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# WEAK*-CONTINUITY OF JORDAN TRIPLE PRODUCTS AND ITS APPLICATIONS

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S. Dineen [3] has shown that if $E$ is a JB*-triple, then so is its bidual $E^{**}$. We observe here that the triple product on $E^{**}$ is in fact separately $w^*$-continuous (Theorem 1.4). This result is used to show that if $E$ is a JB*-triple and a dual Banach space, then $E$ has a unique predual and the triple product an $E$ is separately $w^*$-continuous (Theorem 2.1). From this it will follow that the closed ideals of any JB*-triple are precisely its $M$-ideals, (Theorem 3.2), and that every JB*-triple has a faithful family of triple factor representations (Theorem 3.6).

We refer to Kaup [12], [13] for the definition and general theory of JB*-triples; however, we prefer to write the triple product as $\{xyz\}$ rather than $\{xy^*z\}$. Explicit use is made of the following facts. $\{\cdot,\cdot,\cdot\}$ is jointly continuous, symmetric bilinear in the outer two variables and conjugate linear in the middle variable, and for every $x \in E$ the linear operator $y \to (x \square x)(y) \equiv \{xxy\}$ is a hermitian operator on $E$. Further, $\|\{xxx\}\| = \|x\|^3$ for every $x \in E$, and $\|x \square x\| = \|x\|^2$.

Historically, JB*-triples arose in the study of bounded symmetric domains in Banach spaces and it has been shown by Kaup [12] that every such domain is biholomorphic to the unit ball of a JB*-triple. Also, the range of a contractive projection on a C*-algebra, though not usually a C*-algebra, is a JB*-triple for a suitable triple product (see [5], [19], [13], or Lemma 1.1 below). Every C*-algebra is a JB*-triple in the triple product $\{a, b\, c\} = \frac{1}{2}(ab^*c + cb^*a)$. Some of our results may be viewed as the JB*-triple analogues of similar known results relating C*-algebras to von Neumann algebras.

We refer to Lindenstrauss and Tzafriri [15] for any Banach space facts and notation used. In particular, $B_E$ is the open unit ball of the Banach space $E$, a subspace is a closed linear submanifold, the weak topology on a Banach space induced by a specific predual is denoted $w^*$, and the symbol $\cong$ means isometrically isomorphic.

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**1. On biduals of JB*-triples.**

The proof of Dineen's theorem is based on the ultrapower formulation of the principle of local reflexivity (U.P.L.R.) (see Heinrich [9], whose notation we also employ), which we state here for reference.

U.P.L.R. For each Banach space $E$ there is an ultrafilter $\mathcal{U}$, linear maps $J\colon E^{**} \to E_{\mathcal{U}}$ and $Q\colon E_{\mathcal{U}} \to E^{**}$, and a projection $P$ on $E_{\mathcal{U}}$ such that

(i) $J$ is an isometric embedding and $J|_E$ is the canonical embedding of $E$ into $E_{\mathcal{U}}$,

(ii) $Q((x_i)_{\mathcal{U}}) \equiv w^*\text{-}\lim_{\mathcal{U}} x_i$ and $Q$ has norm 1,

(iii) $QJ = \mathrm{id}_{E^{**}}$ and $QJ|_E$ is the canonical embedding of $E$ into $E^{**}$,

(iv) $P \equiv JQ$ is a norm 1 projection of $E_{\mathcal{U}}$ onto $J(E^{**})$.

Dineen's extension of the triple product on $E$ to $E^{**}$ is given by

$$\begin{aligned}\{xyz\}_{E^{**}} &\equiv Q((\{x_i y_i z_i\}_E)_{\mathcal{U}}) \\ &= Q(\{J(x), J(y), J(z)\}_{E_{\mathcal{U}}}) ,\end{aligned}$$

where $(x_i)_{\mathcal{U}} = J(x)$, $(y_i)_{\mathcal{U}} = J(y)$, and $(z_i)_{\mathcal{U}} = J(z)$.

The construction used to obtain U.P.L.R. from the principle of local reflexivity is required in the proof of Theorem 1.4, so we sketch it here. Let

$$\begin{aligned}I \equiv \{i = (M_i, N_i, \varepsilon_i) \mid\ & M_i \subseteqq E^{**} \\ & \text{and } N_i \subseteqq E^* \text{ are finite dimensional subspaces, } \varepsilon_i > 0\}\end{aligned}$$

and direct $I$ by $i > j$ if $M_i \supseteqq M_j$, $N_i \supseteqq N_j$, $\varepsilon_i < \varepsilon_j$. Let $\mathcal{U}$ be an ultrafilter on $I$ *dominating* $I$, that is, $\mathcal{U}$ contains all sets of the form $\{i \in I \mid i > j\}$, $j \in I$.

By the principle of local reflexivity, for each $i \in I$ there is a linear map $T_i\colon M_i \to E$ which is a $(1+\varepsilon_i)$-isomorphism onto its image and satisfies $T_i|_{M_i \cap E} = \mathrm{id}_{M_i \cap E}$ and $\langle T_i x, f\rangle = \langle x, f\rangle$ for all $x \in M_i$ and $f \in N_i$. For $x \in E^{**}$ define $J(x) = (x_i)_{\mathcal{U}}$, where

$$x_i = \begin{cases} T_i(x) & \text{if } x \in M_i \\ 0 & \text{otherwise} . \end{cases}$$

The properties (i)–(iv) follow readily from the properties of $(T_i)_I$ and the definitions.

