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THE CENTROID OF A JB*-TRIPLE SYSTEM

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The (geometrical) concept of centraliser of a Banach space is defined [3], [4] using extreme points of the dual ball. The center of C*-algebra is the set of elements which commute with all other elements of the spacé. For a C*-algebra with identity the centralizer may be identified with the center. The center of a JB*-algebra is defined by means of operator commutativity (and in fact this coincides with the usual algebraic definition of the center of a non-associtative algebra with identity). In algebras without identity the concept of center may not be very useful and instead the concept of centroid is used. In this article we show that the centroid of a JB*-algebra coincides with its centralizer (and also with its center when the algebra has an identity). See section 3.

We define in section 2 the concept of centroid for a JB*-triple system and show that it coincides with the centralizer. This gives an algebraic interpretation for the centralizer from which we can deduce the JB*-algebra result. Section 4 relates our results to known results on associative JB*-algebras and triples.

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1. Background.

Throughout we consider Banach spaces $E, X, Y, \ldots$ over the complex field and use $E', X', Y', \ldots$ to denote the dual spaces. For L a locally compact Hausdorff space we let $C_0(L)$ denote the C*-algebra of complex-valued continuous functions on L which vanish at infinity (with the understanding that L may actually be compact in which case $C_0(L) = C(L)$).

For the basic theory of JB*-triple systems we refer to [23], [24], [18], but we recall now the definition and some notation.

Definition 1.1. A JB*-*triple system* is a complex Banach space X together

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with a continuous triple product $\{\cdot,\cdot,\cdot\}: X \times X \times X \to X$ which satisfies

(1) $\{x, y, z\}$ is C-bilinear and symmetric in x and z and C-antilinear in y;

(2) the Jordan triple identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}; \tag{1.1}$$

(3) the operators

$$x \square x : X \to X : y \mapsto \{x, x, y\}$$

are Hermitian and have nonnegative spectrum (for $x \in X$);

(4) the JB*-condition

$$\|\{x, x, x\}\| = \|x\|^3.$$

The simplest examples are C*-algebras A with the triple product $\{x, y, x\} = \frac{1}{2}(xy^*z + zy^*x)$. Hilbert spaces are also JB*-triples with triple product defined in terms of the inner product $(\cdot,\cdot)$ by $\{x, y, z\} = \frac{1}{2}((x, y)z + + (z, y)x)$.

Our notation differs slightly from that of some other authors who use $\{x, y^*, x\}$ for our $\{x, y, z\}$, the "*" serving as an indication of conjugate-linearity. We use $x \square y$ to denote the operator

$$x \square y : X \to X : z \mapsto \{x, y, z\}$$

$(x, y \in X)$. An element $e \in X$ is called a *tripotent* if $\{e, e, e\} = e$. A tripotent is *minimal* if the operator $e \square e$ has one-dimensional 1-eigenspace. For any tripotent $e \in X$, we denote the Pierce projection of X onto the eigenspace $X_i(e)$ of $e \square e$ by $P_i(e)$ (for the eigenvalues $i = 0, 1/2, 1$). Recall that the 1-eigenspace $X_1(e)$ is a JB*-algebra for the Jordan product $x \circ y = \{x, e, y\}$.

A closed subspace J of a JB*-triple X is called a JB*-*ideal* if $\{a, u, v\} \in J$ and $\{u, a, v\} \in J$ for each $a \in J$, $u, v \in X$.

A JB*-triple X is called a JBW*-*triple* if X is a dual Banach space and the triple product is separately weak*-weak* continuous.

REMARK 1.2. It is shown in [11, Proposition 4] that if $X = Y'$ is a JBW*-triple, then there is a $1-1$ correspondence between extreme points of the unit ball B_Y of Y and minimal tripotents of X. The correspondence arises as follows. For each $\phi \in Y$, it is shown in [11, Proposition 2] that there exists a unique tripotent $e = e(\phi) \in X$ satisfying both

(i) $\phi = \phi \circ P_1(e)$,

and

(ii) $\phi|X_1(e)$ is a faithful normal positive functional on $X_1(e)$.

Then [11, Proposition 4], e is a minimal tripotent in X if and only if $e = e(\phi)$ for some extreme point ϕ of B_Y.

DEFINITION 1.3 ([3]). If E is a Banach space, then an operator $T: E \to E$ is called a *multiplier* if each extreme point ϕ of the ball $B_{E'}$ is an eigenvector of the transpose ${}^tT$ of T. (We denote the corresponding eigenvalue by $\lambda_T(\phi)$.)

The *centralizer* $C(E)$ of a Banach space E consists of all multipliers $T: E \to E$ with the property that there is another multiplier $S: E \to E$ satisfying $\lambda_S(\phi) = \overline{\lambda_T(\phi)}$ (for all extreme points ϕ of $B_{E'}$).

