delta_p)q}{2q}$$

– When $q \geq \dfrac{2(1-\delta_p)}{2-\delta_p}$, then $L_2\left(\dfrac{1}{q}\right) \leq 0$, thus when $s \in \left(\dfrac{2-\delta_a}{1-\delta_p\bar{q}}, \dfrac{1}{q}\right]$, $L_2(s) \leq 0$. Thus, $\Pi_{2max} \leq \Pi_{3max}$.

– When $q < \dfrac{2(1-\delta_p)}{2-\delta_p}$, we have $L_2\left(\dfrac{1}{q}\right) > 0$. Then we need to further check $s_2$ and $\dfrac{2-\delta_a}{1-\delta_p\bar{q}}$. If $s_2 \leq \dfrac{2-\delta_a}{1-\delta_p\bar{q}}$, then $L_2(s) > 0$ with $s \in \left(\dfrac{2-\delta_a}{1-\delta_p\bar{q}}, \dfrac{1}{q}\right]$; if $s_2 > \dfrac{2-\delta_a}{1-\delta_p\bar{q}}$, then $L_2(s) < 0$ with $s \in \left(\dfrac{2-\delta_a}{1-\delta_p\bar{q}}, s_2\right]$, and $L_2(s) \geq 0$ with $s \in \left(s_2, \dfrac{1}{q}\right]$.

By looking into $s_2/\dfrac{2-\delta_a}{1-\delta_p\bar{q}}$,

we get when $q \leq \dfrac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p} - 2(1-\delta_a)\delta_p}$, $s_2 \leq \dfrac{2-\delta_a}{1-\delta_p\bar{q}}$;

otherwise $s_2 > \dfrac{2-\delta_a}{1-\delta_p\bar{q}}$.

Besides, if $2\delta_a + \delta_p \geq 2$, we have $\dfrac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p} - 2(1-\delta_a)\delta_p} \leq$

$\dfrac{2(1-\delta_p)}{2-\delta_p} \leq \dfrac{1-\delta_p}{2-\delta_a - \delta_p}$;

if $2\delta_a + \delta_p < 2$, $\dfrac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p} - 2(1-\delta_a)\delta_p} > \dfrac{2(1-\delta_p)}{2-\delta_p} >$

$\dfrac{1-\delta_p}{2-\delta_a - \delta_p}$.

Thus, in this case, we have

(a) $2\delta_a + \delta_p \geq 2$:

   i. if $q \in (0, \dfrac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p} - 2(1-\delta_a)\delta_p}]$:

     $\Pi_{2max} > \Pi_{3max}$

   ii. if $q \in (\dfrac{2(1-\delta_a)(1-\delta_p)}{(2-\delta_a)\sqrt{2(1-\delta_a)\delta_p} - 2(1-\delta_a)\delta_p}, \dfrac{2(1-\delta_p)}{2-\delta_p}]$:

     – if $s \in (\dfrac{2-\delta_a}{1-\delta_p\bar{q}}, \dfrac{2}{(1-\delta_p\bar{q}) + \sqrt{(1-\delta_p\bar{q})^2 - 2\delta_p q^2}}]$:

       $\Pi_{2max} \leq \Pi_{3max}$

     – if $s \in (\dfrac{2}{(1-\delta_p\bar{q}) + \sqrt{(1-\delta_p\bar{q})^2 - 2\delta_p q^2}}, \dfrac{1}{q}]$:

       $\Pi_{2max} > \Pi_{3max}$

   iii. if $q \in (\dfrac{2(1-\delta_p)}{2-\delta_p}, \dfrac{1-\delta_a}{2-\delta_a - \delta_p}]$:

     $\Pi_{2max} \leq \Pi_{3max}$

(b) $2\delta_a + \delta_p < 2$:

   $\Pi_{2max} > \Pi_{3max}$

4. $s \in (\dfrac{1}{q}, +\infty)$

Because $s > \dfrac{1}{q}$, we come to compare $\dfrac{(N-Q)Q}{N}\bar{v}$ and $\dfrac{\delta_p(2N-Q)Q}{2N}\bar{v}$.
Similarly, we have that

(a) $2\delta_a + \delta_p \geq 2$:

i. if $q \in (0, \frac{2(1-\delta_p)}{2-\delta_p}]$:

$$\Pi_{2max} > \Pi_{3max}$$

ii. if $q \in (\frac{2(1-\delta_p)}{2-\delta_p}, \frac{1-\delta_a}{2-\delta_a-\delta_p}]$:

– if $s \in (\frac{1}{q}, \frac{2-\delta_p}{2(1-\delta_p)}]$:

$$\Pi_{2max} \leq \Pi_{3max}$$

– if $s \in (\frac{2-\delta_p}{2(1-\delta_p)}, +\infty)$:

$$\Pi_{2max} > \Pi_{3max}$$

(b) $2\delta_a + \delta_p < 2$:

$$\Pi_{2max} > \Pi_{3max}$$

• If $q \in (\frac{1-\delta_p}{2-\delta_a-\delta_p}, 1]$

1. $s \in (0, \frac{\delta_a}{1-\delta_p\bar{q}}]$

Hence $s < \frac{1}{q}$, similarly we get the same conclusion as the case under $q \in (0, \frac{1-\delta_p}{2-\delta_a-\delta_p}]$.