The key to Theorem 1.4 is the observation that any ultrafilter appropriately refining the one defined above can be used to construct maps $J$, $P$, and $Q$ satisfying (i)–(iv). We shall return to this point after two preliminary results.

LEMMA 1.1 *Let $E$ be a* JB*-*triple and $P$ a contractive* (i.e., *norm* 1) *projection on $E$ with $V=P(E)$. Then*

(i) *$V$ is a* JB*-*triple with triple product $\{abc\}_V = P\{abc\}$ for all $a,b,c \in V$.*

*Further, if $a,b \in V$ and $x \in E$, then*

(ii) $P\{axb\} = P\{aP(x)b\}$ .

PROOF. This result is proven for J*-algebras by Friedman and Russo [5], [6]. For JB*-triples (i) is found in Kaup [13] and is also essentially contained in Stachó [19]. For $a=b$, (ii) is contained in the proof of Kaup [13, Theorem], and (ii) itself follows by polarization.

LEMMA 1.2. *Let $f: E^{**} \to E^{**}$ be a bounded linear or conjugate linear function which satisfies the following condition.*

(*) *If $(x_\alpha)_{\mathscr{A}}$ is a bounded net in $E$ converging $w^*$ to $x \in E^{**}$, then $f(x)$ is a $w^*$-accumulation point of $(f(x_\alpha))_{\mathscr{A}}$.*

*Then $f$ is $w^*$-$w^*$ continuous.*

PROOF. It is easy to check that the hypothesis implies that $w^*\text{-}\lim_\alpha f(x_\alpha) = f(x)$, whenever $(x_\alpha)_{\mathscr{A}}$ is a bounded net in $E$ converging $w^*$ to $x \in E^{**}$.

To complete the proof we need only show that for every $x^* \in E^*$ the map $g(x) = \langle x^*, f(x)\rangle$ (or $\overline{\langle x^*, f(x)\rangle}$) is $w^*$-continuous. Since $g \in E^{***}$, by the Krein–Smulian Theorem it is $w^*$-continuous iff $g(x_\alpha) \to 0$ whenever $(x_\alpha)_{\mathscr{A}}$ is a net in $B_{E^{**}}$ converging $w^*$ to $0 \in E^{**}$. Let $(x_\alpha)_{\mathscr{A}}$ be such a net, and let $N$ be any closed neighborhood of 0 in C. Let $\mathscr{B}$ be the $w^*$-neighborhood base at $0 \in E^{**}$ directed by reverse inclusion, and order $\mathscr{A} \times \mathscr{B}$ componentwise, that is, $(\alpha_1, \beta_1) > (\alpha_0, \beta_0)$ if $\alpha_1 > \alpha_0$ and $\beta_1 > \beta_0$. Since $B_E$ is $w^*$-dense in $B_{E^{**}}$, for every $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$ we can choose $y_{(\alpha,\beta)} \in (x_\alpha + \beta) \cap B_E$. One easily checks that $y_{(\alpha,\beta)} \to 0\ w^*$, and so $g(y_{(\alpha,\beta)}) \to 0$ by the first part of the proof. Thus there is $(\alpha_0, \beta_0) \in \mathscr{A} \times \mathscr{B}$ with $g(y_{(\alpha,\beta)}) \in N$ for all $(\alpha,\beta) > (\alpha_0, \beta_0)$. Fix any $\alpha > \alpha_0$. Since

$$x_\alpha = \underset{\beta \in \mathscr{B}}{w^*\text{-}\lim}\, y_{(\alpha,\beta)} ,$$

by the first part of the proof

$$g(x_\alpha) = \underset{\beta \in \mathscr{B}}{w^*\text{-}\lim}\, g(y_{(\alpha,\beta)}) .$$

Hence $g(x_\alpha) \in N$ since $N$ is closed. Thus $g(x_\alpha) \to 0$.

We focus now on the "appropriately refining ultrafilter" mentioned earlier. Let $\mathscr{A}$ and $\mathscr{B}$ be two directed sets and let $\mathscr{U}_{\mathscr{A}}$ and $\mathscr{U}_{\mathscr{B}}$ be ultrafilters dominating $\mathscr{A}$ and $\mathscr{B}$, respectively. Order $\mathscr{A} \times \mathscr{B}$ componentwise. Define the filter base $\mathscr{U}_{\mathscr{A}} \alpha \mathscr{U}_{\mathscr{B}}$ to be the collection of sets of the form

$$\bigcup_{\beta \in B} A_\beta \times \{\beta\} ,$$

where $B \in \mathscr{U}_{\mathscr{B}}$ and $A_\beta \in \mathscr{U}_{\mathscr{A}}$ for all $\beta \in B$. Let $\mathscr{U}$ be any ultrafilter containing the filter generated by $\mathscr{U}_{\mathscr{A}} \alpha \mathscr{U}_{\mathscr{B}}$. Note that this definition is not symmetric in $\mathscr{A}$ and $\mathscr{B}$, and that $\mathscr{U}$ dominates $\mathscr{A} \times \mathscr{B}$. We shall say that $\mathscr{U}$ *mutually refines* $\mathscr{U}_{\mathscr{A}}$ and $\mathscr{U}_{\mathscr{B}}$.