It is shown in [3], [4] that $C(E)$ is a commutative C*-algebra with identity and is therefore *-isomorphic to $C(K)$ for some compact Hausdorff space K. Such a K gives rise to the *maximal function module representation* of E (see [3]) which is a representation

$$\varrho: E \to \left(\prod_{k \in K} E_k\right)_{l_\infty}$$

of E as a subspace of an l_∞-product indexed by K. The representation ϱ is an isometry of E onto $\varrho(E)$ and for a point $x \in E$ with $\varrho(x) = (x_k)_{k \in K}$, and for $f \in C(K)$

$$\varrho(f.x) = (f(k)x_k)_{k \in K} \tag{1.2}$$

(where $f.x$ denotes the result of applying the operator in $C(E)$ corresponding to $f \in C(K) = C(E)$). We denote a (maximal) function module representation of E by $(K, (E_k)_{k \in K}, \tilde{E}, \varrho)$, where $\tilde{E} = \varrho(E)$.

Now let $E = X = Y'$ be a dual space and let $(K, (X_k)_{k \in K}, \tilde{X}, \varrho)$ be the maximal function module representation of X. If ϕ is any extreme point of B_Y, then there is an isolated point $k = k_\phi \in K$ and a corresponding M-decompositon of X

$$X = X_k \oplus_\infty M_k \tag{1.3}$$

satisfying

$$\langle \phi, x \rangle = \langle \phi, x_k \rangle \tag{1.4}$$

where we use x_k to denote the X_k-component of $x \in X$. Moreover $X_k = X_{k_\phi}$ has no (nontrivial) weak*-closed M-ideals. Specializing to the case when X is a JBW*-triple, we find that X_{k_ϕ} is a JBW*-triple which contains a minimal tripotent but has no weak*-closed JB*-ideals (other than $\{0\}$ and X_k). We refer to [7] for the proofs of these facts.

We will several times use (implicitly) the observation that if $X = X_1 \oplus_\infty X_2$

is an M-decomposition of a JB*-triple X, then

$$\{x_1+x_2, y_1+y_2, z_1+z_2\} = \{x_1, y_1, z_1\}+\{x_2, y_2, z_2\}$$

(for $x_1, y_1, z_1 \in X_1$ and $x_2, y_2, z_2 \in X_2$). (This follows easily from Cartan's uniqueness theorem.)

PROPOSITION 1.4. *If J is a closed subspace of a* JB*-*triple system X and $\{a, u, v\} \in J$ whenever $a \in J$, $u,v \in X$, then J is a* JB*-*ideal.*

PROOF. From (1.1) we have, for $a \in J$ and $b,x,y,z \in X$,

$$\{x, \{b, a, y\}, z\} = \{\{a, b, x\}, y, z\} + \{a, y, \{a, b, z\}\} - \{a, b, \{x, y, z\}\}.$$

By hypothesis, the right-hand side belongs to J. To deduce that $\{b, a, y\} \in J$ we need the simple fact that if $\alpha \in X$, then there is a sequence $(x_n)_{n=1}^{\infty}$ in X with

$$\alpha = \lim_{n\to\infty} \{x_n, \alpha, x_n\}.$$

This can be checked using the fact that the subtriple $\langle\alpha\rangle$ of X generated by α is isometric to $C_0(L)$ for some locally compact Hausdorff space L (see [17]). Thus it is sufficient to prove the result for the case $X = C_0(L)$, where we can take $(x_n)_{n=1}^{\infty}$ to be any bounded sequence satisfying $x_n(t) = 1$ for all t with $|\alpha(t)| \geqq 1/n$.

REMARK. 1.5. It is shown in [19, Theorem IV.3.5] that if $J \subset X$ is a weak*-closed subspace of a JBW*-triple X satisfying $\{a, u, u\} \in J$ for all $a \in J$, $u \in X$, then $X = J \oplus_\infty J'$ for some $J' \subset X$. This can be deduced from Proposition 1.4 by appealing to the coincidence of the M-ideals of JBW*-triple with the JBW*-ideals (see [2]) and the fact that weak*-closed M-ideals are M-summands (see [8]).

2. The centroid of a JB*-triple system.

DEFINITION 2.1. We define the *centroid* $Z(X)$ of a JB*-tripele X to be the set of all continuous linear operators $T: X \to X$ satisfying

$$T\{x, y, z\} = \{Tx, y, z\} \tag{2.1}$$

(for all $x,y,z \in X$).