2. $s \in (\frac{\delta_a}{1-\delta_p\bar{q}}, \frac{\delta_a}{1-(2-\delta_p)\bar{q}}]$

With the same as the case under $q \in (0, \frac{1-\delta_p}{2-\delta_a-\delta_p}]$, the comparison of $\frac{(1-\delta_p\bar{q})^2 N^2 - 2(\delta_a + \delta_a\delta_p\bar{q} - 2\delta_p\bar{q})NQ + \delta_a^2 Q^2}{4(1-\delta_a)N}\bar{v}$ and $\bar{q}\delta_p Q\bar{v} + \frac{\delta_p q^2}{2}N\bar{v}$ should be conducted. Thus, we take $L_1$ again.

$$L_1 = \frac{\left[\left(1 - \delta_p\bar{q} - \sqrt{2(1-\delta_a)\delta_p q}\right)N - \delta_a Q\right]\left[\left(1 - \delta_p\bar{q} + \sqrt{2(1-\delta_a)\delta_p q}\right)N - \delta_a Q\right]}{4(1-\delta_a)N}\bar{v}$$

But the range of $q$ and the right side of the range of $s$ are different. Because $q > \frac{1-\delta_p}{2-\delta_a-\delta_p}$, then $1 - \delta_p\bar{q} - \sqrt{2(1-\delta_a)\delta_p q}$ is not necessarily larger than 0. We rewrite $L_1$ as the function of $s$:

$$L_1(s) = \frac{\left[\left(1 - \delta_p \bar{q} - \sqrt{2(1 - \delta_a)\delta_p q}\right)s - \delta_a\right]\left[\left(1 - \delta_p \bar{q} + \sqrt{2(1 - \delta_a)\delta_p q}\right)s - \delta_a\right]}{4(1 - \delta_a)N}Q^2\bar{v}$$

By solving $L_1(s) = 0$, we get the two roots

$$\begin{cases} s_1 &= \dfrac{\delta_a}{(1 - \delta_p\bar{q}) - \sqrt{2(1 - \delta_a)\delta_p q}} \\[4mm] s_2 &= \dfrac{\delta_a}{(1 - \delta_p\bar{q}) + \sqrt{2(1 - \delta_a)\delta_p q}} \end{cases}$$

It's obvious that $0 < s_2 < \dfrac{\delta_a}{1 - \delta_p\bar{q}}$, then we need to discuss the possible cases of $s_1$:

if $s_1 < 0$, then we have $L_1(s) < 0$ with $s \in (\dfrac{\delta_a}{1 - \delta_p\bar{q}}, \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}]$;

if $s_1 > 0$, then $s_1 > s_2$ also holds.

Thus, if $s_1 > 0$, we need to further check whether $s_2 > \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}$

or not. If so, we will have $L_1(s) < 0$ with $s \in (\dfrac{\delta_a}{1 - \delta_p\bar{q}}, \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}]$;

otherwise, $L_1(s) \le 0$ with $s \in (\dfrac{\delta_a}{1 - \delta_p\bar{q}}, s_1]$ and $L_1(s) > 0$ with $s \in$

$(s_1, \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}]$.

By further looking at these thresholds when discussing these conditions, we can summarise the conclusion in this case:

(a) If $2\delta_a + \delta_p \ge 2$:

$$\Pi_{2max} \le \Pi_{3max}$$

(b) If $2\delta_a + \delta_p < 2$:

    i. if $q \in (\dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}, \dfrac{2(1 - \delta_p)}{2(1 - \delta_p) + \sqrt{2(1 - \delta_a)\delta_p}}]$

       – if $s \in (\dfrac{\delta_a}{1 - \delta_p\bar{q}}, \dfrac{\delta_a}{(1 - \delta_p\bar{q}) - \sqrt{2(1 - \delta_a)\delta_p q}})$

      $\Pi_{2max} \le \Pi_{3max}$

- if $s \in (\dfrac{\delta_a}{(1 - \delta_p\bar{q}) - \sqrt{2(1 - \delta_a)\delta_p q}}, \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}]$

$$\Pi_{2max} > \Pi_{3max}$$

ii. if $q \in (\dfrac{2(1 - \delta_p)}{2(1 - \delta_p) + \sqrt{2(1 - \delta_a)\delta_p}}, 1]$

$$\Pi_{2max} \leq \Pi_{3max}$$

3. $s \in (\dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}, \dfrac{1}{q}]$

Also because $s \leq \dfrac{1}{q}$, we compare

$\dfrac{[1 - \delta_p(2 - \delta_p)(1 - q^2)]N^2 - 2[\delta_a - \delta_p(2 - \delta_p)\bar{q}]NQ + \delta_a^2 Q^2}{2(1 - \delta_a)(2 - \delta_p)N}\bar{v}$ and $\bar{q}\delta_p Q\bar{v} +$

$\dfrac{\delta_p q^2}{2}N\bar{v}$.

