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Spinning strings in $AdS_5 \times S^5$ and integrable systems

G. Arutyunov[1,⋆,*], **S. Frolov**[2,⋆], **J. Russo**[3,4] **and A.A. Tseytlin**[2,3,†]

[1] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany*

[2] *Department of Physics, The Ohio State University, Columbus, OH 43210-1106, USA*

[3] *Blackett Laboratory, Imperial College, London, SW7 2BZ, U.K.*

[4] *Departament ECM, Facultat de Física, Universitat de Barcelona Institució Catalana de Recerca i Estudis Avançats (ICREA), Spain*

Abstract

We show that solitonic solutions of the classical string action on the $AdS_5 \times S^5$ background that carry charges (spins) of the Cartan subalgebra of the global symmetry group can be classified in terms of periodic solutions of the Neumann integrable system. We derive equations which determine the energy of these solitons as a function of spins. In the limit of large spins J, the first subleading $1/J$ coefficient in the expansion of the string energy is expected to be non-renormalised to all orders in the inverse string tension expansion and thus can be directly compared to the 1-loop anomalous dimensions of the corresponding composite operators in $\mathcal{N} = 4$ super YM theory. We obtain a closed system of equations that determines this subleading coefficient and, therefore, the 1-loop anomalous dimensions of the dual SYM operators. We expect that an equivalent system of equations should follow from the thermodynamic limit of the algebraic Bethe ansatz for the SO(6) spin chain derived from SYM theory. We also identify a particular string solution whose classical energy exactly reproduces the one-loop anomalous dimension of a certain set of SYM operators with two independent R charges J_1, J_2.

*emails: agleb@aei-potsdam.mpg.de; jrusso@ecm.ub.es; frolov, tseytlin @mps.ohio-state.edu
⋆On leave of absence from Steklov Mathematical Institute, Gubkin str.8, 117966, Moscow, Russia
†Also at Lebedev Institute, Moscow.

1 Introduction and summary

Recently, there was a remarkable progress towards understanding AdS/CFT duality in non-supersymmetric sector of states of string theory on $AdS_5 \times S^5$ [1, 2, 3, 4], generalizing earlier work of [5, 6, 7]. On the string side, one identifies semiclassical states described by solitonic closed-string solutions on a 2-cylinder. They have finite energy and carry SO(6) (and, in general, SO(2,4)) angular momentum components J_i. In the limit of large angular momenta the first subleading term in the expansion of the classical string energies happens to be protected (i.e. is not renormalised by string α' corrections) and thus can be matched onto the dimensions of the corresponding gauge-theory ($\mathcal{N} = 4$ SYM) operators [3]. We refer to [4] for a more detailed discussion.

In general, according to the AdS/CFT duality the $AdS_5 \times S^5$ string sigma model should be equivalent to the $\mathcal{N} = 4$ supersymmetric SU(N) YM theory with $N \to \infty$. The composite primary operators in this theory are classified in terms of UIR's of the superconformal group PSU(2,2,|4), i.e. by the conformal dimension Δ, two spins S_1, S_2 and by the Young tableaux (or, equivalently, by the Dynkin labels) of the R-symmetry group SU(4). At $N = \infty$ only single-trace operators matter. Thus one should expect that the energy of the string solutions considered as the function of the angular momenta J_i should match with the dimensions of the corresponding primary single-trace operators in the SYM theory.

The bosonic part of the classical string action is a combination of SO(2,4) and SO(6) sigma models. The O(n) (or O(p,q)) sigma models are known to be classically integrable [8], and the same should obviously be true also upon imposition of the conformal gauge constraints, i.e. for the corresponding classical string theories (for some related work see [9, 10, 11]). One expects, therefore, a close connection between special classes of string solutions representing particular semiclassical string states and certain integrable models. As was already observed earlier, the folded rotating string solutions with one [12, 6] or two [7, 4] non-vanishing angular momenta are related to the 1-d sine-Gordon model.

Here we will consider a generalization to the case when all the three "Cartan" components of the SO(6) angular momentum are non-zero and will find that in this case the SO(6) sigma model effectively reduces to a special integrable 1-d model – the *Neumann model* [13]. The latter describes a three-dimensional harmonic oscillator with three different frequencies constrained to move on a two- sphere (see, e.g., [14, 15, 16]).

The class of S^5 rotating string solutions we will be discussing is parametrized by the angular momenta $J_i = (J_1, J_2, J_3)$ with the energy being $E = E(J_1, J_2, J_3)$. To be able to compare to perturbative conformal dimensions on the SYM side one needs to assume that

$$\frac{\lambda}{J_i^2} \ll 1 , \qquad \frac{1}{J_i} \ll 1 , \tag{1.1}$$

where λ is the square of string tension (related as usual to 't Hooft coupling). In this case the $AdS_5 \times S^5$ superstring α' corrections to the classical energy will be suppressed by extra powers of $\frac{1}{J_i}$. This happens [2] despite the non-BPS nature of the extended rotating string states (the only BPS state is a point-like string having only one non-zero component of J_i [5]) and is due to the fact that the underlying superstring theory has (i) global supersymmetry and (ii) is effectively massive in this case (with 2-d masses $\sim \frac{1}{J_i})$[1].

Assuming (1.1), the classical energy can be expanded as[2]

$$E = J_{\rm tot} + \frac{\lambda}{J_{\rm tot}} f_1(\frac{J_i}{J_{\rm tot}}) + \frac{\lambda^2}{J_{\rm tot}^3} f_2(\frac{J_i}{J_{\rm tot}}) + \dots \ , \tag{1.2}$$

$$J_{\rm tot} \equiv J_1 + J_2 + J_3 \ . \tag{1.3}$$

One immediate aim is then to determine the coefficient function $f_1(\frac{J_2}{J_{\rm tot}}, \frac{J_3}{J_{\rm tot}})$. Given the analytic dependence of the subleading term in E on λ and its expected "non-renormalizability" on the string side, one may be able, as explained in [2, 4], to compare it directly to the one-loop anomalous dimensions of gauge-theory operators of the type $\mathrm{tr}[(\Phi_1 + i\Phi_2)^{J_1}(\Phi_3 + i\Phi_4)^{J_2}(\Phi_5 + i\Phi_6)^{J_3}] + \dots$ belonging to irreducible representation of SU(4) with Dynkin labels $[J_2 - J_3, J_1 - J_2, J_2 + J_3]$ (we assume for definiteness that $J_3 \leq J_2 \leq J_1$).[3] Note also that the expected non-renormalisation of the $1/J_{\rm tot}$ term to all orders in the inverse string tension predicts (through AdS/CFT duality) that the corresponding term in $\mathcal{N} = 4$ SYM should be one loop exact.

Finding the spectrum of one-loop anomalous dimensions of such scalar operators with all three J_i being non-zero should be possible, as in the two-spin case ($J_3 = 0$) in [3], using the techniques (dilatation operator related to integrable spin chains and Bethe ansatz) developed in a recent remarkable series of papers [19, 20, 3, 21]. Here we will determine f_1 in several special cases with $J_i \neq 0$, thus making string-theory predictions for the corresponding eigenvalues of the anomalous dimension matrix.

One special three-spin solution was found already in [1, 2]: this is a circular string with $J_1 = J_2$ and arbitrary J_3 (the stability condition implies $J_1 + J_2 \leq \frac{4n-1}{(2n-1)^2} J_3$ where n is the winding number; in what follows we set $n = 1$)

$$E = J_1 + J_2 + J_3 + \frac{\lambda(J_1 + J_2)}{2(J_1 + J_2 + J_3)^2} + \dots$$

[1] A similar argument explains [7, 17] why 2- and higher loop corrections to the energy of string states in the BMN [5] sector are suppressed by powers of $1/J$.

[2] In the cases when there is just one non-vanishing component of the spin as in [6] or the conditions (1.1) are not satisfied for at least one of the spins [7, 18], the energy may contain also a constant subleading term $O(\sqrt{\lambda})$. Under the condition $\frac{\sqrt{\lambda}}{J} \ll 1$ such term (which will not be protected against string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$-corrections and thus cannot be easily compared to SYM theory) is much larger that the subleading term in the equation below, and thus the corresponding SYM operators should have larger anomalous dimensions.

[3] The primary operators obtained after diagonalizing the dilatation matrix for the gauge-invariant operators mentioned above are not *super*conformal primaries lying on the unitary bound of the continuous (unprotected) series of UIR's of PSU(2,2|4), as can be seen from the corresponding relation between the conformal dimension and the Dynkin labels. Their parent superconformal primaries can be found by analysing representations of supersymmetry. However, this is immaterial since the anomalous dimensions of primaries and of their susy descendants are equal.

$$= J_{\rm tot} + \frac{\lambda}{2J_{\rm tot}} \ (1 - \frac{J_3}{J_{\rm tot}}) + ... \ , \qquad (1.4)$$

where dots stand for λ^2 and other subleading terms. The special case of $J_3 = 0$, i.e.

$$E = J_1 + J_2 + f_1 \frac{\lambda}{J_1 + J_2} + ... \ , \qquad\qquad f_1 = \frac{1}{2} \qquad (1.5)$$

corresponds to the simplest two-spin *circular* string solution [1]. In spite of being unstable, this solution has its "counterpart" on the SYM side [3]. Another string state with the same quantum numbers $J_1 = J_2, \ J_3 = 0$ but lower energy is represented by the stable folded string solution [4] (which generalizes the single-spin solution of [6] to the two-spin case). In this case the energy is given by (1.5) with[4]

$$f_1 = 0.356... \ ,$$

and, remarkably, can be matched exactly with the corresponding lowest anomalous dimension eigenvalue on the SYM side [3].

As in the two-spin case, in general, there will be several three-spin string solutions for given values of J_1, J_2, J_3 having different values of E. The first subleading term in E will then be expected to correspond to the band of one-loop dimensions of the SYM eigen-operators in the $[J_2 - J_3, J_1 - J_2, J_2 + J_3]$ irrep. In particular, there may be several string solutions with $J_1 = J_2$ and small J_3 generalizing the circular [1, 2] and folded [4] two-spin solutions, and different from the circular string solution of [2] with E given in (1.4).

We shall see that in spite of the formal integrability of the Neumann model, finding the explicit form of the three-spin solutions and, in particular, their energies, turns out to be complicated. Below we shall concentrate on several special cases. In particular, there are two obvious cases generalizing the two-spin solutions mentioned above: (i) generalization of the folded two-spin solution to the case of non-zero $J_3 < J_1 = J_2$; (ii) generalization of the circular two-spin solution to the case of non-zero $J_3 < J_1 = J_2$ which has less energy than the circular three-spin solution of [2] (the latter is unstable for small J_3). In these and similar cases with $J_1, J_2 \gg J_3$ we find the following expression for the energy (to the leading order in $\frac{J_3}{J_{\rm tot}}$)

$$E = J_{\rm tot} + \frac{\lambda}{J_{\rm tot}} (f_1^{(0)} + f_1^{(1)} \frac{J_3}{J_{\rm tot}} + ...) + ... \ . \qquad (1.6)$$

Note that the expression (1.4) for the circular three-spin solution of [2] is thus a special case of (1.6).[5] One of our aims will be to compute the value of the coefficient $f_1^{(1)}$ for

[4] This number has a simple origin in terms of values of elliptic functions as mentioned at the end of Section 3.3.

[5] Let us mention that a reason for considering linear in J_3 terms in E (i.e. leading deformations of the two-spin expressions) is that the corresponding leading terms in SYM anomalous dimensions may be possible to compute by using a perturbation theory near the Heisenberg model Hamiltonian corresponding to the two-spin case, i.e. to the anomalous dimension matrix for the operators in the $[J_2, J_1 - J_2, J_2]$ irrep.

various three-spin solutions. In particular, we will find in Section 3 that the folded string solution that generalizes the $J_1 = J_2,\ J_3 = 0$ solution of [4] has $f_1^{(1)} = 4.79....$

One may try to find also folded string solutions with $J_1 = J_2 = J_3$, which should have less energy than the circular solution of [2]. Though the latter is stable for $J_1 = J_2 = J_3$, it is likely to be only a local minimum of the energy, i.e. there may be another $J_1 = J_2 = J_3$ solution with less energy.[6]

The rest of the paper is organized as follows. In Section 2 we shall present the ansatz for the general three-spin S^5 rotating string solution and explain its relation to the Neumann integrable system. This will allow us to reduce the problem to a pair of first-order differential equations for the two coordinates of S^2 related to 5-th order polynomial defining a hyperelliptic curve of genus 2.

In Section 3.1 we shall argue that to obtain a non-trivial folded string solution with the three non-zero spins the string must be "bent" (i.e. two coordinates of S^2 should have a different number of folding points). In Section 3.2 we shall derive the general system of equations that governs the form of the subleading (or "one-loop") term f_1 in the expression (1.2) for the energy of the bent string. An equivalent system is expected to follow from the thermodynamic limit of the algebraic Bethe ansatz for the SO(6) spin chain determining the one-loop anomalous dimensions on the SYM side [19, 21]. In Section 3.3 we shall study this system in expansion in small J_3 and determine the coefficient $f_1^{(1)}$ in (1.6) in the special case of perturbation near two-spin folded string solution of [4] with $J_1 = J_2$.

Section 4 will be devoted to a different class of three-spin solutions which will have higher energy than folded bent strings for the same values of J_i. Using a combination of analytic and numerical methods we shall again determine the form of the leading correction f_1 in (1.2) in this case.

In Section 5 we shall consider a two-spin solution of a circular type that generalizes the circular solution of [1] to the case of unequal S^5 spins (J_1, J_2). We shall show that, just in the case of the two-spin folded string in [4], the first subleading term in the corresponding expression for the energy matches precisely the one-loop anomalous dimensions of a set of SYM operators with SU(4) Dynkin labels $[J_2, J_1 - J_2, J_2]$ which correspond to solutions of the Bethe ansatz equations in [3] with all Bethe roots lying on the imaginary axis. This complements the results in [3, 4], providing another remarkable test of the AdS/CFT correspondence.

Similar solutions describing string spinning in AdS_5 directions can be analysed in much the same way as described in Section 6. In fact, most of the SO(6) case equations have a direct analog in the SO(2,4) case or are related by an analytic continuation.

In Appendix A we shall explain how the general solution of the Neumann model can be written in terms of θ-functions defined on the Jacobian of a hyperelliptic genus 2 Riemann surface [16]. Appendix B will contain a list of integrals used in Section 3. In

[6] In general, there may be several local minima, i.e. stable solutions with the same quantum numbers.

Appendix C we shall study the vanishing of other ("non-Cartan") components of the SO(6) angular momentum tensor for different three-spin solutions which is crucial [1] for their consistent semiclassical quantum state interpretation (and thus a possibility to establish a correspondence with particular SYM operators with the same quantum numbers). In Appendix D we shall describe the two-spin solution corresponding to the straight folded string without bend points. We will show that such string solution does not allow a deformation towards a non-zero third spin component.

2 Reduction of O(6) sigma-model to the Neumann system

2.1 Rotating string ansatz and integrals of motion

Let us consider the bosonic part of the classical closed string propagating in the $AdS_5 \times S^5$ space-time. The world-sheet action in the conformal gauge is

$$I = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \; [G^{(AdS_5)}_{mn}(x)\partial_a x^m \partial^a x^n \; + \; G^{(S^5)}_{pq}(y)\partial_a y^p \partial^a y^q] \, , \qquad \sqrt{\lambda} \equiv \frac{R^2}{\alpha'} \, . \tag{2.1}$$

The two metrics have the standard form in terms of the 5+5 "angular" coordinates:

$$(ds^2)_{AdS_5} = -\cosh^2\rho \; dt^2 + d\rho^2 + \sinh^2\rho \; (d\theta^2 + \sin^2\theta \; d\phi^2 + \cos^2\theta \; d\varphi^2) \, , \tag{2.2}$$

$$(ds^2)_{S^5} = d\gamma^2 + \cos^2\gamma \; d\varphi_3^2 + \sin^2\gamma \; (d\psi^2 + \cos^2\psi \; d\varphi_1^2 + \sin^2\psi \; d\varphi_2^2) \, . \tag{2.3}$$

It is convenient to represent (2.1) as an action for the O(6)×SO(4,2) sigma-model (we follow the notation of [1])

$$I = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma (L_S + L_{AdS}) \, , \tag{2.4}$$

where

$$L_S = -\frac{1}{2}\partial_a X_M \partial^a X_M + \frac{1}{2}\Lambda(X_M X_M - 1) \, , \tag{2.5}$$

$$L_{AdS} = -\frac{1}{2}\eta_{MN}\partial_a Y_M \partial^a Y_N + \frac{1}{2}\tilde{\Lambda}(\eta_{MN} Y_M Y_N + 1) \, . \tag{2.6}$$

Here X_M, $M = 1, \ldots, 6$ and Y_M, $M = 0, \ldots, 5$ are the the embedding coordinates of R^6 with the Euclidean metric in L_S and with $\eta_{MN} = (-1, +1, +1, +1, +1, -1)$ in L_{AdS} respectively. Λ and $\tilde{\Lambda}$ are the Lagrange multipliers. The action (2.4) is to be supplemented with the usual conformal gauge constraints.

The embedding coordinates are related to the "angular" ones in (2.2),(2.3) as follows:

$$X_1 + iX_2 = \sin\gamma \; \cos\psi \; e^{i\varphi_1} \, , \quad X_3 + iX_4 = \sin\gamma \; \sin\psi \; e^{i\varphi_2} \, , \quad X_5 + iX_6 = \cos\gamma \; e^{i\varphi_3} \, , \tag{2.7}$$

$$Y_1+iY_2 = \sinh\rho\ \sin\theta\, e^{i\phi}\ , \quad Y_3+iY_4 = \sinh\rho\ \cos\theta\, e^{i\varphi}\ , \quad Y_5+iY_0 = \cosh\rho\, e^{it}\ . \quad (2.8)$$

In the next few sections we will be discussing the case when the string is located at the center of AdS_5 and rotating in S^5, i.e. is trivially embedded in AdS_5 as $Y_5+iY_0 = e^{i\kappa\tau}$ with $Y_1, ..., Y_4 = 0$.

The S^5 metric has three commuting translational isometries in φ_i which give rise to three global commuting integrals of motion (spins) J_i. Since we are interested in the periodic motion with three J_i non-zero it is natural to choose the following ansatz for X_M:

$$X_1 + iX_2 = x_1(\sigma)\ e^{iw_1\tau}\ , \quad X_3 + iX_4 = x_2(\sigma)\ e^{iw_2\tau}\ , \quad X_5 + iX_6 = x_3(\sigma)\ e^{iw_3\tau}\ , \quad (2.9)$$

where the real radial functions x_i are independent of time and should, as a consequence of $X_M^2 = 1$, lie on a two-sphere S^2:

$$\sum_{i=1}^{3} x_i^2 = 1\ , \qquad i = 1, 2, 3\ .$$

Then the spins $J_1 = J_{12},\ J_2 = J_{34},\ J_3 = J_{56}$ forming a Cartan subalgebra of SO(6) are

$$J_i = \sqrt{\lambda}\ w_i \int_0^{2\pi} \frac{d\sigma}{2\pi}\ x_i^2(\sigma) \equiv \sqrt{\lambda}\ \mathcal{J}_i\,. \tag{2.10}$$

As discussed in [1], to have a consistent semiclassical string state interpretation of these configurations one should look for solutions for which all other components of the SO(6) angular momentum tensor J_{MN} vanish.