We note that the identity (2.1) may be reformulated in each of the following ways:

$$T\{x, y, z\} = \{x, y, Tz\} \tag{2.2}$$

$$T \circ (x \,\square\, y) = (x \,\square\, y) \circ T \tag{2.3}$$

$$T \circ (x \,\square\, x) = (x \,\square\, x) \circ T. \tag{2.4}$$

The equivalence of (2.1) and (2.2) is clear by symmetry of the triple product, (2.3) is merely (2.2) expressed in operator notation and the equivalence of (2.4) and (2.3) follows easily by the polarization formula

$$2i(x \,\square\, y) = i(x+y) \,\square\, (x+y) - (x+iy) \,\square\, (x+iy) + (1-i)x \,\square\, x + (1-i)y \,\square\, y.$$

LEMMA 2.2. *If X is a JB*-triple system and $x \in X$, then there exist $u,v,w \in X$ (not unique) with $x = \{u, v, w\}$ and $\|x\| = \|u\|\,\|v\|\,\|w\|$.*

PROOF. As in the proof of Proposition 1.4, it is sufficient to consider the case where $X = \langle x \rangle$ = the closed subtriple of X generated by x, i.e. where $X = C_0(L)$ for some locally compact Hausdorff space L. In this situation we can write

$$x = \left\{ |x|^{1/3}, |x|^{1/3}, \frac{x}{|x|^{2/3}} \right\} = \{u, v, w\}.$$

REMARK 2.3. It now follows that if $T \in Z(X)$, X a JB*-triple system, and if M is an M-ideal in X then $T(M) \subset M$. To see this use the fact that the M-ideals of X coincide with the JB*-ideals (see [2]). Hence if $x \in M$, we have $x = \{u, v, w\}$ for some $u,v,w \in M$. Thus $T(x) = \{T(u), v, w\} \in M$.

PROPOSITION 2.4. *If X is a JB*-triple and $T \in Z(X)$, then T is a multiplier.*

PROOF. By [2], [5], [6], the double dual X'' is a JBW*-triple system, and has a triple product which extends that of X (where we consider X as a subspace of X'' in the canonical way). It follows easily from (2.1) by taking weak* limits in each variable separately that

$$''T\{x, y, z\} = \{''Tx, y, z\} \tag{2.5}$$

holds for all $x,y,z \in X''$.

Now let ϕ be an extreme point of $B_{X'}$ and let

$$\begin{aligned} X'' &= X_k'' \oplus_\infty M_k \\ &= Y \oplus_\infty Z \end{aligned} \tag{2.6}$$

be the decomposition of X'' described in (1.3). Let e denote a minimal tripotent in Y. Recall that Y contains only trivial weak*-closed JB*-ideals and $\phi(y+z) = \phi(y)$ for $y \in Y$, $z \in Z$. By (2.5) $''T \in Z(X'')$, and hence Remark 2.3 implies that $''T(Y) \subset Y$ and $''T(Z) \subset Z$. By (2.5)

$$''Te = ''T\{e, e, e\} = \{e, e, ''Te\} = (e \,\square\, e)''Te.$$

Since $e \square e$ has one-dimensional 1-eigenspace it follows that $''Te = \lambda_e e$ (for some $\lambda_e \in \mathbf{C}$). Let $W = \{y \in Y : ''Ty = \lambda_e y\}$. Since $''T$ is weak*-weak* continuous and $0 \neq e \in W$ it follows that W is a nonzero weak*-closed JB*-ideal and is hence equal to Y. Consequently, for $y \in Y$ and $z \in Z$, we have

$$\begin{aligned}\langle 'T\phi, y+z\rangle &= \langle \phi, ''Ty + ''Tz\rangle \\ &= \langle \phi, ''Ty\rangle \\ &= \lambda_e\langle \phi, y\rangle \\ &= \langle \lambda_e\phi, y+z\rangle.\end{aligned}$$

Consequently $'T\phi = \lambda_e\phi$. This completes the proof that T is a multiplier.

LEMMA 2.5. *If X is a* JB**-triple and $T \in Z(X)$, then*

$$T\{x, \{u, v, w\}, z\} = \{x, \{u, Tv, w\}, z\} \tag{2.7}$$

for all $u,v,w,x,z \in X$.

PROOF. From the Jordan triple identity (1.1) we have

$$\{x, \{b, a, y\}, z\} = \{\{a, b, x\}, y, z\} + \{x, y, \{a, b, z\}\} - \{a, b, \{x, y, z\}\}.$$

Replacing a by Ta and using (2.1) repeatedly we deduce that

$$\begin{aligned}\{x, \{b, Ta, y\}, z\} &= T(\{\{a, b, x\}, y, z\} + \{x, y, \{a, b, z\}\} - \{a, b, \{x, y, z\}\}) \\ &= T\{x, \{b, a, y\}, z\}.\end{aligned}$$

LEMMA 2.6. *If X is a* JB**-triple and $T \in Z(X)$, then there is a unique $S : X \to X$ satisfying*

$$S\{u, v, w\} = \{u, Tv, w\} \tag{2.8}$$

(*for all $u,v,w \in X$*).

