Multi-spin string solutions in $AdS_5 \times S^5$

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Abstract

Motivated by attempts to extend AdS/CFT duality to non-BPS states we consider classical closed string solutions with several angular momenta in different directions of AdS_5 and S^5. We find a novel solution describing a circular closed string located at a fixed value of AdS_5 radius while rotating simultaneously in two planes in AdS_5 with equal spins S. This solution is a direct generalization of a two-spin flat-space solution where the string rotates in two orthogonal planes while always lying on a 3-sphere. Similar solution exists for a string rotating in S^5: it is parametrized by the angular momentum J of the center of mass and two equal $SO(6)$ angular momenta $J_2 = J_3 = J'$ in the two rotation planes. The remarkably simple case is of $J = 0$ where the energy depends on J' as $E = \sqrt{(2J')^2 + \lambda}$ (λ is the string tension or 't Hooft coupling). We discuss interpolation of the $E(J')$ formula to weak coupling by identifying the gauge theory operator that should be dual to the corresponding semiclassical string state and utilizing existing results for its perturbative anomalous dimension. This opens up a possibility of studying AdS/CFT duality in this new non-BPS sector. We also investigate small fluctuations and stability of these classical solutions and comment on several generalizations.

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1. Introduction

Generalizing AdS/CFT duality to non-BPS string mode sector can be guided by semiclassical considerations, as suggested in [1,2]. Identifying classical solitonic solutions of $AdS_5 \times S^5$ sigma model carrying basic global charges is important in order to understand the structure of the full string theory spectrum. In general, the states belonging to representations of the isometry group $SO(2,4) \times SO(6)$ are expected to be classified by 6=3+3 charges corresponding to Cartan subalgebra generators, $(E, S_1, S_2; \ J_1, J_2, J_3)$. Here S_1 and S_2 are the two spins of the conformal group (labelling representations of $SO(4)$ isometry of S^3 subspace) and J_i are the three angular momenta of the S^5 isometry group. One may search for classical string solutions which have minimal energy for given values of the 5 charges, $E = E(S_1, S_2, J_1, J_2, J_3)$.[1] The importance of such solutions (in contrast to various other oscillating or pulsating solutions) is that having non-zero global charges simplifies identification of the corresponding dual CFT operators.

Particular classical string solutions with special combinations of these charges were discussed in the past. Point-like string solution (geodesic) lying in AdS_5 does not carry intrinsic spin. Geodesic running in S^5 can carry only *one* component of momentum in S^5 (e.g., $J = J_1$), and expansion of $AdS_5 \times S^5$ string theory near such geodesic was studied in [1]. Extended string solution describing folded closed string rotating in a plane in AdS_5 carries single spin, e.g., $S = S_1$ [3,2]. One can boost the center of mass of the string rotating in AdS_5 along a circle of S^5 getting a solution with two charges (S, J) [4]. Alternatively, one can construct a solution describing folded string rotating about a pole of S^5 [2]; while it carries again only one component (say, $J' = J_2$) of the $SO(6)$ spin it is not equivalent to a point-like orbiting solution.[2] An interpolating solution with the three charges (S, J, J') was constructed in [6]. One may think that while in general there should certainly be extended string solutions with more spins in either or both AdS_5 and S^5 spaces, they may be difficult to construct explicitly, and also their AdS/CFT interpretation may be unclear. Here we would like to point out that such more general solutions are actually easy to find in the special case when the two spins S_1, S_2 in AdS_5 or the two of the three

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[1] The Casimir operators should be functions of these charges. In general, string theory "knows" about many other conserved charges, being integrable. Here we concentrate on most obvious local charges.

[2] The corresponding string vertex operators [5] as well as dual gauge theory operators should be different, with the "point-like" (BPS) one having minimal energy (dimension) for a given value of the angular momentum (R-charge).

angular momenta in S^5 (e.g., J_2 and J_3) are *equal*. The analytic form of the solution with $S_1 = S_2 \equiv S$ turns out to be much simpler than in the single spin case of [3,2]: it is a direct generalization of the flat-space solution describing circular string rotating simultaneously in the two orthogonal spatial planes with equal angular momenta. Such "self-dual" string always lies on an S^3 surface in Minkowski space, and thus it can be easily "embedded" into AdS_5 space using global (or covering) coordinates. The string is positioned at a fixed value of the AdS_5 radius $\rho = \rho_0$, being stabilized by rotation. Similar string solution exists for a more general class of 5-d metrics $ds^2 = -g(\rho)dt^2 + d\rho^2 + h(\rho)d\Omega_3$, but not in AdS_3 or AdS_4: to stabilize the circular string at a fixed value of ρ one needs at least *two* equal spin components.[3]

We will show that this stationary solution is stable under small perturbations if the spins $S_1 = S_2 = S$ are smaller than a critical value. The energy E is an algebraic function of S. For a small-radius string having small $\mathcal{S} \equiv \frac{S}{\sqrt{\lambda}} \ll 1$, i.e. located close to the center of AdS_5, one finds the usual Regge trajectory relation, $E = (\sqrt{\lambda}\ 4S)^{1/2} + ...$ For large string located close to the boundary of AdS_5 we get $E = 2S + c_1(\lambda S)^{1/3} + ...$ $(\mathcal{S} \gg 1)$. [4] The leading $E = 2S$ behaviour is the same as for the single-spin folded string solution [3], but the subleading correction here is proportional to $S^{1/3}$ instead of $\ln S$ in [2].

Modulo the problem of instability of the two-spin solution at large $S \gg \sqrt{\lambda}$, it is natural to conjecture (see section 5) that the Euclidean gauge theory operator in R^4 that should be dual to this two-spin string state should have the form $\mathrm{tr}[\Phi_M(D_1 + iD_2)^S(D_3 + iD_4)^S\Phi_M] + ...$, where Φ_M are SYM theory scalars. It would be interesting to find how its perturbative anomalous dimension Δ depends on large S. An interpolation formula suggested by the semiclassical analysis is $\Delta(S) = S + f(\lambda)S^{1/3} + ...$, $S \gg 1$, where $f(\lambda)_{\lambda \ll 1} = a_1\lambda + a_2\lambda^2 + ...$, and $f(\lambda)_{\lambda \gg 1} = \lambda^{1/3}[c_1 + \frac{c_2}{\sqrt{\lambda}} + ...]$, As we shall see below (following a similar discussion in the single-spin case [4]), if one formally ignores the instability, the 1-loop string correction to the energy of the semiclassical 2-spin solution does scale with spin as $S^{1/3}$. However, it seems implausible that a perturbative anomalous dimension may depend on the spin as a fractional power. We shall comment on this further in section 3.

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[3] Note that a similar (but non-rotating) "winding" string configuration near the boundary of AdS_3 is stabilized by the B_{mn} flux [7].

[4] Interestingly, the power of λ that multiplies a power of S in the semiclassical correction to $E - 2S$ is the same as the power of S only when it is equal to 1/3. One may speculate that a weak-coupling interpolation of this formula is then $E - 2S = (1 + c\lambda S)^{1/3}$.

The AdS/CFT correspondence seems easier to establish in a different sector corresponding to the string solution carrying $SO(6)$ spins, i.e. rotating in S^5. Indeed, there exists a similar circular string solution with two equal angular momenta in S^5: the rotating string moves on S^3 within S^5, with the radius of the string or of S^3 being related to the value of J'. In addition, the center of mass of the string may be rotating along another circle of S^5, leading to a particular string solution with all the three S^5 charges being non-zero ($J_3 = J$, $J_1 = J_2 = J'$). In the most transparent case of $J = 0$ when the string has maximal size (so that $2J' \geq \sqrt{\lambda}$) the energy turns out to depend on J' in a remarkably simple way: $E = \sqrt{(2J')^2 + \lambda}$. While the solution with $J = 0$ turns out to be unstable, there is always a non-trivial region of stability when $J \neq 0$, and there are stable solutions with both J and J' being large compared to $\sqrt{\lambda}$ [8].

We suggest that the corresponding dual CFT operator (having minimal canonical dimension for given values of R-charges J and $J_2 = J_3 = J'$) should be $\mathrm{tr}[(\Phi_1 + i\Phi_2)^J(\Phi_3 + i\Phi_4)^{J'}(\Phi_5 + i\Phi_6)^{J'}] + ...$, where dots stand for appropriate permutations of factors. For $J = 0$ the above semiclassical formula $E(J')$ suggests that for large J' the anomalous dimension of such operator should be $\Delta = 2J' + \frac{f(\lambda)}{4J'} + ...$ We conjecture that $f(\lambda)$ should start with a λ-term both at strong and weak coupling, and we propose to check this against known [9,10] perturbative results. Another interesting direction is to consider the limit $J \gg J'$ and relate the resulting expression for the energy to the dimensions of operators in the sector studied in [1].

The paper is organized as follows. In section 2 we first describe the two-spin closed string solution in flat space and then generalize it to AdS_5 and S^5 spaces viewed as hypersurfaces in $R^{2,4}$ and R^6 respectively. In section 3 we rederive the AdS_5 solution and study it in more detail using explicitly the standard set of global coordinates in AdS_5 space. In particular, we obtain the action for small fluctuations near this solution and find that the solution is stable only if the spin is bounded from above, $S \leq a\sqrt{\lambda}$, where a is of order 1.

In section 4 we present a similar analysis of the S^5 solution. We also note that there exists a "combined" solution where string rotates in both AdS_5 and S^5 and thus carries 5 charges: $S_1 = S_2, J = J_1$ and $J_2 = J_3$. We observe that there are two branches of the solution with $S = 0$ and $J = 0$ with different dependence of the energy on J': one for $J' \leq \frac{1}{2}\sqrt{\lambda}$ and another for $J' \geq \frac{1}{2}\sqrt{\lambda}$. The solution with $J' \leq \frac{1}{2}\sqrt{\lambda}$ is found to be stable for $J' \leq \sqrt{\lambda}\,\frac{3}{8}$, while the solution with large J' and $J = 0$ turns out to be unstable. Given that there are more general stable solutions with both J' and J being large [8], we

shall assume that the instability may not preclude one from using the remarkably simple solution with $J = 0$, $J' \gg \sqrt{\lambda}$ in the context of the AdS/CFT duality. For example, there may exist a more complicated (e.g., pulsating) solution with the same quantum numbers $J = 0$, $J' \neq 0$ whose basic features like energy dependence on J' are the same as of our simple solution. With this motivation in mind we comment on the form of the 1-loop sigma model correction to the energy, and conjecture about the existence of an interpolation formula for $E(J', \lambda)$.

In section 5 we discuss the structure of the (Euclidean) CFT operators which should be dual to the semiclassical two-spin states. We mention, in particular, that the 1-loop results of [9,10] may be used to check our conjecture that the anomalous dimension of the scalar SYM operator with $J = 0$, $J_2 = J_3 = J'$ (with lowest dimension above the BPS bound) should scale with $J' \gg 1$ as $\Delta = 2J' + \frac{\lambda}{4J'} + ...$ not only at strong but also at weak coupling. Section 6 contains some remarks on generalizations and open problems.

In Appendix A we give some details of the stability analysis for both the AdS_5 and S^5 solutions (this will be discussed in more detail in [8]). In Appendix B we derive the quadratic fermionic part of the $AdS_5 \times S^5$ Green-Schwarz action that supplements the bosonic fluctuation actions in sections 3 and 4. The total action should be the starting point for a computation of 1-loop corrections to the energies of our solutions following [4]. We check, in particular, that the fermionic mass matrix contribution to the logarithmic divergences cancels against the bosonic one, in agreement with the conformal invariance of the $AdS_5 \times S^5$ superstring sigma model action. In Appendix C we sketch the derivation of the bosonic fluctuation actions in the conformal gauge, and check consistency with the static gauge results for the fluctuation actions used in sections 3 and 4. Appendix D contains standard facts about relation between Young tableau and Dynkin labels of representations of $SU(4)$ group which is used to identify the operators on the gauge theory side.

2. Two-spin solution in flat space and its AdS_5 or S^5 generalizations

2.1. Flat case

Let us start with closed bosonic string solutions in flat Minkowski space. In orthogonal gauge, string coordinates are given by solutions of free 2-d wave equation, i.e. by combinations of $e^{in(\tau \pm \sigma)}$, subject to the standard constraints $\dot{X}^2 + X'^2 = 0$, $\dot{X} X' = 0$. Let us consider, in particular, a closed string (with its center of mass at rest at the origin of

the cartesian coordinate system) which is rotating in the two orthogonal spatial planes 12 and 34. From the closed string equations on a 2-cylinder $(\tau, \sigma \equiv \sigma + 2\pi)$ with Minkowski signature in both target space and world sheet one finds

$$X_0 = \kappa\tau \ , \qquad X = X_1 + iX_2 = \ r_1(\sigma) \ e^{i\phi(\tau)} \ , \qquad Y = X_3 + iX_4 = \ r_2(\sigma) \ e^{i\varphi(\tau)} \ , \qquad (2.1)$$

$$\phi = n_1\tau \ , \qquad \varphi = n_2\tau \ , \qquad r_1 = a_1 \sin(n_1\sigma) \ , \qquad r_2 = a_2 \sin[n_2(\sigma + \sigma_0)] \ . \qquad (2.2)$$

Here σ_0 is an arbitrary integration constant, and n_i are arbitrary integer numbers. In what follows we assume that n_i are positive. The relation between a_i, n_i and κ follows from the conformal gauge constraint:

$$\kappa^2 = n_1^2 a_1^2 + n_2^2 a_2^2 \ . \qquad (2.3)$$

The energy and the two spins are

$$E = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \ \dot{X}_0 = \frac{\kappa}{\alpha'} \ , \qquad (2.4)$$

$$S_1 = \frac{i}{4\pi\alpha'} \int_0^{2\pi} d\sigma \ (X\dot{\bar{X}} - \bar{X}\dot{X}) = \frac{n_1 a_1^2}{2\alpha'} \ , \qquad S_2 = S_1(X \to Y) = \frac{n_2 a_2^2}{2\alpha'} \ , \qquad (2.5)$$

i.e.

$$E = \sqrt{\frac{2}{\alpha'}(n_1 S_1 + n_2 S_2)} \ . \qquad (2.6)$$

To get the states on the leading Regge trajectory (having minimal energy for given values of the spins) one is to choose $n_1 = n_2 = 1$.