Let

$$L_3 = \dfrac{[1 - \delta_p(2 - \delta_p)(1 - q^2)]N^2 - 2[\delta_a - \delta_p(2 - \delta_p)\bar{q}]NQ + \delta_a^2 Q^2}{2(1 - \delta_a)(2 - \delta_p)N}\bar{v} - \left(\bar{q}\delta_p Q\bar{v} + \dfrac{\delta_p q^2}{2}N\bar{v}\right)$$

$$= \dfrac{[1 - D(1 - \delta_a q^2)]s^2 - 2\delta_a(1 - D\bar{q})s + \delta_a^2}{2(1 - \delta_a)(2 - \delta_p)N}Q^2\bar{v}$$

in which $D = \delta_p(2 - \delta_p)$. We let

$$L_3' = [1 - D(1 - \delta_a q^2)]s^2 - 2\delta_a(1 - D\bar{q})s + \delta_a^2$$

The $\Delta$ of the equation $L_3' = 0$ is

$$\Delta(q) = 4\delta_a^2 \left\{(1 - D\bar{q})^2 - [1 - D(1 - \delta_a q^2)]\right\}$$
$$= 4\delta_a^2 \left\{(D^2 - \delta_a D)q^2 + 2D(1 - D)q - D(1 - D)\right\}$$

It can be proved that when $q \in (\dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}, 1]$, we have $\Delta(q) > 0$. Thus, we can give the roots of the equation $L_3' = 0$ as

$$\begin{cases} s_1 = \dfrac{\delta_a}{(1 - D\bar{q}) - \sqrt{(1 - D\bar{q})^2 - (1 - D + \delta_a D q^2)}} \\[4mm] s_2 = \dfrac{\delta_a}{(1 - D\bar{q}) + \sqrt{(1 - D\bar{q})^2 - (1 - D + \delta_a D q^2)}} \end{cases}$$

It's easy to know that $s_1 > s_2 > 0$. And also because $1 - D\bar{q} > 1 - (2 - \delta_p)\bar{q}$, then we have $s_2 < \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}$. So, the next is to check whether $s_1$ locates in $(\dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}, \dfrac{1}{q}]$ or not.

At first, we can get that when $q < \dfrac{2(1 - \delta_p)}{2(1 - \delta_p) + \sqrt{2(1 - \delta_a)\delta_p}}$, $s_1 < \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}$. And also when $2\delta_a + \delta_p < 2$ we have $\dfrac{1 - \delta_p}{2 - \delta_a - \delta_p} < \dfrac{2(1 - \delta_p)}{2(1 - \delta_p) + \sqrt{2(1 - \delta_a)\delta_p}}$.

Then, we have when $q < \dfrac{1 - \delta_p}{\sqrt{\delta_a(1 - \delta_a)D} + \delta_a(1 - \delta_p)}$, $s_1 < \dfrac{1}{q}$. And also when $2\delta_a + \delta_p < 2$ we have $\dfrac{1 - \delta_p}{\sqrt{\delta_a(1 - \delta_a)D} + \delta_a(1 - \delta_p)} > \dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}$; when $\delta_a > (1 - \delta_p)^2$, we have $\dfrac{1 - \delta_p}{\sqrt{\delta_a(1 - \delta_a)D} + \delta_a(1 - \delta_p)} < 1$.

Thus, the summary of this case is

(a) If $2\delta_a + \delta_p \geq 2$: i.e. $s_1 \geq \dfrac{1}{q}$

$\Pi_{2max} \leq \Pi_{3max}$

(b) If $2\delta_a + \delta_p < 2$:

    i. if $\delta_a \geq (1 - \delta_p)^2$:

        A. if $q \in (\dfrac{1 - \delta_p}{2 - \delta_a - \delta_p}, \dfrac{2(1 - \delta_p)}{2(1 - \delta_p) + \sqrt{2(1 - \delta_a)\delta_p}}]$

        $\Pi_{2max} > \Pi_{3max}$

        B. if $q \in (\dfrac{2(1 - \delta_p)}{2(1 - \delta_p) + \sqrt{2(1 - \delta_a)\delta_p}}, \dfrac{1 - \delta_p}{\sqrt{\delta_a(1 - \delta_a)D} + \delta_a(1 - \delta_p)}]$

        – if $s \in (\dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}}, \dfrac{\delta_a}{(1 - D\bar{q}) - \sqrt{(1 - D\bar{q})^2 - (1 - D + \delta_a D q^2)}}]$

        $\Pi_{2max} \leq \Pi_{3max}$

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$-$ if $s \in (\dfrac{\delta_a}{(1-D\bar{q}) - \sqrt{(1-D\bar{q})^2 - (1-D+\delta_a Dq^2)}}, 1]$

$\Pi_{2max} > \Pi_{3max}$

C. if $q \in (\dfrac{1-\delta_p}{\sqrt{\delta_a(1-\delta_a)D} + \delta_a(1-\delta_p)}, 1]$

$\Pi_{2max} \leq \Pi_{3max}$

ii. if $\delta_a < (1-\delta_p)^2$:

A. if $q \in (\dfrac{1-\delta_p}{2-\delta_a-\delta_p}, \dfrac{2(1-\delta_p)}{2(1-\delta_p) + \sqrt{2(1-\delta_a)\delta_p}}]$

