Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$

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Abstract

Motivated by recent proposals in hep-th/0202021 and hep-th/0204051 we develop semiclassical quantization of superstring in $AdS_5 \times S^5$. We start with a classical solution describing string rotating in AdS_5 and boosted along large circle of S^5. The energy of the classical solution E is a function of the spin S and the momentum J (R-charge) which interpolates between the limiting cases $S = 0$ and $J = 0$ considered previously. We derive the corresponding quadratic fluctuation action for bosonic and fermionic fields from the GS string action and compute the string 1-loop (large $\sqrt{\lambda} = \frac{R^2}{\alpha'}$) correction to the classical energy spectrum in the (S, J) sector. We find that the 1-loop correction to the ground-state energy does not cancel for $S \neq 0$. For large S it scales as $\ln S$, i.e. as the classical term, with no higher powers of $\ln S$ appearing. This supports the conjecture made in hep-th/0204051 that the classical $E - S = a \ln S$ scaling can be interpolated to weak coupling to reproduce the corresponding operator anomalous dimension behaviour in gauge theory.

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1. Introduction

Superstring theory in $AdS_5 \times S^5$ is a highly symmetric yet complicated-looking theory: both bosonic and fermionic sectors of the corresponding 2-d action are non-linear and hard to quantize directly. This precludes one from proving the expected duality to $\mathcal{N}$=4 SYM theory (even in the large N limit) in a direct way. It was recently suggested [1,2,3] that looking at special sectors of states in the full string spectrum where certain quantum numbers are large may allow one to check the AdS/CFT correspondence beyond the supergravity level.

In the simplest and most explicit example [2] one considers the sector of string states localized at the center of AdS_5 and carrying large momentum J along the central circle of S^5. The leading (1-loop) term in the energy spectrum of the corresponding *quantum* string oscillations can be found explicitly in the limit of large string tension or $\sqrt{\lambda} = \frac{R^2}{\alpha'} \gg 1$. Using the near-BPS nature of these string states one is then able to reproduce [2] the same string oscillator spectrum on the gauge-theory side as the spectrum of anomalous dimensions of single-trace operators carrying large R-charge.

In a more complicated but potentially more interesting case [3] one concentrates on states with large angular momentum S in AdS_5 (corresponding to spin in the boundary theory). A remarkable observation [3] is that the *classical* energy of a rotating string in AdS_5 space scales, for large $\frac{S}{\sqrt{\lambda}} \gg 1$, not as in flat (Regge) case $E = \sqrt{2\lambda S}$ but as $E = S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + ...$, which looks the same as the $S + a \ln S$ behaviour of the (canonical+anomalous) dimension of the corresponding bilinear operator in gauge-theory perturbative expansion. Since the string rotating in AdS_5 is not a BPS state, one expects that its energy should receive quantum (sigma-model loop) corrections,

$$E = S + f(\lambda) \ln \frac{S}{\sqrt{\lambda}} + h(\lambda) \ln^2 \frac{S}{\sqrt{\lambda}} + ... \ , \tag{1.1}$$

where $f, h, ...$ are given by series in $\lambda \gg 1$ (large tension) expansion

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + a_1 + \frac{a_2}{\sqrt{\lambda}} + O(\frac{1}{\lambda}) \ , \qquad h(\lambda) = k_1 + \frac{k_2}{\sqrt{\lambda}} + O(\frac{1}{\lambda}) \ , ... \ . \tag{1.2}$$

To be able to establish a correspondence with gauge-theory results $f(\lambda)$ should admit an interpolation to weak coupling, $f(\lambda)_{\lambda \ll 1} = b_1 \lambda + b_2 \lambda^2 + ...$, which should agree with the SYM perturbation theory. For consistency of the proposal of [3] all stronger that $\ln S$, i.e. $\ln^m S$ $(m = 2, 3, ...)$, corrections should cancel, both on the gauge-theory [3] and the string-theory side.