While for a special solution in (2.2) with $\sigma_0 = 0$ the string is folded, for generic values of σ_0 it has the form of an ellipse. Another remarkable special case is when $n_2\sigma_0 = \frac{\pi}{2}$, i.e. when $|X| \sim \sin\sigma$ but $|Y| \sim \cos\sigma$, and

$$n_1 = n_2 = n \ , \qquad a_1 = a_2 = \frac{\kappa}{\sqrt{2n}} \ . \qquad (2.7)$$

Then the string becomes *circular* and, while rotating, it always *lies on* S^3 in R^4 space formed by (X_1, X_2, X_3, X_4). Indeed, the radius in R^4 then remains *constant*

$$|X(\tau, \sigma)|^2 + |Y(\tau, \sigma)|^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 = \frac{\kappa^2}{2n^2} \ . \qquad (2.8)$$

In this case the two spins are equal

$$S_1 = S_2 = \frac{\kappa^2}{4n\alpha'} \equiv S \ , \qquad E = \sqrt{\frac{4n}{\alpha'}}S \ . \qquad (2.9)$$

Thus, the $X \leftrightarrow Y$ symmetric circular string rotating in the two orthogonal planes and corresponding to a state on the leading Regge trajectory $(n_1 = n_2 = 1)$ is described by the following solution

$$X_0 = \kappa\tau \ , \qquad X_1 + iX_2 = \ \frac{\kappa}{\sqrt{2}} \sin\sigma \ e^{i\tau} \ , \qquad X_3 + iX_4 = \ \frac{\kappa}{\sqrt{2}} \cos\sigma \ e^{i\tau} \ . \qquad (2.10)$$

The crucial observation is that such solution can be easily generalised to a solution describing a string rotating in any homogeneous space containing S^3, in particular, AdS_n or S^n with $n \geq 5$.

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2.2. AdS_5 case

Indeed, let us consider the AdS_5 space described as a hypersurface in 6-dimensional space $R^{2,4}$:

$$X_M X_M \equiv \eta_{MN} X^M X^N = -X_5^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -1 . \tag{2.11}$$

The corresponding string sigma model Lagrangian is (Λ is a Lagrange multiplier field)

$$I = -\frac{R^2}{4\pi\alpha'} \int d\tau d\sigma \ L , \qquad L = \partial_a X_M \partial^a X_M + \Lambda(X_M X_M + 1) . \tag{2.12}$$

The equations of motion in the orthogonal gauge then are

$$-\partial^2 X_M + \Lambda X_M = 0 , \qquad X_M X_M = -1 , \qquad \Lambda = \partial_a X_M \partial^a X_M , \tag{2.13}$$

$$\dot{X}_M \dot{X}_M + X'_M X'_M = 0 , \qquad \dot{X}_M X'_M = 0 . \tag{2.14}$$

A special class of solutions of these non-linear equations is characterised by the property $\Lambda = $ const. It is natural to organize the six coordinates X_M into the three 2-planes or complex lines,

$$W \equiv X_5 + iX_0 , \qquad X \equiv X_1 + iX_2 , \qquad Y \equiv X_3 + iX_4 , \qquad |W|^2 - |X|^2 - |Y^2| = 1 . \tag{2.15}$$

Then it is easy to check that the following configuration is an example of the solution of (2.13),(2.14) with $\Lambda = \kappa^2 = $ const (cf. (2.10))[5]

$$W = \cosh\rho_0 \ e^{i\kappa\tau} , \qquad X = \sinh\rho_0 \ \sin\sigma \ e^{i\omega\tau} , \qquad Y = \sinh\rho_0 \ \cos\sigma \ e^{i\omega\tau} , \tag{2.16}$$

where ρ_0 and ω are related to κ as follows

$$\omega^2 = \kappa^2 + 1 , \qquad \sinh^2\rho_0 = \frac{1}{2}\kappa^2 . \tag{2.17}$$

Notice that the expressions for X and Y look exactly the same as in the flat space solution (2.10). Indeed, since $|W|^2 = \cosh^2\rho_0 = 1 + \kappa^2$ the AdS_5 constraint $|W|^2 - |X|^2 - |Y^2| = 1$ is automatically satisfied.

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[5] It would be interesting to find other solutions with $\Lambda = $ const. It might even be possible to classify all such solutions.

This solution describes a circular closed string rotating in the 12 and 34 planes in the global AdS_5 time $t = \kappa\tau$. It will be rederived in the next section using explicitly the standard set of global AdS_5 coordinates $(t, \rho, \theta, \phi, \varphi)$ related to X_M as follows:

$$W = \cosh\rho \; e^{it} \; , \qquad X = \sinh\rho \; \sin\theta \; e^{i\phi} \; , \qquad Y = \sinh\rho \; \cos\theta \; e^{i\varphi} \; . \qquad (2.18)$$

It is clear that the string described by this solution has equal angular momenta in the 12 and 34 planes. In this $R^{2,4}$ embedding representation it is easy to identify the charges of the isometry group $SO(2,4)$ of AdS_5 that are non-vanishing on this solution. In general, the 15 rotation generators J_{MN} of $SO(2,4)$ can be related to the conformal group generators as follows (see, e.g., [11])

$$J_{\mu\nu} = M_{\mu\nu} \; , \qquad J_{\mu 4} = \frac{1}{2}(K_\mu - P_\mu) \; , \qquad J_{\mu 5} = \frac{1}{2}(K_\mu + P_\mu) \; , \qquad J_{54} = D \; , \qquad (2.19)$$

where $\mu, \nu = 0, 1, 2, 3$. We can identify the standard spin with $S_1 = M_{12} = J_{12}$, the second (conformal) spin with $S_2 = J_{34} = \frac{1}{2}(K_3 - P_3)$, and finally the rotation generator in the 05 plane with the global AdS_5 energy, $E = J_{05} = \frac{1}{2}(K_0 + P_0)$.[6] In the present case the only non-vanishing charges are J_{50} and J_{12}, J_{34}, i.e. the energy and the two spins

$$E = \sqrt{\lambda} \; \kappa(1 + \frac{1}{2}\kappa^2) \; , \qquad S \equiv S_1 = S_2 = \frac{1}{4}\sqrt{\lambda} \; \kappa^2 \sqrt{\kappa^2 + 1} \; . \qquad (2.20)$$

The three Casimir operators of $SO(2,4)$ are then expressed in terms of κ. Note also that $E^2 - (2S)^2 = \sqrt{\lambda} \; (\frac{3}{4}\kappa^4 + \kappa^2) \geq 0$. For small κ we get $E \approx \sqrt{4\sqrt{\lambda} \; S}$, i.e. the usual Regge trajectory relation, while for large κ we have $E \approx 2S = S_1 + S_2$, similar to the single-spin case in [2].

It may be worth stressing the following point. We consider classical string solutions with large angular momenta. In quantum theory such a classical solution should correspond to a vector of an irreducible representation labeled by the charges which would then be (half-)integer. Since on our clasical solutions S_1 and S_2 (and E) are the only nonvanishing charges among the relevant generators J_{MN}, we may use them to label representations of the $SO(4) = SU(2) \times SU(2)$ subgroup of the conformal group $SO(2,4)$. Assuming that $S_1 \geq S_2$, the usual labels j_1 and j_2 of $SU(2) \times SU(2)$ can be expressed in terms of S_1

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[6] After the Euclidean continuation $X_0 \to iX_{0E}$ and mapping to $R \times S^3$ it is natural to classify the representations of the conformal group in terms of maximal compact subgroup $SO(4) \times SO(2)$, or $SU(2) \times SU(2) \times SO(2)$. Exchanging X_{0E} with X_4 exchanges the generator $J_{54} = D$ with $J_{05} = \frac{1}{2}(P_0 + K_0) = E$.

and S_2 as $2j_1 = S_1 + S_2$, $2j_2 = S_1 - S_2$.[7] Moreover, the fact that only S_1 and S_2 do not vanish means that in quantum theory the corresponding quantum state should be the highest weight vector of an $SO(4)$ representation. A similar remark will apply to the case of solutions with angular momentum in S^5 we study below: a similar relation will exist between the only non-vanishing $SO(6)$ charges $J_i = (J_{12}, J_{34}, J_{56})$ which are directly (up to permutations) related to the Young tableau labels, and the Dynkin labels of $SU(4)$ representations.

2.3. S^5 case

Let us now consider an S^5 analogue of the flat-space solution (2.1). Here all is similar to the AdS_5 case, apart from the fact that the decoupled time coordinate t is introduced in addition to the S^5 directions X_A

$$X_A X_A = X_1^2 + ... + X_6^2 = |Z|^2 + |X|^2 + |Y|^2 = 1 \ ,$$

$$Z = X_1 + iX_2 \ , \qquad X = X_3 + iX_4 \ , \qquad Y = X_5 + iX_6 \ . \qquad (2.21)$$

The relation to the standard 5 angles $(\gamma, \psi, \varphi_1, \varphi_2, \varphi_3)$ of S^5 is

$$Z = \cos\gamma \ e^{i\varphi_1} \ , \qquad X = \sin\gamma \ \cos\psi \ e^{i\varphi_2} \ , \qquad Y = \sin\gamma \ \sin\psi \ e^{i\varphi_3} \ . \qquad (2.22)$$

A particular solution of the S^5 sigma model equations (which are the direct analogues of (2.13),(2.14)) is (cf. (2.10),(2.16))

$$t = \kappa\tau \ , \qquad Z = \cos\gamma_0 \ e^{i\nu\tau} \ , \qquad X = \sin\gamma_0 \ \cos\sigma \ e^{iw\tau} \ , \qquad Y = \sin\gamma_0 \ \sin\sigma \ e^{iw\tau} \ , \qquad (2.23)$$

where $\Lambda = \nu^2$ and κ and ν are independent parameters while the constants γ_0 and w are expressed in terms of them (cf. (2.17))

$$w^2 = 1 + \nu^2 \ , \qquad \sin^2\gamma_0 = \frac{1}{2}(\kappa^2 - \nu^2) \ . \qquad (2.24)$$

Here the energy of the solution is $E = \sqrt{\lambda} \ \kappa$, and in addition we have 3 non-vanishing components of the $SO(6)$ angular momentum tensor J_{AB} :

$$J_1 = J_{12} \ , \qquad J_2 = J_{34} \ , \qquad J_3 = J_{56} \ ,$$

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[7] The charges S_1 and S_2 are, in fact, directly related to the Gelfand–Zeitlin labels of representations of $SO(4)$.

$$J \equiv J_1 = \sqrt{\lambda} \; \nu[1 - \frac{1}{2}(\kappa^2 - \nu^2)] \;, \qquad J' \equiv J_2 = J_3 = \frac{1}{4}\sqrt{\lambda} \; (\kappa^2 - \nu^2) \; \sqrt{\nu^2 + 1} \;. \quad (2.25)$$

This solution describes a circular closed string rotating (with equal speeds) in the two planes in S^3 within S^5, with its center of mass orbiting along the orthogonal circle of S^5. When embedded into $AdS_5 \times S^5$ it will be located at the origin $\rho = 0$ of AdS_5. We shall return to the discussion of this solution in section 4.1.

Notice that (2.24) implies the bound

$$\nu^2 \leq \kappa^2 \leq \nu^2 + 2 \;. \qquad (2.26)$$

One limiting case is $\kappa = \nu$ when the string is point-like and has no spin J', i.e. moves along the geodesic discussed in [1] and thus has $E = J$. The other is $\nu^2 = \kappa^2 - 2$ so that $\kappa^2 \geq 2$. [8] Here $J = 0$ and the string has maximal size ($\gamma_0 = \frac{\pi}{2}$), while the energy and the two equal $SO(6)$ angular momenta J_1, J_2 take values

$$J' = \frac{1}{2}\sqrt{\lambda} \; \sqrt{\kappa^2 - 1} \; \geq \frac{1}{2}\sqrt{\lambda} \;, \qquad E = \sqrt{\lambda} \, \kappa = \sqrt{(2J')^2 + \lambda} \; \geq \sqrt{2\lambda} \;. \qquad (2.27)$$

This expression for $E(J')$ is very simple and interesting, and we shall return to the discussion of it in sections 4.2 and section 5.

Another special case with $J = 0$ is $\nu = 0$: here $\kappa^2 \leq 2$ and $w = 1$ so that

$$J' = \frac{1}{4}\sqrt{\lambda} \, \kappa^2 \; \leq \; \frac{1}{2}\sqrt{\lambda} \;, \qquad E = \sqrt{4\sqrt{\lambda} \, J'} \; \leq \; \sqrt{2\lambda} \;. \qquad (2.28)$$

Remarkably, here the dependence of the energy on the $SO(6)$ spin is exactly the same as for the leading Regge trajectory in flat space! This is not too surprising since the corresponding string solution (2.23) is then the direct embedding of the flat space solution (2.10) into the S^3 part of S^5 (note that the Lagrange multiplier Λ in the S^5 analog of (2.13) vanishes when $\nu = 0$ and so X_M satisfy the flat-space equations of motion).

One can also construct more general multi-spin solution which has all 5 charges being non-vanishing – $S_1 = S_2$ in AdS_5, and $J = J_1$ and $J_2 = J_3$ in S^5. It will be given by a direct combination of (2.16) and (2.23) with the parameters $\kappa, \nu, \rho_0, \gamma_0$ related by the conformal gauge constraint as $\kappa^2 = \nu^2 + 2\sinh^2 \rho_0 + 2\sin^2 \gamma_0$. The expressions for the

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[8] Here one cannot take the flat-space limit in which $\kappa \to 0$.

$SO(2,4) \times SO(6)$ charges are essentially the same as in (2.20) and (2.25), i.e. (see also section 4.2)

$$E = \sqrt{\lambda} \, \cosh^2 \rho_0 \, \kappa \, , \qquad S = S_1 = S_2 = \frac{1}{2}\sqrt{\lambda} \, \sinh^2 \rho_0 \, \sqrt{\kappa^2 + 1} \, , \qquad (2.29)$$

$$J = J_1 = \sqrt{\lambda} \, \cos^2 \gamma_0 \, \nu \, , \qquad J' = J_2 = J_3 = \frac{1}{2}\sqrt{\lambda} \, \sin^2 \gamma_0 \, \sqrt{\nu^2 + 1} \, . \qquad (2.30)$$

The parameters ρ_0, ν, γ_0 and thus κ can be expressed in terms of S, J and J', giving the general expression for the energy $E = E(S, J, J')$.