$\Pi_{2max} > \Pi_{3max}$

B. if $q \in (\dfrac{2(1-\delta_p)}{2(1-\delta_p) + \sqrt{2(1-\delta_a)\delta_p}}, 1]$

$-$ if $s \in (\dfrac{\delta_a}{1-(2-\delta_p)\bar{q}},$

$\dfrac{\delta_a}{(1-D\bar{q}) - \sqrt{(1-D\bar{q})^2 - (1-D+\delta_a Dq^2)}}]$

$\Pi_{2max} \leq \Pi_{3max}$

$-$ if $s \in (\dfrac{\delta_a}{(1-D\bar{q}) - \sqrt{(1-D\bar{q})^2 - (1-D+\delta_a Dq^2)}}, 1]$

$\Pi_{2max} > \Pi_{3max}$

4. $s \in (\dfrac{1}{q}, \dfrac{2-\delta_a-\delta_p}{1-\delta_p}]$

Because $s > \dfrac{1}{q}$, we need to compare

$\dfrac{(1-\delta_p)^2 N^2 - 2[\delta_a - \delta_p(2-\delta_p)]NQ + [\delta_a^2 - \delta_p(2-\delta_p)]Q^2}{2(1-\delta_a)(2-\delta_p)N}\bar{v}$

and $\dfrac{\delta_p(2N-Q)Q}{2N}\bar{v}$.

Referring to results from Chapter 2, we have:

(a) If $2\delta_a + \delta_p \geq 2$:

$\Pi_{2max} \leq \Pi_{3max}$

(b) If $2\delta_a + \delta_p < 2$:

i. if $q \in (\dfrac{1-\delta_p}{2-\delta_a-\delta_p}, \dfrac{1-\delta_p}{\sqrt{\delta_a(1-\delta_a)D} + \delta_a(1-\delta_p)}]$

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$$\Pi_{2max} > \Pi_{3max}$$

ii. if $q \in (\dfrac{1 - \delta_p}{\sqrt{\delta_a(1 - \delta_a)D} + \delta_a(1 - \delta_p)}, 1]$

– if $s \in (\dfrac{1}{q}, \dfrac{\sqrt{\delta_a(1 - \delta_a)D} + \delta_a(1 - \delta_p)}{1 - \delta_p}]$

$$\Pi_{2max} \leq \Pi_{3max}$$

– if $s \in (\dfrac{\sqrt{\delta_a(1 - \delta_a)D} + \delta_a(1 - \delta_p)}{1 - \delta_p}, \dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}]$

$$\Pi_{2max} > \Pi_{3max}$$

5. $s \in (\dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}, +\infty)$

Because $s > \dfrac{1}{q}$, we need to compare $\dfrac{(N - Q)Q}{N}\overline{v}$ and $\dfrac{\delta_p(2N - Q)Q}{2N}\overline{v}$.

Referring to results from Chapter 2, we have:

(a) If $2\delta_a + \delta_p \geq 2$:

– if $s \in (\dfrac{2 - \delta_a - \delta_p}{1 - \delta_p}, \dfrac{2 - \delta_p}{2(1 - \delta_p)}]$

$$\Pi_{2max} \leq \Pi_{3max}$$

– if $s \in (\dfrac{2 - \delta_p}{2(1 - \delta_p)}, +\infty)$

$$\Pi_{2max} > \Pi_{3max}$$

(b) If $2\delta_a + \delta_p < 2$:

$$\Pi_{2max} > \Pi_{3max}$$

Finally, there are six different kinds of equilibrium solutions in total under different cases. We can get the theorem by re-organising the different cases of these equilibria.

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And for convenience, we let $D = \delta_p(2 - \delta_p)$ and

$$
\begin{cases}
\bar{q} &= 1 - q \\[2mm]
q_{h_1} &= \dfrac{2(1 - \delta_a)(1 - \delta_p)}{(2 - \delta_a)\sqrt{2(1 - \delta_a)\delta_p} - 2(1 - \delta_a)\delta_p} \\[4mm]
q_{h_2} &= \dfrac{2(1 - \delta_p)}{2(1 - \delta_p) + \sqrt{2(1 - \delta_a)\delta_p}} \\[4mm]
q_{h_3} &= \dfrac{1 - \delta_p}{\sqrt{\delta_a(1 - \delta_a)D} + \delta_a(1 - \delta_p)}
\end{cases}
$$

and

$$
\begin{cases}
S_1(q) &= 1/q \\[2mm]
S_2(q) &= \dfrac{\delta_a}{1 - \delta_p\bar{q}} \\[3mm]
S_3(q) &= \dfrac{2 - \delta_a}{1 - \delta_p\bar{q}} \\[3mm]
S_4(q) &= \dfrac{\delta_a}{1 - (2 - \delta_p)\bar{q}} \\[3mm]
S_5(q) &= \dfrac{\delta_a}{(1 - \delta_p\bar{q}) - \sqrt{2(1 - \delta_a)\delta_p q}} \\[3mm]
S_6(q) &= \dfrac{2}{(1 - \delta_p\bar{q}) + \sqrt{(1 - \delta_p\bar{q})^2 - 2\delta_p q^2}} \\[3mm]
S_7(q) &= \dfrac{\delta_a}{(1 - D\bar{q}) - \sqrt{(1 - D\bar{q})^2 - (1 - D + \delta_a D q^2)}}
\end{cases}
$$

$\square$

## A2.4 Proof of Proposition 3.3

*Proof.* Proof of Proposition 3.3

When $p \leq \frac{N-Q}{N}\bar{v}$, advertisers with $v > p$ cannot win an impression in RTB following equation (2.4). They choose to buy guaranteed contracts to secure impressions. Therefore, the number of contracts buyers will exceed $Q$, and all $Q$ impressions will be consumed through guaranteed contracts.