Moreover S is a bounded linear operator and

$$T\{x, y, z\} = \{x, Sy, z\} \tag{2.9}$$

holds for all $x,y,z \in X$.

PROOF. We remark first that JB*-triples X have the following "cancellation property"; if $a,b \in X$ and $\{x, a, z\} = \{x, b, z\}$ holds for all $x,z \in X$ then $a = b$. To see this put $x = z = a-b$ and obtain

$$\|a-b\|^3 = \|\{a-b, a-b, a-b\}\| = \|\{x, a-b, z\}\| = 0.$$

We propose to use (2.8) as a definition of S, but to show S is well-defined by (2.8) (and Lemma 2.2) we need to show that if $\{u, v, w\} = \{\tilde{u}, \tilde{v}, \tilde{w}\}$, then $\{u, Tv, w\} = \{\tilde{u}, T\tilde{v}, \tilde{w}\}$.

From (2.7) we obtain

$$\begin{aligned}\{x, \{u, Tv, w\}, z\} &= T\{x, \{\tilde{u}, \tilde{v}, \tilde{w}\}, z\} \\ &= \{x, \{\tilde{u}, T\tilde{v}, \tilde{w}\}, z\}.\end{aligned}$$

Now the above cancellation property gives

$$\{u, Tv, w\} = \{\tilde{u}, T\tilde{v}, \tilde{w}\}.$$

We now define Sy for $y \in X$ by $Sy = \{u, Tv, w\}$ if $y = \{u, v, w\}$. Clearly we have a well-defined function $S : X \to X$. Using (2.7) we have (for $y = \{u, v, w\}$)

$$\begin{aligned}\{x, Sy, z\} &= \{x, \{u, Tv, w\}, z\} \\ &= T\{x, \{u, v, w\}, z\} \\ &= T\{x, y, z\}\end{aligned}$$

which is (2.9).

Now linearity of S follows easily from (2.9) and the cancellation property. Finally $\|S\| < \infty$ follows because we can express any $y \in X$ as $y = \{u, v, w\}$ with $\|y\| = \|u\|\,\|v\|\,\|w\|$.

LEMMA 2.7. *If X is a JB*-triple, $T \in Z(X)$ and $S : X \to X$ is the unique operator satisfying* (2.8), *then $S \in Z(X)$.*

PROOF. Replacing y by Ty in (1.1) and using (2.2) and (2.8) repeatedly we obtain

$$\begin{aligned}\{a, b, S\{x, y, z\}\} &= \{a, b, \{x, Ty, z\}\} \\ &= \{\{a, b, x\}, Ty, z\} - \{x, \{b, a, Ty\}, z\} + \{x, Ty, \{a, b, z\}\} \\ &= S(\{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}) \\ &= S\{a, b, \{x, y, z\}\}.\end{aligned}$$

By Lemma 2.2, we deduce that

$$\{a, b, Sc\} = S\{a, b, c\}$$

(for all $a, b, c \in X$) which shows that $S \in Z(X)$.

THEOREM 2.8. *For a JB*-triple X, the centroid $Z(X)$ coincides with the centralizer $C(X)$.*

PROOF. The fact that $C(X) \subset Z(X)$ follows by considering the maximal function module representation $(K, (X_k)_{k \in K}, \tilde{X}, \varrho)$ of X. The triple product on X can be calculated coordinate-by-coordinate – i.e. each X_k is a JB*-triple and

$$\varrho(\{x, y, z\})(k) = \{\varrho(x)(k), \varrho(y)(k), \varrho(z)(k)\}$$

(see [7]). By (1.2) it is now clear that the operators in $C(X)$ belong to $Z(X)$.

For the converse $Z(X) \subset C(X)$, we already have by Proposition 2.4 that if $T \in Z(X)$ then T is a multiplier. We claim that the operator S of Lemma 2.5 satisfies $\lambda_S(\phi) = \overline{\lambda_T(\phi)}$ for each extreme point ϕ of $B_{X'}$. For this fix ϕ and let e be the corresponding minimal tripotent of X''. As in (2.6), let $X'' = Y \oplus_\infty Z$. Let $e = e_Y + e_Z$, where $e_Y \in Y$, $e_Z \in Z$. Minimality of e implies that either $e_Y = 0$ or $e_Z = 0$. Since $\phi(e) = 1$, we deduce that $e_Z = 0$ and $e \in Y$. As in the proof of Proposition 2.4, $''Te = \lambda_e e$ and $\lambda_e = \lambda_T(\phi)$. Now weak*-continuity allows us to deduce from (2.8) that

$$''S\{u, v, w\} = \{u, ''Tv, w\}$$

holds for all $u,v,w \in X''$. Consequently

$$''Se = ''S\{e, e, e\} = \{e, ''Te, e\} = \overline{\lambda_e}\{e, e, e\} = \overline{\lambda_e} e.$$

Since $S \in Z(X)$, we can now deduce from the proof of Proposition 2.4 that $\lambda_S(\phi) = \overline{\lambda_e} = \overline{\lambda_T(\phi)}$, as required. This completes the proof of the theorem.