One can also study other similar multi-charge solutions, like an interpolation between the J, J' solution on S^5 and the single-spin $S_1 \neq 0, S_2 = 0$ solution [3,2] in AdS_5 (in this case the radial coordinate ρ will no longer be constant). Then one will find $E = E(S_1, J, J')$ which will be a generalization to $J' \neq 0$ of the expression obtained in [2,4].

3. Two-spin solution in AdS_5 in global coordinates and stability

Here we shall rederive the AdS_5 two-spin solution starting from a more general ansatz with two unequal rotation parameters and then study small fluctuations near the resulting solution.

The string action in AdS_5 written in the conformal gauge in terms of independent global coordinates x^m is

$$I = -\frac{\sqrt{\lambda}}{4\pi} \int d^2\xi \, G_{mn}^{(AdS_5)}(x) \partial_a x^m \partial^a x^n \, , \qquad \sqrt{\lambda} \equiv \frac{R^2}{\alpha'} \qquad (3.1)$$

Here $\xi^a = (\tau, \sigma)$, $\sigma \equiv \sigma + 2\pi$. We shall use the Minkowski signature in both target space and world sheet, so that in conformal gauge $\sqrt{-g}g^{ab} = \eta^{ab} = \text{diag(-1,1)}$. The equations of motion following from the action should be supplemented by the conformal gauge constraints.

We shall use the following explicit parametrization of the (unit-radius) AdS_5 metric (related to the embedding coordinates of the previous section by (2.18))

$$(ds^2)_{AdS_5} = G_{mn}^{(AdS_5)}(x)dx^m dx^n = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3 \, , \qquad (3.2)$$

$$d\Omega_3 = d\theta^2 + \sin^2 \theta \, d\phi^2 + \cos^2 \theta \, d\varphi^2 \, . \qquad (3.3)$$

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This metric has translational isometries in t, ϕ, φ so that a general string solution should possess the following three integrals of motion:

$$E = P_t = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \; \cosh^2 \rho \; \partial_0 t \equiv \sqrt{\lambda} \, \mathcal{E} \; , \tag{3.4}$$

$$S_1 = P_\phi = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \; \sinh^2 \rho \; \sin^2 \theta \; \partial_0 \phi \equiv \sqrt{\lambda} \, \mathcal{S}_1 \; , \tag{3.5}$$

$$S_2 = P_\varphi = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \; \sinh^2 \rho \; \cos^2 \theta \; \partial_0 \varphi \equiv \sqrt{\lambda} \, \mathcal{S}_2 \; . \tag{3.6}$$

The first integral is the space-time energy, and the second and third ones are the spins associated with rotations in ϕ and φ.

3.1. Solution

Our aim is to look for a solution describing a closed string rotating in both ϕ and φ, thus generalizing the single-spin solution of [3,2]. A natural ansatz for such a solution is

$$t = \kappa \tau \; , \quad \phi = \omega_\phi \tau \; , \quad \varphi = \omega_\varphi \tau \; , \quad\quad \kappa, \; \omega_\phi, \; \omega_\varphi = \text{const} \; ,$$

$$\rho = \rho(\sigma) = \rho(\sigma + 2\pi) \; , \quad \theta = \theta(\sigma) = \theta(\sigma + 2\pi) \; , \tag{3.7}$$

where ρ and θ are subject to the corresponding second-order equations (prime denotes derivative over σ)

$$\rho'' = \sinh \rho \; \cosh \rho \; (\kappa^2 + \theta'^2 - \omega_\phi^2 \sin^2 \theta - \omega_\varphi^2 \cos^2 \theta) \; , \tag{3.8}$$

$$(\sinh^2 \rho \; \theta')' = \frac{1}{2}(\omega_\varphi^2 - \omega_\phi^2) \sinh^2 \rho \; \sin 2\theta \quad . \tag{3.9}$$

The first of the conformal gauge constraints

$$G_{mn}^{(AdS_5)}(x)(\dot{x}^m \dot{x}^n + x'^m x'^n) = 0 \; , \quad\quad G_{mn}^{(AdS_5)}(x)\dot{x}^m x'^n = 0 \; , \tag{3.10}$$

then implies that $\rho(\sigma)$ and $\theta(\sigma)$ must satisfy the following 1-st order equation

$$\rho'^2 + \sinh^2 \rho \; \theta'^2 = \kappa^2 \cosh^2 \rho - \sinh^2 \rho \; (\omega_\phi^2 \sin^2 \theta + \omega_\varphi^2 \cos^2 \theta) \; . \tag{3.11}$$

Unfortunately, we do not know how to solve the system of non-linear equations (3.8),(3.11) for generic values of the frequencies ω_φ and ω_ϕ, so in what follows we shall assume that the frequencies are equal:

$$\omega_\varphi \; = \; \omega_\phi \; = \; \omega \; . \tag{3.12}$$

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Then (3.9) implies

$$\theta' = \frac{c}{\sinh^2 \rho} , \qquad c = \text{const.} \tag{3.13}$$

The special solution of (3.9) with $c = 0$, i.e. $\theta = \text{const}$ leads us back to the single-spin case of [3,2]: one can make a global $SO(3)$ rotation (or redefinition of ϕ, φ) to put the rotating string in a single plane. If one assumes that $\theta' \neq 0$, one can show by a detailed analysis that there exists no solution to (3.8),(3.11) with non-constant ρ, i.e. one must set [9]

$$\rho(\sigma) = \rho_0 = \text{const} . \tag{3.14}$$

Then the equations (3.8) and (3.11) take the form

$$\theta'^2 = \omega^2 - \kappa^2 , \qquad \theta'^2 = \coth^2 \rho_0 \, \kappa^2 - \omega^2 . \tag{3.15}$$

The solution to these equations is given by

$$\theta = n\sigma , \quad \kappa^2 = 2n^2 \sinh^2 \rho_0 , \quad \omega^2 = n^2(1 + 2\sinh^2 \rho_0) = \kappa^2 + n^2 . \tag{3.16}$$

Here n is an arbitrary integer representing how many times the string "winds" around the θ-circle (cf. (2.18)). The parameter ρ_0 determines the radius ($\sinh \rho_0$) of a circular string rotating in S^3. It is remarkable that one needs two rotation parameters to be non-zero in order to stabilize the size of the string at fixed value of the AdS_5 radius ρ.

In what follows we shall consider the case of (cf.(2.17))

$$n = 1 , \qquad \text{i.e.} \qquad \kappa = \sqrt{2} \sinh \rho_0 , \qquad \omega^2 = \kappa^2 + 1 . \tag{3.17}$$

In the flat space limit ($\kappa \to 0$, $\rho_0 \to 0$) this corresponds to a state on the leading Regge trajectory, i.e. having minimal energy for a given spin. The dependence on the "winding number" n can be easily restored in all the equations below.

The integrals of motion (3.4), (3.5), (3.6) on this solution are given by the same expressions as in (2.20) (we consider the values rescaled by the string tension $\sqrt{\lambda}$ as defined in (3.4),(3.5),(3.6))

$$\mathcal{E} = \kappa \cosh^2 \rho_0 = \kappa(1 + \frac{1}{2}\kappa^2) , \tag{3.18}$$

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[9] Using (3.13) we get from (3.11): $\rho'^2 = -V(\rho)$, $V(\rho) \equiv \frac{c^2}{\sinh^2 \rho} - \kappa^2 \cosh^2 \rho + \omega^2 \sinh^2 \rho$. From the form of the effective "potential" V in this equation one finds that one cannot satisfy the closed string periodicity condition in σ (3.7) unless ρ is fixed to be at zero of V.

$$S_1 = S_2 \equiv S , \qquad S = \frac{1}{2}\omega \sinh^2 \rho_0 = \frac{1}{4}\kappa^2 \sqrt{\kappa^2 + 1} , \tag{3.19}$$

One can easily solve the cubic equation for κ^2 as a function of S to find the dependence of $\mathcal{E}$ on S. In the case of a small string with $\rho_0 \to 0, \kappa \to 0$ (i.e. a string near the center of AdS_5) we get

$$\mathcal{E} = \sqrt{4S}\left[1 + S + O(S^2)\right] , \qquad S \ll 1 . \tag{3.20}$$

This is the usual Regge trajectory relation in flat space plus the first correction due to the curvature of AdS_5. In the case of a large string with $\rho_0 \gg 1$, $\kappa \gg 1$ (i.e. a long string close to the boundary of AdS_5) we get

$$\mathcal{E} = 2S + \frac{3}{4}(4S)^{1/3} + O(S^{-1/3}) , \qquad S \gg 1 . \tag{3.21}$$

Note that here the first correction to $\mathcal{E} - 2S$ goes as $S^{1/3}$, which is different from the $\ln S$ behavior in the single-spin (folded rotating closed string) case in [2]. However, as we explain in the next section, the solution with large S turns out to be unstable.

3.2. Fluctuations, stability and 1-loop correction

To compute the quadratic action for fluctuations near the above solution it is useful to start with the Nambu-Goto analog of the action (3.1) and choose the static gauge

$$t = \kappa\tau , \qquad \theta = \sigma . \tag{3.22}$$

Let us note that the induced metric on our solution is flat:

$$ds_2^2 = \sinh^2 \rho_0 \left(-d\tau^2 + d\sigma^2\right) . \tag{3.23}$$

Shifting the remaining three fields away from their classical values

$$\rho \to \rho_0 + \tilde{\rho} , \quad \phi \to \omega\tau + \tilde{\phi} , \quad \varphi \to \omega\tau + \tilde{\varphi} , \tag{3.24}$$

and expanding the Nambu-Goto action up to the second order in the fluctuation fields, we get the quadratic Lagrangian for the fluctuations $\tilde{\rho}$, $\tilde{\phi}$ and $\tilde{\varphi}$

$$L = -\frac{1}{2}(\partial_a\tilde{\rho})^2 - \frac{1}{4}\cos^2\sigma \ \kappa^2[1 + \cos^2\sigma \ (1 + \kappa^2)](\partial_a\tilde{\varphi})^2$$

$$-\frac{1}{4}\sin^2\sigma \ \kappa^2[1 + \sin^2\sigma(1 + \kappa^2)](\partial_a\tilde{\phi})^2 - \frac{1}{2}\cos^2\sigma \sin^2\sigma \ \kappa^2(1 + \kappa^2)\partial_a\tilde{\phi}\partial^a\tilde{\varphi}$$

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$$+ 2\sqrt{\kappa^2(1+\kappa^2)(2+\kappa^2)}\tilde{\rho}(\partial_0\tilde{\varphi}\cos^2\sigma + \partial_0\tilde{\phi}\sin^2\sigma) + (2+\kappa^2)\tilde{\rho}^2 \ . \tag{3.25}$$

This Lagrangian takes simpler form after making the following change of variables $(\tilde{\phi}, \tilde{\varphi}) \rightarrow (\alpha, \beta)$ [10]

$$\alpha = a \ (\tilde{\varphi}\cos^2\sigma + \tilde{\phi}\sin^2\sigma) \ , \qquad \beta = b \ \sin 2\sigma \ (\tilde{\phi} - \tilde{\varphi}) \ , \tag{3.26}$$

$$\tilde{\phi} = \frac{\alpha}{a} + \frac{\beta}{2b}\tan\sigma \ , \qquad \tilde{\varphi} = \frac{\alpha}{a} - \frac{\beta}{2b}\cot\sigma \ , \tag{3.27}$$

$$a = \frac{1}{\sqrt{2}\kappa}\sqrt{2+\kappa^2} \ , \qquad b = -\frac{1}{\sqrt{2}\kappa} \ . \tag{3.28}$$

An apparent singularity of the transformation (3.26) at $\sigma = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ is not physical because it is a reflection of the obvious coordinate singularity of the AdS_5 metric (3.2),(3.3) at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.[11] In terms of the new fields (3.25) takes the following simple form

$$L = -\frac{1}{2}(\partial_a\tilde{\rho})^2 - \frac{1}{2}(\partial_a\alpha)^2 - \frac{1}{2}(\partial_a\beta)^2$$

$$+ 2\sqrt{2(1+\kappa^2)}\partial_0\alpha \ \tilde{\rho} - 2\sqrt{2+\kappa^2}\partial_1\alpha \ \beta - 2(1+\kappa^2)\beta^2 + (2+\kappa^2)\tilde{\rho}^2 \ . \tag{3.29}$$

The Lagrangian (3.29) can be rewritten as (omitting total derivative)

$$L = -\frac{1}{2}(D_a\eta^s)^2 - \frac{1}{2}M_{sr}\eta^s\eta^r \ , \qquad D_a\eta^s = \partial_a\eta^s + A_a^{sr}\eta_r \ , \qquad \eta^s = (\tilde{\rho}, \alpha, \beta) \ , \tag{3.30}$$

$$A_0^{\alpha\tilde{\rho}} = -A_0^{\tilde{\rho}\alpha} = \sqrt{2(1+\kappa^2)} \ , \qquad A_1^{\alpha\beta} = -A_1^{\beta\alpha} = \sqrt{2+\kappa^2} \ ,$$

$$M_{sr}\eta^s\eta^r = -2\tilde{\rho}^2 + \kappa^2\alpha^2 + (2+3\kappa^2)\beta^2 \ , \tag{3.31}$$

where the 2-d non-abelian $SO(3)$ gauge field A_a^{sr} has a constant but non-vanishing field strength $F_{01}^{\beta\tilde{\rho}} = \sqrt{2(1+\kappa^2)(2+\kappa^2)}$. It is remarkable that in spite of the τ and σ dependence of our background solution, the fluctuation action is quite simple, having constant coefficients; in particular, it is simpler than the corresponding action in the one-spin case in

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[10] This transformation has a very simple interpretation in terms of fluctuations of the complex fields X and Y in (2.18): expanding near their classical values (2.16) in the static gauge where θ is not fluctuating and ignoring fluctuations of ρ we get: $\tilde{X} = i\sinh\rho_0 \ \sin\sigma \ e^{i\omega\tau} \ \tilde{\phi}, \quad \tilde{Y} = i\sinh\rho_0 \ \cos\sigma \ e^{i\omega\tau} \ \tilde{\varphi}$. Then α, β expressed in terms of $\tilde{X}$ and $\tilde{Y}$ take the form (at each given τ) of an $O(2)$ rotation with angle σ.