The space-time energy E of the string (related to a generator of the compact SO(2) subgroup of SO(4,2)) is simply

$$E = \sqrt{\lambda}\ \kappa \equiv \sqrt{\lambda}\ \mathcal{E}\,. \tag{2.11}$$

The only non-trivial Virasoro constraint is then (dot and prime are derivatives over τ and σ)

$$\kappa^2 = \dot{X}_M \dot{X}_M + X'_M X'_M\,. \tag{2.12}$$

As a consequence of this relation the energy becomes a function of the SO(6) spins:

$$E = E(J_1, J_2, J_3)\,. \tag{2.13}$$

On the string theory side of the AdS/CFT duality the problem is thus to classify the solutions of the string sigma-model subject to the Virasoro constraints with further determination of their space-time energy as a function of the spins. Below we shall derive a closed system of equations which, in principle, allows one to find the energy as a function of the spins for generic three-spin solutions and therefore to determine the dimensions of the corresponding gauge theory operators.

Substituting the ansatz (2.9) into the SO(6) Lagrangian (2.5) we get the following 1-d ("mechanical") system

$$L = \frac{1}{2}\sum_{i=1}^{3}(x_i'^2 - w_i^2 x_i^2) + \frac{1}{2}\Lambda(\sum_{i=1}^{3} x_i^2 - 1)\,. \tag{2.14}$$

It describes an $n = 3$ dimensional harmonic oscillator constrained to remain on a unit $n-1 = 2$ sphere. This is the special case of the n-dimensional *Neumann* dynamical system [13] which is known to be integrable [14].

The Virasoro constraint implies that the energy H of the Neumann system is given by

$$H = \frac{1}{2}\sum_{i=1}^{3}(x_i'^2 + w_i^2 x_i^2) = \frac{1}{2}\kappa^2\,. \tag{2.15}$$

Solving the equation of motion for the Lagrange multiplier Λ we obtain the following non-linear equations for x_i:

$$x_i'' = -w_i^2 x_i - x_i \sum_{j=1}^{3}\left(x_j'^2 - w_j^2 x_j^2\right). \tag{2.16}$$

The canonical momenta conjugate to x_i are

$$\pi_i = x_i'\ , \qquad \sum_{i=1}^{3}\pi_i x_i = 0\ .$$

One can think about this dynamical system as being originally defined on the cotangent bundle T^*R^3 . Imposing the constraints reduces the phase space to T^*S^2. The Dirac bracket obtained from the canonical structure $\{\pi_i, x_j\} = \delta_{ij}$ is

$$\{\pi_i,\pi_j\}_D = x_i\pi_j - x_j\pi_i\,,\quad \{\pi_i, x_j\}_D = \delta_{ij} - x_i x_j\,,\quad \{x_i, x_j\}_D = 0\,. \tag{2.17}$$

One can check that (2.16) follows from the Poisson structure (2.17) and the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{3}(\pi_i^2 + w_i^2 x_i^2) \tag{2.18}$$

supplemented with the two constraints $\sum_{i=1}^3 x_i^2 = 1,\ \sum_{i=1}^3 \pi_i x_i = 0$.

The crucial point allowing to solve this model is that the n-dimensional ($n = 3$ in the present case) Neumann system has the following n integrals of motion [22]:

$$F_i = x_i^2 + \sum_{j\neq i}\frac{(x_i\pi_j - x_j\pi_i)^2}{w_i^2 - w_j^2}\,, \qquad \sum_{i=1}^{n} F_i = 1\ . \tag{2.19}$$

In the present case $n = 3$ and thus only *two* of the three integrals of motion are independent. Moreover, these integrals are in involution with respect to the Poisson bracket (2.17) and the Hamiltonian is

$$H = \frac{1}{2}\sum_{i=1}^{3} w_i^2 F_i \ . \tag{2.20}$$

Thus, any two of these three integrals of motion are enough to integrate this dynamical system since the motion occurs on a surface of constant integrals.

In order to find the relevant closed string solutions we need also to impose the periodicity conditions on x_i:

$$x_i(\sigma) = x_i(\sigma + 2\pi) \ , \tag{2.21}$$

i.e. we are interested in "periodic" version of the Neumann model.

2.2 First-order system for the ellipsoidal coordinates

It is convenient to describe the phase space of this model in terms of independent 2+2 canonical variables rather than the 3+3 constrained variables x_i, π_i. One natural coordinate system on a two-sphere is the angular (γ, ψ) one implied by (2.7) and (2.9):

$$x_1 = \sin\gamma\ \cos\psi \ , \qquad x_2 = \sin\gamma\ \sin\psi \ , \qquad x_3 = \cos\gamma \ . \tag{2.22}$$

However, if all the frequencies w_i are different and so the Hamiltonian is not spherically symmetric, it appears advantageous to use the so called ellipsoidal coordinates [15]. The ellipsoidal coordinates are introduced as the two real roots ζ_1 and ζ_2 of the following quadratic equation

$$\frac{x_1^2}{\zeta - w_1^2} + \frac{x_2^2}{\zeta - w_2^2} + \frac{x_3^2}{\zeta - w_3^2} = 0\,. \tag{2.23}$$

Assuming $w_1 < w_2 < w_3$ we can define the range of ζ_a $(a = 1, 2)$ as

$$w_1^2 \le \zeta_1 \le w_2^2 \le \zeta_2 \le w_3^2\,. \tag{2.24}$$

With this range ζ_a cover $\frac{1}{8}$-th of the two-sphere corresponding to $x_i \ge 0$. One can think of the whole sphere as covering the domain (2.24) and branching along its boundary. For $x_i \ge 0$ we have

$$x_1 = \sqrt{\frac{(\zeta_1 - w_1^2)(\zeta_2 - w_1^2)}{w_{21}^2 w_{31}^2}}\ , \qquad x_2 = \sqrt{\frac{(w_2^2 - \zeta_1)(\zeta_2 - w_2^2)}{w_{21}^2 w_{32}^2}}\,, \tag{2.25}$$

$$x_3 = \sqrt{\frac{(w_3^2 - \zeta_1)(w_3^2 - \zeta_2)}{w_{31}^2 w_{32}^2}}\,, \qquad\qquad w_{ij}^2 \equiv w_i^2 - w_j^2 \ . \tag{2.26}$$

One can check that $\sum_{i=1}^{3} x_i^2 = 1$, i.e. we indeed get a parametrization of a two-sphere. Substituting now this parametrization for x_i into eq. (2.14) we get the following sigma-model Lagrangian

$$L = \frac{1}{2} g_{ab}(\zeta)\ \zeta_a' \zeta_b' - U(\zeta)\,, \tag{2.27}$$

where the non-zero components of the two-sphere metric are

$$g_{11} = \frac{\zeta_2 - \zeta_1}{4(\zeta_1 - w_1^2)(w_2^2 - \zeta_1)(w_3^2 - \zeta_1)}\ , \quad g_{22} = \frac{\zeta_2 - \zeta_1}{4(\zeta_2 - w_1^2)(\zeta_2 - w_2^2)(w_3^2 - \zeta_2)}\ , \tag{2.28}$$

and the potential U is very simple

$$U = \frac{1}{2}(w_1^2 + w_2^2 + w_3^2 - \zeta_1 - \zeta_2)\,. \tag{2.29}$$

Note that in the domain (2.24) the metric g_{ab} is non-negative.

Expressing the integrals of motion (2.19) in terms of ζ_a one finds a system of two 1-st order equations which can also be obtained by solving directly the associated Hamiltonian-Jacobi problem

$$\left(\frac{d\zeta_1}{d\sigma}\right)^2 = -4\frac{P(\zeta_1)}{(\zeta_2 - \zeta_1)^2}\,, \qquad \left(\frac{d\zeta_2}{d\sigma}\right)^2 = -4\frac{P(\zeta_2)}{(\zeta_2 - \zeta_1)^2}\,. \tag{2.30}$$

Here the function $P(\zeta)$ is a 5-th order polynomial

$$P(\zeta) = (\zeta - w_1^2)(\zeta - w_2^2)(\zeta - w_3^2)(\zeta - b_1)(\zeta - b_2)\ . \tag{2.31}$$

The parameters b_1, b_2 are the two constants of motion which can be expressed in terms of integrals F_i in (2.19) by solving the system of equations

$$\begin{aligned} b_1 + b_2 &= (w_2^2 + w_3^2)F_1 + (w_1^2 + w_3^2)F_2 + (w_1^2 + w_2^2)F_3\,, \\ b_1 b_2 &= w_2^2 w_3^2 F_1 + w_1^2 w_3^2 F_2 + w_1^2 w_2^2 F_3\,. \end{aligned} \tag{2.32}$$

In terms of variables b_i the Hamiltonian (2.15) reads as

$$H = \frac{1}{2}\Big(w_1^2 + w_2^2 + w_3^2 - b_1 - b_2\Big) = \frac{1}{2}\kappa^2 = \frac{1}{2}\mathcal{E}^2\,. \tag{2.33}$$

In what follows we shall assume that

$$b_1 \le b_2\ . \tag{2.34}$$

In this case (2.30) implies that

$$b_1 \le \zeta_1 \le b_2\ , \qquad b_2 \le \zeta_2\ . \tag{2.35}$$

Let us note also that the polynomial $P(\zeta)$ in (2.31) can be interpreted as defining a hyperelliptic curve of genus 2

$$s^2 + P(\zeta) = 0\,, \tag{2.36}$$

with s and ζ being two complex coordinates. Thus, we have found that the most general three-spin string solutions are naturally associated with hyperelliptic curves.

The system (2.30) allows one to achieve the full separation of the variables: dividing one equation in (2.30) by the other one can integrate, e.g., ζ_2 in terms of ζ_1 and then obtain a closed equation for ζ_1 as the function of σ. In finding the solutions we need also to take into account the periodicity conditions (2.21) now viewed as conditions on ζ_1, ζ_2.

The spins $J_i = \sqrt{\lambda}\ \mathcal{J}_i$ in (2.10) expressed in terms of ζ_1, ζ_2 satisfy the following relations

$$\frac{\mathcal{J}_1}{w_1} + \frac{\mathcal{J}_2}{w_2} + \frac{\mathcal{J}_3}{w_3} = 1\ , \tag{2.37}$$

$$w_1\mathcal{J}_1 + w_2\mathcal{J}_2 + w_3\mathcal{J}_3 = w_1^2 + w_2^2 + w_3^2 - \int_0^{2\pi} \frac{d\sigma}{2\pi}\ (\zeta_1 + \zeta_2)\ , \tag{2.38}$$

$$\frac{\mathcal{J}_1}{w_1^3} + \frac{\mathcal{J}_2}{w_2^3} + \frac{\mathcal{J}_3}{w_3^3} = \frac{1}{w_1^2 w_2^2 w_3^2} \int_0^{2\pi} \frac{d\sigma}{2\pi}\ \zeta_1\zeta_2\ \ . \tag{2.39}$$

To find the energy (2.11) in terms of the spins J_i we need to express the frequencies w_i and the Neumann integrals of motion parameters b_a in terms of $\mathcal{J}_i$ and use (2.33). After finding a periodic solution of (2.30), this reduces to the problem of computing the two independent integrals on the r.h.s. of eqs. (2.38) and (2.39).

2.3 Moduli space of the multi-spin string solutions

We are thus interested in finding periodic finite-energy solitonic solutions of O(6) sigma model defined on a 2-cylinder that carry three global charges J_i. They can be parametrised by the three frequencies w_i (or J_i) as well by the two integrals of motion b_a. The five parameters (w_i, b_a) may be viewed as coordinates on a moduli space of these solitons.

Because of the closed string periodicity condition in σ, general solutions will be classified by two integer "winding number" parameters n_a which will be related to w_i and b_a after solving the periodicity condition (2.21). In general, there will be several different solutions for given values of J_1, J_2, J_3, i.e. the energy of the string E will be a function not only of J_1, J_2, J_3 but also of the values of n_a.

Depending on the values of these parameters (i.e. location in the moduli space) one may find different geometric types of the resulting rotating string solutions. In particular, the string may be "folded" (with topology of an interval) or "circular" (with topology of a circle). A folded string may be straight (as in the one- and two-spin examples considered in [6] and [4]) or bent (at one or several points) as in the general three-spin case discussed below in Section 3. A "circular" string may have the form of a round circle as in the two-spin and three-spin solutions of [1, 2] or a more general "bent circle" shape as in the three-spin solutions of Section 4 below.

Before turning to the S^5 rotation case, it is useful to review how these different string shapes appear in the case of a closed string rotating in flat $R^{1,5}$ Minkowski space. In orthogonal gauge, string coordinates are given by solutions of free 2-d wave equation, i.e. by combinations of $e^{in(\tau\pm\sigma)}$, subject to the standard constraints $\dot{X}^2 + X'^2 = 0$, $\dot{X}X' = 0$. For a closed string rotating in the two orthogonal spatial planes 12 and 34 and moving along the 5-th spatial direction we find (cf. (2.9))

$$X_0 = \kappa\tau \ , \quad X_1+iX_2 = \ x_1(\sigma)\, e^{iw_1\tau} \ , \quad X_3+iX_4 = \ x_2(\sigma)\, e^{iw_2\tau} \ , \quad X_5 = p_5\tau \ , \tag{2.40}$$

with

$$w_1 = n_1 \ , \quad w_2 = n_2 \ , \quad x_1 = a_1 \sin(n_1\sigma) \ , \qquad x_2 = a_2 \sin[n_2(\sigma+\sigma_0)] \ . \tag{2.41}$$

Here σ_0 is an arbitrary integration constant, and n_a are arbitrary integer numbers and the conformal gauge constraint implies that

$$\kappa^2 = p_5^2 + n_1^2 a_1^2 + n_2^2 a_2^2 \ . \tag{2.42}$$

Then the energy, the two spins and the 5-th component of the linear momentum are

$$E = \frac{\kappa}{\alpha'} \ , \qquad J_1 = \frac{n_1 a_1^2}{2\alpha'} \ , \qquad J_2 = \frac{n_2 a_2^2}{2\alpha'} \ , \quad P_5 = \frac{p_5}{\alpha'} \ , \tag{2.43}$$

i.e.

$$E = \sqrt{p_5^2 + \frac{2}{\alpha'}(n_1 J_1 + n_2 J_2)} \ . \tag{2.44}$$

To get the two-spin states on the leading Regge trajectory (having minimal energy for given values of the *two* non-zero spins) one is to choose $n_1 = n_2 = 1$ with $\sigma_0 = \frac{\pi}{2}$.

The shape of the string depends on the values of σ_0 and n_1, n_2. If σ_0 is irrational then the string always has a "circular" (loop-like) shape. In general, the "circular" string will not be lying in one plane, i.e. will have one or several bends. For rational values of σ_0 the string can be either circular or folded, depending on the values of n_1, n_2.

Consider as the first example the case of $\sigma_0 = 0$. If $n_1 = n_2$ the string is folded and straight, i.e. have no bends. Indeed, then $X_1 + iX_2$ is proportional to $X_3 + iX_4$ and thus one may put the string in a single 2-plane by a global O(4) rotation. If both n_1 and n_2 are either even or odd and different then the string is folded and has several bends (in the 13 and 24 planes). For example, if $n_2 = 3n_1$ then the folded string is wound n_1 times and has two bends (in this case $x_2 = x_1(4x_1^2 - 3)$).

Next let us consider the case of $\sigma_0 = \frac{\pi}{2n_2}$. If $n_1 = n_2$ the string is an ellipsoid, becoming a round circle in the special case of $a_1 = a_2$ [1]. The string is also circular if n_1 is even and n_2 is odd. If, however, n_1 is odd and n_2 is even the string is folded. For example, if $n_2 = 2n_1$ then the folded string is wound n_1 times and has a single bend at one point (in this case $x_2 = 1 - 2x_1^2$).

The structure of the soliton string solutions in curved S^5 case is analogous. Indeed, the equations of motion of the Neumann system are linearized on the Jacobian of the hyperelliptic curve (2.36). The image of the string in the Jacobian whose real connected part is identified with the Liouville torus can wind around two non-trivial cycles with the winding numbers n_1 and n_2 respectively (see Appendix A).

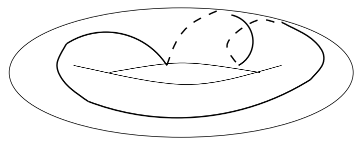

Fig.1: Image of a physical string at fixed moment of time in the Jacobian (the Liouville torus). The string winds around the fundamental cycles with winding numbers n_1 and n_2.

The size and the shape of the Liouville torus are governed by the moduli (w_i, b_a). Specifying the winding numbers n_1, n_2, two of the five parameters (w_i, b_a) are then uniquely determined by the periodicity conditions. The actual shape of the physical string at fixed moment of time lying on two-sphere will depend on the numbers n_1, n_2 and on the remaining moduli parameters and may be of (bent) folded type or of circular type.

Let us now study the folded and circular string solutions in turn.

3 Folded string solutions

3.1 Folded bent strings with three spins

Our aim here will be to analyse folded string solutions of (2.30). We shall see that to have three non-zero spins the folded string must be bent at least at one point.

By definition, a folded closed string configuration is such that for all of the coordinates $X_M(\sigma, \tau) = X_M(2\pi - \sigma, \tau)$, i.e. $X'_M(0, \tau) = X'_M(\pi, \tau) = 0$ (we choose $\sigma = \pi$ as a middle point). In the case of our rotating ansatz (2.9) this leads to

$$x_i(\sigma) = x_i(2\pi - \sigma) \ , \qquad x'_i(0) = x'_i(\pi) = 0, \qquad i = 1, 2, 3 \ .$$

If x'_i vanishes for all i only at the two points, then the string has no bends. Such straight folded string can carry only one non-trivial component of the spin in flat space, but in the case of rotation in S^5 it may carry two non-zero spins [4].