While $p \in (\frac{N-Q}{N}\bar{v}, \frac{N-\delta_a Q}{N}\bar{v}]$, if no extra advertisers exist, the original batch of advertisers with valuations in $(p, \frac{p - \delta_a \frac{N-Q}{N}\bar{v}}{1 - \delta_a}]$ will join RTB, and their bidding prices

decrease from $p$ to $\frac{N-Q}{N}\overline{v}$. Note that if advertisers' bidding prices are $\frac{N-Q}{N}\overline{v}$, they would be the last ones that can win an impression in RTB. Therefore, after the publisher introduces $M$ extra advertisers to join RTB, some advertisers with bidding prices close to $\frac{N-Q}{N}\overline{v}$ cannot win impressions anymore. Because extra advertisers will bid truthfully by their valuations since they have missed the guaranteed contracts. Then advertisers with valuations close to but less than $\frac{p-\delta_a\frac{N-Q}{N}\overline{v}}{1-\delta_a}$ may quit to seek more utilities by bidding close to $p$. On the contrary, they can get a promised $v-p$ payback if they turn back to buy a guaranteed contract. Consequently, the former threshold value $v' = \frac{p-\delta_a\frac{N-Q}{N}\overline{v}}{1-\delta_a}$ get smaller because of the existence of $M$ extra advertisers.

Denoted the new threshold as $v'_M$, then the consumers of impressions consist of four parts: a) the number of contracts buyers is $\frac{\overline{v}-v'_M}{\overline{v}}N$, b) the number of untruth-telling bidders from the original batch in RTB is $\frac{v'_M-p}{\overline{v}}N$, c) the number of truth-telling bidders from the original batch in RTB is $\frac{p-b(v'_M)}{\overline{v}}N$, d) the number of truth-telling bidders from the extra batch is $\frac{v-b(v'_M)}{\overline{v}}M$. Accordingly, we have

$$\frac{\overline{v}-v'_M}{\overline{v}}N + \frac{v'_M-p}{\overline{v}}N + \frac{p-b(v'_M)}{\overline{v}}N + \frac{v-b(v'_M)}{\overline{v}}M = Q$$

Solving the equation we get

$$\begin{cases} v'_M &= \dfrac{p-\delta_a\frac{N+M-Q}{N+M}\overline{v}}{1-\delta_a} \\ b(v'_M) &= \dfrac{N+M-Q}{N+M}\overline{v} \end{cases} \tag{A2.1}$$

If the publisher sets contract price in $(\frac{N-Q}{N}\overline{v}, \frac{N+M-Q}{N+M}\overline{v}]$, then $v'_M \le p$ holds. This means no advertisers with valuations larger than $p$ choose to join RTB. On the other hand, if the price $p > \frac{N+M-\delta_a Q}{N+M}\overline{v}$, then $v'_M > \overline{v}$. In this case, all advertisers with valuations larger than $p$ will join RTB and bid untruthfully. Finally, if the price exceeds $\overline{v}$, all advertisers go to RTB and follow the truth-telling bidding strategy.

In summary, we can get advertisers' behavioural modes under different pricing segments.

# A2.5   Proof of Corollary 3.5

*Proof.* Proof of Corollary 3.5

First, we substitute $q \to 0$ into the result of Theorem 3.2, the publisher's decisions are only depend on $\delta_a$, $\delta_p$ and $s$:

1. $s \in (1, +\infty) \cap (0, \frac{\delta_a}{1-\delta_p}]$ Max revenue:

$$\Pi_{max} = \delta_p Q \overline{v}$$

with optimal price $p^* = \overline{v}$;

2. $s \in (1, +\infty) \cap (\frac{\delta_a}{1-\delta_p}, \frac{2-\delta_a}{1-\delta_p}]$

Max revenue:

$$\Pi_{max} = \frac{(1-\delta_a)^2 N^2 - 2(\delta_a + \delta_a\delta_p - 2\delta_p)NQ + \delta_a^2 Q^2}{4(1-\delta_a)N}\overline{v}$$

with optimal price $p^* = \frac{(1+\delta_p)N - \delta_a Q}{2N}\overline{v}$;

3. $s \in (\frac{2-\delta_a}{1-\delta_p}, +\infty)$

Max revenue:

$$\Pi_{max} = \frac{(N-Q)Q}{N}\overline{v}$$

with optimal price $p^* = \frac{N-Q}{N}\overline{v}$.