[11] While to cover S^3 once one is usually assuming $0 \leq \theta \leq \frac{\pi}{2}$ with ϕ and φ changing from 0 to 2π, here to embed the closed string in S^3 at each fixed moment of time (3.7) we need to consider θ in the interval from 0 to 2π.

[4]. The absence of explicit dependence on σ will allow us to solve the linearized equations of motion for the fluctuations.

Note that the radial fluctuation $\tilde{\rho}$ has a negative mass term in (3.31), suggesting an instability (the Hamiltonian corresponding to (3.30) is not positive definite). However, since it is coupled to a gauge field (i.e. is mixing with other fluctuations) the latter may, in principle, stabilize the $\tilde{\rho}$ evolution.[12] Thus the stability issue needs to be carefully studied. This is done in Appendix A. It is found there that the solution is stable only for a certain range of values of κ, i.e. for not very large values of $\mathcal{S}$ (see (3.19))

$$0 \leq \kappa^2 \leq \frac{5}{2} \ , \qquad \text{i.e.} \qquad 0 \leq \mathcal{S} \leq \frac{5}{8}\sqrt{\frac{7}{2}} \ . \tag{3.32}$$

Note that for the maximal value of $\mathcal{S}$, i.e. $\mathcal{S} \approx 1.17$ the value of the spin $S = \sqrt{\lambda}\,\mathcal{S}$ is still large since in the semiclassical approximation it is assumed that $\sqrt{\lambda} \gg 1$.

It is of interest to compute the 1-loop superstring sigma model correction to the energy of the two-spin solution. In principle, it can be done following the same approach as was used in [4] in the single-spin case. It is straightforward to supplement (3.29) with the Green-Schwarz quadratic fermionic term as in [12,4] (see Appendix B). The fermionic contribution cancels the 2-d logarithmic divergence coming from the mass term in (3.30) (which is proportional to $\sum_r M_{rr} = 4\kappa^2$). If we ignore the instability of the solution for large κ and formally consider the limit $\kappa \gg 1$ in (3.29) we will get

$$L_{\kappa \gg 1} \to -\frac{1}{2}(\partial_a\tilde{\rho})^2 - \frac{1}{2}(\partial_a\alpha)^2 - \frac{1}{2}(\partial_a\beta)^2 + 2\sqrt{2}\kappa\partial_0\alpha \ \tilde{\rho} - 2\kappa\partial_1\alpha \ \beta \ -2\kappa^2\beta^2 + \kappa^2\tilde{\rho}^2 \ . \tag{3.33}$$

Since κ is the only non-trivial parameter in (3.33), the 1-loop correction to the energy on the 2-d cylinder is expected to scale as κ (see the discussion [4]). That would imply that the large $\mathcal{S}$ expansion of the energy in (3.21) is corrected at the one loop order by (cf. (3.19)) $\mathcal{E}_1 \sim \frac{\kappa}{\sqrt{\lambda}} \sim \frac{1}{\sqrt{\lambda}}\mathcal{S}^{1/3}$, $\mathcal{S} \gg 1$. This may be consistent with the following conjecture for the general behaviour of $E(S, \lambda)$

$$E = 2S + [h(\lambda) + f(\lambda)S]^{1/3} + ... \ , \qquad S \gg 1 \ , \tag{3.34}$$

where

$$f(\lambda)_{\lambda \gg 1} = \lambda(c_0 + \frac{c_1}{\sqrt{\lambda}} + ...) \ , \qquad f(\lambda)_{\lambda \ll 1} = \lambda(b_0 + b_1\lambda + ...) \ . \tag{3.35}$$

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[12] Examples of similar situations are a charged (inverted) harmonic oscillator in a magnetic field, and a "tachyon" mode in AdS space.

We caution, however, that the instability of the solution for large κ or large $\mathcal{S}$ may preclude interpolation from strong to weak coupling in the large spin limit. One could still hope that since the solution with large $\mathcal{S}$ should evolve into a solution which will still carry the same spin, one may still find the classical energy behaving with spin as in (3.21). However, one may not be able then to compute the sigma model loop corrections to the energy in a reliable way.

4. Multi-spin string solutions in $AdS_5 \times S^5$

Let us now find a similar rotating string solution in S^5 and its generalizations having spins in both AdS_5 and S^5 factors. This was already discussed in terms of the embedding coordinates in section 2. Here we will rederive these solutions in terms of angles of S^5 and study some of their properties in more detail.

The bosonic part of the $AdS_5 \times S^5$ string action is

$$I = -\frac{\sqrt{\lambda}}{4\pi} \int d^2\xi \left[G_{mn}^{(AdS_5)}(x)\partial_a x^m \partial^a x^n \ + \ G_{pq}^{(S^5)}(y)\partial_a y^p \partial^a y^q \right] , \qquad \sqrt{\lambda} \equiv \frac{R^2}{\alpha'} . \quad (4.1)$$

We shall use the following explicit parametrization of the unit-radius metric on S^5:

$$(ds^2)_{S^5} = G_{pq}^{(S^5)}(y)dy^p dy^q = d\gamma^2 + \cos^2\gamma \, d\varphi_1^2 + \sin^2\gamma \, (d\psi^2 + \cos^2\psi \, d\varphi_2^2 + \sin^2\psi \, d\varphi_3^2) . \quad (4.2)$$

This metric has three translational isometries in φ_i, so that in addition to the three AdS_5 integrals of motion (3.4),(3.5),(3.6), a general solution should also have the following three integrals of motion depending on the S^5 part of the action:

$$J_1 = P_{\varphi_1} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \, \cos^2\gamma \, \partial_0\varphi_1 \equiv \sqrt{\lambda} \, \mathcal{J}_1 , \qquad (4.3)$$

$$J_2 = P_{\varphi_2} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \, \sin^2\gamma \, \cos^2\psi \, \partial_0\varphi_2 \equiv \sqrt{\lambda} \, \mathcal{J}_2 , \qquad (4.4)$$

$$J_3 = P_{\varphi_3} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \, \sin^2\gamma \, \sin^2\psi \, \partial_0\varphi_3 \equiv \sqrt{\lambda} \, \mathcal{J}_3 . \qquad (4.5)$$

4.1. Circular string rotating in S^5

Let us look for a solution describing a closed string located at the center $\rho = 0$ of AdS_5 and at a fixed value of one of the S^5 angles $\gamma = \gamma_0 = $ const, rotating within S^3 part of S^5 (with equal frequences as in the AdS_5 case), with its center of mass orbiting along a circle of S^5. A natural ansatz for such a solution is

$$t = \kappa\tau , \quad \rho = 0 , \quad \gamma = \gamma_0 , \quad \varphi_1 = \nu\tau , \quad \varphi_2 = \varphi_3 = \mathrm{w}\tau , \quad \psi = \sigma , \quad (4.6)$$

where $\kappa, \gamma_0, \nu, \mathrm{w} = $ const. The equations of motion for the fields and the conformal gauge constraints then lead to the following relations between γ_0, κ, ν and w

$$\mathrm{w}^2 = \nu^2 + 1 , \quad \kappa^2 = \nu^2 + 2\sin^2\gamma_0 . \quad (4.7)$$

Just as in the case of the two-spin solution in AdS_5 the induced metric here is flat

$$ds_2^2 = \sin^2\gamma_0 \left(-d\tau^2 + d\sigma^2\right) . \quad (4.8)$$

Taking into account that

$$\mathcal{J} \equiv \mathcal{J}_1 = \cos^2\gamma_0 \, \nu , \quad \mathcal{J}_2 = \mathcal{J}_3 = \mathcal{J}' , \quad \mathcal{J}' \equiv \frac{1}{2}\sin^2\gamma_0 \, \mathrm{w} , \quad (4.9)$$

we find the following equation for $\nu = \nu(\mathcal{J}, \mathcal{J}')$

$$\nu\sqrt{\nu^2 + 1} = \mathcal{J}\sqrt{\nu^2 + 1} + 2\mathcal{J}'\nu . \quad (4.10)$$

Since the energy $\mathcal{E}$ in (3.4) is equal to κ, we can use eq.(4.10) to find the dependence of the energy on the R-charges $\mathcal{J}$ and $\mathcal{J}'$

$$\mathcal{E}^2 = \nu^2 + \frac{4\mathcal{J}'}{\sqrt{\nu^2 + 1}} , \quad \mathcal{E} = \mathcal{E}(\mathcal{J}, \mathcal{J}') . \quad (4.11)$$

It is instructive to restore the λ-dependence in the formulas (4.10) and (4.11), i.e. to rewrite them in terms of the energy E, the R-charges J, J' and the auxiliary "charge" $\mathcal{V} = \sqrt{\lambda}\,\nu = \cos^{-2}\gamma\, J$:

$$\mathcal{V}\sqrt{\mathcal{V}^2 + \lambda} = J\sqrt{\mathcal{V}^2 + \lambda} + 2J'\mathcal{V} , \quad \mathcal{V} \equiv \sqrt{\lambda}\,\nu , \quad (4.12)$$

$$E^2 = \mathcal{V}^2 + \frac{4\lambda J'}{\sqrt{\mathcal{V}^2 + \lambda}} = \mathcal{V}^2 + 2\lambda(1 - \frac{J}{\mathcal{V}}) , \quad E = E(J, J') . \quad (4.13)$$

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A nice feature of this representation is that if $\mathcal{V} \gg \sqrt{\lambda}$ then the expression for the energy takes the form of a perturbative expansion in λ because $\mathcal{V}$ can be found from (4.12) as a series in $\frac{\lambda}{(J+2J')}$. In particular, we get the following expression for the energy at the first order in λ

$$E^2 \approx (J + 2J')^2 + \frac{2\lambda J'}{J + 2J'} , \qquad E \approx J + 2J' + \frac{\lambda J'}{(J + 2J')^2} , \qquad (4.14)$$

where $\mathcal{V} \approx J + 2J' - \frac{\lambda J'}{(J+2J')^2}$. Note that here there is no restriction on values of J and J' apart from the requirement that $J + 2J' \gg \sqrt{\lambda}$.

One may be tempted to conjecture that the formula (4.14) may be valid at *small* values of λ if $J + 2J'$ is very large. However, (4.14) was obtained in the strong coupling $\lambda \gg 1$ regime, and we expect it to receive $\frac{1}{\sqrt{\lambda}}$ string sigma model corrections. In particular, even the coefficient in front of $J + 2J'$ may get changed by the corrections, and, if so, the one-loop perturbative correction to the dimension of the corresponding CFT operator dual to the string solution will not be of order $\frac{J'}{(J+2J')^2}$ but of order $J + 2J'$.

If $J \gg J'$ the energy (4.14) takes the form

$$E \approx J + 2J' + \lambda \frac{J'}{J^2} . \qquad (4.15)$$

This expression for E is consistent with the string oscillation spectrum in the sector with large $J \gg \sqrt{\lambda}$ [1,13], i.e. with the plane-wave spectrum (similar comparison was done in [4]). From the plane-wave spectrum point of view, J' represents the angular momentum carried by string oscillations. Since the linear term in J is not renormalized in the BMN limit, one may conjecture that the same should happen here.

If we set $J = 0$ in (4.15) we get

$$E^2 = (2J')^2 + \lambda , \qquad \text{i.e.} \qquad E \approx 2J' + \frac{\lambda}{4J'} . \qquad (4.16)$$

Thus at large J' the correction goes as $\frac{1}{J'}$, instead of a constant shift found in the case of the single-spin folded string rotating in S^5 [2].

We can also consider the case with $\mathcal{V} \ll \sqrt{\lambda}$ (when $\mathcal{V} \approx J$)

$$E^2 \approx 4\sqrt{\lambda} J' + J^2 . \qquad (4.17)$$

Setting $J = 0$ we reproduce the usual Regge trajectory relation.

It is interesting to note that the limit $J \to 0$ depends on the value of the second angular momentum J' (see also the discussion of this case in sections 2 and 4.3). When $J' = \frac{1}{2}\sqrt{\lambda}$ the dependence of the energy on the angular momentum changes its form, i.e. the system undergoes a kind of "second order phase transition" (the second derivative of the energy over the orbital momentum has a discontinuity at $J' = \frac{1}{2}\sqrt{\lambda}$). This happens because for $\nu = 0$ one has $\mathcal{J}' = \frac{1}{2}\sin^2 \gamma_0$, so that the value $\mathcal{J}' = \frac{1}{2}$ is found when the string reaches its maximal size ($\gamma_0 = \frac{\pi}{2}$), i.e. when it rotates on the maximal-size 3-sphere within S^5.

4.2. Circular string rotating in both AdS_5 and S^5

As already discussed in section 2, it is straightforward to combine the two-spin solution in AdS_5 with the three angular momenta solution in S^5. For completeness, let us summarize the resulting solution depending on 3 different parameters in terms of the global coordinates of AdS_5 and S^5 used in section 3 and in this section:

$$\rho = \rho_0 , \qquad t = \kappa\tau , \qquad \varphi = \phi = \omega\tau , \qquad \theta = \sigma , \qquad (4.18)$$

$$\gamma = \gamma_0 , \qquad \varphi_1 = \nu\tau , \qquad \varphi_2 = \varphi_3 = w\tau , \qquad \psi = \sigma . \qquad (4.19)$$

The equations of motion and the conformal gauge constraint lead to the following relations

$$\omega^2 = \kappa^2 + 1 , \qquad w^2 = \nu^2 + 1 , \qquad \kappa^2 = \nu^2 + 2\sinh^2\rho_0 + 2\sin^2\gamma_0 , \qquad (4.20)$$

and the energy and the 5 conserved charges are given by the same relations as in (2.29),(2.30)

$$\mathcal{E} = \kappa \ \cosh^2\rho_0 = \kappa \ [1 + \frac{1}{2}(\kappa^2 - 2\sin^2\gamma_0 - \nu^2)] , \qquad (4.21)$$

$$\mathcal{S} = \omega \sinh^2\rho_0 = \frac{1}{2}(\kappa^2 - 2\sin^2\gamma_0 - \nu^2)\sqrt{\kappa^2 + 1} , \qquad (4.22)$$

$$\mathcal{J} = \cos^2\gamma_0 \ \nu , \qquad \mathcal{J}' = \frac{1}{2}\sin^2\gamma_0 \ w = \frac{1}{2}\sin^2\gamma_0\sqrt{\nu^2 + 1} . \qquad (4.23)$$

One can use these equations to analyse the dependence of the energy on the spins and $SO(6)$ charges. In particular, when $\mathcal{J}$ is very large while the string size is small, one reproduces the corresponding part of the oscillator plane-wave string spectrum (with spin $\mathcal{S}$ and angular momentum $\mathcal{J}'$ here carried by the semiclassical string instead of being produced by string oscillations as in [1]).