Corollary 2.9. *Let X be a JB*-triple system. Then the centralizer $C(X)$ consists of all bounded linear operators $T: X \to X$ which commute with all Hermitian operators on X.*

Proof. If $T \in C(X)$, then T commutes with all Hermitians (see [7]). Conversely if T commutes with all Hermitians, then T satisfies (2.4) because $x \square x$ is Hermitian. Hence $T \in Z(X)$.

Corollary 2.10. *Let X be a JB*-triple system. Denote by $\mathscr{L}(X)$ the algebra of all bounded operators on X and by $\mathscr{A}$ the subalgebra of $\mathscr{L}(X)$ generated by the Hermitian operators. Then $C(X)$ is the center of $\mathscr{A}$.*

Proof. By Corollary 2.9, $C(X)$ contains the centre of $\mathscr{A}$. Conversely, if we identify $C(X)$ with $C(K)$ for some compact Hausdorff space K, then every function $f \in C(K)$ may be written as $f = u + iv$ with u,v real-valued functions in $C(K)$. Since u and v correspond to Hermitian operators in $C(X)$, it follows that $C(X) \subset \mathscr{A}$. Hence Corollary 2.9 implies the result.

Corollary 2.11. *Let X be a JB*-triple system and a dual Banach space. Then the centroid of X is one-dimensional if and only if X is irreducible* (i.e. *if and only if it is not possible to express X as a l_∞ direct sum $X_1 \oplus_\infty X_2$ of two nonzero subspaces X_1, X_2*).

PROOF. Suppose the C*-algebra $Z(X) = C(X)$ is isomorphic to $C(K)$ for K a compact Hausdorff space. Then K is extremely disconnected and moreover X is irreducible if and only if K has no proper open subsets (i.e. if and only if K is a singleton) – (see [3], [7]). The result follows.

3. The case of a JB*-algebra.

DEFINITION 3.1. A JB*-*algebra* is a complex Banach space X with a (nonassociative) product $x \circ y$ and an involution $x \to x^*$ satisfying the Jordan algebra axioms

(1) $x \circ y = y \circ x$
(2) $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$

and the JB*-condition

(3) $\|\{x, x, x\}\| = \|x\|^3$

where the triple product $\{\cdot, \cdot, \cdot\}$ on X is defined in terms of the product and involution by

$$\{x, y, z\} = x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*) \tag{3.1}$$

With the triple product given by (3.1) (and the JB*-algebra norm) every JB*-algebra is a JB*-triple system. We refer to [14], [23] for background information on JB-algebras. Note that our $\{x, y, z\}$ would often be denoted $\{x, y^*, z\}$ in the literature on Jordan algebras.

For X a JB*-algebra and $x \in X$, we denote by M_x the multiplication operator $M_x: X \to X$, $M_x(y) = x \circ y$. We now define the center and centroid of a JB*-algebra. We will see that the center is a useful concept in the case of a JB*-algebra with identity but that the centroid is more appropriate in general (as observed in [16, 7.6.1]).

DEFINITION 3.2. The center of a JB*-algebra X is defined as the collection of all $a \in X$ satisfying

$$M_a M_x = M_x M_a \tag{3.2}$$

(for all $x \in X$).

In view of the commutativity of the Jordan product we note that (3.2) is equivalent to the associativity property

$$a \circ (x \circ y) = (a \circ x) \circ y \tag{3.3}$$

(all $x, y \in X$).

DEFINITION 3.3. The centroid of a JB*-algebra X is defined as the set of all bounded operators $T: X \to X$ satisfying

$$T(x \circ y) = (Tx) \circ y \tag{3.4}$$

(for all $x, y \in X$).

Clearly (3.4) is equivalent to $T(x \circ y) = x \circ (Ty)$.

PROPOSITION 3.4. *Let X be a* JB*-*algebra. Then the centroid of the* JB*-*algebra X coincides with the centroid of the* JB*-*triple X (as defined in* (2.1)).

PROOF. If $T: X \to X$ satisfies (3.4), then it is easily seen (from (3.1)) that T satisfies (2.1).