To analyse when the folded string can carry three non-zero spins let us use the angular (ψ, γ) parametrization of two-sphere formed by x_i (2.22). Let us consider a folded string stretched along γ and ψ,

$$-\psi_0 \le \psi(\sigma) \le \psi_0 \ , \qquad \frac{\pi}{2} - \gamma_0 \le \gamma(\sigma) \le \frac{\pi}{2} + \gamma_0 \ , \tag{3.1}$$

and assume that

$$\psi(\frac{\pi}{2}) = \psi(\frac{3\pi}{2}) = 0 \ , \qquad \psi(0) = -\psi(\pi) = \psi_0, \qquad \psi'(0) = \psi'(\pi) = 0 \ , \qquad (3.2)$$

$$\gamma(\frac{\pi}{2}) = \gamma(\frac{3\pi}{2}) = \frac{\pi}{2} \ , \qquad \gamma(0) = \frac{\pi}{2} - \gamma_0 \ , \qquad \gamma(\pi) = \frac{\pi}{2} + \gamma_0 \ , \qquad \gamma'(0) = \gamma'(\pi) = 0 \ . \qquad (3.3)$$

This configuration is a folded string without bends. The case with two spins considered in [4] corresponds to $\gamma_0 = 0$, so one may expect that to have a string with a small non-zero J_3 one needs to consider a case with small γ_0. However, it is possible to see that this no-bend case is not a genuine three-spin case – there is a global SO(3) rotation that can be used to eliminate J_3. Indeed, for a consistent semiclassical state interpretation one has to check that only three components, $J_1 \equiv J_{12},\ J_2 \equiv J_{34},\ J_3 \equiv J_{56}$ of the SO(6) angular momentum are nonvanishing. Assuming that the above string configuration exists when all the frequencies w_i are different, the angular momentum conservation of J_{36} requires the vanishing of the following integral (cf. (2.22))

$$\int_0^{2\pi} d\sigma\ x_2(\sigma)\ x_3(\sigma) = \int_0^{2\pi} d\sigma\ \ \sin\gamma\ \ \cos\gamma\ \ \sin\psi\ . \qquad (3.4)$$

However, it is easy to see that for the folded string configuration (3.3) the integrand here is positive for any value of σ. Thus J_{36} (and also J_{45}) do not vanish. We conclude, therefore, that for the case of all w_i different, the above string solution does not exist. Now consider the case $w_2 = w_3$. As soon as $w_2 = w_3$, one can rotate the folded string to place it entirely on the equator ($\gamma = \frac{\pi}{2}$) of S^2 inside S^5. Then $x_3(\sigma) = 0$ and $J_3 = 0$ for the transformed configuration. The conclusion is therefore that the folded straight string configuration can only correspond to a two-spin case, i.e. the periodicity conditions should imply $w_3 = w_2$.

The reason why in the no-bend case one is able to rotate the string by a global SO(3) transformation to set $\gamma = \frac{\pi}{2}$ is that the folded string should be stretching along a geodesic (i.e. some oblique big circle) of S^2 (ψ, γ) part of S^5. This follows from the fact that both of the derivatives γ' and ψ' vanish only at the same (two ending) points ($\sigma = 0, \pi$) of the folded string (having a wiggle or a bend at some intermediate point of the string would mean the vanishing of one of the derivatives ψ' or γ' there).

We conclude that to find a non-trivial folded string solution with three spins we need to admit a possibility of bends, i.e. the points on the string where one of the two coordinates has zero σ derivative while the other does not (see also Appendix D). For example, if γ' changes its sign not only at $\sigma = 0, \pi$ but also at some $2n$ other points while ψ' does not that would mean one has n folds in γ but only one fold in ψ, implying the existence of n bend points. In what follows we will be considering the simplest and most symmetric case of a single bend point located in the middle of the folded string, i.e. with γ' vanishing at $\sigma_0 = \frac{\pi}{2}$ and $\sigma_0 = \frac{3\pi}{2}$; such configuration is expected to have minimal energy for given values of the three spins.

To have a folded string with a single bend we will thus require

$$x_i'(0) = x_i'(\pi) = 0 \ , \qquad x_3'(\frac{\pi}{2}) = x_3'(\frac{3\pi}{2}) = 0 \ .$$

In terms of the coordinates ζ_1 and ζ_2 in (2.25) these conditions can be satisfied if

$$\zeta_1(\frac{\pi}{2}) = \zeta_1(\frac{3\pi}{2}) = w_2^2 \ , \qquad \zeta_1'(\frac{\pi}{2}) = \zeta_1'(\frac{3\pi}{2}) = 0 \ . \tag{3.5}$$

In view of equations (2.30) the second condition is equivalent to

$$\zeta_2(\frac{\pi}{2}) = \zeta_2(\frac{3\pi}{2}) = b_2 \ . \tag{3.6}$$

The conditions (3.5), (3.6) mean that there should exist 4 points $\sigma_1, ..., \sigma_4$ located as

$$0 < \sigma_1 < \frac{\pi}{2} \ , \quad \frac{\pi}{2} < \sigma_2 < \pi \ , \quad \pi < \sigma_3 = 2\pi - \sigma_2 < \frac{3\pi}{2} \ , \quad \frac{3\pi}{2} < \sigma_4 = 2\pi - \sigma_1 < 2\pi \tag{3.7}$$

at which $\zeta_2 = w_3$, and $x_3 = 0$. Passing through these points x_3 changes its sign. The exact positions of $\sigma_1, ..., \sigma_4$ are determined by the values of the parameters w_i and b_a of the solution. Note that the middle of the folded string at $\sigma = \frac{\pi}{2}, \frac{3\pi}{2}$ is not located at $x_1 = 1, \ x_2 = x_3 = 0$. Thus, at $\sigma = 0$ we have $\zeta_1 = b_1$ and $\zeta_2 = b_2$. Increasing σ, both ζ_1 and ζ_2 increase until at $\sigma = \sigma_1$ the coordinate ζ_2 reaches the point w_3^2. Then ζ_2' changes its sign and ζ_2 begins to decrease. At $\sigma = \frac{\pi}{2}$ the coordinate ζ_1 becomes w_2^2 and ζ_2 reaches the turning point b_2. Next, ζ_1 begins to decrease and ζ_2 begins to increase until at $\sigma = \sigma_2$ the coordinate ζ_2 reaches the point w_3^2 where ζ_2' changes its sign again. After that both ζ_1 and ζ_2 decrease and at $\sigma = \pi$ they reach the turning points $\zeta_1 = b_1$ and $\zeta_2 = b_2$ (see Fig.2).

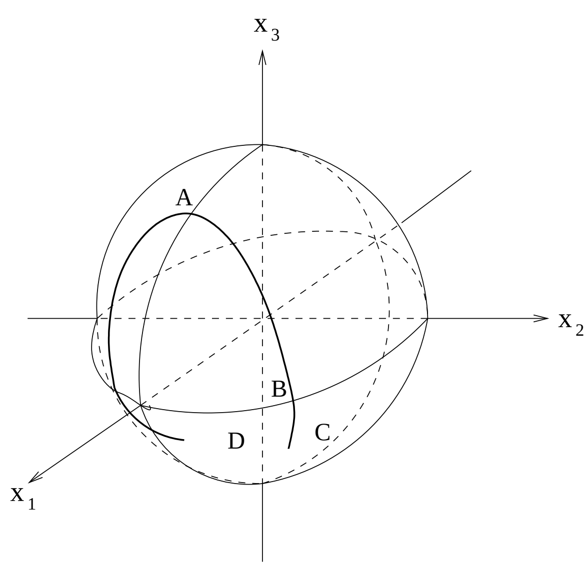

Fig.2: Bent string. The point A corresponds to $\sigma = \frac{\pi}{2}, \frac{3\pi}{2}$, the point B - to $\sigma = \sigma_1, 2\pi - \sigma_1$ and the points C and D are the turning points where $\sigma = 0$ and $\sigma = \pi$. The turning points C and D are symmetric w.r.t. to the plane 13. When $b_2 \to w_3^2$ the bend point A tends to equator and the string itself concentrates around $\gamma = \frac{\pi}{2}$, i.e. we recover a straight string representing the two-spin solution.

Thus, two derivatives ζ_1' and ζ_2' are both positive on the interval $0 < \sigma < \sigma_1$, and the equations of motion (2.30) take the form

$$\frac{d\zeta_1}{\sqrt{-P(\zeta_1)}} = 2\frac{d\sigma}{\zeta_2 - \zeta_1} \ , \qquad \frac{d\zeta_2}{\sqrt{-P(\zeta_2)}} = 2\frac{d\sigma}{\zeta_2 - \zeta_1} \ . \tag{3.8}$$

The periodicity conditions (2.21) follow from (3.8) and from the range (2.24), (2.35) of $\zeta_1, \ \zeta_2$ (these conditions do not depend on σ_1)

$$\int_{b_1}^{w_2^2} \frac{d\zeta_1(\zeta_2 - \zeta_1)}{\sqrt{-P(\zeta_1)}} = \pi \ , \qquad 2\int_{b_2}^{w_3^2} \frac{d\zeta_2(\zeta_2 - \zeta_1)}{\sqrt{-P(\zeta_2)}} = \pi \ . \tag{3.9}$$

The presence of the coefficient 2 in the second equation reflects the fact that we are considering the single-bend solution. As a consequence of (2.30) we have also the following relation which is valid for any point σ from the interval $0 \le \sigma \le 2\pi$

$$|\frac{d\zeta_1}{d\zeta_2}| = \sqrt{\frac{P(\zeta_1)}{P(\zeta_2)}} \ . \tag{3.10}$$

Then

$$\begin{aligned} \int_{c_1}^{w_2^2} \frac{d\zeta_1}{\sqrt{-P(\zeta_1)}} &= \int_{b_2}^{w_3^2} \frac{d\zeta_2}{\sqrt{-P(\zeta_2)}} \ , \qquad c_1 \equiv \zeta_1(\sigma_1) \ , \\ \int_{b_1}^{c_1} \frac{d\zeta_1}{\sqrt{-P(\zeta_1)}} &= \int_{b_2}^{w_3^2} \frac{d\zeta_2}{\sqrt{-P(\zeta_2)}} \end{aligned} \tag{3.11}$$

and, therefore,

$$2\int_{b_2}^{w_3^2} \frac{d\zeta_2}{\sqrt{-P(\zeta_2)}} = \int_{b_1}^{w_2^2} \frac{d\zeta_1}{\sqrt{-P(\zeta_1)}} \ . \tag{3.12}$$

The conditions (3.9) and equations of motion for ζ_1, ζ_2 (3.8) also imply

$$2\int_{b_2}^{w_3^2} \frac{\zeta_2 d\zeta_2}{\sqrt{-P(\zeta_2)}} - \int_{b_1}^{w_2^2} \frac{\zeta_1 d\zeta_1}{\sqrt{-P(\zeta_1)}} = \pi \tag{3.13}$$

It will be convenient for the analysis in next subsections to make the following change of variables $\zeta_{1,2} \to \xi_{1,2}$

$$\zeta_1 = w_2^2 - (w_2^2 - b_1)\xi_1 \ , \qquad \zeta_2 = w_3^2 - (w_3^2 - b_2)\xi_2 \ , \tag{3.14}$$

Then eqs. (2.30) take the form

$$\left(\frac{d\xi_1}{d\sigma}\right)^2 = 4\frac{w_{21}^2(b_2 - w_2^2)}{w_{32}^2}\frac{\xi_1(1-\xi_1)(1-t_1\xi_1)(1-u_1\xi_1)(1-v_1\xi_1)}{(1-u_1\xi_1-u_2\xi_2)^2} \ , \tag{3.15}$$

$$\left(\frac{d\xi_2}{d\sigma}\right)^2 = 4\frac{w_{31}^2(w_3^2 - b_1)}{w_{32}^2}\frac{\xi_2(1-\xi_2)(1-t_2\xi_2)(1-u_2\xi_2)(1-v_2\xi_2)}{(1-u_1\xi_1-u_2\xi_2)^2} \ . \tag{3.16}$$

Here we introduced the parameters

$$t_1 = \frac{w_2^2 - b_1}{w_{21}^2} > 0 \ , \quad u_1 = -\frac{w_2^2 - b_1}{w_{32}^2} < 0 \ , \quad v_1 = -\frac{w_2^2 - b_1}{b_2 - w_2^2} < 0 \ . \qquad (3.17)$$

$$t_2 = \frac{w_3^2 - b_2}{w_{31}^2} > 0 \ , \quad u_2 = \frac{w_3^2 - b_2}{w_{32}^2} > 0 \ , \qquad v_2 = \frac{w_3^2 - b_2}{w_3^2 - b_1} > 0 \ . \qquad (3.18)$$

3.2 System of equations for the leading correction to the energy

Let us now concentrate on the case of solutions which have all three spins very large (1.1). The energy for such solutions is expected to scale as in (1.2). Our aim is to derive a closed system of equations which allows one to find the leading-order function f_1 in (1.2), thus determining the one-loop (first order in λ) correction to conformal dimensions of the corresponding dual SYM operators with free-field theory dimension $\Delta_0 = J_1 + J_2 + J_3$ in SU(4) irreps with Dynkin labels $[J_2 - J_3, J_1 - J_2, J_2 + J_3]$. The same system of equations should then be expected to emerge from the thermodynamic limit of the algebraic Bethe ansatz for the SO(6) spin chain derived from the SYM theory [19, 3, 21].

The two-spin case [4] and the relation (2.33) for the energy of the string suggest that one should look for solutions which have the following moduli parameters

$$w_i^2 = \mathcal{J}_{\rm tot}^2 + \omega_i(\frac{\mathcal{J}_i}{\mathcal{J}_{\rm tot}}) + O(\frac{1}{\mathcal{J}_{\rm tot}^2}) \ , \qquad b_a = \mathcal{J}_{tot}^2 + \beta_a(\frac{\mathcal{J}_i}{\mathcal{J}_{\rm tot}}) + O(\frac{1}{\mathcal{J}_{\rm tot}^2}) \ , \qquad (3.19)$$

where

$$J_{\rm tot} = \sqrt{\lambda}\ \mathcal{J}_{\rm tot} = \sqrt{\lambda}\ (\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3) \ , \qquad (3.20)$$

is the total spin of the string. We shall assume that in the large $\mathcal{J}_{\rm tot}$ limit the "corrections" ω_i and β_a depend only on the ratios $\frac{\mathcal{J}_i}{\mathcal{J}_{\rm tot}} = \frac{J_i}{J_{\rm tot}}$. Note that the presence of a linear $O(\mathcal{J}_{\rm tot})$ term in ω_i^2 and b_a in (3.19) can be ruled out by "parity" ($w_i \to -w_i$, etc.) considerations.

Then the relation (2.33) implies that

$$E^2 = J_{\rm tot}^2 + \lambda(\omega_1 + \omega_2 + \omega_3 - \beta_1 - \beta_2) + O(\frac{1}{J_{\rm tot}}) \ , \qquad (3.21)$$

i.e.

$$E = J_{\rm tot} + \frac{\lambda}{J_{\rm tot}} f_1(\frac{J_i}{J_{\rm tot}}) + O(\frac{1}{J_{\rm tot}^3}) \ , \qquad f_1 = \frac{1}{2}(\omega_1 + \omega_2 + \omega_3 - \beta_1 - \beta_2) \ . \qquad (3.22)$$

We want therefore to find the expressions for ω_i and β_a and thus for f_1 in terms of the current ratios $\frac{J_i}{J_{\rm tot}}$.

The five parameters ω_i, β_i can be found by solving a system of five equations that follows in the limit $\mathcal{J}_{\rm tot} \to \infty$ from the periodicity conditions (3.12) and (3.13), equations (2.37)-(2.38), and one of the equations (2.10) expressing spins through the parameters. This system can be written as follows in terms of the "hyperelliptic" integrals $I_{10}, I_{20}, \mathcal{I}_{11}, \mathcal{I}_{21}, \mathcal{I}_{22}$ defined in Appendix B

$$2I_{20} = I_{10} \ , \tag{3.23}$$

$$2\mathcal{I}_{21} = \omega_{32} I_{10} + \mathcal{I}_{11} - \pi \ , \tag{3.24}$$

$$\omega_1 j_1 + \omega_2 j_2 + \omega_3 j_3 = 0 \ , \qquad j_i \equiv \frac{J_i}{J_{\rm tot}} \ , \tag{3.25}$$

$$\omega_1 + \omega_2 - \omega_3 + \frac{2}{\pi}\left[\frac{1}{2}\omega_{32}^2 I_0 + \omega_{32}\mathcal{I}_{11} - \frac{1}{2}(2\mathcal{I}_{22} - \mathcal{I}_{12})\right] = 0 \ , \tag{3.26}$$

$$j_3 = \frac{2t_2}{\pi}\int_0^1 d\xi_1 \, \frac{d\sigma}{d\xi_1}\,(1-u_1\xi_1)\ \xi_2(\xi_1) \ . \tag{3.27}$$

Here $\omega_{ij} \equiv \omega_i - \omega_j$ and

$$\frac{d\sigma}{d\xi_1} = \frac{\sqrt{\omega_{32}}\,(1-u_1\xi_1-u_2\xi_2)}{2\sqrt{\omega_{21}(\beta_2-\omega_2)}}\frac{1}{\sqrt{\xi_1(1-\xi_1)(1-t_1\xi_1)(1-u_1\xi_1)(1-v_1\xi_1)}} \ , \tag{3.28}$$

which follows from eq.(3.8). We could use any of the other two equations in (2.10) instead of eq.(3.27).[7] All of the parameters in (3.17),(3.18) and in the above system of equations depend only on the ratios $j_i = \frac{\mathcal{J}_i}{\mathcal{J}_{\rm tot}} = \frac{J_i}{J_{\rm tot}}$.

Since the integral in (3.27) can not be computed analytically, there are at least two ways to proceed for a generic three-spin case. One may try to compute this integral and therefore to solve the whole system (3.23)–(3.27) numerically (see Section 3.4). One may also try to develop a perturbation theory around the two-spin solution of [4] assuming that the third spin component J_3 is small as compared to the total spin $J_{\rm tot}$. Since the two-spin solution was expressed in terms of the elliptic functions, one could expect that the same should be true for a perturbative expansion at the vicinity of the two-spin solution. This is indeed what happens as will be explained in the next subsection.

3.3 Two-spin solution and small J_3 expansion

As was mentioned above, the limit $b_2 \to w_3^2$ should correspond to the two-spin solution. In this case the folded string stretches along the equator, i.e. it is straight (without bends). This limit corresponds to a degeneration of the hyperelliptic curve (2.36) governing the string dynamics into an elliptic one.