4.3. Fluctuations and stability of S^5 solution with $J = 0$, $J' \neq 0$

Let us now discuss the stability of the simplest S^5 solution with $\mathcal{J} = 0$ and $\mathcal{J}' \neq 0$. There are two different cases that should be discussed separately.

$\mathcal{J}' \leq \frac{1}{2}$ case

The solution with $\mathcal{J}' \leq \frac{1}{2}$ is found by setting $\nu = 0$ in (4.7),(4.9) (see also (2.28)). Then (see (4.17))

$$\nu = 0 , \qquad w = 1 , \qquad \kappa^2 = 2\sin^2\gamma_0 \leq 2 , \qquad \mathcal{J}' = \frac{1}{4}\kappa^2 , \qquad \mathcal{E} = \sqrt{4\mathcal{J}'} . \qquad (4.24)$$

As was already mentioned in section 2 (below (2.28)), this solution is essentially the embedding of the flat space solution into S^5. However, the fluctuation spectrum will of course be different from the flat space case.

The computation of the quadratic fluctuation action is a repetition of the one done in the AdS_5 case in section 3.2. We choose the static gauge

$$t = \kappa\tau , \qquad \psi = \sigma$$

and expand the Nambu-Goto action up to the second order in fluctuations. As in the case of point-like string orbiting in S^5 [4], the Lagrangian for the AdS_5 fluctuations will be represented by the 4 massive field contributions[13]

$$L_{AdS_5} = -\frac{1}{2}(\partial_a\tilde{\eta}_k)^2 - \frac{1}{2}\kappa^2\tilde{\eta}_k^2 , \qquad k = 1, 2, 3, 4 . \tag{4.25}$$

The additional contribution of S^5 fluctuations is (cf.(3.29))

$$L_{S^5} = -\frac{1}{2}(\partial_a\hat{\varphi}_1)^2 - \frac{1}{2}(\partial_a\tilde{\gamma})^2 - \frac{1}{2}(\partial_a\alpha)^2 - \frac{1}{2}(\partial_a\beta)^2 - 2\mu\partial_0\alpha \ \tilde{\gamma}$$

$$-2\sqrt{2}\partial_1\alpha \ \beta + \ \mu^2\tilde{\gamma}^2 - 2\beta^2 , \qquad \mu^2 \equiv 2 - \kappa^2 . \tag{4.26}$$

Here $\hat{\varphi}_1 = \frac{1}{\sqrt{2}}\mu\tilde{\varphi}_1$ and the fields α and β are defined as in (3.26)

$$\alpha = -\kappa(\tilde{\varphi}_2\cos^2\sigma + \tilde{\varphi}_3\sin^2\sigma) , \qquad \beta = -\frac{\kappa}{2\sqrt{2}}\sin 2\sigma \ (\tilde{\varphi}_2 - \tilde{\varphi}_3) . \tag{4.27}$$

This Lagrangian is similar to the one (3.29) for fluctuations around the two-spin solution in AdS_5. Its $\tilde{\gamma}, \alpha, \beta$ part can be written in the form (3.30) as follows

$$L(\tilde{\gamma}, \alpha, \beta) = \frac{1}{2}(\partial_0\tilde{\gamma} + \mu\alpha)^2 - \frac{1}{2}(\partial_1\tilde{\gamma})^2 + \frac{1}{2}(\partial_0\alpha - \mu\tilde{\gamma})^2 - \frac{1}{2}(\partial_1\alpha + \sqrt{2}\beta)^2$$

$$+ \frac{1}{2}(\partial_0\beta)^2 - \frac{1}{2}(\partial_1\beta - \sqrt{2}\alpha)^2 + \frac{1}{2}\mu^2\tilde{\gamma}^2 + \frac{1}{2}(2 - \mu^2)\alpha^2 - \beta^2 . \tag{4.28}$$

Note that the sum of squares of masses here vanishes, in agreement with the discussion of fluctuation Lagrangian in conformal gauge in Appendix C.

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[13] Expanding near the $\rho = 0$ point in (3.2) one needs to introduce the 4 cartesian-type coordinates, e.g., writing the AdS_5 metric as $ds^2 = -\frac{(1+\frac{1}{4}\eta^2)^2}{(1-\frac{1}{4}\eta^2)^2}dt^2 + \frac{d\eta_k d\eta_k}{(1-\frac{1}{4}\eta^2)^2}$.

The Hamiltonian corresponding to (4.28) does not appear to be positive definite, and so the stability of the solution is a priori in question. It is shown in Appendix A that the solution is, in fact, stable if

$$\frac{1}{2} \leq \mu^2 \leq 2 , \qquad \text{i.e.} \qquad 0 \leq \mathcal{J}' \leq \frac{3}{8} .$$

$\mathcal{J}' \geq \frac{1}{2}$ case

The solution with $\mathcal{J}' \geq \frac{1}{2}$ is found by setting $\gamma_0 = \frac{\pi}{2}$. Since $\cos \gamma_0 = 0$, the value of ν in this case is undetermined (cf. (4.2)), and the conformal gauge constraint gives (cf. (4.7),(4.9),(4.16))

$$\kappa^2 = \mathrm{w}^2 + 1 \geq 2 , \qquad \mathcal{J}' = \frac{1}{2}\mathrm{w} , \qquad \mathcal{E} = \kappa , \qquad \text{i.e.} \qquad \mathcal{E} = \sqrt{(2\mathcal{J}')^2 + 1} . \qquad (4.29)$$

The coordinates γ and φ_1 are not suitable for studying fluctuations around $\gamma = \frac{\pi}{2}$ (which is a center of the "2-sphere" part $d\gamma^2 + \cos^2 \gamma \ d\varphi_1^2$ of the S^5 metric (4.2)). Introducing instead the "cartesian-type" coordinates X_1, X_2 as in (2.22) (which have zero values on the classical solution)

$$Z = X_1 + iX_2 = \cos \gamma \ \mathrm{e}^{i\varphi_1} , \qquad (4.30)$$

we find that the quadratic fluctuation action (obtained in the static gauge $t = \kappa\tau, \ \psi = \sigma$) is then given by the sum of the AdS_5 part (4.25) and (we ignore total derivative terms)

$$L_{S^5} = -\frac{1}{2}|\partial_a Z|^2 - \frac{1}{2}(\kappa^2 - 2)|Z|^2 + L(\alpha, \beta) , \qquad (4.31)$$

$$L(\alpha, \beta) = -\frac{1}{2}(\partial_a \alpha)^2 - \frac{1}{2}(\partial_a \beta)^2 - 2\kappa\partial_1\alpha \ \beta - 2(\kappa^2 - 1)\beta^2$$

$$= \frac{1}{2}(\partial_0 \alpha)^2 - \frac{1}{2}(\partial_1 \alpha + \kappa\beta)^2 + \frac{1}{2}(\partial_0 \beta)^2 - \frac{1}{2}(\partial_1 \beta - \kappa\alpha)^2 + \frac{1}{2}\kappa^2\alpha^2 - \frac{1}{2}(3\kappa^2 - 4)\beta^2 , \ (4.32)$$

where, as in (3.26),(4.27),

$$\alpha = -\kappa(\cos^2 \sigma \ \tilde{\varphi}_2 + \sin^2 \sigma \ \tilde{\varphi}_3) , \qquad \beta = -\frac{1}{2}\sin 2\sigma \ (\tilde{\varphi}_2 - \tilde{\varphi}_3) . \qquad (4.33)$$

The Lagrangian (4.32) is simpler than (4.26), describing a collection of 2 coupled fields with a constant abelian connection; however, the mass matrix is not $O(2)$ invariant, so after the rotation there will be a remaining σ-dependence in the mass matrix. Note that the sum of mass-squared terms for S^5 fluctuations is equal to $2(\kappa^2 - 2) - \kappa^2 + 3\kappa^2 - 4 = 4(\kappa^2 - 2)$

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in correspondence with the results in the conformal gauge and with the cancellation of divergences between the bosonic and fermionic sectors (see Appendices B and C).

The negative mass term for α in (4.32) raises again the question about stability. To analyze the stability of this solution it is sufficient to consider only the α, β part (4.32) of the Lagrangian. Following the procedure explained for the AdS_5 case in Appendix A, i.e. expanding the fluctuations α and β in Fourier series in σ, and then looking for solutions in the form $e^{i\omega_n \tau}$, we find the following frequency spectrum

$$\omega_n^2 = n^2 + 2(\kappa^2 - 1) \pm 2\sqrt{(\kappa^2 - 1)^2 + \kappa^2 n^2} \ . \tag{4.34}$$

It is clear that the ω_n spectrum is real if

$$[n^2 + 2(\kappa^2 - 1)]^2 - 4[(\kappa^2 - 1)^2 + \kappa^2 n^2] = n^2(n^2 - 4) \geq 0 \ . \tag{4.35}$$

This condition does not depend on κ and is not satisfied for the mode with $n = \pm 1$.[14] A possible interpretation of this mode is that in the frame rotating together with the string where string is at rest, it describes the obvious instability of a circular string wound around large circle of S^3 [8].[15]

We conclude that the rotating solution with $J = 0$ is not stable for any value of the angular momentum $J' \geq \frac{1}{2}\sqrt{\lambda}$.

As was already mentioned in the introduction, to get a stable solution with large J' one needs also to switch on a non-zero (and large) value of the angular momentum J [8]. If one could ignore the instability, the solution with $J = 0$, $J' \geq \frac{1}{2}\sqrt{\lambda}$ would be the most simple and interesting case for the study of the AdS/CFT correspondence in a novel sector of states. One could try to compute the 1-loop string sigma model correction to the classical energy in (4.29) by starting with the sum of the bosonic fluctuation action (4.32) (assuming one could formally omit the unstable mode absent at large J) and the fermionic action derived in Appendix B. This will be discussed (for the general stable case of $J, J' \neq 0$) in [8]. To estimate this correction at large values of $\mathcal{J}'$ one may note that for

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[14] The stability is obvious for the $e^{in\sigma}$ fluctuation modes with $n \geq 2$. Indeed, (4.32) can be written as $L(\alpha, \beta) = \frac{1}{2}(\partial_0 \alpha)^2 - \frac{1}{2}(\partial_1 \alpha + 2\kappa\beta)^2 + \frac{1}{2}(\partial_0 \beta)^2 - \frac{1}{2}(\partial_1 \beta)^2 + 2\beta^2$, and thus the corresponding Hamiltonian is non-negative for the modes with $n \geq 2$. Let us note also that the problem of analysing the small fluctuation spectrum of this theory is similar to the one of solving string theory in "non-diagonal" (metric and 2-form field) plane-wave background (cf. [14]).

[15] Such unstable mode would be absent in the S^5/Z_2 case.

large κ all non-zero masses of the 2-d fields are equal to κ, and thus (see [4]) the 1-loop correction to the energy should be expected to scale as

$$\mathcal{E}_1 \sim \frac{\kappa}{\sqrt{\lambda}} \sim \frac{1}{\sqrt{\lambda}} \mathcal{J}' \ , \qquad \mathcal{J}' \approx \frac{1}{2}\kappa \gg 1 \ . \tag{4.36}$$

That seems to suggest the following interpolation formula for the energy (cf. (4.16))

$$E = 2h(\lambda)J' + f(\lambda)\frac{\lambda}{4J'} \ , \qquad J' \gg 1 \ , \tag{4.37}$$

$$h(\lambda)_{\sqrt{\lambda} \gg 1} = 1 + \frac{a_1}{\sqrt{\lambda}} + ... \ , \qquad f(\lambda)_{\sqrt{\lambda} \gg 1} = 1 + \frac{b_1}{\sqrt{\lambda}} + ... \ . \tag{4.38}$$

In spite of the absence of 2-d supersymmetry in the corresponding quadratic part of GS action, it may actually happen that the coefficients a_1 and b_1 vanish, i.e. the first two terms in the energy are not renormalized at the leading order in $\frac{1}{\sqrt{\lambda}}$ expansion. That would support the conjecture, prompted by the appearance of the first power of λ in the calssical expression for E, that $E = 2J' + \frac{\lambda}{4J'} + ...$ is actually true also at weak coupling, i.e. is the exact result for the first two terms in the anomalous dimension of the corresponding dual gauge theory operator. We shall discuss this conjecture further in the next section.

5. Towards testing AdS/CFT duality in non-supersymmetric multi-spin sectors

Let us now discuss the gauge-theory operators that should correspond to the string states represented by the classical solutions found above. The eventual aim is to try to compare their anomalous dimensions as functions of spins and R-charges to the semiclassical expressions for the energies found above.

5.1. AdS_5 rotation case

The semiclassical closed string states found in global coordinates in $AdS_5 \times S^5$ should be dual to SYM states on $R \times S^3$. Going through the usual argument of Euclidean continuation and conformal mapping to R^4 (cf. [15]) they should correspond to local operators in Euclidean 4-d space. One can then rotate back to Minkowski space, but here we prefer not to do that. The $SO(4)$ isometry of S^3 in AdS_5 is then becoming the "Lorentz" symmetry of R^4. Thus the Euclidean gauge theory operators will be classified by its representations, i.e. will be labelled by the values (S_1, S_2) of the two $SO(4) = SU(2) \times SU(2)$ spins. In addition, they will carry also the three quantum numbers of $SO(6)$ R-symmetry group.