Conversely, if $T: X \to X$ satisfies (2.1), then we consider the double transpose $''T: X'' \to X''$. By [26] the Jordan product and involution on X extend to X'' to make X'' a JB*-algebra with identity e. Moreover the product on X'' is separately weak*-weak* continuous. Now, via (3.1), we have a separately weak*-weak* continuous triple product on X''. Thus it is easy to deduce from (2.1) that

$$''T\{x, y, z\} = \{''Tx, y, z\}$$

holds for all $x, y, z \in X''$ (by taking weak*-limits in each variable separately). Since $x \circ y = \{x, e, y\}$ it follows that

$$''T(x \circ y) = (''Tx) \circ y$$

holds for all $x, y \in X''$. Restricting to $x, y \in X$, we obtain (3.4), which completes the proof.

From Proposition 3.4 and Theorem 2.8 it follows instantly that the centralizer of a JB*-algebra X coincides with its centroid. For unital JB*-algebras, we can make a further identification with the center of X.

PROPOSITION 3.5. *Let X be a* JB*-*algebra with unit element e. Then an operator $T: X \to X$ belongs to the centroid if and only if $T = M_a$ for some element a in the center of X.*

PROOF. If $T: X \to X$ is in the centroid of X, then

$$T(x) = T(e \circ x) = T(e) \circ x = a \circ x = M_a(x),$$

where $a = T(e)$. Also a must be in the center of X because

$$M_a M_x(y) = M_a(x \circ y) = T(x \circ y) = x \circ Ty = M_x M_a(y).$$

Conversely, if a is the center, then (3.3) yields (3.4) for $T = M_a$ and thus M_a is in the centroid.

4. Associative JB*-algebras and triples.

In this section we characterize associative JB*-algebras and triples and relate them to the centroid and to the "commutative J*-algebras" of Friedman and Russo [9] and also to some unpublished results of T. Barton.

We recall that a *state* ϕ on a JB*-algebra X with identity is a continuous linear functional ϕ on X such that $\|\phi\| = \phi(1) = 1$. An extreme point of the set of states, $S(X)$, is called a *pure state*. The set of pure states is denoted by $P(X)$.

The *numerical radius* of $x \in X$, $\vartheta(x)$, is defined by

$$\vartheta(x) = \sup\{\|\phi(x)\| : \phi \in P(X)\}$$

and, by the Hahn-Banach theorem, this equals

$$\sup\{\|\phi(x)\| : \phi \in S(X)\}.$$

PROPOSITION 4.1. *If X is a JB*-algebra with identity, then the following are equivalent*:

(1) *X is associative (and hence $X \cong Z(X)$)*;
(2) *X is a commutative C^*-algebra*;
(3) *All pure states on X are multiplicative linear functionals.*

PROOF. (1) $\Rightarrow$ (2) by Proposition 3.5. Since the pure states on the commutative C*-algebra, $C(K)$, are point evaluations we have (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). If $x,y,z \in X$ then, for all $\phi \in P(X)$,

$$|\phi(x \circ (y \circ z) - (x \circ y) \circ z)| = |\phi(x)\phi(y)\phi(z) - \phi(x)\phi(y)\phi(z)| = 0.$$

Hence

$$\vartheta(x \circ (y \circ z) - (x \circ y) \circ z) = 0.$$

By [26, Theorem 2(iv)], $\vartheta(a) \leqq \|a\| \leqq e\vartheta(a)$ for all $a \in X$ and hence $(x \circ y) \circ z = x \circ (y \circ z)$. This completes the proof.

We now consider JB*-triple systems.

DEFINITION 4.2 ([23]). A JB*-triple system X is *associative* if

$$\{x, y, \{z, u, v\}\} = \{x, \{y, z, u\}, v\} = \{\{x, y, z\}, u, v\}$$

for all $x,y,z,u,v \in X$.

Using the Jordan triple identity one sees that X is associative if and only if $\{x, y, \{z, u, v\}\} = \{\{x, y, z\}, u, v\}$ for all x,y,z,u,v in X (see [23, p. 324]). By Definition 2.1, we thus have that X is associative if and only if $x \,\square\, y \in Z(X)$ for all $x,y \in X$ (and if $T = x \,\square\, y$ in Lemma 2.1, then $S = y \,\square\, x$). A J*-*homomorphism* (respectively J*-*anti-homomorphism*) of a JB*-triple is a linear functional $\phi \in X'$ such that $\phi\{x, y, z\} = \phi(x)(y)\phi(z)$ (respectively $-\phi(x)\phi(y)\phi(z)$) for all $x,y,z \in X$.