Then we let $M \to +\infty$ in Theorem 3.4. The conditions of different cases are depend on $\frac{N+M-Q}{N+M}$. Because $M \to +\infty$, we have

$$\lim_{M\to+\infty} \frac{N+M-Q}{N+M} = 1.$$

Thus, no matter $2\delta_a + \delta_p \geq 2$ or $2\delta_a + \delta_p < 2$, the publisher's optimal decision is

$$p^* = \lim_{M\to+\infty} \frac{N+M-Q}{N+M}\overline{v} = \overline{v}$$

and its max revenue is

$$\Pi_{max} = \lim_{M\to+\infty} \frac{2Q(N+M-Q)(N+\delta_p M)}{2(N+M)^2}\overline{v} = \delta_p Q \overline{v}$$

The comparison is obvious that the publisher takes more advantage of revenue space with information dominance. Because the upper bound of the revenue from the RTB is $\delta_p Q \overline{v}$. Thus, if the original scarcity is high enough, the publisher can make more profit by selling some impressions to guaranteed contracts. $\square$

## A2.6  Proof of Proposition 3.6.

*Proof.* Proof of Proposition 3.6

We prove this proposition from the side of the original advertisers and the publisher, respectively.

From the original advertisers' side, if the advertiser with valuation $v = \frac{N+x-Q}{N+x}\overline{v}$ chooses to attend the RTB and loses, then the better decision for it is to buy a guaranteed contract in period 1. This means a quota of impressions that can be won by extra advertisers in RTB would be transferred to guaranteed contracts. Therefore, the publisher wastes a unit of cost to introduce an extra advertiser. Consequently, the current $x$ is not an optimal one for them. Thus, condition a) must hold for an optimal $x^*(p)$

From the publisher side, it's natural that an optimal $x^*(p)$ must maximise its revenue compared with other $x$ under current $p$.

$\square$

## A2.7  Proof of Equation (3.3)

*Proof.* Proof of Equation (3.3)

1. $p \in [0, \frac{N-Q}{N}\overline{v}]$

   All impressions will be consumed by original advertisers through guaranteed contracts. Thus, the revenue function of the publisher is

   $$\Pi_1(p, x) = Qp - cx$$

2. $p \in (\frac{N-Q}{N}\overline{v}, \frac{N-\delta_a Q}{N}\overline{v}]$

   (a) $x \in (\frac{Np-(N-Q)\overline{v}}{\overline{v}-p}, +\infty)$

   In this case, $p < \frac{N+x-Q}{N+x}\overline{v}$. Thus, original advertisers with valuations $v > p$ will all buy guaranteed contracts because they cannot win impressions in the RTB under the strategy of $b(v)$. And the left impressions will all be won by the extra advertisers. The corresponding revenue of the publisher is

   $$\Pi_2^1(p, x) = \frac{\overline{v} - p}{\overline{v}}Np + \delta_p \int_{\frac{x-(Q-\frac{\overline{v}-p}{\overline{v}}N)}{x}\overline{v}}^{\overline{v}} xv\frac{1}{\overline{v}}\,\mathrm{d}v - cx$$

   (b) $x \in [0, \frac{Np-(N-Q)\overline{v}}{\overline{v}-p}]$

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In this case, $p > \frac{N+x-Q}{N+x}\overline{v}$ and $p \le \frac{N-\delta_a Q}{N}\overline{v} < \frac{N+x-\delta_a Q}{N+x}\overline{v}$. We know that the original advertisers with valuations $v = b^{-1}(\frac{N+x-Q}{N+x}\overline{v})$ can join the RTB by bidding $\frac{N+x-Q}{N+x}\overline{v}$. Then we get the threshold value in this case is

$$v'_x = \frac{p - \delta_a \frac{N+x-Q}{N+x}\overline{v}}{1 - \delta_a}.$$

Original advertisers with valuations $v \in (v'_x, \overline{v}]$ will buy guaranteed contracts; other advertisers from the original batch will attend the RTB by bidding $b(v)$. So the revenue function in this case is

$$\Pi_2^2(p, x) = \frac{\overline{v} - v'_x}{\overline{v}} N p$$
$$+ \delta_p \left\{ \int_p^{v'_x} Nb(v)\frac{1}{\overline{v}}dv + \int_{b(v'_x)}^p Nv\frac{1}{\overline{v}}dv + \int_{b(v'_x)}^{\overline{v}} xv\frac{1}{\overline{v}}dv \right\} - cx$$

3. $p \in (\frac{N-\delta_a Q}{N}\overline{v}, \overline{v})$

(a) $x \in (\frac{Np-(N-Q)\overline{v}}{\overline{v}-p}, +\infty)$

Because $p < \frac{N+x-Q}{N+x}\overline{v}$, from the case above we get the revenue function under the current case is

$$\Pi_3^1(p, x) = \frac{\overline{v} - p}{\overline{v}} N p + \delta_p \int_{x-(Q-\frac{\overline{v}-p}{\overline{v}}N)\frac{\overline{v}}{x}}^{\overline{v}} xv\frac{1}{\overline{v}}dv - cx$$

(b) $x \in (\frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}, \frac{Np-(N-Q)\overline{v}}{\overline{v}-p}]$

Because $p \in [\frac{N+x-Q}{N+x}\overline{v}, \frac{N+x-\delta_a Q}{N+x}\overline{v})$, from the case above we get the revenue function under this case is

$$\Pi_3^2(p, x) = \frac{\overline{v} - v'_x}{\overline{v}} N p$$
$$+ \delta_p \left\{ \int_p^{v'_x} Nb(v)\frac{1}{\overline{v}}dv + \int_{b(v'_x)}^p Nv\frac{1}{\overline{v}}dv + \int_{b(v'_x)}^{\overline{v}} xv\frac{1}{\overline{v}}dv \right\} - cx$$