LEMMA 1.3. *Let $\mathscr{A}, \mathscr{B}, \mathscr{U}_{\mathscr{A}}, \mathscr{U}_{\mathscr{B}}$, and $\mathscr{U}$ be as above. Let K be any compact Hausdorff space and $(x_{(\alpha,\beta)}) \subseteqq K$. Then*

$$\lim_{\mathscr{U}} x_{(\alpha,\beta)} = \lim_{\mathscr{U}_{\mathscr{B}}} \lim_{\mathscr{U}_{\mathscr{A}}} x_{(\alpha,\beta)} .$$

PROOF. First recall that $y = \lim_{\mathscr{U}_{\mathscr{A}}} y_\alpha$ means $\{\alpha \in \mathscr{A} \mid y_\alpha \in N\} \in \mathscr{U}_{\mathscr{A}}$ for all neighborhoods $N$ of $y$, and that $y$ exists and is unique since $K$ is compact and Hausdorff.

Let

$$x = \lim_{\mathscr{U}_{\mathscr{B}}} \lim_{\mathscr{U}_{\mathscr{A}}} x_{(\alpha,\beta)}$$

and let $N$ be any neighborhood of $x$. Then

$$B \equiv \left\{ \beta \in \mathscr{B} \,\middle|\, \lim_{\mathscr{U}_{\mathscr{A}}} x_{(\alpha,\beta)} \in N \right\} \in \mathscr{U}_{\mathscr{B}} ,$$

and for each $\beta \in B$

$$A_\beta \equiv \{\alpha \in \mathscr{A} \mid x_{(\alpha,\beta)} \in N\} \in \mathscr{U}_{\mathscr{A}} ,$$

since $N$ is a neighborhood of $\lim_{\mathscr{U}_{\mathscr{A}}} x_{(\alpha,\beta)}$. Since

$$\bigcup_{\beta \in B} A_\beta \times \{\beta\} \subseteqq \{(\alpha,\beta) \in \mathscr{A} \times \mathscr{B} \mid x_{(\alpha,\beta)} \in N\}$$

and the left-hand set above is in $\mathscr{U}_{\mathscr{A}} \alpha \mathscr{U}_{\mathscr{B}} \subseteqq \mathscr{U}$, the right-hand set is also in $\mathscr{U}$.

THEOREM 1.4. *Let E be a* JB*-*triple. Then E** is a* JB*-*triple whose triple product extends that on E and is separately w*-continuous.*

PROOF. Let $I, \mathscr{U}_I, J_I, P_I$, and $Q_I$ be as in U.P.L.R. Then the triple product on $E^{**}$ obtained in Dineen's theorem is

$$(**) \qquad \{xyz\}_{E^{**}} = Q_I\{J_I(x), J_I(y), J_I(z)\}_{E_{\mathscr{U}_I}}.$$

Let $\mathscr{B}$ be the $w^*$-neighborhood base at $0 \in E^{**}$ directed by reverse inclusion, let $\mathscr{U}_{\mathscr{B}}$ be an ultrafilter dominating $\mathscr{B}$, and let $\mathscr{U}$ be an ultrafilter which mutually refines $\mathscr{U}_I$ and $\mathscr{U}_{\mathscr{B}}$ (in that order). For each $x \in E^{**}$ define $J(x) = (x_{(i,\beta)})_{\mathscr{U}}$, where

$$x_{(i,\beta)} = \begin{cases} T_i(x) & \text{if } x \in M_i \\ 0 & \text{otherwise} \end{cases},$$

where $(T_i)_I$ are as in the U.P.L.R. construction. Using this construction and Dineen's proof we obtain maps $J$, $P$, and $Q$ satisfying (i)–(iv) of U.P.L.R. and a triple product $Q\{J(x), J(y), (z)\}_{E_{\mathscr{U}}}$ on $E^{**}$. Since the triple product of a JB*-triple is uniquely determined by the norm (Kaup [12]), this triple product agrees with (**).

Fix $a, b \in E^{**}$ and define $f\colon E^{**} \to E^{**}$ by $f(x) = \{axb\}_{E^{**}}$. Note that $f$ is a bounded conjugate linear operator by norm continuity of the triple product. Let $(x_\beta)_{\mathscr{B}}$ be a bounded net in $E$ converging $w^*$ to $x \in E^{**}$. Let $\tilde{x}_{(i,\beta)} = x_\beta$ for all $i \in I$. Note that

$$\begin{aligned} Q\big((\tilde{x}_{(i,\beta)})_{\mathscr{U}}\big) &= w^*\text{-}\lim_{\mathscr{U}} \tilde{x}_{(i,\beta)} \\ &= w^*\text{-}\lim_{\mathscr{U}_{\mathscr{B}}} w^*\text{-}\lim_{\mathscr{U}_I} \tilde{x}_{(i,\beta)} \\ &= w^*\text{-}\lim_{\mathscr{U}_{\mathscr{B}}} x_\beta \\ &= x \end{aligned}$$

by Lemma 1.3. Then

$$\begin{aligned} f(x) = \{axb\}_{E^{**}} &= Q\{Ja, Jx, Jb\}_{E_{\mathscr{U}}} \\ &= Q\{Ja, JQ\big((\tilde{x}_{(i,\beta)})_{\mathscr{U}}\big), Jb\}_{E_{\mathscr{U}}} \\ &= QP\{Ja, P\big((\tilde{x}_{i,\beta)})_{\mathscr{U}}\big), Jb\}_{E_{\mathscr{U}}} \\ &= Q\{Ja, (\tilde{x}_{i,\beta)})_{\mathscr{U}}, Jb\}_{E_{\mathscr{U}}} \\ &= Q\{(a_{i,\beta)})_{\mathscr{U}}, (\tilde{x}_{(i,\beta)})_{\mathscr{U}}, (b_{(i,\beta)})_{\mathscr{U}}\}_{E_{\mathscr{U}}} \\ &= Q\big((\{a_i, x_\beta, b_i\}_E)_{\mathscr{U}}\big) \end{aligned}$$