Let us first recall the form of the gauge theory operators that are expected to be dual to the single-spin string state in AdS_5 [2]. If Φ_M ($M = 1, ..., 6$) are the adjoint scalars of $N = 4$ SYM theory and D_μ is the covariant derivative, a representative operator with canonical dimension $\Delta_0 = S + 2$ is the standard gauge-invariant minimal twist operator $O_{\{\mu_1...\mu_S\}} = \text{tr}\left(\Phi_M D_{\{\mu_1}...D_{\mu_S\}}\Phi_M\right)$, where $\{\mu_1...\mu_S\}$ denotes symmetrization and subtraction of traces. This operator will in general mix with similar operators obtained by replacing the scalars Φ_M by the gauge field strength $F_{\mu\nu}$ or by the fermions so to find its perturbative anomalous dimension one would need to diagonalise the corresponding anomalous dimension matrix (see, e.g., [16,17,18] and references there).[16] The equivalent form of the above operator is

$$O_S = \text{tr}\left(\Phi_M D_{\mu_1}...D_{\mu_S}\Phi_M\right)\ n^{\mu_1}...n^{\mu_S} \ = \ \text{tr}\left[\Phi_M(n^\mu D_\mu)^S\Phi_M\right] \ , \qquad (5.1)$$

where the multiplication by the product of constant null vector n^μ ($n^\mu n_\mu = 0$) factors implements the symmetrization and subtraction of traces. In Minkowski $R^{1,3}$ theory one may choose $n^\mu = (1, 0, 0, 1)$, getting $O_S = \text{tr}\left(\Phi_M(D_+)^S\Phi_M\right)$, $D_+ = D_0 + D_3$. In the Euclidean version which we use here one is to choose n^μ to be complex, e.g., $n^\mu = (1, i, 0, 0)$, getting

$$O_S = \text{tr}\left[\Phi_M(D_X)^S\Phi_M\right] \ , \qquad D_X \equiv D_1 + iD_2 \ . \qquad (5.2)$$

It is now clear how to generalize this discussion to the case of operators carrying two spins of $SO(4)$: a representative operator will be

$$O_{S_1,S_2} = \text{tr}\left[\Phi_M(D_X)^{S_1}(D_Y)^{S_2}\Phi_M\right] \ , \qquad (5.3)$$

$$D_X \equiv D_1 + iD_2 \ , \qquad D_Y \equiv D_3 + iD_4 \ , \qquad \Delta_0 = S_1 + S_2 + 2 \ . \qquad (5.4)$$

Since the covariant derivatives D_X and D_Y do not commute, this operator will be mixing, in particular, with various other operators containing permutations of S_1 derivatives D_X and S_2 derivatives D_Y and having the same canonical dimension, e.g.,

$$\text{tr}\left[\Phi_M(D_X)^{k_1}(D_Y)^{m_2}...(D_X)^{k_l}(D_Y)^{m_l}\Phi_M\right] \ , \qquad \sum_i k_i = S_1 \ , \qquad \sum_i m_i = S_2 \ . \quad (5.5)$$

The eigenvector of anomalous dimension matrix is expected to be a particular combination of such operators (in addition to others involving gauge field strength and fermions).

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[16] Two-loop anomalous dimensions for some higher spin currents were found in [19].

An irreducible representation of the rotation group is represented by a particular Young tableau with two rows with S_1 and S_2 as numbers of boxes, i.e. should contain additional antisymmetrizations of D_X and D_Y factors in (5.3).[17]

The equivalent "covariant" form of (5.3) is found by introducing two independent null vectors n^μ and m^ν and generalising (5.1) as follows

$$O_{S_1,S_2} = \text{tr}(\Phi_M D_{\mu_1}...D_{\mu_{S_1}} D_{\nu_1}...D_{\nu_{S_2}} \Phi_M) \, n^{\mu_1}...n^{\mu_{S_1}} m^{\nu_1}...m^{\nu_{S_2}} , \quad n^\mu n_\mu = 0 , \quad m^\mu m_\mu = 0 . \tag{5.6}$$

This operator can be readily extended to the Minkowski version of the theory by choosing, e.g., $n^\mu = (1, 0, 0, 1)$ and $m^\mu = (0, 1, i, 0)$ with $\eta_{\mu\nu} = (-1, 1, 1, 1)$.

As is well known [17], for $S_1 = S \gg 1$, $S_2 = 0$ (or vice versa) the perturbative anomalous dimension of such operators scales as $\ln S$, which is the same as the scaling of the energy of the single-spin rotating string solution in AdS_5; this strongly supports the existence of an interpolation formula $\Delta = S + f(\lambda) \ln S$ between the weak coupling and the strong coupling regions [2,4].

To compare with the string solution found in the present paper where $S_1 = S_2 = S \gg 1$ one needs to know the perturbative (one-loop) anomalous dimension of the operators like

$$O_{S,S} = \text{tr}\left[\Phi_M (D_X)^S (D_Y)^S \Phi_M\right] + ... \tag{5.7}$$

where dots stand for appropriate permutations of D_X and D_Y. We are not aware of computations of anomalous dimension of such operators in the literature, and here can only speculate about possible interpolation formula for $\Delta(S)$ in this case (see also section 3.2). Assuming one can trust the expression (3.21) for the energy for large $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ (in spite of instability of the solution for large $\mathcal{S}$) one would expect to find the order $S^{1/3}$ correction in the anomalous dimension replacing the familiar $\ln S$ correction for the single-spin operators (5.2). However, it seems hard to imagine how such fractional-power term could appear in the 1-loop SYM computation for the anomalous dimension of (5.7). We suspect that the interpolation formula (3.34),(3.35) may be a more plausible alternative, implying that at large S but small λ (with $\lambda S \ll 1$) one should expect to find the anomalous dimension of (5.7) going as

$$\Delta = 2k(\lambda)S + ... , \qquad k(\lambda) = 1 + a_1\lambda + a_2\lambda^2 + ... \tag{5.8}$$

It would be very interesting to check this by direct perturbative computations on the gauge theory side.

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[17] Due to these antisymmetrizations the resulting operators may not be superconformal primary operators.

5.2. S^5 **rotation case**

The construction of operators that carry several $SO(6)$ "spins" and thus should be dual to the string states represented by the solutions in section 4 describing closed strings rotating in S^5 is somewhat similar. Let us introduce the notation for the three complex scalars of $N = 4$ SYM theory (cf. (2.21))

$$\Phi_Z = \Phi_1 + i\Phi_2 , \qquad \Phi_X = \Phi_3 + i\Phi_4 , \qquad \Phi_Y = \Phi_5 + i\Phi_6 . \tag{5.9}$$

Among the operators with the minimal canonical dimension for given (J_1, J_2, J_3) values of $SO(6)$ charges there are operators with holomorphic dependence of the three complex scalars:

$$O_{J_1, J_2, J_3} = \text{tr}[(\Phi_Z)^{J_1}(\Phi_X)^{J_2}(\Phi_Y)^{J_3}] + ... , \qquad \Delta_0 = J_1 + J_2 + J_3 , \tag{5.10}$$

where dots stand for permutations of Φ_X, Φ_Y, Φ_Z factors needed to form an irreducible representation of $SO(6)$ that is expected to be an eigenvector of the anomalous dimension matrix. The 1-loop anomalous dimension matrix for generic scalar operators of the form

$$\mathcal{O}_{M_1....M_j} = \text{tr}(\Phi_{M_1}....\Phi_{M_j}) \tag{5.11}$$

was computed in [9,10]. Taking its symmetric traceless part (i.e. multiplying (5.11) by a null vector, e.g., $n^M = (1, i, 0, 0, 0, 0)$) one finds a chiral primary operator whose dimension is protected. This case is equivalent to (5.10) with $J_1 = J$ and $J_2 = J_3 = 0$, i.e. $O_{J,0,0} = \text{tr}(\Phi_Z)^J$, which is dual to the ground state of the string theory expanded near the point-like string orbiting in S^5 [1].[18]

The operator that should be dual to the string solution with $J_1 = J$, $J_2 = J_3 = J'$ found above should then be

$$O_{J,J',J'} = \text{tr}[(\Phi_Z)^J(\Phi_X)^{J'}(\Phi_Y)^{J'}] + ... , \qquad \Delta_0 = J + 2J' . \tag{5.12}$$

This operator belongs to the irreducible representation of $SU(4)$ with Young tableau labels (J, J', J') or with Dynkin labels $[0, J - J', 2J']$ (see Appendix D)[19] if $J \geq J'$, and to

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[18] We are considering only single-trace operators as seems appropriate for the elementary string state – gauge theory operator correspondence in the large N limit. Note that there exist also multi-trace operators which may carry the same quantum numbers and may be 1/4 or 1/8 BPS (see, e.g., [20] and refs. there).

[19] We are grateful to M. Staudacher and N. Beisert for correcting a mistake in this identification in the original version of this paper.

the representation $(J', J', J) = [J' - J, 0, J' + J]$ if $J' \geq J$, and does not seem to be a superconformal primary operator. For example, it is known that the operator with $J = 0$, $J' = 2$ is a superconformal descendant of the Konishi operator $K = \text{tr}\,(\Phi_M \Phi_M)$. In the near-BPS limit $J \gg J'$ the operators of the form (5.12) are examples of the BMN operators [1] with a small number of impurities, and one can, in principle, make detailed comparison between semiclassical predictions (4.15) and the perturbative results of [1] and [9,10]. If J' is comparable to or much larger than J, we are very far from the BPS operator $\text{tr}(\Phi_Z)^J$, and the conformal dimensions of the operators cannot be computed from the plane-wave string spectrum.

The semiclassical results obtained in section 4 are the only source of nonperturbative predictions for the dimensions of these operators. Let us stress again that among a large number of operators in these representations only the one with the lowest conformal dimension should be dual to the string solution we found. It is also interesting to point out that one and the same formula (4.15) should be giving the conformal dimensions of the operators from the two different ($[0, J - J', 2J']$ or $[J' - J, 0, J' + J]$) representations. This should be true not only in the large λ limit but also in the weak-coupling perturbation theory.

As discussed in section 4.3, the simplest novel case for a non-trivial check of the AdS/CFT duality in a non-supersymmetric sector is when the circular string orbiting in S^5 has maximal size, i.e. has $J = 0$, $J' \geq \frac{1}{2}\sqrt{\lambda}$. The exact expression for its classical energy given in (2.27),(4.16) is $E = \sqrt{(2J')^2 + \lambda}$. Provided the instability of this solution could be ignored (e.g., by embedding it into a class of stable solutions with $J \neq 0$) for large $\mathcal{J}' = \frac{J'}{\sqrt{\lambda}}$ the expression for the energy $E(J')$ gives the following prediction for the strong-coupling behaviour of the anomalous dimension of the corresponding operator

$$O_{0,J',J'} = \text{tr}[(\Phi_X)^{J'}(\Phi_Y)^{J'}] + \dots , \tag{5.13}$$

$$\Delta = 2J' + \frac{\lambda}{4J'} + \dots , \qquad J' \gg \sqrt{\lambda} \gg 1 . \tag{5.14}$$

As was already discussed in section 4.3, our conjecture is that this expression is actually valid also at *weak* coupling, i.e. the first two terms in Δ are not renormalized.

One can check this against the results of [21] applied to the operators transforming in the $[J', 0, J']$ representation for $J' = 4$ and $J' = 5$. At $J' = 4$ one finds that the lowest anomalous dimension of the operators in $[4, 0, 4]$ is $\gamma = 0.0411\lambda$ while eq.(5.14) predicts the

anomalous dimension to be $\gamma = 0.0625\lambda$, i.e. the deviation is about 34%.[20] However, at $J' = 5$ one gets the lowest anomalous dimension to be $g = 0.042\lambda$ while eq.(5.14) predicts $\gamma = 0.05\lambda$, i.e. the deviation in this case is just 16%. It would be useful to compute one-loop anomalous dimensions of the $[J', 0, J']$ operators for $J' = 6$ to see if the agreement with eq.(5.14) is really getting better at large J'.

Since it is sufficient to consider the operators with holomorphic dependence on the fields Φ_X and Φ_Y, the Hamiltonian of the integrable $SO(6)$ spin chain considered in [10] reduces to the Hamiltonian of the simplest $\mathrm{XXX}_{1/2}$ spin model [22]. It would be very interesting to find the corresponding one-loop anomalous dimension and the explicit form of the associated operator (5.13) in the large J' limit by utilizing this connection to the $\mathrm{XXX}_{1/2}$ model.

6. Concluding remarks

There are various generalizations of the AdS_5 and S^5 solutions we have found. For example, the special S^5 solution considered in section 4.3 with maximal size of the string ($\gamma_0 = \frac{\pi}{2}$) is also a solution of string theory on $R_t \times S^3$:

$$ds^2 = -dt^2 + d\psi^2 + \cos^2 \psi \; d\varphi_2^2 + \sin^2 \psi \; d\varphi_3^2 \; ,$$

$$t = \kappa\tau \; , \quad \psi = \sigma \; , \quad \varphi_2 = \varphi_3 = \mathrm{w}\tau \; , \quad \kappa^2 = \mathrm{w}^2 + 1 \; . \tag{6.1}$$

It is plausible, therefore, that it can be embedded into various other spaces containing S^3 factors, in particular into $AdS_5 \times T^{1,1}$ space related via AdS/CFT to an $N = 2$ supercon-formal theory [23].

Similarly, analogs of AdS_5 two-spin solution of section 3 exist for a more general class of 5-d (or higher-dimensional) metrics with $SO(4)$ isometry, e.g., $ds^2 = -g(\rho)dt^2 + d\rho^2 + h(\rho)d\Omega_3$. As in the AdS_5 case, the two equal rotations in S^3 allow one to stabilize a circular string at a fixed value of $\rho = \rho_0$ (stability under small fluctuations will depend on the explicit form of $g(\rho)$ and $h(\rho)$ and on the value of the spin). In particular, such solution will exist for an AdS black hole metric.[21]

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[20] Diagonalizing the matrix of anomalous dimensions from [9,10] one finds also an operator with much closer value $\gamma \approx 0.0619\lambda$, but comparison with the results of [9] shows that it belongs to [2,4,2] representation. We are grateful to G. Arutyunov and J. Minahan for explanations related to this point.

[21] Single-spin solutions in this and similar "non-conformal" cases were discussed in [3,24].