To see explicitly how this happens let us change the variables $\zeta_{1,2} \to \xi_{1,2}$ as in (3.14). In the limit

$$\epsilon \equiv w_3^2 - b_2 \to 0$$

[7] Let us also note that we cannot use eq.(2.38) to determine the leading order correction f_1 because at this order eq.(2.38) is a consequence of (3.25) and (3.26).

one finds that $t_2, u_2, v_2 \to 0$ and $u_1 \to v_1$ so that eqs. (3.15), (3.16) simplify to

$$\left(\frac{d\xi_1}{d\sigma}\right)^2 = 4w_{21}^2\xi_1(1-\xi_1)(1-t_1\xi_1)\,, \tag{3.29}$$

$$\left(\frac{d\xi_2}{d\sigma}\right)^2 = 4w_{21}^2(1-u_1)\Big(1-\frac{t_1}{u_1}\Big)\frac{\xi_2(1-\xi_2)}{(1-u_1\xi_1)^2}\,. \tag{3.30}$$

Now one can recognize in eq.(3.29) the differential equation for the elliptic sn function, i.e. its solution is

$$\xi_1 = \mathrm{sn}^2\left(\mathrm{K}(t_1) - \sqrt{w_{21}^2}\sigma, t_1\right)\,. \tag{3.31}$$

Here K denotes the complete elliptic integral of the first kind.[8] In writing eq. (3.31) we have also used the fact that ξ_1' is negative on the interval $0 < \sigma < \frac{\pi}{2}$ and that $\xi_1(0) = 1$ due to the identity $\mathrm{sn}(\mathrm{K}(t_1), t_1) = 1$. Eq. (3.31) defines an elliptic curve with modulus t_1. Then the second equation (3.30) can be integrated to find ξ_2. Note that in the two-spin case the variable ξ_2 is an auxiliary one as the physical coordinates x_1, x_2 ($x_3 = 0$, $\gamma = \frac{\pi}{2}$) are parametrized in terms of ξ_1 only (cf. (2.25),(3.14)). The simplicity of the two-spin case is thus related to the fact that the equations for ζ_1 and ζ_2 decouple when $b_2 = w_3^2$.

To determine the parameters (t_1, u_1) entering eqs. (3.29) and (3.30) from the periodicity conditions (3.12) and (3.13) let us study in more detail the relation (3.10). In general, the latter implies the following two equalities

$$I_1(\zeta_1) = I_2(\zeta_2) + I_2(b_2)\,, \qquad b_1 \le \zeta_1 \le c_1\,, \quad b_2 \le \zeta_2 \le w_3^2\,, \tag{3.32}$$

$$I_1(\zeta_1) = -I_2(\zeta_2) + I_2(b_2)\,, \qquad c_1 \le \zeta_1 \le w_2^2\,, \quad b_2 \le \zeta_2 \le w_3^2\,, \tag{3.33}$$

where we have introduced the two period integrals of the Abelian differential of the first kind:

$$I_1(\zeta_1) = \int_{\zeta_1}^{w_2^2} \frac{dz}{\sqrt{-P(z)}}\,, \qquad I_2(\zeta_2) = \int_{\zeta_2}^{w_3^2} \frac{dz}{\sqrt{-P(z)}}\,. \tag{3.34}$$

Making the change of integration variable $z \to w_2^2 - (w_2^2 - b_1)\xi_1\, z$ for I_1, and $z \to w_3^2 - (w_3^2 - b_2)\xi_2\, z$ for I_2, and using (3.14) we obtain a more useful form of I_1, I_2 given in Appendix B. In the limit $\epsilon = w_3^2 - b_2 \to 0$ both integrals can be easily computed and one finds

$$I_1(\epsilon \to 0) = \frac{2\Pi[u_1, \arcsin\sqrt{\xi_1}, t_1]}{\sqrt{w_{21}^2 w_{32}^2 (b_2 - w_2^2)}}\,, \qquad I_2(\epsilon \to 0) = \frac{2\,\arcsin\sqrt{\xi_2}}{\sqrt{w_{31}^2 w_{32}^2 (w_3^2 - b_1)}}\,, \tag{3.35}$$

where Π denotes the incomplete elliptic integral of the third kind. Now (3.32) allows one to solve for ξ_2 in terms of ξ_1, namely

$$\xi_2(\xi_1) = \cos^2\left[\sqrt{(1-u_1)\Big(1-\frac{t_1}{u_1}\Big)}\;\Pi(u_1, \arcsin\sqrt{\xi_1}, t_1)\right]\,. \tag{3.36}$$

[8] We recall the definitions of the elliptic integrals in Appendix B.

Since $\zeta_a \to b_a$ implies that $\xi_a \to 1$ we find the following transcendental equation relating the parameters u_1 and t_1:

$$\Pi(u_1, t_1) \;=\; \pi\sqrt{\frac{u_1}{(1-u_1)(u_1-t_1)}}\,. \tag{3.37}$$

Then eq. (3.36) can be written as

$$\xi_2(\xi_1) = \cos^2\left[\pi\frac{\Pi(u_1, \arcsin\sqrt{\xi_1}, t_1)}{\Pi(u_1, t_1)}\right]\,, \tag{3.38}$$

This relation is valid for all values of ξ_2 from the interval $[0,1]$. In particular, one recognizes that $\xi_2(1) = \xi_2(0) = 1$ and $\xi_2(c_1) = 0$, where $c_1 \approx 0.56862$. This dependence of ξ_2 on ξ_1 is a reflection of the U-shaped form of our string.

Let us now consider the periodicity condition (3.13):

$$2I_{21} - I_{11} = \pi \tag{3.39}$$

where we have introduced the following period integrals of the other Abelian differential of the first kind:

$$I_{11} = \int_{b_1}^{w_2^2}\frac{z dz}{\sqrt{-P(z)}}\,, \qquad I_{21} = \int_{b_2}^{w_3^2}\frac{z dz}{\sqrt{-P(z)}}\,. \tag{3.40}$$

The same change of variables as for (3.32) allows one to compute these integrals in the limit $\epsilon \to 0$ with the result

$$\frac{\pi w_3^2}{\sqrt{w_{31}^2 w_{32}^2(w_3^2 - b_1)}} - \frac{w_2^2}{u_1\sqrt{w_{21}^2 w_{32}^4}}\Big[p\mathrm{K}(t_1) + (u_1 - p)\Pi(u_1, t_1)\Big] = \frac{\pi}{2}\,. \tag{3.41}$$

where $p = \frac{w_2^2 - b_1}{w_2^2} > 0$. With the help of (3.37) the last equation reduces to

$$\mathrm{K}(t_1) = \frac{\pi}{2}\sqrt{w_{21}^2}\,. \tag{3.42}$$

Eqs. (3.37) and (3.42) completely determine the parameters t_1, u_1 of the two-spin solution in terms of the frequencies w_1, w_2. Note that the requirement $\xi_1(\frac{\pi}{2}) = \xi_1(\frac{3\pi}{2}) = 0$ produces the same eq. (3.42) for t_1.

In the limit $\epsilon \to 0$ we can compute the integral

$$\int_0^{\frac{\pi}{2}}(\zeta_1 + \zeta_2)\,d\sigma = \frac{1}{2}(2I_{22} - I_{12})\,, \tag{3.43}$$

where

$$I_{12} = \int_{b_1}^{w_2^2}\frac{z^2 dz}{\sqrt{-P(z)}}\,, \qquad I_{22} = \int_{b_2}^{w_3^2}\frac{z^2 dz}{\sqrt{-P(z)}}\,. \tag{3.44}$$

As a result,

$$\int_0^{\frac{\pi}{2}} d\sigma \ (\zeta_1 + \zeta_2) \ = \frac{\pi}{2}(w_1^2 + w_3^2) + \sqrt{w_{21}^2}\mathrm{E}(t_1) \, , \tag{3.45}$$

where E is the complete elliptic integral of the second kind. Then in the limit $\epsilon \to 0$ eqs. (2.37) and (2.38) reduce to the following equations

$$\frac{\mathcal{J}_1}{w_1} + \frac{\mathcal{J}_2}{w_2} = 1 \ , \tag{3.46}$$

$$w_1\mathcal{J}_1 + w_2\mathcal{J}_2 - w_2^2 = -\frac{2}{\pi}\sqrt{w_{21}^2}\mathrm{E}(t_1) \ , \tag{3.47}$$

with eq. (2.39) being their consequence. Summarizing, we have shown how the three-spin hyperelliptic solution degenerates to the two-spin elliptic one, the later being completely determined by the system (3.37), (3.42), (3.46) and (3.47). The energy of the two-spin solution is determined from

$$\mathcal{E}^2 = \kappa^2 = w_1^2 + w_{21}^2 t_1 \, . \tag{3.48}$$

In the two-spin case with $\mathcal{J}_1 = \mathcal{J}_2 \equiv \mathcal{J}$, $\mathcal{J}_3 = 0$ we know that [4]

$$t_1 = 0.826115 + ... \, , \quad w_1 = 2\mathcal{J} - \frac{0.272922}{\mathcal{J}} \, , \quad w_2 = 2\mathcal{J} + \frac{0.272922}{\mathcal{J}} \ . \tag{3.49}$$

We can then use (3.37) to determine w_3. The result is

$$u_1 = -0.777383 \ , \quad w_{32}^2 = 2.32025 \ , \quad w_3 = 2\mathcal{J} + \frac{1.70597}{2\mathcal{J}} \ . \tag{3.50}$$

Finally, it is now easy to compute the energy of the three-spin string up to the term linear in $j_3 \equiv \frac{J_3}{J_{\rm tot}}$. To this end we should expand the system of equations (3.23)–(3.27) in ϵ and use the two-spin solution as the zero-order approximation. Then this system reduces to a linear system which can be readily solved. We find that the parameters $\omega_i, \beta_1, u_1, t_1$ in (3.17), (3.19) have the following expansion in j_3:

$$\begin{aligned}
&\epsilon \equiv w_3^2 - b_2 = \omega_3 - \beta_2 = 6.12528\, j_3 \ , \qquad j_3 \equiv \frac{J_3}{J_{\rm tot}} \\
&u_1 = -0.777383 + 0.424835\, j_3 \ , \quad t_1 = 0.826115 + 0.183849\, j_3 \ , \\
&\omega_1 = -1.09169 - 3.86759\, j_3 \ , \quad \omega_2 = 1.09169 - 2.95629\, j_3 \ , \\
&\omega_3 = 3.41194 - 0.20351\, j_3 \ , \quad \beta_1 = -0.712032 - 4.11054 j_3 \ .
\end{aligned} \tag{3.51}$$

Using these values of the parameters we find for the energy

$$\mathcal{E}^2 = \mathcal{J}_{\rm tot}^2 + \omega_1 + \omega_2 - \beta_1 + \epsilon \ = \mathcal{J}_{\rm tot}^2 + 0.712032 + 3.41194\, j_3 \ , \tag{3.52}$$

i.e.

$$E^2 = J_{\rm tot}^2 + 0.712032\,\lambda + 3.41194\,\frac{J_3}{J_{\rm tot}}\,\lambda \ . \tag{3.53}$$

Thus the energy has the form (1.6) with positive coefficients

$$E = J_{\rm tot} + 0.356016 \frac{\lambda}{J_{\rm tot}} \left(1 + 4.79183 \frac{J_3}{J_{\rm tot}} \right) \ . \tag{3.54}$$

The coefficient $f_1^{(0)} = 0.356016$ is the same as in the two-spin solution [4],[9] while $f_1^{(1)} = 4.79183$ is the string-theory prediction for the term linear in J_3 in the one-loop anomalous dimension of the dual CFT operator.

3.4 Comments on folded string solutions with general values of J_1, J_2, J_3

Let us now study more general folded string solutions which can be far from the two-spin configuration. This can be done numerically as follows. One starts with three values w_1, w_2, w_3 as input parameters and solves the two periodicity conditions (3.12), (3.13), for the unknowns b_1 and b_2. Then one determines the parameter $c_1 = \zeta_1(\pi/4)$ by solving numerically eq.(3.12) (or, equivalently, eq.(3.11)). With these values $w_1, w_2, w_3, b_1, b_2, c_1$ one can then compute

$$\int_0^{2\pi} \frac{d\sigma}{2\pi} (\zeta_1 + \zeta_2) = \frac{2}{\pi} \int_{b_2}^{w_3^2} \frac{z^2 dz}{\sqrt{-P(z)}} - \frac{1}{\pi} \int_{b_1}^{w_2^2} \frac{z^2 dz}{\sqrt{-P(z)}} \ , \tag{3.55}$$

$$\int_0^{2\pi} \frac{d\sigma}{2\pi} \zeta_1 \zeta_2 = \frac{1}{\pi} \int_{b_1}^{w_2^2} \frac{z \ z_2(z)(z_2(z) - z) dz}{\sqrt{-P(z)}} \tag{3.56}$$

and find J_1, J_2, J_3 from eqs. (2.37)-(2.39). The parameters of different solutions obtained in this way are shown in Table 1.

w_1^2	w_2^2	w_3^2	b_1	b_2	c_1	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\Delta\mathcal{E}^2$
23.52	25.80	28.29	23.87	27.81	24.70	2.20	2.32	0.48	1.06
34.65	36.89	39.40	35.01	39.00	35.82	2.75	2.86	0.38	0.94
47.63	49.88	52.40	47.99	51.97	48.80	3.21	3.32	0.47	0.96
1	4	9	1.14	4.64	2.31	0.26	0.59	1.33	3.45
25	28	33	25.14	28.64	26.31	1.30	1.56	2.55	2.88
49	52	57	49.14	52.64	50.31	1.82	2.13	3.35	2.83
49	53	59	49.06	53.36	50.61	1.50	1.96	3.97	3.42
49	55	64	49.01	55.09	51.33	1.16	1.59	4.96	4.47
49	51	55	49.36	52.27	50.08	2.69	2.18	2.30	1.92

Table 1: Parameters for string configurations with different values of angular momenta.

[9] This coefficient has a simple origin: $f_1^{(0)} = \frac{1}{\pi^2}(2t-1)[\mathrm{K}(t)]^2$, where $t = 0.826115$ is the (unique) solution of $\mathrm{K}(t) = 2\mathrm{E}(t)$.

The first three entries are cases close to the two-spin case, which are indeed consistent with eq.(3.53) for the energy. The input values of w_1, w_2, w_3 are obtained from eqs. (3.51) using $\mathcal{J}_3 = 0.5$, $\mathcal{J}_{\rm tot} = 5$ for the first entry, and $\mathcal{J}_3 = 0.4$, $\mathcal{J}_{\rm tot} = 6$, $\mathcal{J}_3 = 0.5$, $\mathcal{J}_{\rm tot} = 7$, for the second and third entries. According to eq. (3.53), the perturbation-theory values for the correction to the energy

$$\Delta E^2 \equiv E^2 - J_{\rm tot}^2 = \lambda \Delta \mathcal{E}^2$$

found in the expansion in powers of $\frac{J_3}{J_{\rm tot}}$ are, respectively, $\Delta\mathcal{E}^2 \cong 1.04$, $\Delta\mathcal{E}^2 \cong 0.96$, $\Delta\mathcal{E}^2 \cong 0.94$. They agree with the results in Table 1.

The values of J_1, J_2, J_3 are also in good agreement with direct perturbation theory results. Small differences are expected, in view of the higher order corrections in powers of $\frac{J_3}{J_{\rm tot}}$ and in view of the fact that $J_{\rm tot}$ is not very large (the results of Table 1 represent summation of all terms in $\frac{1}{J_{\rm tot}}$ expansion, while the (3.53) contains only the leading correction term). Other cases in Table 1 are far from the two-spin case, i.e. have J_3 of the same order as (or larger than) J_1, J_2.

In general, a random choice of w_1, w_2, w_3 may not correspond to a folded string solution. For large J_i, there are no folded string solutions when the w_{ij}^2 are not small compared to the w_i^2.

We have considered some cases with the same values of w_{31}^2, w_{21}^2, but increasing values of w_i. They exhibit the following interesting fact. The differences $b_1 - w_1^2$ and $b_2 - w_2^2$, $c_1 - w_1^2$ are always the same. The difference in the energy ΔE^2 approaches some asymptotic value as J_i increase.[10] For the particular entries of Table 1, one observes that as the differences $w_i^2 - w_j^2$ increase, b_1 gets closer to w_1^2 and b_2 gets closer to w_2^2.

In conclusion, there exist folded string solutions for diverse values of J_1, J_2, J_3. In the case when J_3 is smaller than J_1, J_2, the numerical calculation reproduces the perturbation theory results of the previous subsection.

4 Three-spin string solutions of circular type

4.1 String solutions of circular type in ellipsoidal coordinates

Let us start with recalling that our parameters are assumed to satisfy in general the conditions (2.24),(2.35), i.e.

$$w_1^2 \le \zeta_1 \le w_2^2 \le \zeta_2 \le w_3^2 \ , \qquad b_1 \le \zeta_1 \le b_2 \le \zeta_2 \ . \tag{4.1}$$

As was discussed in Section 3, to describe a folded string we should consider b_1 lying in the same range as ζ_1, and b_2 lying in the same range as ζ_2, i.e.

$$w_1^2 \le b_1 \le w_2^2 \ , \qquad w_2^2 \le b_2 \le w_3^2 \ . \tag{4.2}$$

[10]For very large values of J_i the difference $\Delta E^2 \equiv E^2 - J_{\rm tot}^2$ should depend only on the ratios $\frac{J_i}{J_{\rm tot}}$.

To find a "circular" string solution we should relax at least one of the conditions (4.2), i.e. to assume that either b_1 or b_2 do not belong to the corresponding intervals. Thus there are two different cases to be considered

$$I. \quad w_1^2 < b_2 < w_2^2 \ , \qquad\qquad II. \quad b_1 < w_1^2 \ .$$

Let us start with the case I. We have then two options for the value of b_1:

$$(i) \quad w_1^2 < b_1 < w_2^2 \ , \quad \text{or} \qquad (ii) \quad b_1 < w_1^2 \ .$$

In what follows we shall consider the case (i), because the case (ii) appears to similar to II.