$$= w^*\text{-}\lim_{\mathcal{U}_{\mathcal{B}}} w^*\text{-}\lim_{\mathcal{U}_I} \{a_i, x_\beta, b_i\}_E$$

$$= w^*\text{-}\lim_{\mathcal{U}_{\mathcal{B}}} Q_I\{J_I(a), J_I(x_\beta), J_I(b)\}_{E_{\mathcal{U}}}$$

$$= w^*\text{-}\lim_{\mathcal{U}_{\mathcal{B}}} \{ax_\beta b\}_{E^{**}}$$

$$= w^*\text{-}\lim_{\mathcal{U}_{\mathcal{B}}} f(x_\beta),$$

where we have used $JQ = P$, $QP = Q$, Lemma 1.1 (ii), Lemma 1.3, and the fact that $a_{(i,\beta)}$ (respectively $b_{(i,\beta)}$) does not depend upon $\beta$. Thus $f(x)$ is a $w^*$-accumulation point of $(f(x_\beta))_{\mathcal{B}}$. Since any net in $E^{**}$ which is $w^*$-convergent to $x$ has a subnet indexed by $\mathcal{B}$, Lemma 1.2 implies $f$ *is* $w^* - w^*$ continuous. The proof of $w^*$-continuity in the other variables of the triple product is furnished by the next lemma.

LEMMA 1.5. *Let $E$ be a* JB*-*triple and let $E_*$ be any predual of $E$ for which the map $x \to \{axb\}$ is $\sigma(E, E^*)$-continuous for all $a, b \in E$. Then the maps $x \to \{abx\}$ and $x \to \{xab\}$ are $\sigma(E, E_*)$-continuous for all $a, b \in E$.*

PROOF. By polarization and symmetry in the outer variables it suffices to show that $x \to D_a(x) \equiv \{aax\}$ is $\sigma(E, E_*)$-continuous for all $a \in E$. We first show that $D_a$ is "locally" $\sigma(E, E_*)$-continuous.

Fix any regular tripotent $e \in E$ and let $Q_e(x) \equiv \{exe\}$. Then $Q_e^2$ is the projection onto the Pierce 1-eigenspace belonging to $e$. Define $U_t \equiv \exp(it D_a)$ for any $t \in \mathbb{R}$, and let $\hat{e} = U_t(e)$. $U_t$ is an isometry of $E$, so $U_t$ is a J*-homomorphism and $\hat{e}$ is also a regular tripotent (recall that by Kaup and Upmeier [14], the regular tripotents are exactly the (real or complex) extreme points of the unit ball). Thus

$$x = \{eex\} \quad \text{iff } U_t(x) = \{\hat{e}\hat{e}U_t(x)\}$$

and so the image of $U_t Q_e^2$ is $Q_{\hat{e}}^2(E)$.

Denote $A = Q_e^2(E)$. $A$ is a JB*-algebra which is a $w^*$-closed subspace of $E$ since $Q_e^2$ is a $w^*$-continuous projection. Further,

$$(***) \qquad (E^*/A_\perp)^* = (A_\perp)^\perp = \bar{A}^{\sigma(E,E^*)} = A,$$

so $A$ is also a dual space. Hence $A$ is a JBW*-algebra and so has, in particular, a unique predual $A_*$ (Edwards [4]). Likewise $B = Q_{\hat{e}}^2(E)$ has a unique predual $B_*$. Since $U_t Q_e^2|_A : A \to B$ is a surjective isometry, it is $\sigma(A, A_*)$-$\sigma(B, B_*)$ continuous. By (***) the $\sigma(A, A_*)$-topology and the relative $\sigma(E, E_*)$-topology on $A$ coincide, and similarly for $B$. Thus $U_t Q_e^2$ is $\sigma(E, E_*)$-$\sigma(E, E_*)$ continuous for all $t \in \mathrm{R}$ and all regular tripotents $e$.

Now,

$$\begin{aligned} iD_a Q_e^2 &= \frac{\partial}{\partial t} \exp(it\, D_a) Q_e^2 |_{t=0} \\ &= \lim_{t \to 0} \frac{1}{t} (U_t Q_e^2 - Q_e^2) \end{aligned}$$

is a limit of $\sigma(E,E_*)$-continuous operators, the convergence being in operator norm. Thus $D_a Q_e^2$ is $\sigma(E,E_*)$-continuous, and so $D_a Q_e = D_a Q_e^2 Q_e$ is, too.

Using the main identity for Jordan triple systems we obtain

$$\begin{aligned} D_a Q_e(x) &= \{aa\{exe\}\} \\ &= 2\{\{aae\}xe\} - \{e\{aax\}e\} \,. \end{aligned}$$

Thus $Q_e D_a$ is the difference of two $\sigma(E, E_*)$-continuous operators and so is itself $\sigma(E, E^*)$-continuous.

Finally, let $f \in E_*$. It remains to show that $fDa$ is $\sigma(E,E_*)$-continuous. $f$ attains its norm on some extreme point $e$ of the unit ball of $E$, which is a regular tripotent. Thus

$$\|f\| = |\langle f, e\rangle| = |\langle fQ_e^2, e\rangle| \leqq \|f\| \|Q_e^2\| \leqq \|f\| \,,$$

and so $f = fQ_e^2$ by [7, Proposition 1]. Then

$$fD_a = fQ_e^2 D_a = fQ_e \circ Q_e D_a$$

is a composition of $\sigma(E,E_*)$-continuous operators, and the lemma is proven.