A compact Hausdorff space together with a continuous mapping σ: $T \times K \to K$ satisfying $\sigma(\alpha, \sigma(\beta, x)) = \sigma(\alpha\beta, x)$ and $\sigma(1, x) = x$ all $x \in K$ and

$$\alpha, \beta \in T = \{e^{i\theta} : \theta \in \mathrm{R}\}$$

is called a T_σ-space (see [2]). Let

$$C_\sigma(K) = \{f \in C(K) : f(\sigma(\alpha, x)) = \alpha f(x) \text{ for all } (\sigma, x) \in T \times K\}.$$

A Banach space X isometrically isomorphic to $C_\sigma(K)$ is called a C_σ-space.

THEOREM 4.3. *If X is a* JB*-*triple system, then the following are equivalent*:

(1) *X is associative*;
(2) *$x \square y \in Z(X)$ for all $x,y \in X$*;
(3) *X is a C_σ-space*;
(4) *All extreme points of the unit ball of X are either* J*-*homomorphisms or* J*-*anti-homomorphisms*;
(5) *If $(K, (X_k)_{k \in K}, \tilde{X}, \varrho)$ is a maximal function module representation of X, then X_k has dimension at most one for each $k \in K$*;
(6) *The centroid $Z(X)$ coincides with $J(X) = \{T_1 + iT_2 : T_1, T_2$ are Hermitian operators on $X\}$*;
(7) *For each $\phi \in X'$, the tripotent $e = e(\phi) \in X''$ corresponding to ϕ (see Remark* 1.2) *has $X_{1/2}(e) = \{0\}$*;
(8) *For each extreme point ϕ of $B_{X'}$, the Pierce space $X_{1/2}(e)$ is $\{0\}$, where $e = e(\phi) \in X''$ is the corresponding minimal tripotent.*

PROOF. We have already noted (1) $\Leftrightarrow$ (2). (1) $\Rightarrow$ (3) $\Rightarrow$ (4) is due to Friedman and Russo [9]. (4) $\Rightarrow$ (1) is proved in a fashion similar to the implication (3) $\Rightarrow$ (1) in Proposition 4.1 using the Krein-Milman theorem in place of the numerical radius.

(3) $\Rightarrow$ (5): This is known for the counterparts of C_σ-spaces over the reals (see [21]). We include a proof for the complex case.

Suppose $X = C_\sigma(K_0)$, where K_0 is a T_σ-space. We can assume that K_0 is the set of extreme points of $B_{X'}$ (possibly union $\{0\}$) in the weak*-topology and that $\sigma(\alpha, k) = \alpha k$ (see for example [9]). Hence, given any two linearly independent elements $k_1, k_2 \in K_0$ and any two scalar values $a_1, a_2 \in \mathrm{C}$, we can use the Tietze extension theorem to find $f \in C(K_0)$ satisfying $f(\alpha k_i) = \alpha a_i$ for $i = 1,2$. Using the projection $Q : C(K_0) \to C_\sigma(K_0)$ given by

$$(Qf)(k) = \int_T \alpha^{-1} f(\alpha k) d\alpha$$

($d\alpha$ denotes Haar measure on $\boldsymbol{T}$), we find that there is an $f \in C_\sigma(K_0)$ with $f(k_i) = a_i$, $(i = 1, 2)$.

Let K_1 denote the set $K_0/\boldsymbol{T}$ of $\boldsymbol{T}$-orbits in K_0, with the quotient topology. Then K_1 is a compact Hausdorff topological space and we can identify functions $g \in C(K_1)$ with the functions $g \in C(K_0)$, which satisfy $g(\alpha k) = g(k)$ for all $\alpha \in \boldsymbol{T}$, $k \in K_0$). These functions act on $C_\sigma(K_0)$ by pointwise multiplication and it is easy to see that $C_\sigma(K_0)$ is then a reduced locally $C(K_1)$-convex $C(K_1)$-module (see [13, §7]). Consequently the $C(K_1)$-action on $C_\sigma(K_0)$ corresponds to a function module representation of $C_\sigma(K_0)$ with base space K_1.

For fixed $k_0 \in K_0$ we claim that

$$N_{k_0} = \{f \in C_\sigma(K_0) : f(k_0) = 0\}$$

has codimension one in $C_\sigma(K_0)$ – unless $k_0 = 0$, when $N_{k_0} = C_\sigma(K_0)$. To see this, fix $0 \neq k_0 \in K_0$. As above, there exists $f_0 \in C_\sigma(K_0)$ with $f_0(k_0) = 1$. Now, for arbitrary $f \in C_\sigma(K_0)$ we can write