When $w_1^2 \le b_1 \le w_2^2$, we have

$$b_1 \le \zeta_1 \le b_2 \ , \qquad\qquad w_2^2 \le \zeta_2 \le w_3^2$$

and there are many different circular string configurations with these values of parameters. Let us first consider the simplest one corresponding to

$$b_1 = b_2 \equiv b \ , \qquad \text{i.e.} \qquad \zeta_1 = b \ .$$

Making the change of variable $\zeta_2 \to \xi_2$,

$$\zeta_2 = w_3^2 - w_{32}^2 \xi_2 \ ,$$

we get from (2.30) the following equation for ξ_2

$$\left(\frac{d\xi_2}{d\sigma}\right)^2 = 4w_{31}^2 \xi_2 (1-\xi_2)(1-t\xi_2) \ , \qquad t \equiv \frac{w_{32}^2}{w_{31}^2}. \tag{4.3}$$

Assuming the initial condition $\xi_2(0) = 0$, the solution of this equation is

$$\sqrt{\xi_2} = \mathrm{sn}\Big(\sqrt{w_{31}^2}\sigma, \ t\Big) \,. \tag{4.4}$$

According to this formula the function $\sqrt{\xi_2}$ oscillates between -1 and 1 as σ goes around the string. Hence, if we want our solution to describe a circular type string with a winding number n, the real period of the function sn should be equal to $\frac{2\pi}{n}\sqrt{w_{31}^2}$, i.e.

$$2\pi\sqrt{w_{31}^2} = 4n\mathrm{K}(t) \,. \tag{4.5}$$

We used the fact that the elliptic function 4K is the real period of the function sn. Since eq.(4.5) appears to be similar to (3.42) it may not be apparent why we are dealing with strings of circular type rather then with multifolded strings. To clarify this point let us write down the expressions for the physical-space coordinates x_i in (2.25):

$$x_1 = \sqrt{\frac{b - w_1^2}{w_{21}^2}} \Big(\frac{w_{31}^2 - w_{32}^2 \xi_2}{w_{31}^2}\Big)^{1/2}, \quad x_2 = \sqrt{\frac{w_2^2 - b}{w_{21}^2}} (1 - \xi_2)^{1/2}, \quad x_3 = \sqrt{\frac{w_3^2 - b}{w_{31}^2}} \xi_2^{1/2}.$$

When ξ_2 changes from 0 to 1 all x_i remain non-negative. However, x_2, x_3 can acquire zero value, i.e. can reach the boundary of the coordinate patch $x_i \geq 0$ and, therefore, they may change sign. Change of the sign means that we go to another coordinate patch on S^2. This is indeed the case as one can see by computing the derivatives $x'_{2,3}$: One finds that x_2 changes its sign when ξ_2 crosses 1, while the sign of x_3 changes when ξ_2 crosses zero. Thus, the shape of the resulting string configuration appears to be of a *circular* type.

Conversely, for the folded string configurations of Section 3, the coordinate x_3 is zero, while x_1 remains strictly positive as ξ_1 changes from 0 to 1. We see once again that the shape of the string depends essentially on the relation between the parameters of the moduli space (cf. Section 2.3) which explicitly occur in the expressions for the coordinates x_i in terms of the variable ξ_2: $x_i = x_i(\xi_2; w_i, b_a)$.

It is useful to note that in the limit $w_3 \to w_2$ the periodicity condition (4.5) reduces to $w_{31}^2 = n^2$ and thus we recover the circular three-spin solution found in [1]:

$$\sqrt{\xi_2} \quad \to \quad \mathrm{sn}(n\sigma, 0) \ = \ \sin n\sigma\,. \tag{4.6}$$

We want to emphasize that neither the solution (4.4) nor the periodicity condition (4.5) depend on b. This parameter occurs only in the expressions of x_i through ξ_2 and it tells us how many spins our solution has. For generic b we have a three-spin solution. A two-spin solution arises in the limit $b \to w_1^2$ (or $b \to w_2^2$). In this limit, $x_1 = 0$, i.e. $\mathcal{J}_1 = 0$, while

$$x_2 = \sqrt{1 - \xi_2}\,, \qquad x_3 = \sqrt{\xi_2}\,. \tag{4.7}$$

The spins $\mathcal{J}_2$ and $\mathcal{J}_3$ are then easy to compute

$$\mathcal{J}_2 \ = \ w_2 \int_0^{2\pi} \frac{d\sigma}{2\pi}(1 - \xi_2) = \frac{w_2}{t}\left[t - 1 + \frac{\mathrm{E}(t)}{\mathrm{K}(t)}\right], \tag{4.8}$$

$$\mathcal{J}_3 \ = \ w_3 \int_0^{2\pi} \frac{d\sigma}{2\pi}\xi_2 = \frac{w_3}{t}\left[1 - \frac{\mathrm{E}(t)}{\mathrm{K}(t)}\right], \tag{4.9}$$

where we made use of the explicit solution (4.4). These two equations can be used to express $w_{2,3}$ through $\mathcal{J}_{2,3}$, while the parameter $t = \frac{w_{32}^2}{w_{31}^2}$ can be found from (4.5). The energy is then given by

$$\mathcal{E}^2 = w_2^2 + \frac{w_{32}^2}{t}\,. \tag{4.10}$$

We will not go here into the detailed analysis of the two-spin circular type solution above, we will do it in Section 5, where a comparison with the gauge theory results will be presented.

Let us consider now the limiting case when $w_3 \to w_2$. We also assume that $w_1^2 < b_1$ and therefore $b_1 < \zeta_1 < b_2$. First we perform the following change of variables

$$\zeta_1 = b_2 - b_{21}\xi_1 \ , \qquad \zeta_2 = w_3^2 - w_{32}^2\xi_2 \ , \qquad b_{21} \equiv b_2 - b_1 \ .$$

Then taking the limit $w_3 \to w_2$ we get from (2.30) the following equations for ξ_1, ξ_2

$$\left(\frac{d\xi_1}{d\sigma}\right)^2 = 4h\xi_1(1-\xi_1)(1-t_1\xi_1) \ , \tag{4.11}$$

$$\left(\frac{d\xi_2}{d\sigma}\right)^2 = 4h(1-u_1)\Big(1-\frac{t_1}{u_1}\Big)\frac{\xi_2(1-\xi_2)}{(1-u_1\xi_1)^2} \ , \tag{4.12}$$

where we introduced the parameters

$$0 < t_1 = \frac{b_{21}}{b_2 - w_1^2} < 1 \, , \qquad u_1 = -\frac{b_{21}}{w_2^2 - b_2} < 0 \, , \qquad h = b_2 - w_1^2 > 0 \ . \tag{4.13}$$

These equations have the same form as (3.29) and (3.30) governing the two-spin solutions for the folded string but there is an important difference. Now the variable ξ_2 enters the parametrization of the physical coordinates x_i:

$$\begin{aligned} x_1^2 &= \frac{b_2 - w_1^2}{w_{21}^2}(1-t_1\xi_1) \, , \qquad x_2^2 = \frac{w_2^2 - b_2}{w_{21}^2}(1-u_1\xi_1)(1-\xi_2) \, , \\ x_3^2 &= \frac{w_2^2 - b_2}{w_{21}^2}(1-u_1\xi_1)\xi_2 \, . \end{aligned} \tag{4.14}$$

It is easy to see that the corresponding shape of the string is again of a circular type.

As was pointed out in Section 2.3, generic winding string solitons are parametrized by two integers (n_1, n_2). To construct a corresponding solution we assume that the variable $\sqrt{\xi_1}$ oscillates between -1 and 1 with period $\frac{2\pi}{n_1}\sqrt{h}$. As above, this gives an equation that determines the parameter t_1:

$$\mathrm{K}(t_1) = \frac{\pi}{2n_1}\sqrt{h} \, . \tag{4.15}$$

Then eq.(4.12) can be integrated to give

$$\sqrt{\xi_2} = \sin\Big[\sqrt{h(1-u_1)\Big(1-\frac{t_1}{u_1}\Big)}\int_0^\sigma \frac{d\sigma'}{1-u_1\xi_1(\sigma')}\Big] \, , \tag{4.16}$$

where the initial condition $\xi_2(0) = 0$ was assumed.

Suppose now that the variable $\sqrt{\xi_2}$ performs n_2 oscillations between -1 and 1 as σ runs from 0 to 2π. For $\sigma = 2\pi$ the integral on the r.h.s. of (4.16) can be easily evaluated by a change of variables, and we obtain the second elliptic function equation to determine the parameter u_1:

$$\Pi(u_1,t_1)=\frac{\pi}{2}\frac{n_2}{n_1}\sqrt{\frac{u_1}{(1-u_1)(u_1-t_1)}}\,. \tag{4.17}$$

Equations (4.15) and (4.17) generalize (3.42) and (3.37) for arbitrary winding numbers n_1 and n_2. At this point it should be mentioned that imposition of the relation $w_3=w_2$ implies that the three spins $\mathcal{J}_i$ are not any more independent but rather satisfy an additional constraint. Indeed, our solution is governed by four parameters w_1,w_2,b_1,b_2 which obey the five equations, three of them determine the spins $\mathcal{J}_i$ through these parameters and the other two are the periodicity conditions (4.15) and (4.17). Therefore, one of the equations is in fact a constraint for $\mathcal{J}_i$.

The case of the "round-circle" string can be recovered by taking first the limit $t_1\to 0$ and then $u_1\to 0$. In this double limit eqs. (4.15) and (4.17) reduce to

$$b_2-w_1^2=n_1^2\,,\qquad\qquad n_1=n_2\,. \tag{4.18}$$

Let us consider now the case II and show that it is quite similar to the case I. This time we have the following range of parameters

$$b_1\le w_1^2<w_2^2<b_2<w_3^2\,. \tag{4.19}$$

and thus

$$w_1^2\le\zeta_1\le w_2^2\,,\qquad\qquad b_2\le\zeta_2\le w_3^2\,. \tag{4.20}$$

To demonstrate that we are dealing with string solutions of circular type we consider the limit $b_2\to w_3^2$. Performing a change of variables

$$\zeta_1=w_1^2+w_{21}^2\xi_1\ ,\qquad\qquad \zeta_2=w_3^2-(w_3^2-b_2)\xi_2\ . \tag{4.21}$$

and taking the limit $b_2\to w_3^2$ one obtains the same system of equations (4.11) and (4.12) parametrised by

$$t_1=-\frac{w_{21}^2}{w_1^2-b_1}<0\,,\qquad u_1=\frac{w_{21}^2}{w_{31}^2}>0\,,\qquad h=w_1^2-b_1>0. \tag{4.22}$$

The coordinates x_i depend only on the variable ξ_1

$$x_1=\sqrt{\xi_1}\,,\qquad\qquad x_2=\sqrt{1-\xi_1}\,,\qquad\qquad x_3=0\,. \tag{4.23}$$

Applying the same arguments as above we conclude that the form of the string is of a circular type.

One can easily develop a perturbation theory around the circular type solutions by using the equations expressing spins through w_i and b_a. Assuming the same ansatz for these parameters as in (3.19) one can obtain a system of equations which should determine the leading-order correction to the energy in (1.2), i.e. a one-loop anomalous dimension for the corresponding SYM operators. Again, an equivalent system of equations should follow from the Bethe ansatz approach on the field-theory side.

4.2 String solutions of circular type in spherical coordinates

As was already discussed, when all the frequencies w_i are different, the ellipsoidal coordinates allow one to separate the variables and formulate a closed system of equations that determines the energy E as a function of spins J_i. However, when two of the three frequencies coincide, the U(1) subgroup of O(3) in (2.1) is restored and it is useful to adopt the spherical coordinates. To make a contact with [1] in this and the following subsection we relabel $x_1 \to x_3$, $x_3 \to x_2$ and consider the case when

$$w_1 = w_2 \ > \ w_3 \ . \tag{4.24}$$

The spherical coordinates are global and thus particularly appropriate for the study of solutions with non-minimal energy that represent strings of circular type. Here we shall first present the equations for the spherical coordinates and then relate them to the above discussion of the circular type strings in the ellipsoidal coordinates.

The equations of motion for the spherical coordinates ψ and γ follow from the action (2.1) and the metric (2.3), and in the case $w_1 = w_2$ take the form

$$(\sin^2\gamma\ \psi')' = 0 \ , \qquad\qquad \gamma'' + \frac{1}{2}\sin 2\gamma\ (w_{13}^2 - \psi'^2) = 0 \ . \tag{4.25}$$

Then

$$\psi' = \frac{c}{\sin^2\gamma} \ , \qquad\qquad c = \text{const} \ , \tag{4.26}$$

$$\gamma'' - c^2 \frac{\cos\gamma}{\sin^3\gamma} + \frac{1}{2} w_{13}^2 \sin 2\gamma = 0 \ , \tag{4.27}$$

and the conformal gauge constraint reduces to

$$\gamma'^2 \ = \ \kappa^2 - \frac{c^2}{\sin^2\gamma} - w_1^2\sin^2\gamma - w_3^2\cos^2\gamma \, . \tag{4.28}$$

This can be rewritten as

$$x'^2 = w_{13}^2(x^2 - a_-)(a_+ - x^2) \, , \qquad\qquad x \equiv x_3 = \cos\gamma \ , \tag{4.29}$$

where the constants $a_\pm$ are

$$a_\pm = \frac{1}{2w_{13}^2}\Big[w_{13}^2 + w_1^2 - \kappa^2 \pm \sqrt{(\kappa^2 - w_3^2)^2 - 4c^2 w_{13}^2}\ \Big] \, . \tag{4.30}$$

Clearly, to have two different turning points a_- and a_+ we have to impose a condition[11] $0 < a_\pm \le 1$. Note also that $a_+ = 1$ for $c = 0$. The expressions of the integral of motion c and the energy of the system in terms of the turning points $a_\pm$ are

$$c^2 = w_{13}^2(1 - a_+)(1 - a_-) \, , \qquad \kappa^2 = w_1^2 + (1 - a_- - a_+) w_{13}^2 \, . \tag{4.31}$$

[11] This implies $w_3 < w_1, \ \ w_3 < \kappa$.

To establish a connection with the discussion of the circular type strings in terms of the ellipsoidal coordinates, we introduce a new variable

$$\xi = \frac{x^2 - a_-}{a_+ - a_-} \ .$$

Then eq. (4.29) takes the same form as (4.11) with parameters

$$h = a_- w_{13}^2 \,, \qquad t_1 = -\frac{a_+ - a_-}{a_-} < 0 \,. \tag{4.32}$$

As to eq. (4.26), we rewrite it as

$$\psi'^2 = \frac{c^2}{(1-a_-)^2} \frac{1}{(1-u_1\xi)^2} \tag{4.33}$$

with

$$0 < u_1 = \frac{a_+ - a_-}{1 - a_-} < 1 \,. \tag{4.34}$$

Using (4.31), one can establish a direct correspondence between eq. (4.33) and eq. (4.12). The periodicity conditions are then given by (4.15), where $n_1 = 1$ and by (4.17). In this way one recovers the ellipsoidal coordinate description of the circular type strings.

4.3 Energy of circular type strings

In this subsection we shall find numerical solutions of the periodicity conditions for the circular strings and calculate their energy in the spacial case of two equal frequencies (4.24).

Let us define the two "turning" angles $\gamma_1, \ \gamma_2$ by

$$a_- = \cos^2\gamma_1 \ , \qquad a_+ = \cos^2\gamma_2 \ ,$$

so that

$$c = w_{13} \sin\gamma_1 \sin\gamma_2 \ , \qquad w_{13} = \sqrt{w_1^2 - w_3^2} \ .$$

The string extends from $\gamma = \gamma_1$ (which is the value closer to the equator point $\gamma = \frac{\pi}{2}$) to $\gamma = \gamma_2$.

The string energy can be determined from

$$E^2 = \lambda\kappa^2 = \lambda[w_{13}^2(\sin^2\gamma_2 - \cos^2\gamma_1) + w_1^2] \ . \tag{4.35}$$

Our aim is to express E as a function of J_3 and $J_1 + J_2$, i.e. of

$$\mathcal{J}_3 \equiv \frac{J_3}{\sqrt{\lambda}} \ , \qquad \mathcal{J} \equiv \frac{\mathrm{J}}{\sqrt{\lambda}} \ , \qquad \mathrm{J} \equiv J_1 + J_2 \ . \tag{4.36}$$

The parameters $w_1,\ w_{13}$ are given by

$$w_1 = \frac{\mathcal{J} w_3}{w_3 - \mathcal{J}_3}\ , \qquad\qquad w_{13} = w_3 \sqrt{\frac{\mathcal{J}^2}{(w_3 - \mathcal{J}_3)^2} - 1}\ , \tag{4.37}$$

so we have three unknowns $w_3,\ \gamma_1,\ \gamma_2$ and three equations:

$$\pi w_{13} = K_1(\gamma_1, \gamma_2)\ , \quad \frac{J_3}{\sqrt{\lambda}} = \frac{w_3}{\pi w_{13}}\ K_2(\gamma_1, \gamma_2)\ , \quad n\pi = K_3(\gamma_1, \gamma_2)\ , \tag{4.38}$$

where $K_1(\gamma_1, \gamma_2),\ K_2(\gamma_1, \gamma_2),\ K_3(\gamma_1, \gamma_2)$ are the integrals which in the previous subsection where computed in terms of the elliptic functions,

$$K_1(\gamma_1, \gamma_2) = \int_{\gamma_1}^{\gamma_2} d\gamma \frac{\sin\gamma}{\sqrt{(\cos^2\gamma_2 - \cos^2\gamma)(\cos^2\gamma - \cos^2\gamma_1)}}\ , \tag{4.39}$$

$$K_2(\gamma_1, \gamma_2) = \int_{\gamma_1}^{\gamma_2} d\gamma \frac{\sin\gamma\ \cos^2\gamma}{\sqrt{(\cos^2\gamma_2 - \cos^2\gamma)(\cos^2\gamma - \cos^2\gamma_1)}}\ , \tag{4.40}$$

$$K_3(\gamma_1, \gamma_2) = \sin\gamma_1 \sin\gamma_2 \int_{\gamma_1}^{\gamma_2} d\gamma \frac{(\sin\gamma)^{-1}}{\sqrt{(\cos^2\gamma_2 - \cos^2\gamma)(\cos^2\gamma - \cos^2\gamma_1)}}\ . \tag{4.41}$$

The system (4.38) can be reduced to a system of two equations and two unknowns $\gamma_1,\ \gamma_2$ by noting that

$$\frac{\mathcal{J}_3}{w_3} = \frac{K_2(\gamma_1, \gamma_2)}{K_1(\gamma_1, \gamma_2)}\ .$$

Using (4.37), the second and third equations of (4.38) become:

$$n\pi = K_3(\gamma_1, \gamma_2)\ , \qquad K_2(\gamma_1, \gamma_2) = \pi \mathcal{J}_3 \sqrt{\frac{\mathcal{J}^2}{\mathcal{J}_3^2 [\frac{K_1(\gamma_1, \gamma_2)}{K_2(\gamma_1, \gamma_2)} - 1]^2} - 1} \tag{4.42}$$

These can be solved numerically for $\gamma_1,\ \gamma_2$. Note that for large $\mathcal{J}_3$ the square root on the right hand side of the second equation must be very small. This gives a hint for the values of the parameters $\gamma_1,\ \gamma_2$ which solve these equations. We shall always consider the case when $\mathcal{J}_3, \mathrm{J} \gg 1$.

The general features which result from numerical analysis are the following. The above system has solutions for all possible $n = 1, 2, 3, ...$ For given n, there is a minimal value of J_3/J. A lower bound can be obtained for large n. For $n \gg 1$, K_3 must be large, which implies that the angles γ_1, γ_2 are close to $\frac{\pi}{2}$. In this region, one can approximate the integrals, leading to the bound $J_3/\mathrm{J} > 1/4n^2$.

When J_3/J is small, the solution has maximum eccentricity, with γ_1 near $\pi/2$ and γ_2 near 0. As J_3/J increases, the angle γ_1 increases and γ_2 decreases, until some critical value of J_3/J where $\gamma_1 = \gamma_2$. For higher values of J_3/J, there is no solution.