## 2. JBW*-triples.

Our goal in this section is to prove the JB*-triple analogue of a theorem of Sakai for C*-algebras.

If $E$ is a dual Banach space, then a Banach space $F$ is a *predual* of $E$ if $F^* \cong E$. $F$ is the *unique predual* of $E$ if every predual of $E$ is isometrically isomorphic to $F$, and if every surjective isometry of $E$ is $w^*$-$w^*$ continuous. Equivalently, $F$ is the unique predual of $E$ if the canonical image of any predual of $E$ in $E^*$ coincides with the canonical image of $F$ in $E^*$.

DEFINITION. A JBW*-*triple* is a JB*-triple which is a dual space possessing a predual with respect to which the triple product is separately $w^*$-continuous.

Recently, Horn [11] has shown that a JBW*-triple has a unique predual, and that a JB*-triple with a unique predual has a separarely $w^*$-continuous triple product. We shall use Theorem 1.4 to conclude that a JB*-triple which is dual Banach space is a JBW*-triple, and so obtain the following result.

THEOREM 2.1. *Let E be a* JB*-*triple which is a dual Banach space. Then E has a unique predual* $E_*$, *and the triple product on E is separately* $\sigma(E, E_*)$-*continuous.*

We begin with some definitions. Let $E$ be a JB*-triple. A *tripotent* is a non-zero element $e \in E$ with $\{eee\} = e$. Since $\|e\| = \|\{eee\}\| = \|e\|^3$, every tripotent has norm 1. In Kaup and Upmeier [14, Proposition 3.5] it is shown that every complex extreme point of $\bar{B}_E$ is a (regular) tripotent. In general, $E$ may contain no tripotents, but if $E$ is a dual space then it has many due to the Krein–Milman theorem. Two tripotents $e_1$ and $e_2$ are *orthogonal* if $e_1 \square e_2 = 0 = e_2 \square e_1$.

The following lemma is almost wholly contained in Horn [11]. We offer an elementary proof.

LEMMA 2.2. *Let E be a* JBW*-*triple and* $(e_\alpha)_{\mathscr{A}}$ *a family of pairwise orthogonal tripotents in E. Then*

(i) *for every finite subset A of* $\mathscr{A}$ *and scalars* $|\lambda_\alpha| = 1$, $\alpha \in A$, *we have* $\|\sum_{\alpha \in A} \lambda_\alpha e_\alpha\| = 1$.

(ii) $\sum_{\alpha \in \mathscr{A}} e_\alpha$ *converges in the* $w^*$-*topology to a tripotent.*

PROOF.

(i) Let $\mathscr{B}$ be the family of finite subsets of $\mathscr{A}$, and let $A \in \mathscr{B}$ and $\lambda_\alpha$, $\alpha \in A$, with $|\lambda_\alpha| = 1$ be chosen. It is easily verified that $\sum_{\alpha \in A} \lambda_\alpha e_\alpha$ is a tripotent, and thus has norm 1.

(ii) Let $f \in E^*$. By (i)

$$\|f\| \geqq \sup_{|\lambda_\alpha| = 1} \left| \langle f, \sum_{\alpha \in A} \lambda_\alpha e_\alpha \rangle \right| = \sum_{\alpha \in A} |\langle f, e_\alpha \rangle| \quad \forall A \in \mathscr{B} .$$

Thus, given $\varepsilon > 0$ there is an $A \in \mathscr{B}$ with $|\langle f, e_B - e_C \rangle| < \varepsilon$ for all $B, C \in \mathscr{B}$ with $B \cap C \supseteqq A$. Hence, $(e_A)_{\mathscr{B}}$ is weakly (and also $w^*$) Cauchy. Since $\bar{B}_E$ is $w^*$-compact, $(e_A)_{\mathscr{B}}$ converges $w^*$ to an element $e \in \bar{B}_E$.

It remains only to verify that $e$ is a tripotent. Fix $\alpha \in \mathscr{A}$. By Theorem 1.4

$$\{ee e_\alpha\} = \underset{A \in \mathscr{B}}{w^*\text{-lim}} \{ee_A e_\alpha\} = \{ee_\alpha e_\alpha\} = \underset{A \in \mathscr{B}}{w^*\text{-lim}} \{e_A e_\alpha e_\alpha\} = e_\alpha .$$

Consequently, $\{eee_A\} = e_A$ for all $A \in \mathscr{B}$. Hence,

$$\{eee\} = \underset{A \in \mathscr{B}}{w^*\text{-lim}} \{eee_A\} = \underset{\mathscr{B}}{w^*\text{-lim}}\, e_A = e .$$

We recall two further results from Horn [11].

LEMMA 2.3. *Let $E$ be a* JBW*-*triple and let $E_*$ be a predual of $E$ such that the triple product in $E$ is separately $\sigma(E, E_*)$-continuous. Then $f \in E^* \setminus E_*$ iff there is an orthogonal family of tripotents $(e_\alpha)_{\mathscr{A}}$ in $E$ with $f(\sum_{\mathscr{A}} e_\alpha) \neq \sum_{\mathscr{A}} f(e_\alpha)$.*

LEMMA 2.4. *Let $E$ be a* JB*-*triple which is dual Banach space. If $E$ has a unique predual, then the triple product in $E$ is separately $w^*$-continuous.*

Finally, we summarize some of the results in Godefroy [8] relevant to our purpose. In his paper, Godefroy studies the property of being the unique predual of a dual space by means of a property he terms *well-framed* ("bien-encadre").