$$f = f(k_0)f_0 + (f - f(k_0)f_0),$$

which proves our claim. If now we take $f \in N_{k_0}$, then we can write

$$f = |f|^{1/2} \frac{f}{|f|^{1/2}},$$

which shows that

$$N_{k_0} = \{gf : g \in C(K_1), f \in C_\sigma(K_0), g(k_0) = 0\}.$$

Now let $(K, (X_k)_{k \in K}, \tilde{X}, \varrho)$ denote a maximal function module representation of $X = C_\sigma(K_0)$. Then there is a continuous surjection $\tau : K \to K_1$ such that

$$(g \circ \tau) \cdot f = g \cdot f$$

for $g \in C(K_1)$, and $f \in X$. Since $X_k \cong X/M_k$, where M_k is the closure of

$$\{G.f : G \in C(K), G(k) = 0, f \in X\}$$

and since $M_k \supset N_{\tau(k)}$, it follows that X_k has dimension at most one for each $k \in K$.

(5) $\Rightarrow$ (6): By [7, Proposition 25], if $T : X \to X$ is Hermitian, then there are Hermitian operators $T_k : X_k \to X_k$ so that

$$\varrho(Tx) = (T_k(x_k))_{k \in K}.$$

It follows then from (5) that all Hermitians commute and thus, from Corollary 2.9 that $J(X) \subset C(X)$. Since $Z(X) = C(X) \subset J(X)$ is true in general (see the proof of Corollary 2.9), (6) follows.

$(6) \Rightarrow (2)$: Since $x \square x$ is Hermitian (for $x \in X$), it follows from (6) and polarization that $x \square y \in J(X) = Z(X)$ for each $x, y \in X$.

We have now established the equivalence of (1)–(6).

$(3) \Rightarrow (7)$: If X is a C_σ-space, then so is X'' (in fact X'' is a commutative von Neumann algebra – see [9, Remark 2.7] for instance). Representing elements of X'' as functions on a compact Hausdorff space K, we find that $e \in X''$ is a tripotent if and only if $|e|$ has values in $\{0, 1\}$. Then $e \square e$ is multiplication by $|e|^2$ which can only have eigenvalues 0 and 1.

$(7) \Rightarrow (8)$: This is obvious.

$(8) \Rightarrow (1)$: For this part of the proof only, we let $(K, (X''_k)_{k \in K}, \tilde{X}'', \varrho)$ denote a maximal function module representation of X''. For each extreme point ϕ of $B_{X'}$ we have an isolated point $k = k_\phi \in K$ so that (as in (1.3) and (1.4))

$$X'' = X''_k \oplus_\infty M_k$$

and

$$\langle \phi, x_k + m_k \rangle = \langle \phi, x_k \rangle.$$

We also have a minimal tripotent $e = e(\phi)$ corresponding to ϕ as in Remark 1.2. As in the proof of Theorem 2.8, we can show that $e \in X''_k$.

Since $P_{1/2}(e) = 0$, it follows from [11, Lemma 1.3] that

$$X'' = X''_0(e) \oplus_\infty X''_1(e).$$

Since $e \in X''_k$ and X''_k is a weak*-closed minimal JB*-ideal, we deduce that $X''_1(e) = X''_k$, and hence that X''_k is one-dimensional.

Now let K_e denote the isolated points of K which arise from extreme points of $B_{X'}$ as above. Let K_0 denote the closure of K_e. Then K_0 is a clopen subset of K. We write

$$X''_{K_0} = \{x \in X'' : \varrho(x)(k) = 0 \quad \text{for all } k \notin K_0\}$$

and

$$P : X'' \to X''_{K_0}$$

for the M-projection of X' onto X''_{K_0}. (P is induced by multiplication by the characteristic function of K_0.)

By [7, Theorem 33, Example 37] or [12]

$$X''_{K_0} \cong \left(\prod_{k \in K_e} X''_k \right)_{l_\infty}$$

and, since each X''_k $(k \in K_e)$ is one-dimensional, X''_{K_0} is an associative JB*-triple.

If $J: X \to X''$ denotes the canonical embedding of X in its double dual, then $PJ: X \to X''_{K_0}$ is an injective isometry (see [7, Example 37]) which is easily seen to be a JB*-homomorphism. Hence X is associative. This completes the proof of the theorem.

REMARK 4.4. Banach spaces satisfying condition (5) of Theorem 4.3 are called square Banach spaces (see [3]). Condition (6) of Theorem 4.3 is equivalent to each of the following

(6)′ $J(X)$ is commutative.
(6)″ Every pair of Hermitian operators on X commutes.

J*-algebras (= JB*-subtriples of C*-algebras, also called JC*-triples) satisfying condition (7) above were called "commutative" by Friedman and Russo ([9]). (The operator they denote by G coincides with the Pierce $\frac{1}{2}$-projection $P_{1/2}$ (see [10]).)

Some of the equivalences in Theorem 4.3 are due to T. Barton (private communication).

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