At the critical value, one has a circular string. Indeed, since $\gamma(\sigma)$ in this case is constant, then

$$\psi' = n = \text{const}$$

and the solution reduces to the round-circle string solution of [1, 2]. The critical value of J_3/J depends on n. To compute it, we take the limit $\gamma_2 \to \gamma_1$ and large J_3 of the equations in (4.42). This implies $\frac{\mathrm{J}}{J_3} \to \frac{K_1}{K_2} - 1$. For $\gamma_1 = \gamma_2 \equiv \gamma_0$, one has $K_2/K_1 \to \cos^2\gamma_0$, and $K_3 \to \pi/(2\cos\gamma_0)$, so the equations become

$$\cos\gamma_0 = \frac{1}{2n} \ , \qquad \tan^2\gamma_0 = \frac{\mathrm{J}}{J_3} \ . \tag{4.43}$$

One also has $w_{13} = n$. The second equation can also be written as $\sin^2\gamma_0 = \mathrm{J}/(\mathrm{J}+J_3)$, which coincides with the expression for the angle γ_0 of the circular solutions obtained in [2]. The first equation $\cos\gamma_0 = \frac{1}{2n}$ implies that for given n, there is a single value of J/J_3 which gives circular solutions

$$\frac{\mathrm{J}}{J_3} = 4n^2 - 1 \ . \tag{4.44}$$

This is in contrast to [2], where there are circular solutions for any J/J_3 at fixed n. The origin of this extra condition can be traced back to the condition $K_1 = \pi w_{13}$ coming from imposing periodicity of $\gamma(\sigma)$. In the circular solution of [2] with $\gamma = \gamma_0$ =const, the condition $\cos\gamma_0 = 1/(2n)$ does not appear, since constant γ is already periodic without need to impose extra conditions. The origin of this extra condition can also be understood by considering perturbations around the γ =const solution. Expanding $\psi = n\sigma + \tilde{\psi}(\sigma)$, $\gamma(\sigma) = \gamma_0 + \tilde{\gamma}(\sigma)$, and substituting into the general equation (4.27), we find to first order

$$\tilde{\gamma}'' + (2n\cos\gamma_0)^2\tilde{\gamma} = 0 \ . \tag{4.45}$$

This has solution $\tilde{\gamma}(\sigma) = a\sin(m\sigma)$, $m \equiv 2n\cos\gamma_0$. Imposing periodicity of γ_1, we find that m must be an integer. The solution with a single winding in γ is in fact $m = 1$, or $\cos\gamma_0 = 1/(2n)$, which is precisely the condition obtained above by taking the limit $\gamma_2 \to \gamma_1$ on the general solution.

Thus, for a given winding n in $\psi(\sigma)$, there exist perturbations of the round-circle solutions only if a circular string is located at a special angle $\cos\gamma_0 = 1/(2n)$ (i.e. for special values of J/J_3): only such discrete set of circular strings admit regular deformations.

Returning to the general case, an important feature of the solution is that in the infinite J_3, J limit with fixed J/J_3, the correction to the energy

$$\Delta E^2 \equiv E^2 - (\mathrm{J} + J_3)^2 \ , \qquad \Delta\mathcal{E}^2 = \lambda^{-1}\Delta E^2$$

is a function (as in (3.21)) of the ratio J/J_3 only

$$E^2 = (\mathrm{J} + J_3)^2 + \lambda \ \mathrm{f}(\frac{\mathrm{J}}{J_3}) \ . \tag{4.46}$$

Just as in the folded string case in Section 3.2, this limit effectively singles out the leading correction to the energy (4.35) (cf. (1.2))

$$E = \mathrm{J} + J_3 + \frac{\lambda}{2(\mathrm{J}+J_3)}\ \mathrm{f}(\frac{\mathrm{J}}{J_3}) + ... \tag{4.47}$$

A summary of numerical results for $n = 1$ is given in Table 2.

$\mathcal{J}$	$\mathcal{J}_3$	$\frac{\mathcal{J}}{\mathcal{J}_3}$	γ_1	γ_2	$\Delta\mathcal{E}^2$
925	100	9.25	1.57061	0.17367	1.316
900	100	9	1.57055	0.17874	1.291
850	100	8.5	1.57038	0.18987	1.242
600	75	8	1.57010	0.20253	1.194
400	50	8	1.57010	0.20253	1.194
525	75	7	1.56879	0.23406	1.097
350	50	7	1.56879	0.23406	1.097
1500	250	6	1.56485	0.27827	1.002
500	100	5	1.56894	0.34622	0.910
450	100	4.5	1.56740	0.39740	0.866
2000	500	4	1.56427	0.47176	0.824
800	200	4	1.56427	0.47176	0.824
700	200	3.5	1.43237	0.59732	0.785
1500	500	3	1.04763	1.04677	0.750

Table 2: Energies of string configurations with $n = 1$, lying between angles γ_1 and γ_2. A few cases with equal J/J_3 are included to illustrate explicitly that for these large values of J, J_3 the angles and energies depend only on the ratio J/J_3.

For $\mathrm{J}/J_3 = 3$, we see that the angles γ_1 and γ_2 are nearly equal. This means that γ is approximately constant, so this is the case of the circular string mentioned above lying at $\gamma = 1.0476 = 0.3334\ \pi \approx \frac{\pi}{3}$.

For the circular string solution with $\psi' = n$ const, the energy and the angle γ are given by [2]

$$E^2_{\rm circ} = (\mathrm{J}+J_3)^2 + \frac{\lambda n^2 \mathrm{J}}{\mathrm{J}+J_3} + ... \ , \qquad \sin^2\gamma_0 = \frac{\mathrm{J}}{\mathrm{J}+J_3} + ... \ , \tag{4.48}$$

where dots stand for terms which vanish at large J_3, J. In the case $\mathrm{J}/J_3 = 3$, $n = 1$, we find $\Delta\mathcal{E}^2_{\rm circ} = 0.75$, and $\gamma_0 = \pi/3$, which is in full agreement with the case $\mathrm{J}/J_3 = 3$ of Table 2.

As a function of J/J_3, ΔE^2 can be roughly approximated by a straight line. A more accurate fitting is by adding a $(\mathrm{J}/J_3)^2$ correction (see fig. 3):

$$\Delta\mathcal{E}^2 = \ \mathrm{f}(\frac{\mathrm{J}}{J_3}) \cong c_0 + c_1\frac{\mathrm{J}}{J_3} + c_2\frac{\mathrm{J}^2}{J_3^2}\ , \tag{4.49}$$

$$c_0 \cong 0.524\ , \qquad c_1 \cong 0.068\ , \qquad c_2 = 0.0002\ . \tag{4.50}$$

This formula has a different structure as compared to the round-circle case, eq. (4.48).[12] The string configurations are in general very different. In particular, as explained above, for given n there is a minimum value of J_3/J, so these solutions do not include the circular solution with $J_3 = 0$ surrounding the equator $\gamma = \pi/2$ as a limit. The present solution with single ($n = 1$) winding exists only for J/J_3 lying in the interval between the minimal value at $\mathrm{J}/J_3 = 3$ and maximal value $\mathrm{J}/J_3 \cong 10$.

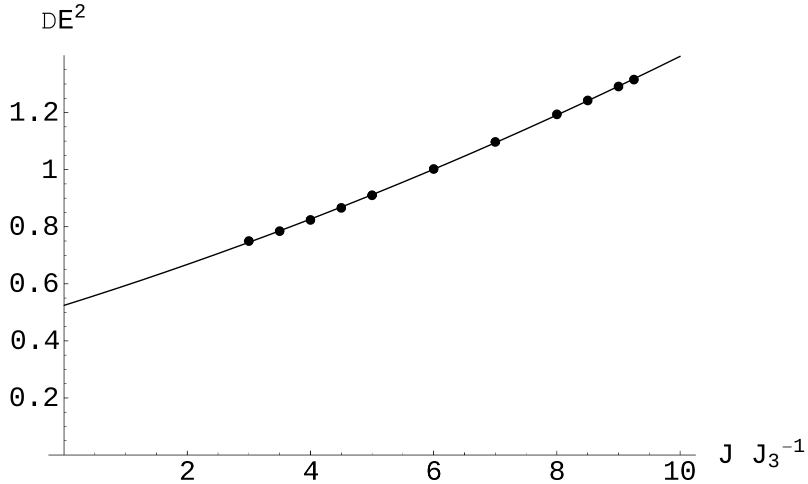

Fig.3: Fitting of ΔE^2 as a function of J/J_3 by $\Delta\mathcal{E}^2 = c_0 + c_1\frac{\mathrm{J}}{J_3} + c_2\frac{\mathrm{J}^2}{J_3^2}$.

As a result, we get the following expansion for the energy of the corresponding string state with charges J and J_3:

$$E = \mathrm{J} + J_3 + \frac{\lambda}{2(\mathrm{J}+J_3)}\left(c_0 + c_1\frac{\mathrm{J}}{J_3} + c_2\frac{\mathrm{J}^2}{J_3^2}\right) + O(\lambda^2) \tag{4.51}$$

Let us comment on solutions with higher winding numbers.[13] For $n = 2$ the critical value at which the string becomes circular is $\mathrm{J}/J_3 = 4n^2 - 1 = 15$, and $\gamma_0 = 1.32$ (this can be compared with the value $\gamma_0 = \pi/3 \cong 1.047$ appearing at $n = 1$, $\mathrm{J}/J_3 = 3$). For $n = 3$, the critical value is $\mathrm{J}/J_3 = 35$, and $\gamma_0 = 1.40$, etc. The larger is n, the angles γ_1, γ_2 get closer to the value $\gamma = \pi/2$. In particular, for $n = 2$ one has solutions with:

(i) $\mathrm{J}/J_3 = 10,\ \gamma_1 = 1.559,\ \gamma_2 = 0.791,\ \Delta\mathcal{E}^2 \cong 3.62$;

(ii) $\mathrm{J}/J_3 = 12,\ \gamma_1 = 1.495, \gamma_2 = 1.053,\ \Delta\mathcal{E}^2 \cong 3.69$.

[12]It is also possible to fit the data in terms of an expansion in, say, $\mathrm{J}/J_{\rm tot}$ or $J_3/J_{\rm tot}$, $J_{\rm tot} = \mathrm{J} + J_3$. The formula (4.49) is given as a simple book-keeping of the numerical data.

[13]In general, one expects that for a string with fixed shape (which requires fixed γ_1, γ_2) the energy grows as $\Delta\mathcal{E}^2 \sim n^2$.... In the present case, different winding numbers give different angles γ_1, γ_2, so this simple n^2 dependence does not appear.

The case (i) may be compared with the case $\mathrm{J}/J_3 = 4$ in Table 1. For a similar value of γ_1, the string is closer to the equator, i.e. the higher is n, the less is the eccentricity of the circular string.

5 Circular string with two angular momenta: A non-trivial test of AdS/CFT duality

Here we would like to complement the work of Ref. [4], which considered the case of the *folded* string solution with two spins in S^5, by discussing the analogous case of the circular type two-spin solution and comparing it with the corresponding SYM results of [3].

In Section 4 of Ref. [3] a set of SYM operators with SU(4) Dynkin labels $[J_2, J_1 - J_2, J_2]$ was found, whose one-loop anomalous dimensions correspond to solutions of the Bethe equations with all Bethe roots lying on the imaginary axis. The one-loop anomalous dimension was determined to be

$$\Delta = J_1 + J_2 + \frac{\lambda}{2J_{\rm tot}} f(\epsilon) \ , \qquad \epsilon \equiv \frac{1}{2} - \frac{J_2}{J_{\rm tot}} = \frac{J_1 - J_2}{2(J_1 + J_2)} \ , \tag{5.1}$$

where

$$f(\epsilon) = 1 + 8\ \epsilon^2 + 24\ \epsilon^4 + 96\ \epsilon^6 + 408\ \epsilon^8 + ... \tag{5.2}$$

Our aim here is to demonstrate that the corresponding dual string configuration is described by a circular type solution (4.3), (4.4).

To make a comparison with the SYM results straightforward it is convenient to redefine the variables w_i which obeyed $w_1 \le w_2 \le w_3$ as follows: $w_1 \to w_3$, $w_3 \to w_1$. Then the expressions (4.8), (4.9) and (4.10) transform into

$$\mathcal{J}_1 = \frac{w_1}{t}\left[1 - \frac{\mathrm{E}(t)}{\mathrm{K}(t)}\right] \ , \qquad \mathcal{J}_2 \ = \ \frac{w_2}{t}\left[t - 1 + \frac{\mathrm{E}(t)}{\mathrm{K}(t)}\right] \tag{5.3}$$

and

$$\mathcal{E}^2 = w_2^2 + \frac{w_{12}^2}{t} \ , \tag{5.4}$$

where the parameter $t = \frac{w_{12}^2}{w_{13}^2} < 1$. These equations are supplemented by the periodicity condition (4.5) with $n = 1$, which now reads as

$$\mathrm{K}(t) = \frac{\pi}{2}\sqrt{\frac{w_{12}^2}{t}}. \tag{5.5}$$

Note that in spherical coordinates (γ, ψ) the equations of motion describing our circular type string are $\gamma = \frac{\pi}{2}$ and $\psi'' + \frac{1}{2} w_{12}^2 \sin 2\psi = 0$. It is worthwhile to mention that after the obvious rescaling $\psi \to \frac{1}{2}\psi$ the last equation describes a motion of the plane pendulum in the gravitational field. Integrating it once, we get $\psi'^2 = w_{12}^2(1/t - \sin^2\psi)$, where t appears as an integration constant. If $t > 1$, then $1/t = \sin^2\psi_0$ and this solution describes the *folded* string extending from $-\psi_0$ to ψ_0. For $t < 1$, when there is no turning point where $\psi' = 0$, the solution will describe a circular string extending all the way around the equator $\gamma = \frac{\pi}{2}$ with ψ from 0 to 2π. In the limit $t \to 0$, this solution will approach the circular type string with $J_1 = J_2$. Thus the parameter t provides an interpolation between the circular and the folded string configurations. Clearly, the rotatory motion of the pendulum requires more total energy than the oscillatory phase and this explains why the energy of the circular string is bigger than that of the folded one.

We want to find the leading correction to the energy (5.4) in the limit of the large total spin $J_{\rm tot}$ for a generic circular type configuration with $J_1 \neq J_2$ and to determine thereby the one-loop anomalous dimension of the corresponding SYM operators. To this end we assume the following large $J_{\rm tot} = \sqrt{\lambda}\mathcal{J}_{\rm tot}$ expansions

$$w_1^2 = \mathcal{J}_{\rm tot}^2 + \omega_1\,, \quad w_2^2 = \mathcal{J}_{\rm tot}^2 + \omega_2\,, \quad t \equiv t(\epsilon), \quad \omega_a = \omega_a(\epsilon)\,, \tag{5.6}$$

where the variable ϵ is defined in (5.1). In the limit of large $J_{\rm tot}$ the energy (5.4) is

$$\mathcal{E} = \mathcal{J}_{\rm tot} + \frac{1}{2\mathcal{J}_{\rm tot}}\Big[\omega_2 + \frac{1}{t}(\omega_1 - \omega_2)\Big]\,. \tag{5.7}$$

Expanding eqs. (5.3) and (5.5) it is easy to determine the $\omega_{1,2}$ as the functions of the variable t:

$$\begin{aligned} \omega_1 &= \frac{4}{\pi^2}\mathrm{K}(t)\Big[(t-1)\mathrm{K}(t) + \mathrm{E}(t)\Big]\,, \\ \omega_2 &= \frac{4}{\pi^2}\mathrm{K}(t)\Big[\mathrm{E}(t) - \mathrm{K}(t)\Big]\,, \end{aligned} \tag{5.8}$$

while t is determined as the function of ϵ from the following equation:

$$\epsilon = \frac{1}{t} - \frac{1}{2} - \frac{\mathrm{E}(t)}{t\mathrm{K}(t)}\,. \tag{5.9}$$

The last equation can be easily solved by the power series ansatz and one obtains

$$\begin{aligned} t(\epsilon) &= 16\epsilon - 128\epsilon^2 + 736\epsilon^3 - 3584\epsilon^4 + 15808\epsilon^5 \\ &- 65024\epsilon^6 + 253888\epsilon^7 - 951296\epsilon^8 + 3446272\epsilon^9 + ... \end{aligned} \tag{5.10}$$

Inserting now this expansion in eqs. (5.8) we first determine ω_1 and ω_2 and then the energy (5.7). The final result reads as

$$E = J_{\rm tot} + \frac{\lambda}{2J_{\rm tot}}\Big[\frac{4}{\pi^2}\mathrm{K}(t)\mathrm{E}(t)\Big] \tag{5.11}$$

and one obtains the following well-defined perturbative expansion of the energy

$$E = J_{\rm tot} + \frac{\lambda}{2J_{\rm tot}}(1 + 8\ \epsilon^2 + 24\ \epsilon^4 + 96\ \epsilon^6 + 408\ \epsilon^8 + ...) + ... \ . \tag{5.12}$$

Quite remarkably, as in the folded string case in [4], this exactly reproduces the full series (5.1), (5.2) for the anomalous dimension of the set of operators corresponding to imaginary Bethe roots in [3].[14]

For $\epsilon = 0$ (i.e. for $J_1 = J_2$), one recovers the expression for the correction to the energy (anomalous dimension) $\frac{\lambda}{2J_{\rm tot}}$ corresponding to the circular string solution discussed in [1]. The opposite, "BPS-like" limit $\epsilon \to \frac{1}{2}$, i.e. $J_2 \to 0$, is ill-defined. This is in contrast to the folded string case, where $J_2 \to 0$ leads to shrinking of the folded string to the BPS particle (the corresponding dual operator with $\Delta = J_1$ transforms in the BPS irrep $[0, J_1, 0]$). Obviously the circular string winding around the equator of S^2 can not be contracted to a point particle.

We conclude this section by emphasizing that the Neumann dynamical system and its string-like solutions encode the information about the full (all-loop) anomalous dimension of the corresponding gauge theory operators. In fact, our treatment above can be extended in a straightforward manner to determine the subleading terms in the energy which should correspond to the two- (and higher-) loop corrections to the anomalous dimensions of the dual SYM operators. This might help to understand, from the field-theoretic point of view, the integrability of the quantum SYM theory beyond the one-loop level.

6 Strings rotating in AdS_5

In this section we briefly discuss multi-spin strings rotating in AdS_5 [1], emphasizing the similarity with the case of rotation in S^5.