LEMMA 2.5. *Let $E$ be a Banach space.*

(i) *If $E$ is well-framed, then $E$ is the unique predual of $E^*$.*
(ii) *Suppose that for each $x \in E^{**} \setminus E$ there is a subspace $F$ of $E^*$ which does not contain a subspace isomorphic to $l_1$ such that $x|_{B_F}$ is not $w^*$-continuous. Then $E$ is well-framed.*
(iii) *If $E$ is well-framed, then so is any subspace of $E$.*

PROOF. (i) is part of Théorème 15 of [8]. By [17], if $y \in B_{F^{**}}$, then $y|_L$ has a point of $\sigma(F^*, F)$-continuity for each $\sigma(F^*, F)$-compact subset $L$ of $F^* = E^{**}/F^\perp$. It follows that if $M$ is a $\sigma(E^{**}, E^*)$-compact subset of $E^{**}$, then $y|_M$ has a point of $\sigma(E^{**}, E^*)$-continuity. Thus $B_F$ is a "*-admissible" subset of $E^*$ [8, Definition 13]. Now (ii) follows by Proposition 17 of [8]. Finally, (iii) is part of Théorème 16 of [8].

PROOF OF THEOREM 2.1. Let $E$ be a JB*-triple and a dual space. Let $E_*$ be any predual of $E$. By Theorem 1.4, $E^{**}$ is a JBW*-triple (with respect to the $w^*$-topology on $E^{**}$ induced by $E^*$). Let $f \in E^{***} \setminus E^*$. By Lemma 2.3 there is an orthogonal family of tripotents $(e_\alpha)_{\mathscr{A}}$ in $E^{**}$ with $f(e) \neq \lim_{A \in \mathscr{B}} f(e_A)$, where the notation is that used in the proof of Lemma 2.2.

Let $K$ be the closed unit ball of the norm closed span, $\overline{\text{span}}\{e, e_\alpha \mid \alpha \in \mathscr{A}\}$. $K$ is a bounded subset of $E^{**}$ and $f|_K$ is not $w^*$-continuous.

We claim that $K$ contains no $l_1$-sequences. Assuming this, then Lemma 2.5 implies in turn that $E^*$ is well-framed, $E_*$ is well-framed, and thus $E_*$ is the unique predual of $E$. Applying Lemma 2.4 completes the proof.

To prove the claim we proceed as follows. Let $A \in \mathscr{B}$ and let $(\lambda_\alpha)_{\alpha \in A}$ satisfy $\max_A |\lambda_\alpha| = 1$. Choose $\alpha_0 \in A$ with $|\lambda_{\alpha_0}| = 1$. Then

$$1 = \|\lambda_{\alpha_0} e_{\alpha_0}\| = \left\|\left\{e_{\alpha_0}, e_{\alpha_0}, \sum_A \lambda_\alpha e_\alpha\right\}\right\| \leqq \left\|\sum_A \lambda_\alpha e_\alpha\right\|,$$

since

$$\|e_{\alpha_0} \square e_{\alpha_0}\| = \|e_{\alpha_0}\|^2 = 1 .$$

Also, since every element of the unit ball of $l_\infty^n$ ($= \mathbf{C}^n$ with the max norm) can be expressed as a convex combination of extreme points (i.e., points with $|\lambda_1| = \ldots = |\lambda_n| = 1$), Lemma 2.2 (i) implies $\|\sum_A \lambda_\alpha e_\alpha\| \leqq 1$.

We have shown that

$$\left\|\sum_A \lambda_\alpha e_\alpha\right\| = \max_{\alpha \in A} |\lambda_\alpha| \quad \forall A \in \mathscr{B} .$$

It follows that the closed linear span of any countable subset of $(e_\alpha)_{\mathscr{A}}$ is isometrically isomorphic to $c_0$.

Suppose $K$ contains an $l_1$-sequence. Then there is a countable subset of $\{e, e_\alpha \mid \alpha \in \mathscr{A}\}$ whose closed linear span $F$ contains a subspace isomorphic to $l_1$. But $F$ is isomorphic to

$$\mathbf{C}e \oplus_\infty \overline{\operatorname{span}}\{e_n \mid e_n \in (e_\alpha)_{\mathscr{A}} \ \forall n\} \cong \mathbf{C}e \oplus_\infty c_0 \cong c_0 ,$$

and $c_0$ contains no subspace isomorphic to $l_1$. This contradiction concludes the proof of the claim.

## 3. Ideals and factor representations of a JB*-triple.

The results in this section were motivated by the articles of Smith and Ward [18] and Paya et al. [16], where the $M$-ideal structure in Banach algebras and non-commutative JB*-algebras, respectively, was investigated.

Throughout this section, the $\sigma(E^{**}, E^*)$-closure of a set $A \subset E$ will be denoted by $\tilde{A}$, and the unique predual of a JBW*-triple $E$ will be written $E_*$. The triple product in the JB*-triple $E$ and in $E^{**}$ will be written without distinguishing affixes where no confusion seems likely to occur.