(c) $x \in [0, \frac{Np-(N-\delta_a Q)\overline{v}}{\overline{v}-p}]$

We know that $p \ge \frac{N+x-\delta_a Q}{N+x}\overline{v}$ such that $v'_x \ge \overline{v}$. So all original advertisers will go to the RTB with the bidding strategy $b(v)$.

$$\Pi_3^3(p, x) = \delta_p \left\{ \int_p^{\overline{v}} Nb(v)\frac{1}{\overline{v}}dv + \int_{\frac{N+x-Q}{N+x}\overline{v}}^p Nv\frac{1}{\overline{v}}dv + \int_{\frac{N+x-Q}{N+x}\overline{v}}^{\overline{v}} xv\frac{1}{\overline{v}}dv \right\} - cx$$

4. $p \in [\overline{v}, +\infty)$

   If the publisher sets the price higher than $\overline{v}$, all advertisers' behaviour remains the same as that case in which the price equals $\overline{v}$. All advertisers, no matter from the original batch or not, will join the RTB and bid truthfully.

$$\Pi_4(p, x) = \delta_p \left\{ \int_{\frac{N+x-Q}{N+x}\overline{v}}^{\overline{v}} Nv\frac{1}{\overline{v}}\mathrm{d}v + \int_{\frac{N+x-Q}{N+x}\overline{v}}^{\overline{v}} xv\frac{1}{\overline{v}}\mathrm{d}v \right\} - cx$$

   In summary, we obtain the whole revenue function on the domain of $p$ and $x$. $\qquad\square$

## A2.8 Proof of Lemma 3.7.

*Proof.* Proof of Lemma 3.7

The first-order of $\Pi_p(x)$ on this interval is

$$\frac{\mathrm{d}\Pi_p(x)}{\mathrm{d}x} = -c(1 - \delta_a)(N + x)^3 - (1 - \delta_a)\delta_p Q\overline{v}x^2 + (\delta_a p - \delta_p\overline{v})NQx$$
$$+ \delta_a NQ[Np - \delta_p(N - Q)\overline{v}],$$

and the second order is

$$\frac{\mathrm{d}^2\Pi_p(x)}{\mathrm{d}x^2} = -3c(1 - \delta_a)(N + x)^2 - 2(1 - \delta_a)\delta_p Q\overline{v}x + (\delta_a p - \delta_p\overline{v})NQ.$$

First, we have $\dfrac{\mathrm{d}\Pi_p(0)}{\mathrm{d}x} > 0$ for $p > \frac{N-Q}{N}\overline{v} > \delta_p\frac{N-Q}{N}\overline{v}$. Then, by observing $\dfrac{\mathrm{d}^2\Pi_p(x)}{\mathrm{d}x^2}$, we know that there are three possible cases of its roots: a) no real roots; b) two negative roots; c) one negative root and one positive root. No matter in which case, we can guarantee that $\dfrac{\mathrm{d}\Pi_p(x)}{\mathrm{d}x} > 0$ first and $\dfrac{\mathrm{d}\Pi_p(x)}{\mathrm{d}x} \leq 0$ then when $x$ increasing from 0. $\qquad\square$

# A3 Appendix for Study 3

## A3.1 Summary of Key notations

**Table A3.1:** Summary of Key notations

| Notation | Type | Description |
|---|---|---|
| $Q$ | Parameter | Total number of impressions |
| $\Theta, \theta$ | Parameter | Random parameter and its realisation for impression quality |
| $A, \alpha$ | Parameter | Random parameter and its realisation for matching degree |
| $i$ | Parameter | Advertiser index, $i \in \{1, 2\}$ |
| $t$ | Parameter | Impressions index, $t \in \{0, 1, 2, \ldots, Q\}$ |
| $v_{i,t}$ | Parameter | Advertiser $i$'s valuation for impression $t$ |
| $f(\cdot)$ | Function | Probability density function for impression quality |
| $g(\cdot)$ | Function | Probability density function for matching degree |
| $\mathbb{E}(\cdot)$ | Function | Expectation for an random variable |
| $\mathbb{E}_\Omega(\cdot)$ | Function | Conditional expectation for an random variable on $\Omega$ |
| $p$ | Decision Variable | Price of guaranteed contracts |
| $U$ | Decision Variable | Available number of guaranteed contracts |
| $x_i$ | Decision Variable | Advertiser $i$'s demand for guaranteed contracts |
| $U'$ | Decision Variable | Agreed number of guaranteed contracts |
| $\Theta_{GC}$ | Decision Variable | Quality criteria for guaranteed contracts |
| $\theta'$ | Decision Variable | The threshold quality in threshold type allocation policies |
| $b_i$ | Decision Variable | Bid from advertiser $i$ |

## A3.2 Proof of Proposition 4.1

*Proof.* Proof of Proposition 4.1

From equation (4.8), we know these two advertisers' utility functions are symmetry, then if one advertiser gets more utility with a distinguished number of guar-

anteed contracts, this solution would also be better for another one. Thus, if one advertiser gets optimal utility when $x_i = x^*$, then $x^*$ will also be at least one of the optimal demands. $\square$