There are several directions in which the present work needs to be completed or extended. In [8] we shall study the S^5 solution with $J \neq 0, J' \neq 0$ in detail, identifying the range of its stability, and computing the 1-loop superstring sigma model correction to the classical energy. One should then be able to compare the string results (e.g., for $J \sim J'$) to the perturbative results for anomalous dimensions of the corresponding gauge theory operators. It would be very interesting also to compute the leading-order perturbative contributions to the anomalous dimensions of the operators discussed in section 5. This applies, in particular, to the operator (5.13) whose dimension should be possible to find using the integrable-model connection suggested in [10].

Appendix A. Stability analysis

Here we analyze the stability of the AdS_5 and S^5 two-spin solutions discussed above under small perturbations.

A.1. Stability of the two-spin AdS_5 solution

The equations of motion for fluctuations that follow from (3.29) are

$$\ddot{\tilde{\rho}} - \tilde{\rho}'' - 2(2+\kappa^2)\tilde{\rho} - 2\sqrt{2(1+\kappa^2)}\dot{\alpha} = 0 \ , \tag{A.1}$$

$$\ddot{\alpha} - \alpha'' - 2\sqrt{2+\kappa^2}\beta' + 2\sqrt{2(1+\kappa^2)}\dot{\tilde{\rho}} = 0 \ , \tag{A.2}$$

$$\ddot{\beta} - \beta'' + 4(1+\kappa^2)\beta + 2\sqrt{2+\kappa^2}\alpha' = 0 \ . \tag{A.3}$$

To solve the system (A.1)–(A.3) of linear differential equations we expand the fluctuations in Fourier series in σ

$$\tilde{\rho} = \sum_n \rho_n(\tau)\mathrm{e}^{in\sigma} \ , \quad \alpha = \sum_n \alpha_n(\tau)\mathrm{e}^{in\sigma} \ , \quad \beta = \sum_n \beta_n(\tau)\mathrm{e}^{in\sigma} \ . \tag{A.4}$$

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Then we get the following system of equations for the n-th modes of the fluctuations

$$\ddot{\rho}_n + n^2\rho_n - 2(2+\kappa^2)\rho_n - 2\dot{\alpha}_n\sqrt{2(1+\kappa^2)} = 0 \ , \tag{A.5}$$

$$\ddot{\alpha}_n + n^2\alpha_n + 2\dot{\rho}_n\sqrt{2(1+\kappa^2)} - 2in\beta_n\sqrt{2+\kappa^2} = 0 \ , \tag{A.6}$$

$$\ddot{\beta}_n + n^2\beta_n + 4(1+\kappa^2)\beta_n + 2in\alpha_n\sqrt{2+\kappa^2} = 0 \ . \tag{A.7}$$

We start the analysis of the system (A.5)-(A.7) by looking for solutions of the form

$$\rho_n(\tau) = C_n^\rho e^{i\omega_n\tau} \ , \quad \alpha_n(\tau) = C_n^\alpha e^{i\omega_n\tau} \ , \quad \beta_n(\tau) = C_n^\beta e^{i\omega_n\tau} \ . \tag{A.8}$$

The stability of the two-spin string solution requires the frequencies ω_n to be real. Substituting (A.8) into (A.5),(A.6),(A.7) we get

$$[n^2 - 2(2+\kappa^2) - \omega_n^2]C_n^\rho - 2i\omega_n\sqrt{2(1+\kappa^2)}C_n^\alpha = 0 \ , \tag{A.9}$$

$$(n^2 - \omega_n^2)C_n^\alpha + 2i\omega_n\sqrt{2(1+\kappa^2)}C_n^\rho - 2in\sqrt{2+\kappa^2}C_n^\beta = 0 \ , \tag{A.10}$$

$$[n^2 + 4(1+\kappa^2) - \omega_n^2]C_n^\beta + 2in\sqrt{2+\kappa^2}C_n^\alpha = 0 \ . \tag{A.11}$$

This is a system of linear equations for the coefficients C_n^ρ, C_n^α, C_n^β which can be written in the form $A_{ij}C^j = 0$; it has nontrivial solutions only for such values of ω_n for which the determinant of the matrix A_{ij} vanishes. This gives the equation for ω_n:

$$f_n(z) = 0 \ , \qquad z \equiv \omega_n^2 \ ,$$

$$f_n(z) \equiv z^3 - (8+10\kappa^2+3n^2)z^2 + (16+40\kappa^2+24\kappa^4+8\kappa^2n^2+3n^4)z - n^2(n^2-4)(n^2-4-2\kappa^2) \ . \tag{A.12}$$

We need to find the values of κ such that all roots of this cubic equation are positive. It is not difficult to show that the two extrema z_- and z_+ $(z_- < z_+)$ of $f_n(z)$ are positive, and $f_n(z_-) > 0$ and $f_n(z_+) < 0$. Therefore, all roots are positive if

$$f_n(0) = -n^2(n^2-4)(n^2-4-2\kappa^2) \leq 0 \ . \tag{A.13}$$

We see that $f_0(0) = f_2(0) = 0$, $f_1(0) < 0$ and $f_3(0) = -45(5-2\kappa^2)$. From the expression for $f_3(0)$ we conclude that the solution is stable only if

$$\kappa^2 \leq \frac{5}{2} \ . \tag{A.14}$$

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We also see that for $n = 0$ and $n = 2$ the determinant of the matrix A_{ij} in (A.12) vanishes at $\omega_n = 0$. That means that for these modes there is a solution of the form

$$\rho_n(\tau) = \tilde{C}_n^\rho \tau + C_n^\rho , \qquad \alpha_n(\tau) = \tilde{C}_n^\alpha \tau + C_n^\alpha , \qquad \beta_n(\tau) = \tilde{C}_n^\beta \tau + C_n^\beta . \qquad (\text{A.15})$$

Substituting (A.15) into the equations of motion (A.5), (A.6) and (A.7), one can easily find the corresponding solutions for $n = 0$ and $n = 2$

$$\rho_0(\tau) = C_0^\rho , \qquad \alpha_0(\tau) = -\frac{2 + \kappa^2}{\sqrt{2(1 + \kappa^2)}} C_0^\rho \, \tau + C_0^\alpha , \qquad \beta_0(\tau) = 0 , \qquad (\text{A.16})$$

$$\rho_2(\tau) = C_2^\rho , \qquad \alpha_2(\tau) = -\frac{\kappa^2}{\sqrt{2(1 + \kappa^2)}} C_0^\rho \, \tau + C_2^\alpha , \qquad \beta_2(\tau) = -\frac{i}{\sqrt{2 + \kappa^2}} \alpha_2(\tau) . \qquad (\text{A.17})$$

Since the fluctuation $\tilde{\rho}$ corresponding to (A.16),(A.17) does not depend on τ, such solutions do not lead to instability of the two-spin string solution. In fact, both solutions (A.16) and (A.17) have simple interpretation. The $n = 0$ solution reflects the fact that the radius ρ_0 (or κ) of the two-spin string solution (3.12)–(3.16) is a free parameter. This parameter can be changed, and this leads to the existence of the zero mode in the spectrum of fluctuations. The $n = 2$ solution appears because the two-spin string solution was found only for equal frequencies. We expect that the general solution will depend on the two independent frequencies, and, therefore, on the two parameters; the existence of the zero mode fluctuation with $n = 2$ is related to this (cf. (3.26)).

A.2. Stability of the two-spin S^5 solution

The analysis of stability in the S^5 case follows closely the procedure explained above in the AdS_5 case. Here we shall consider explicitly only the case of the solution with $J = 0$ and $\mathcal{J}' \leq \frac{1}{2}$ discussed in section 4.3. The equations of motion for fluctuations that follow from (4.26) are

$$\ddot{\tilde{\gamma}} - \tilde{\gamma}'' - 2\mu^2 \tilde{\gamma} + 2\mu\dot{\alpha} = 0 , \qquad (\text{A.18})$$

$$\ddot{\alpha} - \alpha'' - 2\sqrt{2}\beta' - 2\mu\dot{\tilde{\gamma}} = 0 , \qquad (\text{A.19})$$

$$\ddot{\beta} - \beta'' + 4\beta + 2\sqrt{2}\alpha' = 0 . \qquad (\text{A.20})$$

Expanding the fluctuations $\tilde{\gamma}$, α and β in Fourier series in σ, and then looking for solutions in the form $\mathrm{e}^{i\omega_n \tau}$, we find the following equation for the frequency spectrum ($\mu^2 \equiv 2 - \kappa^2$)

$$f_n(z) = 0 , \qquad z \equiv \omega_n^2 ,$$

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$$f_n(z) \equiv z^3 - (4 + 2\mu^2 + 3n^2)z^2 + (8\mu^2 + 3n^4)z - n^2(n^2 - 4)(n^2 - 2\mu^2) . \qquad \text{(A.21)}$$

Just as in the AdS_5 case, the two extrema $z_- < z_+$ of $f_n(z)$ are positive, with $f_n(z_-) > 0$ and $f_n(z_+) < 0$. Therefore, all roots are positive if

$$f_n(0) = -n^2(n^2 - 4)(n^2 - 2\mu^2) \leq 0 . \qquad \text{(A.22)}$$

Since $\mu^2 \leq 2$, the solution is stable only if $f_1(0) = 3(1 - 2\mu^2) \leq 0$, i.e. if

$$\frac{1}{2} \leq \mu^2 \leq 2 .$$

Appendix B. Quadratic fermionic part of the superstring action

The quadratic part of the $AdS_5 \times S^5$ Green-Schwarz superstring action [25] expanded near a particular bosonic string solution can be found as described, e.g., in [26,12] and used in a similar context in [4]. Assuming the induced metric is flat, the relevant part of the fermionic action is

$$L_F = i(\eta^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})\bar{\theta}^I \varrho_a D_b \theta^J , \qquad \varrho_a \equiv \Gamma_A e_a^A , \quad e_a^A = E_M^A \partial_a X^M , \qquad \text{(B.1)}$$

where $I, J = 1, 2$, $s^{IJ} =$ diag(1,-1), and ϱ_a are projections of the 10-d Dirac matrices. Here X^M are the string coordinates (given functions of τ and σ for a particular classical solution) corresponding to the AdS_5 ($M = 0, 1, 2, 3, 4$) and S^5 ($M = 5, 6, 7, 8, 9$) factors. The covariant derivative D_a can be put into the following form

$$D_a\theta^I = (\delta^{IJ}\mathrm{D}_a - \frac{i}{2}\epsilon^{IJ}\Gamma_* \varrho_a)\theta^J , \qquad \Gamma_* \equiv i\Gamma_{01234} , \quad \Gamma_*^2 = 1 , \qquad \text{(B.2)}$$

where

$$\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB} , \qquad \omega_a^{AB} \equiv \partial_a X^M \omega_M^{AB} , \qquad \text{(B.3)}$$

and the "mass term" originates from the R-R 5-form coupling [25].

B.1. AdS_5 case

In the case of the AdS_5 two-spin solution (4.18) the 2-d projections of Γ-matrices that enter the fermionic action are (the indices $A = 0, 1, 2, 3, 4$ here will label the $t, \rho, \theta, \phi, \varphi$ directions in the tangent space):

$$\varrho_0 = \kappa \cosh \rho_0 \; \Gamma_0 + \omega \sinh \rho_0 \tilde{\Gamma}_4 \;, \qquad \varrho_1 = \sinh \rho_0 \; \Gamma_2 \;, \qquad \varrho_{(a} \varrho_{b)} = \sinh^2 \rho_0 \; \eta_{ab} \;, \quad \text{(B.4)}$$

$$\tilde{\Gamma}_3 \equiv \cos \sigma \; \Gamma_3 - \sin \sigma \; \Gamma_4 \;, \qquad \tilde{\Gamma}_4 \equiv \cos \sigma \; \Gamma_4 + \sin \sigma \; \Gamma_3 \;. \tag{B.5}$$

The projected Lorentz connection $\omega_a^{AB} = \partial_a X^M \omega_M^{AB}$ has the following components

$$\omega_0^{01} = \kappa \sinh \rho_0 \;, \qquad \omega_0^{31} = \omega \cosh \rho_0 \sin \sigma \;, \qquad \omega_0^{41} = \omega \cosh \rho_0 \cos \sigma \;,$$

$$\omega_0^{32} = -\omega \cos \sigma \;, \qquad \omega_0^{42} = \omega \sin \sigma \;, \qquad \omega_1^{21} = \cosh \rho_0 \;. \tag{B.6}$$

Then

$$D_0 = \partial_0 + \frac{1}{2}(\kappa \sinh \rho_0 \Gamma_0 + \omega \cosh \rho_0 \tilde{\Gamma}_4)\Gamma_1 + \frac{1}{2}\omega \Gamma_2 \tilde{\Gamma}_3 \;, \qquad D_1 = \partial_1 - \frac{1}{2} \cosh \rho_0 \; \Gamma_1 \Gamma_2 \;. \tag{B.7}$$

After the σ-dependent rotation of θ^I in the 34-plane, $\theta^I \to \Psi^I = S^{-1}\theta^I$, $S = \exp(\frac{1}{2}\sigma\Gamma_3\Gamma_4)$, we find $\tilde{\Gamma}_{3,4} \to \Gamma_{3,4}$ in (B.4) and (B.7), at the expense of getting an additional constant $\Gamma_3\Gamma_4$ term in D_1. This eliminates the σ-dependence from the fermionic action. Note that Γ_* is invariant under this rotation.

To interpret the resulting fermionic action as a collection of massive 2-d fermions with standard kinetic terms it is useful to make as in [4] a further "rotation" (Lorentz boost) in the 04-plane, with $S = \exp(\frac{1}{2}\alpha\Gamma_0\Gamma_4)$, where $\cosh \alpha = \kappa \cosh \rho_0$. Then we get $\varrho_a = \sinh \rho_0 \; \tau_a$, $\tau_a = (\Gamma_0, \Gamma_2)$, $\tau_{(a}\tau_{b)} = \eta_{ab}$. There is the corresponding change in D_a while Γ_* remains invariant. Fixing the kappa-symmetry gauge by $\Psi^1 = \Psi^2$ and rescaling the fermions by $\sinh \rho_0$ we can interpret the resulting action

$$L_F = 2i \left(\bar{\Psi}\tau^a D_a \Psi + i\bar{\Psi}M\Psi \right) \;, \tag{B.8}$$

$$M = \frac{1}{2} \sinh \rho_0 \; \epsilon^{ab} \tau_a \Gamma_* \tau_b = im_F \Gamma_{134} \;, \qquad m_F = \sinh \rho_0 = \frac{1}{\sqrt{2}}\kappa \tag{B.9}$$

as describing a collection of 2-d massive Majorana fermions on a flat 2-d background coupled to a constant non-abelian 2-d gauge field (represented by constant Lorentz connection ω_a^{AB} terms). Indeed, in the representation for Γ_A where Γ_0 and Γ_2 are 2-d Dirac matrices times a unit 8×8 matrix we get as in [4] 4+4 species of 2-d Majorana fermions with masses $\pm m_F$.