We use the same rotation ansatz as in the S^5 case (2.9) written in terms of the embedding coordinates (2.8) as follows

$$Y_1 + iY_2 = y_1(\sigma)\ e^{iw_1\tau}\ , \quad Y_3 + iY_4 = y_2(\sigma)\ e^{iw_2\tau}\ , \quad Y_5 + iY_0 = y_3(\sigma)\ e^{iw_3\tau}\ . \tag{6.1}$$

The real radial functions y_i are independent of time and should, as a consequence of $\eta_{MN}Y_MY_N = -1$, lie on a two-dimensional hyperboloid:

$$\eta_{ij}y_iy_j \equiv -y_1^2 - y_2^2 + y_3^2 = 1\ . \tag{6.2}$$

Note that because of the definition of $Y_5 + iY_0$ in (2.8), $t = w_3\tau$, and, therefore, the 3-rd frequency is now $w_3 = \kappa$. Just as it was in the S^5 case, the three O(2,4) spins

$$S_1 = S_{12}\ , \qquad S_2 = S_{34}\ , \qquad S_3 \equiv E = S_{56}$$

[14] The relative position of the parameters (w_i, b_a) describing the moduli space of string solitons is therefore reflected on the field theory side in the way the Bethe roots are distributed on the complex plane. Folded and circular type strings correspond to double contour roots and imaginary roots respectively.

forming a Cartan subalgebra of SO(2,4) are given by

$$S_i = \sqrt{\lambda}\; w_i \int_0^{2\pi} \frac{d\sigma}{2\pi}\; y_i^2(\sigma) \equiv \sqrt{\lambda}\; \mathcal{S}_i\,, \tag{6.3}$$

and satisfy, because of (6.2), the following relation

$$\frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}_1}{w_1} - \frac{\mathcal{S}_2}{w_2} = 1\ , \qquad \kappa = w_3\ . \tag{6.4}$$

The effective 1-d mechanical system describing this class of rotating solutions in AdS_5 has the following Lagrangian (obtained from (2.6) after an overall sign change, $L_{AdS} \to -\tilde{L}$)

$$\tilde{L} = \frac{1}{2}\eta_{ij}(y_i' y_j' - w_i^2 y_i y_j) + \frac{1}{2}\tilde{\Lambda}(\eta_{ij} y_i y_j - 1)\ . \tag{6.5}$$

Comparing eqs. (6.5),(6.3) and (6.4) with the corresponding eqs. (2.14),(2.10) and (2.37) for the S^5 case, we see that the relation to the S^5 case is through the analytic continuation[15]

$$x_1 \to i y_1\ , \qquad x_2 \to i y_2\ . \tag{6.6}$$

The ellipsoidal coordinates ζ_a defined as in (2.23) now provide the parametrization of the two-sheeted hyperboloid $y_3^2 - y_1^2 - y_2^2 = 1$. Taking into account the analytical continuation, we get the following relations between y_i and ζ_a

$$y_1^2 = \frac{(w_1^2-\zeta_1)(w_1^2-\zeta_2)}{w_{21}^2 w_{13}^2}\,, \quad y_2^2 = \frac{(w_2^2-\zeta_1)(w_2^2-\zeta_2)}{w_{21}^2 w_{32}^2}\,, \quad y_3^2 = \frac{(w_3^2-\zeta_1)(w_3^2-\zeta_2)}{w_{31}^2 w_{32}^2}\ .$$

It is not difficult to check that the equations of motion for ζ_a and the Hamiltonian of the effective 1-d system have the same form (2.30), (2.33) as in the S^5 case. In terms of ζ_a the only difference between the S^5 and AdS_5 cases is in the range of their allowed values.

Another essential feature of the string motion in AdS_5 space is that in this case the Hamiltonian (2.33) coincides with the r.h.s. of the only non-trivial Virasoro constraint $\eta_{MN}(\dot{Y}_M \dot{Y}_N + Y_M' Y_N')$, and, therefore, has to vanish (recall that $\kappa = w_3$)

$$\kappa^2 + w_1^2 + w_2^2 - b_1 - b_2 = 0\ . \tag{6.7}$$

This constraint, together with (6.3) and periodicity conditions will lead to the expression for the energy as a function of the two (in general, unequal) SO(2,4) spins,

$$E = E(S_1, S_2)\ .$$

The previously known examples of the one-spin folded string solution (with $w_1 \neq 0, w_2 = 0$) [12, 6] and the circular string solution (with $w_1 = w_2 \neq 0$) [1] suggest that the frequencies w_i should be chosen as

$$\kappa = w_3 \leq w_1 \leq w_2\ . \tag{6.8}$$

[15] The "angular" coordinates in (2.2),(2.3) are related by $\gamma \to i\rho$.

We shall also assume for definiteness that

$$\zeta_1 < \zeta_2 \ , \qquad\qquad b_1 < b_2 \ . \tag{6.9}$$

Then one can show that

$$w_1^2 \le \zeta_1 \le w_2^2 \le \zeta_2 \ , \qquad\qquad \zeta_1 \le b_1 \le \zeta_2 \le b_2 \ . \tag{6.10}$$

Just as it was in the S^5 case, a folded string solution exists if b_1 and b_2 belong to the same w_i^2 intervals as ζ_1 and ζ_2, respectively, i.e. if

$$w_1^2 \le \zeta_1 \le b_1 \le w_2^2 \le \zeta_2 \le b_2 \ . \tag{6.11}$$

A two-spin folded string solution exists only if the string is bent. The periodicity conditions for a bent folded string are similar to the ones in the S^5 case. The relations between spins and the energy also have the same form as the S^5 case relations (2.37-2.39), with the replacement

$$\mathcal{J}_1 \to -\mathcal{S}_1 \ , \qquad \mathcal{J}_2 \to -\mathcal{S}_2 \ , \qquad \mathcal{J}_3 \to \mathcal{E} \ .$$

It would be interesting to analyze the resulting system of equations in the limit of large spins S_1, S_2. This limit seems to correspond to a long folded string with a large bend.

Two-spin string solutions of circular type exist only if both b_1 and b_2 belong to the same interval as ζ_2, i.e.

$$w_1^2 \le \zeta_1 \le w_2^2 \le b_1 \le \zeta_2 \le b_2 \ . \tag{6.12}$$

Again, there are two simple cases in which the solutions can be analysed in detail: (i) $w_1 = w_2$, and (ii) $b_1 = b_2$. The simplest round-circle string solution found in [1] corresponds to the case $w_1 = w_2$ and $b_1 = b_2$.

7 Concluding remarks

In this paper we have developed a unified treatment of the rotating string solutions in $AdS_5 \times S^5$ based on the integrability of the Neumann dynamical system. We have shown that generic multi-spin solutions are naturally associated to the hyperelliptic genus 2 Riemann surface. The shape of the closed string at fixed time may be of a folded (straight or bent) string and of a circular type. This depends on the two winding numbers n_1, n_2 and on the relative values of the parameters describing the solution moduli space.

We have also studied perturbation theory around the simplest two-spin solutions in the direction of the non-zero third spin component and derived a leading correction to the energy in this case. This enabled us to make an explicit prediction for the 1-loop anomalous dimensions of the corresponding gauge-invariant operators in $\mathcal{N} = 4$

SYM theory (see (3.54)). One may hope that a simple picture of the constrained harmonic oscillator motion linearizing on the Liouville torus may be also uncovered in the equations governing the algebraic Bethe ansatz for gauge-invariant operators on the SYM side (cf. [3, 21]).

It would be interesting also to see if and how the "hyperellipticity" of the general three-spin solutions is related to the more complicated nature of the dual CFT operators. Indeed, in comparison to the "elliptic" two-spin solutions where the dual operators ($\mathrm{tr}[(\Phi_1 + i\Phi_2)^{J_1}(\Phi_3 + i\Phi_4)^{J_2}]+...$) are made out of hypermultiplets (in the $\mathcal{N} = 2$ language), the operators dual to genuine three-spin hyperelliptic string solitons will also mix (beyond the one-loop level) with the operators from the field-strength multiplet.

As we have shown above, a rotating string in the AdS_5 space is described by a "non-compact" version of the Neumann dynamical system which has a simple interpretation in terms of the harmonic oscillator constrained to move on the 2-d hyperboloid. It is desirable to study integrability of this system in more detail, and, in particular, to determine the energy as a function of spins $E = E(S_1, S_2)$ for "long string" configurations. A most natural interpretation of the corresponding dual SYM operators will be in terms of non-local Wilson loops [23] , and one may hope to shed some light on their integrable structures (see, e.g., [24] and refs. there). One specific open problem is to study a folded bent string solution with two equal spins $S_1 = S_2$ which should have lower energy than the circular solution found in [1].

It would be very important to try to go beyond semiclassics and identify the string states and field-theoretic operators for small values of the spins and energy (dimension). One possible direction could be to develop a string-bit model type approximation of continuous string world-sheet (cf. [25]). Indeed, in the CFT an elementary field contributes a "quantum" of dimension and spin to a composite operator. Long composite operators are then viewed as made of many quanta, and a wave approximation corresponds to considering excitations of the continuous string world-sheet. To understand the AdS/CFT correspondence beyond semiclassical approximation one needs to find an analogue of "quantum" of energy and spin on the string side.

A related problem is to see if and how the integrability of the classical bosonic SO(2,4)×SO(6) sigma model can be extended to the $AdS_5 \times S^5$ Green-Schwarz superstring sigma model (for a recent work in this direction addressing integrability of the classical supercoset [26] sigma model see [27]).

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Appendix A General solution of the Neumann system in terms of θ-functions

The general solution of the $n = 3$ Neumann system can be formulated in terms of the theta functions defined on the Jacobian of the hyperelliptic genus 2 Riemann surface [16]. Introduce the theta-functions with characteristics η:

$$\theta[\eta](z) = \sum_{n\in Z^2} \exp\Big[2\pi i z(n+\eta') + i\pi\tau(n+\eta')^2 + 2\pi i\eta''(n+\eta')\Big] , \tag{A.1}$$

where $z = (z_1, z_2)$ and $n = (n_1, n_2)$. Here τ is a period 2×2 symmetric matrix with the positive definite imaginary part. The characteristic η is a 2×2 matrix $\eta = (\eta', \eta'')$ made of two columns η' and η''.

The two normalized Abelian differentials of the 1-st kind (here $a, b = 1, 2$)

$$\int_{A_a} \Omega_b = \delta_{ab} , \qquad \int_{B_a} \Omega_b = \tau_{ab} \tag{A.2}$$

can be written in the form

$$\Omega_a = p_a \Xi_1 + e_a \Xi_2 , \tag{A.3}$$

where

$$\Xi_1 = \frac{d\zeta}{s} , \qquad \Xi_2 = \frac{\zeta d\zeta}{s} , \tag{A.4}$$

and $s(\zeta)$ is determined by (2.36). The differentials $\Xi_a,\ a = 1, 2$, satisfy

$$A_{ab} = \int_{A_a} \Xi_b , \qquad B_{ab} = \int_{B_a} \Xi_b \tag{A.5}$$

with $\tau_{ab} = (BA^{-1})_{ab}$. The normalization condition (A.2) relates the coefficients $e_a,\ p_a$ to A_{ab}, B_{ab} (see also below).

In particular, when $\zeta \to \infty$, the leading coefficient of the differentials Ω_a are e_a which are the frequencies of oscillation on the Jacobian. The solution in [16] reads ($i = 1, 2, 3$)

$$x_i^2(\sigma) = \frac{\theta^2[\eta_{2i-1}](z_0 + \frac{1}{2}ie\sigma)\theta^2[\eta_{2i-1}](0)}{\theta^2[0](z_0 + \frac{1}{2}ie\sigma)\theta^2[0](0)} . \tag{A.6}$$

Here

$$\eta_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \eta_5 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \tag{A.7}$$

are the half-periods and z_0 is a constant vector (the initial condition) which, without loss of generality, we choose to be zero. The specific coefficient 1/2 in front of e in eq. (A.6) is related to the fact that the canonical variable ψ_a conjugate to b_a obeys the linear equation of motion

$$\psi_a' = \{H, \psi_a\} = \frac{1}{2}, \tag{A.8}$$

where H is given by eq. (2.33).

Finally, the identity $\sum_{k=1}^{3} x_k^2 = 1$ is due to the Frobenius formula (here $g = 2$)

$$\sum_{k=1}^{g+1} \frac{\theta^2[\eta_{2k-1}](z)\theta^2[\eta_{2k-1}](0)}{\theta^2[0](z)\theta^2[0](0)} = 1 . \tag{A.9}$$

We do not know the constant vector e_a explicitly, but we can impose the periodicity condition: $x_i(\sigma + 2\pi) = x_i(\sigma)$. In general, solutions (A.6) are not periodic functions of σ and achieve periodicity we have to require

$$\pi e_a = \tau_{ab} m_b + n_a \tag{A.10}$$

where m_a and n_a are vectors with integer components. In addition, if we are interested in the real solutions, then the motion occurs on the real connected component of the Jacobian identified with the Liouville torus. In this case e_a is real and we obtain the condition

$$\pi e_a = n_a \tag{A.11}$$

Since $e_1 = -A_{21}/(\det A)$, $e_2 = A_{11}/(\det A)$, the latter condition reduces to

$$n_1 A_{11} + n_2 A_{21} = 0 , \qquad n_1 A_{12} + n_2 A_{22} = \pi . \tag{A.12}$$

Written in the integral form these conditions are

$$n_1 \int_{A_1} \frac{d\zeta}{s} + n_2 \int_{A_2} \frac{d\zeta}{s} = 0 , \tag{A.13}$$

and

$$n_1 \int_{A_1} \frac{\zeta d\zeta}{s} + n_2 \int_{A_2} \frac{\zeta d\zeta}{s} = \pi . \tag{A.14}$$

In this way we have demonstrated that the general periodic solitons are characterized by two integers (n_1, n_2).

Thus, when σ goes from 0 to 2π the image of the string in the Jacobian winds around the real circles A_1 and A_2 with winding numbers n_1 and n_2 respectively. The different periodicity conditions discussed in the main text can be obtained by picking in eqs. (A.13) and (A.14) the concrete values for (n_1, n_2) and specifying the cycles in terms of the branch points (w_i, b_a). It would be interesting to see how the various elliptic solutions discussed in the paper can be directly obtained from (A.6) by degenerating the period matrix τ.

Appendix B Basic integrals

Here we recall the definitions of $\mathrm{K}(t)$, $\mathrm{E}(t)$ and $\Pi(u,t)$ that are the complete elliptic integrals of the first, the second and the third kind respectively:

$$\mathrm{K}(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}} = \frac{\pi}{2}\ {}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;t\Big)\,, \tag{B.1}$$

$$\mathrm{E}(t) = \int_0^1 \frac{\sqrt{1-tx^2}}{\sqrt{1-x^2}}dx = \frac{\pi}{2}\ {}_2F_1\Big(-\frac{1}{2},\frac{1}{2};1;t\Big)\,, \tag{B.2}$$

$$\Pi(u,t) = \int_0^1 \frac{dx}{(1-ux^2)\sqrt{(1-x^2)(1-tx^2)}}\,, \tag{B.3}$$

where for $\mathrm{K}(t)$ and $\mathrm{E}(t)$ we also provided their expressions in terms of the Gauss hypergeometric function.

The incomplete elliptic integral of the third kind is given by

$$\Pi(u,\phi,t) = \int_0^{\sin\phi} \frac{dx}{(1-ux^2)\sqrt{(1-x^2)(1-tx^2)}}\,. \tag{B.4}$$

In Section 3.3 we used the following integrals

$$I_1 = \frac{\sqrt{\xi_1}}{\sqrt{w_{21}^2 w_{32}^2 (b_2 - w_2^2)}} \int_0^1 \frac{dz}{\sqrt{z(1-\xi_1 z)(1-t_1\xi_1 z)(1-u_1\xi_1 z)(1-v_1\xi_1 z)}}\,, \tag{B.5}$$

$$I_2 = \frac{\sqrt{\xi_2}}{\sqrt{w_{31}^2 w_{32}^2 (w_3^2 - b_1)}} \int_0^1 \frac{dz}{\sqrt{z(1-\xi_2 z)(1-t_2\xi_2 z)(1-u_2\xi_2 z)(1-v_2\xi_2 z)}}\,, \tag{B.6}$$

as well as

$$\begin{aligned} I_{10} &= \int_{b_1}^{w_2^2} \frac{dz}{\sqrt{-P(z)}} = \frac{1}{\sqrt{w_{21}^2 w_{32}^2 (w_{32}^2 - \epsilon)}} \int_0^1 \frac{dz}{\sqrt{z(1-z)(1-t_1 z)(1-u_1 z)(1-v_1 z)}} \\ &\approx \frac{1}{\sqrt{w_{21}^2}\, w_{32}^2} \int_0^1 \frac{dz}{\sqrt{z(1-z)(1-t_1 z)}(1-u_1 z)} \\ &+ \frac{\epsilon}{2\sqrt{w_{21}^2} w_{32}^4} \int_0^1 \frac{dz}{\sqrt{z(1-z)(1-t_1 z)}(1-u_1 z)^2}\,, \end{aligned} \tag{B.7}$$

$$\begin{aligned} I_{20} &= \int_{b_2}^{w_3^2} \frac{dz}{\sqrt{-P(z)}} = \frac{1}{\sqrt{w_{31}^2 w_{32}^2 (w_3^2 - b_1)}} \int_0^1 \frac{dz}{\sqrt{z(1-z)(1-t_2 z)(1-u_2 z)(1-v_2 z)}} \\ &\approx \frac{\pi}{\sqrt{w_{31}^2 w_{32}^2 (w_3^2 - b_1)}} + \frac{\epsilon\pi((w_3^2 - b_1)(w_{31}^2 + w_{32}^2) + w_{31}^2 w_{32}^2)}{4(w_{31}^2 w_{32}^2 (w_3^2 - b_1))^{\frac{3}{2}}}\,, \end{aligned} \tag{B.8}$$

$$
\begin{aligned}
I_{11} &= \int_{b_1}^{w_2^2} \frac{z\ dz}{\sqrt{-P(z)}} = \frac{1}{\sqrt{w_{21}^2 w_{32}^2 (w_{32}^2-\epsilon)}} \int_0^1 \frac{[w_2^2 - (w_2^2 - b_1)z]\ dz}{\sqrt{z(1-z)(1-t_1 z)(1-u_1 z)(1-v_1 z)}} \\
&= w_2^2 I_{10} - \mathcal{I}_{11}\ . \qquad \text{(B.9)}
\end{aligned}
$$

Here $\epsilon = w_3^2 - b_2$ and

$$
\begin{aligned}
\mathcal{I}_{11} &= \frac{w_2^2 - b_1}{\sqrt{w_{21}^2 w_{32}^2 (w_{32}^2-\epsilon)}} \int_0^1 \frac{dz\, z}{\sqrt{z(1-z)(1-t_1 z)(1-u_1 z)(1-v_1 z)}} \\
&\approx -\frac{u_1}{\sqrt{w_{21}^2}} \int_0^1 \frac{dz\, z}{\sqrt{z(1-z)(1-t_1 z)}(1-u_1 z)} \\
&- \frac{\epsilon\, u_1}{2\sqrt{w_{21}^2}w_{32}^2} \int_0^1 \frac{dz\, z}{\sqrt{z(1-z)(1-t_1 z)}(1-u_1 z)^2}\ . \qquad \text{(B.10)}
\end{aligned}
$$