A contractive projection $P$ on a Banach space $E$ is an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\| \quad \text{for all } x \in E ,$$

and the image of an $L$-projection is called an *L-summand* of $E$. An *M-ideal* of $E$ is a subspace $M$ of $E$ such that $M^{\perp}$ is an $L$-summand in $E^*$. A contractive projection $P$ on $E$ is an *M-projection* if

$$\|x\| = \max(\|Px\|, \|x - Px\|) \quad \text{for all } x \in E ,$$

and the image of an $M$-projection is called an *M-summand*. We make explicit use of the following facts. If $T$ is a hermitian operator on $E$ and $M$ is an $M$-ideal in $E$, then $T(M) \subseteq M$ (Paya et. al. [16]). An $M$-projection on a dual space is the adjoint of an $L$-projection (Cunningham et. al. [2]). Also, a straightforward verification shows that an $M$-projection is a hermitian operator. We refer to Alfsen and Effros [1] and Hirsberg [10] for further background on $M$-ideals.

A *closed ideal* in a JB*-triple $E$ is a complex subspace $F$ satisfying $\{xyF\} \subseteq F$ and $\{xFy\} \subseteq F$ for all $x, y \in E$. Observe that by polarization it is enough to take $x = y$ in this definition. We shall make frequent use of the following result.

LEMMA 3.1. (Horn [11]). *Let $E$ be a* JBW*-*triple and $J \subseteq E$ a $w^*$-closed ideal. Then $J$ is an $M$-summand of $E$.*

THEOREM 3.2. *The closed ideals in a* JB*-*triple $E$ are precisely the $M$-ideals of $E$.*

PROOF. Let $M$ be an $M$-ideal in $E$. Since $x \square x$ is a hermitian operator, $\{xxM\} \subseteq M$ for every $x \in E$.

$\tilde{M} = M^{\perp\perp}$ is an $M$-summand in $E^{**}$, so there is an $M$-projection $P: E^{**} \to \tilde{M}$. Fix $x \in E$ and $a \in M$. Write $\{xax\} = y + z$, where $y \in \tilde{M}$ and $P(z) = 0$. Using Kaup [12, Proposition 5.5] and the fact that $P$ is hermitian, we obtain

$$\begin{aligned} y = P\{xax\} &= 2\{P(x)ax\} - \{xax\} \\ &= 2\{P(x)ax\} - (y + z) . \end{aligned}$$

Thus $z = 2\{P(x)ax\} - 2y \in \tilde{M}$, since $\{P(x)ax\} \in \tilde{M}$ by the first part of the proof (and polarization). Consequently $z = 0$. Hence $\{xax\} \in \tilde{M} \cap E = M$, so $M$ is an ideal in $E$.

Conversely, let $J$ be a closed ideal in $E$. Then $\tilde{J}$ is a $w^*$-closed ideal in $E^{**}$. By Lemma 3.1, $\tilde{J}$ is an $M$-summand of $E^{**}$. Let $P: E^{**} \to \tilde{J}$ be an $M$-projection. $P$ is the adjoint of an $L$-projection $Q: E^* \to F$, where $E^* = J^{\perp} \oplus_1 F$. Thus $J$ is an $M$-ideal.

REMARK. If $E$ is a C*-algebra, it follows immediately from Theorem 3.2, [16], and [18] that the closed two-sided ideals (with respect to the associative product), the closed Jordan ideals (with respect to the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$), and the closed JB*-triple ideals (with respect to the triple product $\{abc\} = \frac{1}{2}(ab^*c + cb^*a)$) all agree. If $E$ is a JB*-algebra, then the closed Jordan ideals concide with the closed ideals with respect to the triple product $\{abc\} = (ab^*)c - (ca)b^* + (b^*c)a$.

DEFINITION (Horn [11]). A JBW*-triple $E$ is a *triple factor* if it is irreducible, i.e., not isometric to an $l_\infty$-sum of JB*-triples.

In [11] it is also shown that $E$ is a triple factor iff it has no proper $w^*$-closed ideals.

The next series of results will be used to explore *triple factor representations* of a JB*-triple $E$, i.e., linear homomorphisms preserving the triple product which take $E$ onto a $w^*$-dense subset of a triple factor.

LEMMA 3.3. *Let $E$ be a* JB**-triple and let $F$ be a subspace of $E^*$ such that $D_x^*(F) \subseteqq F$ and $Q_x^*(F) \subseteqq F$ for all $x \in E$, where $D_x(y) \equiv \{xxy\}$ and $Q_x(y) \equiv \{xyx\}$. Then $F^\perp$ is a $w^*$-closed ideal in $E^{**}$.*

PROOF. Let $x \in E$. Since $\langle D_x^{**}(y), x^* \rangle = \langle \{xxy\}_{E^{**}}, x^* \rangle$ for all $y \in E$ and $x^* \in E^*$, the $w^*$-continuity of $\{xx\cdot\}_{E^{**}}$ yields $D_x^{**} = \{xx\cdot\}_{E^{**}}$. Let $f \in F$ and $a \in F^\perp$. Then

$$\langle D_x^{**}(a), f \rangle = \langle a, D_x^*(f) \rangle = 0 ,$$

since $D_x^*(f) \in F$, and so $\{xxa\} \in F^\perp$. By polarization, $\{xya\} \in F^\perp$ for all $x, y \in E$. Using $w^*$-continuity and $w^*$-closure of $F^\perp$ it follows that $\{xya\} \in F^\perp \ \forall x \in E^{**}$, $y \in E$, and in turn that $\{xya\} \in F^\perp \ \forall x, y \in E^{**}$. Thus $\{xyF^\perp\} \subseteqq F^\perp$ for all $x, y \in E^{**}$.