We will not go into a detailed analysis of this action here and just mention that, as expected on the general grounds of conformal invariance of the $AdS_5 \times S^5$ string action [25], the fermionic contribution to the divergent part of the 1-loop effective action (which is proportional to the sum of mass-squared terms, i.e. $8 \times \frac{\kappa^2}{2}$) cancels the logarithmic divergence coming from the bosonic fluctuation action (3.30) (connection terms in both the bosonic and fermionic actions do not contribute to logarithmic divergences in 2 dimensions).

B.2. S^5 case

In the case of the S^5 solution (4.6) we shall label the tangent space coordinates by $A = 0, 5, 6, 7, 8, 9$ corresponding to the t direction of AdS_5 and $\gamma, \varphi_1, \psi, \varphi_2, \varphi_3$ directions of S^5. Then

$$\varrho_0 = \kappa \Gamma_0 + \nu \cos \gamma_0 \, \Gamma_6 + \text{w} \sin \gamma_0 \, \tilde{\Gamma}_8 \, , \qquad \varrho_1 = \sin \gamma_0 \, \Gamma_7 \, , \qquad \varrho_{(a} \varrho_{b)} = \sin^2 \gamma_0 \, \eta_{ab} \, , \quad \text{(B.10)}$$

$$\tilde{\Gamma}_8 \equiv \cos \sigma \, \Gamma_8 + \sin \sigma \, \Gamma_9 \, , \qquad \tilde{\Gamma}_9 \equiv \cos \sigma \, \Gamma_9 - \sin \sigma \, \Gamma_8 \, . \qquad \text{(B.11)}$$

The projected Lorentz connection has the following non-zero components

$$\omega_0^{65} = -\nu \sin \gamma_0 \, , \qquad \omega_0^{85} = \text{w} \cos \gamma_0 \, \cos \sigma \, , \qquad \omega_0^{95} = \text{w} \cos \gamma_0 \, \sin \sigma \, ,$$

$$\omega_0^{87} = -\text{w} \sin \sigma \, , \qquad \omega_0^{97} = \text{w} \cos \sigma \, , \qquad \omega_1^{75} = \cos \gamma_0 \, . \qquad \text{(B.12)}$$

As above, we first do local Lorentz rotation in the 89-plane to eliminate the σ-dependence; as a result, $\tilde{\Gamma}_{8,9} \to \Gamma_{8,9}$. Then (for generic ν and γ_0) we need to do two rotations – in the 68 and 06 planes – to put ϱ_0 into the form $\varrho_0 = \sin \gamma_0 \, \Gamma_0$. After the rotation in the 68-plane (under which Γ_* in (B.2) is invariant) we get $\varrho_0 = \kappa \Gamma_0 + a \Gamma_6$, $a^2 = \nu^2 \cos^2 \gamma_0 + \text{w}^2 \sin^2 \gamma_0 = \nu^2 + \sin^2 \gamma_0$. Under the boost in the 06-plane $S = \exp(\frac{1}{2} \beta \Gamma_0 \Gamma_6)$, where $\cosh \beta = \frac{\kappa}{\sin \gamma_0}$, the expression for Γ_* becomes

$$\Gamma'_* = S^{-1} \Gamma_* S = i(\cosh \beta \, \Gamma_0 - \sinh \beta \, \Gamma_6) \Gamma_{1234} \, . \qquad \text{(B.13)}$$

Then fixing the kappa-symmetry gauge by $\Psi^1 = \Psi^2$ and rescaling the fermions by $\sin \gamma_0$ we finish with the same action as in (B.8) with $\tau_a = (\Gamma_0, \Gamma_7)$ and

$$M = \frac{1}{2} \sin \gamma_0 \, \epsilon^{ab} \tau_a \Gamma'_* \tau_b = i m_{_F} \Gamma_{07} \Gamma_{12346} \, , \qquad m_{_F} = \sin \gamma_0 \, \sinh \beta = \frac{1}{\sqrt{2}} \sqrt{\kappa^2 + \nu^2} \, . \quad \text{(B.14)}$$

The contribution to the divergences is then proportional to $8 \times \frac{1}{2}(\kappa^2 + \nu^2) = 4\kappa^2 + 4\nu^2$ which is indeed the same as coming from the bosonic sector (see also Appendix C).

The same result is found also in the special cases discussed in section 4.3. When $\nu = 0$ (see (4.24)) we have $\varrho_0 = \kappa \Gamma_0 + \sin \gamma_0 \, \tilde{\Gamma}_8$, $\varrho_1 = \sin \gamma_0 \, \Gamma_7$, $\sin \gamma_0 = \frac{\kappa}{\sqrt{2}}$. Here we need a boost in 08-plane with parameter $\cosh \beta = \sqrt{2}$ to get $\varrho_a = \sin \gamma_0 \, \tau_a$. Then $m_{_F}$ is the same as in (B.14). When $\gamma_0 = \frac{\pi}{2}$, $\text{w}^2 = \kappa^2 - 1$ (see (4.29))[22] we get $\varrho_0 = \kappa \Gamma_0 + \text{w} \tilde{\Gamma}_8$, $\varrho_1 = \Gamma_7$ and thus the required 08-boost parameter has $\cosh \beta = \kappa$. That gives

$$\gamma_0 = \frac{\pi}{2} \, : \qquad M = i m_{_F} \Gamma_{07} \Gamma_{12348} \, , \qquad m_{_F} = \sinh \beta = \sqrt{\kappa^2 - 1} \, . \qquad \text{(B.15)}$$

Note that the fermionic and bosonic masses are different, reflecting the absence of the 2-d supersymmetry. The fermionic contribution to the logarithmic divergence is proportional to $8 \times (\kappa^2 - 1) = 4\kappa^2 + 4(\kappa^2 - 2)$ which indeed cancels the contribution from the bosonic fluctuations: 4 massive AdS_5 fields (4.25) and 4 massive S^5 fields in (4.31).

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[22]   Note that for $\cos \gamma_0 = 0$ the connection (B.12) simplifies substantially.

Appendix C. Bosonic fluctuation action in conformal gauge

In discussing fluctuation actions in the main part of the paper we used the static gauge. Let us note for completeness that similar conclusions can be reached also if one uses the conformal gauge (see, e.g., [12,4]). Let us recall that the general form of the quadratic fluctuation action for a sigma model in the conformal gauge written in terms of tangent-space fluctuations ($X^M \to X^M + \zeta^M$, $\zeta^A = E_M^A(X)\zeta^M$) is

$$I_B^{(2)} = -\frac{1}{2}\int d^2\xi \left(\sqrt{-g}g^{ab}\mathrm{D}_a\zeta^A \mathrm{D}_b\zeta^B \ \eta_{AB} + M_{AB}z^A z^B\right), \tag{C.1}$$

$$M_{AB} = -\sqrt{-g}g^{ab}e_a^C e_b^D R_{ACBD}, \qquad e_a^A \equiv \partial_a X^M E_M^A(X), \qquad g_{ab} = e_a^A e_b^B \eta_{AB}, \tag{C.2}$$

where $\mathrm{D}_a\zeta^A = \partial_a\zeta^A + \omega_a^{AB}(X)\zeta^B$ with the same projected Lorentz connection as in (B.3). For the AdS_5 part the curvature is $R_{ACBD} = -\eta_{AB}\eta_{CD} + \eta_{AD}\eta_{CB}$ while for the S^5 part it has the opposite sign, $R_{ACBD} = \delta_{AB}\delta_{CD} - \delta_{AD}\delta_{CB}$. If the induced metric is flat (as in all examples discussed in the present paper), the divergent part of the 1-loop action is determined simply by the trace of M_{AB} and should be the same as found in the static gauge, cf. [4].

Let list the expression for the mass matrix in (C.2) for the solutions discussed above (the expressions for the corresponding connections can be found in Appendix B). In the AdS_5 case (4.18) one gets:

$$e_0^0 = \kappa\cosh\rho_0, \qquad e_1^2 = \sinh\rho_0, \qquad e_0^2 = \omega\sinh\rho_0 \ \cos\sigma,$$

$$e_0^3 = \omega\sinh\rho_0 \ \sin\sigma, \qquad g_{ab} = \sinh^2\rho_0 \ \eta_{ab},$$

so that $\eta_{AB}M^{AB} = M_{AdS_5}^2$,

$$M_{AdS_5}^2 = 4\eta^{ab}e_a^C e_b^D \eta_{CD} = 4(\kappa^2\cosh^2\rho_0 + \sinh^2\rho_0 - \omega^2\sinh^2\rho_0) = 4\kappa^2. \tag{C.3}$$

This gives the same contribution to divergences as coming from the fermions (cf. (B.9)).

For the S^5 solution (4.6) one has both the AdS_5 and S^5 fluctuations, and

$$e_0^0 = \kappa, \qquad e_0^6 = \nu\cos\gamma_0, \qquad e_1^7 = \sin\gamma_0,$$

$$e_0^8 = \mathrm{w}\sin\gamma_0 \ \cos\sigma, \qquad e_0^9 = \mathrm{w}\sin\gamma_0 \ \sin\sigma, \qquad g_{ab} = \sin^2\gamma_0 \ \eta_{ab},$$

so that here

$$\eta_{AB}M^{AB} = M_{AdS_5}^2 + M_{S^5}^2, \qquad M_{AdS_5}^2 = 4\eta^{ab}e_a^C e_b^D \eta_{CD} = 4\kappa^2, \tag{C.4}$$

$$M_{S^5}^2 = -4\eta^{ab}e_a^C e_b^D \delta_{CD} = -4(\sin^2\gamma_0 - \nu^2\cos^2\gamma_0 - \mathrm{w}^2\sin^2\gamma_0) = 4\nu^2. \tag{C.5}$$

This is in agreement with the static gauge result (4.26) for $\nu = 0$ and is the same as the divergent contribution coming from the fermionic sector with mass matrix (B.14).

In the special case $\gamma_0 = \frac{\pi}{2}$, $\mathrm{w}^2 = \kappa^2 - 1$ we get instead

$$M_{S^5}^2 = 4(\kappa^2 - 2), \tag{C.6}$$

which is also in agreement with the static gauge result (4.31) and again cancels the divergences coming from the fermionic sector (4.31).

Appendix D. Dynkin labels and Young tableau labels of $SU(4)$ irreps

In this Appendix we recall (see, e.g., [27]) how the Dynkin labels of representations of the algebra $so(6)$ or, equivalently, of $su(4)$, are expressed in terms of the Young labels (numbers of boxes in raws of a Young tableau). Recall that the generators $J_1 = J_{12}$, $J_2 = J_{34}$, $J_3 = J_{56}$ form a basis of Cartan generators of $so(6)$. The simple roots can be chosen as

$$\alpha_1^{so(6)} = e_1 - e_2 = \{1, -1, 0\}, \quad \alpha_2^{so(6)} = e_2 - e_3 = \{0, 1, -1\}, \quad \alpha_3^{so(6)} = e_2 + e_3 = \{0, 1, 1\}. \tag{D.1}$$

Since $\alpha_2^{so(6)} \cdot \alpha_1^{so(6)} = \alpha_1^{so(6)} \cdot \alpha_3^{so(6)} = -1$ and $(\alpha_a^{so(6)})^2 = 2$ the root system is equivalent to the $su(4)$ root system with the following identification of the simple roots

$$\alpha_1^{su(4)} = \alpha_2^{so(6)}, \quad \alpha_2^{su(4)} = \alpha_1^{so(6)}, \quad \alpha_3^{su(4)} = \alpha_3^{so(6)}, \tag{D.2}$$

i.e. the two algebras are isomorphic.

An irreducible representation of $su(4)$ can be labelled by the eigenvalues of the Cartan generators J_i on the highest weight vector:

$$J_i |j_1, j_2, j_3\rangle = j_i |j_1, j_2, j_3\rangle, \qquad j_1 \geq j_2 \geq j_3 \geq 0, \tag{D.3}$$

where j_i is the number of boxes in the i-th row of the Young tableau associated with the representation which can then be denoted as (j_1, j_2, j_3). The same representation can be also labeled by the Dynkin labels d_a that are related to the Young labels j_i as follows:

$$j = \sum_{a=1}^{3} d_a \lambda_a, \qquad j \equiv \{j_1, j_2, j_3\}, \tag{D.4}$$

where λ_a are the fundamental weights defined by

$$\frac{2\lambda_a \cdot \alpha_b}{\alpha_b^2} = \lambda_a \cdot \alpha_b = \delta_{ab}. \tag{D.5}$$

By using (D.1) and (D.2), we can easily solve (D.5) to get

$$\lambda_1 = \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}, \qquad \lambda_2 = \{1, 0, 0\}, \qquad \lambda_3 = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}. \tag{D.6}$$

Then from (D.4) and (D.6) we find

$$j_1 = \frac{1}{2}(d_1 + 2d_2 + d_3), \qquad j_2 = \frac{1}{2}(d_1 + d_3), \qquad j_3 = \frac{1}{2}(-d_1 + d_3). \tag{D.7}$$

Solving the system, we get the relation between the Dynkin labels and the Young labels

$$d_1 = j_2 - j_3, \qquad d_2 = j_1 - j_2, \qquad d_3 = j_2 + j_3. \tag{D.8}$$

The Dynkin labels have to be non-negative. The representation associated with the Dynkin labels d_i is denoted as $[d_1, d_2, d_3]$.

Coming back to the string solution (2.23), we see that if $J \geq J'$ then the representation is (J, J', J') or $[0, J - J', 2J']$. If $J' \geq J$ the representation is (J', J', J) or $[J' - J, 0, J' + J]$. Note that in the last case we also have to rearrange the Cartan generators in such an order that $j_1 \geq j_2 \geq j_3 \geq 0$.

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