Also,

$$
\begin{aligned}
I_{21} &= \int_{b_2}^{w_3^2} \frac{dz\, z}{\sqrt{-P(z)}} = \frac{1}{\sqrt{w_{31}^2 w_{32}^2 (w_3^2-b_1)}} \int_0^1 \frac{dz(w_3^2 - \epsilon z)}{\sqrt{z(1-z)(1-t_2 z)(1-u_2 z)(1-v_2 z)}} \\
&= w_3^2 I_{20} - \mathcal{I}_{21}\ , \qquad \text{(B.11)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_{21} &= \frac{\epsilon}{\sqrt{w_{31}^2 w_{32}^2 (w_3^2-b_1)}} \int_0^1 \frac{dz\, z}{\sqrt{z(1-z)(1-t_2 z)(1-u_2 z)(1-v_2 z)}} \qquad \text{(B.12)} \\
&\approx \frac{\epsilon\pi}{2\sqrt{w_{31}^2 w_{32}^2 (w_3^2-b_1)}}\ . \qquad \text{(B.13)}
\end{aligned}
$$

Similarly, we define

$$
\begin{aligned}
I_{12} &= \int_{b_1}^{w_2^2} \frac{dz\, z^2}{\sqrt{-P(z)}} = \frac{1}{\sqrt{w_{21}^2 w_{32}^2 (w_{32}^2-\epsilon)}} \int_0^1 \frac{dz(w_2^2 - (w_2^2 - b_1)z)^2}{\sqrt{z(1-z)(1-t_1 z)(1-u_1 z)(1-v_1 z)}} \\
&= w_2^4 I_{10} - 2w_2^2 \mathcal{I}_{11} + \mathcal{I}_{12}\ , \qquad \text{(B.14)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_{12} &= \frac{(w_2^2 - b_1)^2}{\sqrt{w_{21}^2 w_{32}^2 (w_{32}^2-\epsilon)}} \int_0^1 \frac{dz\, z^2}{\sqrt{z(1-z)(1-t_1 z)(1-u_1 z)(1-v_1 z)}} \\
&\approx \frac{u_1^2\, w_{32}^2}{\sqrt{w_{21}^2}} \int_0^1 \frac{dz\, z^2}{\sqrt{z(1-z)(1-t_1 z)}(1-u_1 z)} \\
&+ \frac{\epsilon\, u_1^2}{2\sqrt{w_{21}^2}} \int_0^1 \frac{dz\, z^2}{\sqrt{z(1-z)(1-t_1 z)}(1-u_1 z)^2}\ , \qquad \text{(B.15)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{22} &= \int_{b_2}^{w_3^2} \frac{dz\, z^2}{\sqrt{-P(z)}} = \frac{1}{\sqrt{w_{31}^2 w_{32}^2 (w_3^2 - b_1)}} \int_0^1 \frac{dz (w_3^2 - \epsilon z)^2}{\sqrt{z(1-z)(1-t_2 z)(1-u_2 z)(1-v_2 z)}} \\
&= w_3^4 I_{20} - 2 w_3^2 \mathcal{I}_{21} + \mathcal{I}_{22} \ , \qquad \text{(B.16)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{22} &= \frac{\epsilon^2}{\sqrt{w_{31}^2 w_{32}^2 (w_3^2 - b_1)}} \int_0^1 \frac{dz\, z^2}{\sqrt{z(1-z)(1-t_2 z)(1-u_2 z)(1-v_2 z)}} \\
&\approx \frac{3\epsilon^2 \pi}{8\sqrt{w_{31}^2 w_{32}^2 (w_3^2 - b_1)}} \approx 0 \ . \qquad \text{(B.17)}
\end{aligned}
$$

Then the periodicity conditions and the integral $\int d\sigma(\zeta_1 + \zeta_2)$ in Section 3.2 take the form

$$
2 I_{20} = I_{10} \ , \qquad \text{(B.18)}
$$

$$
2 I_{21} - I_{11} = w_{32}^2 I_{10} - 2\mathcal{I}_{21} + \mathcal{I}_{11} = \pi \ , \qquad \text{or} \qquad 2\mathcal{I}_{21} = w_{32}^2 I_{10} + \mathcal{I}_{11} - \pi \ , \qquad \text{(B.19)}
$$

$$
\int_0^{\frac{\pi}{2}} d\sigma \ (\zeta_1 + \zeta_2) = \frac{1}{2} I_{22} - \frac{1}{2} I_{12} = \pi w_3^2 - \frac{1}{2} w_{32}^4 I_0 - w_{32}^2 \mathcal{I}_{11} + \frac{1}{2}(2\mathcal{I}_{22} - \mathcal{I}_{12}) \ . \qquad \text{(B.20)}
$$

We also have the following periodicity condition

$$
\begin{aligned}
\pi &= \int_{b_1}^{w_2^2} \frac{d\zeta_1 \left[\zeta_2(\zeta_1) - \zeta_1\right]}{\sqrt{-P(\zeta_1)}} = w_{32}^2 I_0 + \mathcal{I}_{11} \\
&- \frac{\epsilon}{\sqrt{w_{21}^2 w_{32}^2 (w_{32}^2 - \epsilon)}} \int_0^1 \frac{d\xi_1\, \xi_2(\xi_1)}{\sqrt{\xi_1 (1-\xi_1)(1-t_1\xi_1)(1-u_1\xi_1)(1-v_1\xi_1)}} \\
&\approx \frac{1}{\sqrt{w_{21}^2}} \int_0^1 \frac{dz}{\sqrt{z(1-z)(1-t_1 z)}} = \frac{2}{\sqrt{w_{21}^2}} \mathrm{K}(t_1) \ . \qquad \text{(B.21)}
\end{aligned}
$$

Appendix C Vanishing of "non-Cartan" components of angular momentum

The SO(6) momentum components J_{MN} written in terms of the 6 embedding coordinates X_M in (2.7) are

$$
J_{MN} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \ (X_M \partial_\tau X_N - X_N \partial_\tau X_M) \ . \qquad \text{(C.1)}
$$

They are conserved in time by virtue of the SO(6) symmetry of the Lagrangian (2.5). The "diagonal" components $J_1 = J_{12}, \ J_2 = J_{34}$ and $J_3 = J_{56}$) are clearly non-zero for the rotating string ansatz (2.9) (as long as all $w_i \neq 0$), being proportional to an integral (2.10) of a positive definite quantity x_i^2.

The question we would like to address is what are the conditions for the vanishing of all other components of J_{MN}. First, assume that $w_1 \neq w_2$. Then one has, in particular,

$$J_{13} = \sqrt{\lambda}\,[w_1 \sin(w_1\tau)\cos(w_2\tau) - w_2 \sin(w_2\tau)\cos(w_1\tau)] \int_0^{2\pi} \frac{d\sigma}{2\pi} x_1(\sigma) x_2(\sigma) \ . \quad \text{(C.2)}$$

Since J_{13} must be time-independent on shell, it follows that $\int_0^{2\pi} \frac{d\sigma}{2\pi} x_1(\sigma)x_2(\sigma)$ must vanish on the solution of the string equations of motion. Similarly, $J_{14},\ J_{23},\ J_{24}$ must also vanish, since they are proportional to the same integral over σ. One reaches analogous conclusions in the cases $w_1 \neq w_3$ and $w_2 \neq w_3$.

The solution of Section 3 where the string is folded in ψ and γ is possible only if all w_i are different, and so all extra J_{MN} components are necessarily zero there.

The only case that may in principle lead to non-zero values for the "non-Cartan" components of J_{MN} is when some two of the three frequencies happen to be equal. If $w_1 = w_2$, then J_{13} and J_{24} are automatically zero. For the components J_{23} and J_{14} one finds (using (2.7))

$$J_{23} = -J_{14} = -\sqrt{\lambda}\ w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} x_1(\sigma)x_2(\sigma) = -\sqrt{\lambda}\ w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2\gamma \cos\psi \sin\psi \ , \quad \text{(C.3)}$$

which may potentially give a non-zero result. Similarly, if $w_1 = w_3$,

$$J_{25} = -J_{16} = -\sqrt{\lambda}\ w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} x_1(\sigma)x_3(\sigma) = -\sqrt{\lambda}\ w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin\gamma \cos\gamma \cos\psi \ , \quad \text{(C.4)}$$

and if $w_2 = w_3$

$$J_{45} = -J_{36} = -\sqrt{\lambda}\ w_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} x_2(\sigma)x_3(\sigma) = -\sqrt{\lambda}\ w_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin\gamma \cos\gamma \sin\psi \ . \quad \text{(C.5)}$$

Let us study the values of these components in some special cases. First, let us consider a string configuration which is folded in γ: when σ goes from 0 to π let γ vary between γ_1 and γ_2 and ψ between 0 to π, and when σ goes from π to 2π let γ vary from γ_2 to γ_1 and ψ from π to 2π. Then the integral (C.3) vanishes because the contribution of the integration region $0 < \sigma < \pi/2$ cancels against the contribution of the integration region from $3\pi/2 < \sigma < 2\pi$, while the contribution of $\pi/2 < \sigma < \pi$ cancels against that of $\pi < \sigma < 3\pi/2$. In the same way, the integral (C.5) also gives zero as the integration region $0 < \sigma < \pi$ cancels with the region $\pi < \sigma < 2\pi$, where $\sin\psi$ has the opposite sign. The only non-obvious case is the vanishing of the integral (C.4). If there is an extra symmetry such as, e.g., $\gamma(\pi - \sigma) = \frac{\pi}{2} - \gamma(\sigma)$ (as in the cases where the string extends from the equator to both sides in a symmetrical way), then the integral vanishes by similar symmetry arguments. If $\gamma(\pi - \sigma)$ is not directly related to $\gamma(\sigma)$ in a simple way, this integral needs to be performed explicitly. For the "circular" solution considered in Section 4, with $\psi' = c/\sin^2\gamma$, J_{25} and thus the integral (C.4) must vanish because there $w_1 \neq w_3$. We have checked independently that (C.4) is indeed zero in this case.

Appendix D Straight folded string

Here we describe the two-spin solution realized by the folded string without bend points. As will follow from our consideration, such solution is rigid in a sense that it does not allow a deformation in the direction of the non-zero third spin component.

We assume the same range of parameters as in Section 3, i.e.

$$w_1^2 < b_1 \le \zeta_1 \le w_2^2 < b_2 \le \zeta_2 \le w_3^2 \tag{D.1}$$

and perform the same change of variables (3.14) with the subsequent two-spin limit $b_2 \to w_3^2$, so that equations (2.30) will be reduced to the system (3.29), (3.30). However, this time we assume the existence of the two turning points, at $\sigma = 0$ and at $\sigma = \pi$, and no bend points. This implies that both derivatives ζ'_a are negative on the interval $0 \le \sigma \le \frac{\pi}{2}$ and positive for $\frac{\pi}{2} \le \sigma \le \pi$. The periodicity conditions describing this situation are

$$\begin{aligned} I_1(\zeta_1) &= I_2(\zeta_2) \ , && \text{(D.2)} \\ I_{21} &- I_{11} = \pi \ , && \text{(D.3)} \end{aligned}$$

where $I_{1,2}$ are defined in (3.34) while I_{11} and I_{21} are given by (3.40). Considerations similar to those in Section 3.3. allow one to determine ξ_2 as function of ξ_1:

$$\xi_2(\xi_1) = \sin^2\left[\frac{\pi}{2}\frac{\Pi(u_1, \arcsin\sqrt{\xi_1}, t_1)}{\Pi(u_1, t_1)}\right] . \tag{D.4}$$

This time ξ_2 is the monotonic function of ξ_1 on the interval [0,1]. In particular, $\xi_2(0) = 0$ and $\xi_2(1) = 1$.

In addition, the parameters u_1, t_1 should obey the following two equations

$$\begin{aligned} \Pi(u_1, t_1) &= \frac{\pi}{2}\sqrt{\frac{u_1}{(1-u_1)(u_1-t_1)}} , && \text{(D.5)} \\ \mathrm{K}(t_1) &= \frac{\pi}{2}\sqrt{w_{21}^2} . && \text{(D.6)} \end{aligned}$$

Comparison with (4.15) and (4.17) shows that the image of our solution on the Liouville torus is characterized by the winding numbers $n_1 = n_2 = 1$.

The relations between w_i and $\mathcal{J}_i$ retain the same form (3.46) and (3.47) as they do not depend on ξ_2. Solution of the eqs. (3.42), which is the same as (D.6), and (3.46), (3.47) for w_1, w_2 and t_1, is given by eq. (3.49). By using this solution we can now infer the value of u_1 from eq. (D.5). One finds $u_1 = -\infty$, which implies that $w_2 = w_3$.

It turns out that the perturbation theory around $u_1 = -\infty$ is ill-defined as this is essential singularity. Moreover, the function $\xi_2(\xi_1)$ ceases to be regular in the limit $u_1 \to -\infty$ as $\xi_2(\xi_1) = 1$ for $0 < \xi_1 \le 1$ and $\xi_2(\xi_1) = 0$ for $\xi_1 = 0$. Therefore, we conclude that our folded string is "rigid" in the sense that it cannot be bent to acquire a small amount of the third spin component $\mathcal{J}_3$.

It should be emphasized that the variable u_1 does not enter either the relations between w_i and $\mathcal{J}_i$ or the expression (3.48) for the energy. This parameter and the function ξ_2 play only an auxiliary rôle for the two-spin solutions. However, they both arise in degeneration of a certain hyperelliptic solution and, therefore, they point out a direction in which an elliptic two-spin solution can (not) be deformed.

References

[1] S. Frolov and A. A. Tseytlin, "Multi-spin string solutions in $AdS_5 \times S^5$," hep-th/0304255.

[2] S. Frolov and A. A. Tseytlin, "Quantizing three-spin string solution in $AdS_5 \times S^5$," hep-th/0306130.

[3] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, "Stringing spins and spinning strings," hep-th/0306139.

[4] S. Frolov and A. A. Tseytlin, "Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors," hep-th/0306143.

[5] D. Berenstein, J. M. Maldacena and H. Nastase, "Strings in flat space and pp waves from N = 4 super Yang Mills," JHEP **0204**, 013 (2002) [hep-th/0202021].

[6] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "A semi-classical limit of the gauge/string correspondence," Nucl. Phys. B **636**, 99 (2002), hep-th/0204051.

[7] S. Frolov and A. A. Tseytlin, "Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$," JHEP **0206**, 007 (2002), hep-th/0204226.

[8] K. Pohlmeyer, "Integrable Hamiltonian Systems And Interactions Through Quadratic Constraints," Commun. Math. Phys. **46**, 207 (1976). M. Luscher and K. Pohlmeyer, "Scattering Of Massless Lumps And Nonlocal Charges In The Two-Dimensional Classical Nonlinear Sigma Model," Nucl. Phys. B **137**, 46 (1978). H. Eichenherr and M. Forger, "On The Dual Symmetry Of The Nonlinear Sigma Models," Nucl. Phys. B **155**, 381 (1979).

[9] B.M. Barbashov and V.V. Nesterenko, "Relativistic String Model In A Space-Time Of A Constant Curvature, Commun. Math. Phys. **78**, 499 (1981). B.M. Barbashov and V.V. Nesterenko, "Introduction To The Relativistic String Theory," *Singapore, World Scientific* (1990) 249 p.

[10] H. J. De Vega and N. Sanchez, "Exact integrability of strings in D-Dimensional De Sitter space-time," Phys. Rev. **D47**, 3394 (1993). M. Zyskin, "On gauge fields: Strings duality as an integrable system," hep-th/0004105.

[11] G. Mandal, N. V. Suryanarayana and S. R. Wadia, "Aspects of semiclassical strings in AdS(5)," Phys. Lett. B **543**, 81 (2002), hep-th/0206103.

[12] H. J. de Vega and I. L. Egusquiza, "Planetoid String Solutions in 3 + 1 Axisymmetric Spacetimes," Phys. Rev. D **54**, 7513 (1996), hep-th/9607056.

[13] C. Neumann, "De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur," J. Reine Angew. Math. **56**, (1859) 46-63.

[14] A. M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Springer Verlag, 1990.

[15] O. Babelon and M. Talon, "Separation Of Variables For The Classical And Quantum Neumann Model," Nucl. Phys. B **379**, 321 (1992), hep-th/9201035.

[16] D. Mumford, Tata Lecture Notes on Theta Functions, Springer Verlag, 1994.

[17] A. A. Tseytlin, "Semiclassical quantization of superstrings: $AdS_5 \times S^5$ and beyond," Int. J. Mod. Phys. A **18**, 981 (2003), hep-th/0209116.

[18] J. G. Russo, "Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$," JHEP **0206**, 038 (2002), hep-th/0205244.

[19] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," JHEP **0303**, 013 (2003), hep-th/0212208.

[20] N. Beisert, C. Kristjansen and M. Staudacher, "The dilatation operator of N = 4 super Yang-Mills theory," hep-th/0303060. N. Beisert, "The complete one-loop dilatation operator of N = 4 super Yang-Mills theory," hep-th/0307015.

[21] N. Beisert and M. Staudacher, "The N = 4 SYM integrable super spin chain," hep-th/0307042.

[22] K. Uhlenbeck, Equivariant harmonic maps into spheres, Proc. Tulane Conference on Harmonic Maps.

[23] M. Kruczenski, "A note on twist two operators in N = 4 SYM and Wilson loops in Minkowski signature," JHEP **0212**, 024 (2002), hep-th/0210115. Y. Makeenko, "Light-cone Wilson loops and the string / gauge correspondence," JHEP **0301**, 007 (2003), hep-th/0210256.

[24] A. V. Kotikov, L. N. Lipatov and V. N. Velizhanin, "Anomalous dimensions of Wilson operators in N = 4 SYM theory," Phys. Lett. B **557**, 114 (2003), hep-ph/0301021. A. V. Belitsky, A. S. Gorsky and G. P. Korchemsky, "Gauge / string duality for QCD conformal operators," hep-th/0304028.

[25] H. Verlinde, "Bits, matrices and 1/N," hep-th/0206059. D. Vaman and H. Verlinde, "Bit strings from N = 4 gauge theory," hep-th/0209215. U. Danielsson, F. Kristiansson, M. Lubcke and K. Zarembo, "String bits without doubling," hep-th/0306147.

[26] R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in $AdS_5 \times S^5$ background," Nucl. Phys. B **533**, 109 (1998), hep-th/9805028.

[27] I. Bena, J. Polchinski and R. Roiban, "Hidden symmetries of the $AdS_5 \times S^5$ superstring," hep-th/0305116.