## A3.3 Proof of Lemma 4.2.

*Proof.* Proof of Lemma 4.2

Because $\sum_{y=0}^{U'-1} h(y) + \sum_{z=U'}^{Q-1} k(z) = 1$, the publisher's revenue function can be written as

$$
\begin{aligned}
\Pi^{pub} = {} & U'p \\
& + \mathbb{E}(\min\{\alpha_1, \alpha_2\})\mathbb{E}_{RTB}(\theta)(Q - U') \\
& + \mathbb{E}(\min\{\alpha_1, \alpha_2\})(\mathbb{E}(\theta) - \mathbb{E}_{RTB}(\theta)) \sum_{z=U'}^{Q-1} k(z)(Q - z)
\end{aligned}
$$

**guaranteed contracts prior policy**   In this case, we have

$$
\frac{\partial \Pi^{pub}}{\partial \theta'} =
$$

$$
\frac{\bar{\alpha}}{6}\left\{(Q - U') - \sum_{z=U'}^{Q-1}\left[\frac{(z-1)!}{(U'-1)!(z-U')!}\left(\frac{\bar{\theta}-\theta'}{\bar{\theta}}\right)^{U'}\left(\frac{\theta'}{\bar{\theta}}\right)^{z-U'}(Q-z)\left(1+U-\frac{\bar{\theta}-\theta'}{\theta'}(z-U')\right)\right]\right\}
$$

Let $x = \dfrac{\theta'}{\bar{\theta}} \in [0, 1]$, $n = z - U'$, and

$$
f(x) = \sum_{n=0}^{Q-U-1}\left[\frac{(U+n-1)!}{(U'-1)!n!}(1-x)^{U'}x^n(Q-U-n)(1+U-(1-x)n)\right],
$$

we have
$$
\frac{\partial \Pi^{pub}}{\partial \theta'} = \frac{\bar{\alpha}}{6}\left[(Q-U') - f(x)\right].
$$

Because $f(0) = (Q - U')(1 + U) \geq (Q - U')$, $f(1) = 0$ and $f(x)$ is continuous on $[0, 1]$. Therefore, there must be an odd number of solutions of $\dfrac{\partial \Pi^{pub}}{\partial \theta'} = 0$. If the number is 1, then we need to compare $\Pi^{pub}(\theta' = 0)$ and $\Pi^{pub}(\theta' = \bar{\theta})$ to get the local maximum on $[0, \bar{\theta}]$. Otherwise, if the number of solutions for $\Pi^{pub}(\theta' = 0)$ is $m = 2k + 1(k > 0)$, then we also need to compare the value on the $k$ local max extrema[1].

---

[1]Note that we have implemented some numerical experiments to illustrate the property of $f(x)$,

**RTB prior policy** We have

$$
\frac{\partial \Pi^{pub}}{\partial \theta'} =
$$

$$
\frac{\overline{\alpha}}{6} \left\{ (Q - U') - \sum_{z=U'}^{Q-1} \left[ \frac{(z-1)!}{(U'-1)!(z-U')!} \left( \frac{\theta'}{\overline{\theta}} \right)^{U'} \left( \frac{\overline{\theta} - \theta'}{\overline{\theta}} \right)^{z-U'} (Q-z) \left( 1 + U - \frac{\theta'}{\overline{\theta} - \theta'}(z - U') \right) \right] \right\}
$$

Let $x = \frac{\theta'}{\overline{\theta}} \in [0, 1]$, $n = z - U'$, and

$$
g(x) = \sum_{n=0}^{Q-U-1} \left[ \frac{(U+n-1)!}{(U'-1)!n!} x^{U'} (1-x)^n (Q - U - n) \left( 1 + U - \frac{x}{1-x}n \right) \right],
$$

Then we have

$$
\frac{\partial \Pi^{pub}}{\partial \theta'} = \frac{\overline{\alpha}}{6} \left[ (Q - U') - g(x) \right].
$$

Similarly, because $g(0) = 0$, $g(1) = Q$ and $g(x)$ is continuous on $[0, 1]$. Therefore, there must be an odd number of solutions of $\frac{\partial \Pi^{pub}}{\partial \theta'} = 0$. If the number is 1, the local maximum is obtained at $\theta'^*$ such that $\frac{\partial \Pi^{pub}}{\partial \theta'} = 0$. Otherwise, if the number of solutions for $\Pi^{pub}(\theta' = 0)$ is $m = 2k + 1(k > 0)$, then we also need to compare the value on the $k + 1$ local max extrema[2].

In summary, no matter whether in the guaranteed contracts prior policy or in the RTB prior policy, $\Pi^{pub}$ of $\theta'$ only contains infinite extrema on $[0, \overline{\theta}]$.

$\square$

# A3.4   Proof of Lemma 4.3.

*Proof.* Proof Lemma 4.3

As advertisers' utility from one impression via the guaranteed contracts is $u_{i,t}^{GC} = \alpha_{i,t}\theta_t - p$, then we have $u_{i,t}^{GC} < 0$ always holds when $p > \overline{\alpha}\overline{\theta}$. Therefore, the pricing of guaranteed contracts over $\overline{\alpha}\overline{\theta}$ makes all advertisers take participate in the RTB. $\square$

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we find that $f(x)$ is monotonic on $[0, 1]$ in all of our tests.

[2]Numerical experiments also illustrate that $g(x)$ is monotonic on $[0, 1]$.