The demonstration of $\{xF^\perp y\} \subseteqq F^\perp \, \forall x,y \in E^{**}$ is similar.

PROPOSITION 3.4. *If $E$ is a* JBW**-triple, then $E_*$ is an $L$-summand in $E^*$.*

PROOF. Consider $E_*$ as a subspace of $E^*$. Fix $x \in E$. $D_x$ is $w^*$-continuous since $E$ is a JBW*-triple, and so $D_x^*(E_*) \subseteqq E_*$. Similarly $Q_x^*(E_*) \subseteqq E_*$. By Lemma 3.3 and Lemma 3.1, $E_*^\perp$ is an $M$-summand in $E^{**}$. As in the proof of Theorem 3.2, it follows that $E_*$ is an $L$-summand in $E^*$.

In the next two arguments it will be necessary to distinguish between an element $x \in E$ and its canonical image in $E^{**}$. $j$ will denote the canonical embedding of any Banach space into its bidual. We will also employ the following notation. If $x \in E^*$, then $M_x$ is the largest $M$-ideal in $E$ contained in Ker $(x)$. In case $x$ is an extreme point of $\bar{B}_{E^*}$ (written ext $(E^*)$), then $M_x$ is called a *primitive M-ideal.*

LEMMA 3.5. *Let $E$ be a* JB*-*triple and $x \in$ ext $(E^*)$. Then $M_{j(x)}$ is a $w^*$-closed primitive M-ideal in $E^{**}$.*

PROOF. Since $E^{**}$ is a JBW*-triple, Proposition 3.4 implies that $E^{***} = j(E^*) \oplus_1 L$. Let $y \in E^{***}$ and suppose $\|j(x)+\lambda y\| \leqq 1$ for all $|\lambda| \leqq 1$, $\lambda \in \mathbb{R}$. Let $y = y_1 + y_2$ with $y_1 \in j(E^*)$ and $y_2 \in L$. Then

$$\|j(x)+\lambda y_1\| + |\lambda|\,\|y_2\| \leqq 1 \quad \forall |\lambda| \leqq 1 .$$

$y_1 = 0$ since $x \in \text{ext}\,(E^*)$. But $\|j(x)\| = 1$, so $y_2 = 0$ also. Hence $j(x) \in \text{ext}\,(E^{***})$ and $M_{j(x)}$ is a primitive $M$-ideal.

Let $M$ be the $w^*$-closure of $M_{j(x)}$, let $m \in M$, and let $(m_\alpha)_{\mathscr{A}} \subset M_{j(x)}$ with $m_\alpha \to m$ $w^*$. Then for every $y, z \in E^{**}$,

$$\{yzm_\alpha\} \to \{yzm\}\ w^* .$$

Since $M_{j(x)}$ is an $M$-ideal, it is an ideal and so $\{yzm_\alpha\} \in M_{j(x)}\ \forall\, \alpha \in \mathscr{A}$. Thus $\{yzm\} \in M$. Similarly $\{ymz\} \in M$, and so $M$ is an ideal and thus an $M$-ideal. Since $\ker(j(x))$ is $w^*$-closed, $M \subseteqq \ker(j(x))$ and hence $M_{j(x)} = M$ by maximality of $M_{j(x)}$.

THEOREM 3.6. *Every* JB*-*triple $E$ has a faithful family of triple factor representations.*

PROOF. Let $x \in$ ext $(E^*)$. By Lemma 3.5, Theorem 3.2, and Lemma 3.1, $M_{j(x)}$ is an $M$-summand in $E^{**}$. Write $E^{**} = M_{j(x)} \oplus_\infty G$ and let $P$: $E^{**} \to M_{j(x)}$ be an $M$-projection.

$G$ is itself a JBW*-triple. Suppose it has a proper $w^*$-closed ideal $M_1$. By Lemma 3.1 there is a $w^*$-closed ideal $M_2$ in $G$ with $G = M_1 \oplus_\infty M_2$. By Theorem 3.2, $M_1$ and $M_2$ are also $M$-ideals in $E^{**}$. Since $M_1 \cap M_2 = \{0\} \subseteqq M_{j(x)}$ and $M_{j(x)}$ is a primitive $M$-ideal, either $M_1 \subseteqq M_{j(x)}$ or $M_2 \subseteqq M_{j(x)}$ (see [1, 3.1]). Both are impossible. Hence, $G$ is a triple factor.

Define $\varphi_x = (\text{id-}P) \circ j$: $E \to G$. Both $M_{j(x)}$ and $G$ are ideals in $E^{**}$, so

$$\{abc\} \in \begin{cases} M_{j(x)} & \text{if } a,b,c \in M_{j(x)} \\ G & \text{if } a,b,c \in G\,. \\ \{0\} & \text{otherwise} \end{cases}$$

Then $\varphi_x\{abc\} = \{\varphi_x(a), \varphi_x(b), \varphi_x(c)\}$. $P$ is hermitian and $E^{**}$ has an unique predual, so $P$ and hence id-$P$ are $w^* - w^*$ continuous by Paya et. al. [16].

$j(E)$ is $w^*$-dense in $E^{**}$, so $\varphi_x(E)$ is $w^*$-dense in $G$. Thus $\varphi_x$ is a triple factor representation.

Because

$$\ker(\varphi_x) = M_x \subseteqq \ker(x) \quad \text{and} \quad \bigcap_{x \in \operatorname{ext}(E^*)} \ker(x) = \{0\} ,$$

$\{\varphi_x | x \in \operatorname{ext}(E^*)\}$ is a faithful family.

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