Here we shall compute the 1-loop string correction to the energy of the classical rotating AdS_5 string solution [4] considered in [3] and show that while $f(\lambda)$ in (1.2) does receive a correction (a_1 is non-zero), $h(\lambda)$ indeed vanishes ($k_1 = 0$). We shall also argue that the same remains to be true to all orders in string $\frac{1}{\sqrt{\lambda}}$ expansion. This provides strong support to the suggestion of [3], and thus another highly non-trivial check of the string theory – gauge theory correspondence.

Below we shall develop the general formalism for semiclassical quantisation near classical sigma-model solution describing rotating string in $AdS_5 \times S^5$, with the aim to compute the leading (1-loop) term in the energy of excited quantum string states in the sector with given angular momenta. We will use the Green-Schwarz formulation [5], and to the 1-loop order, will need only the simple quadratic term in the full fermionic action. Part of the discussion will be very similar to the one in [6,7,8] where the 1-loop correction to the $AdS_5 \times S^5$ string partition function near a "long" string configuration relevant for the Wilson loop average [9] was computed.

We shall consider the classical rotating solution that generalises both special cases considered in [3] and [2], i.e. describes a folded closed string stretched along radial direction ρ of AdS_5 and rotating along large circle ϕ of the boundary S^3, with its center of mass moving along the large circle φ of S^5. The classical energy is a complicated function of the two angular momenta S and J. It interpolates between $E = J$ for $S = 0$, $E \approx S + \frac{\sqrt{\lambda}}{\pi} \ln S$ for $J = 0$ and large S, and $E \approx \sqrt{J^2 + 2\sqrt{\lambda} S}$ for $\frac{J}{\sqrt{\lambda}} \ll 1$, $\frac{S}{\sqrt{\lambda}} \ll 1$ (which is the familiar flat-space expression $E^2 = J^2 + \frac{2}{\alpha'} S$). Consideration of this 2-parameter case is illuminating as it allows to interpolate between part of the quantum string spectrum in [2] (with spin represented by small string oscillations in AdS_5 directions) and the classical spectrum of a short string rotating in AdS_5 and boosted along the circle of S^5 (see section 3).

We shall start the discussion of the 1-loop quantum correction to the classical energy spectrum in section 4 with the case of $J \neq 0$ but $S = 0$, i.e. with non-zero S^5 boost but no spin in AdS_5 . We shall explain in detail (following suggestions of [2] and [3]) that the semiclassical quantization near the boosted point-like string solution gives directly the same oscillator string spectrum as obtained by exact quantization [10,11] of the closed superstring in the plane-wave [12] background. Since in the 1-loop approximation near a bosonic string solution one uses only the quadratic term in the fermionic part of the action, one does not need to go through the derivation [10] of the exact form of the GS action in the plane wave background, and the choice of the light-cone kappa-symmetry gauge is essentially imposed on us by the form of the classical string background.

In section 5 we shall derive the action for small fluctuations near the general (boosted and rotated) string solution, by considering bosonic fields in both static (sect. 5.1) and conformal (sect. 5.2) gauges and obtaining (in sect. 5.3) the fermionic part of the action from the quadratic part of the GS action in $AdS_5 \times S^5$ background (for non-zero S we can fix covariant kappa-symmetry gauge as in [8]). The resulting action describes a collection of 2-d bosonic and fermionic fields on a flat cylinder $(\tau, \sigma) = (\tau, \sigma + 2\pi)$ having, in general, non-constant (σ-dependent) masses.

In section 6 we shall study the 1-loop quantum corrections to the energy of the classical solution using certain approximations. We shall first compute the 1-loop shift of the ground-state energy ΔE (sect. 6). It cancelled out in the BPS case $E = J,\ S = 0$ [2] but does not vanish in the case of non-zero spin. In the limit of large S and $J = 0$ we shall find that the 1-loop coefficients in (1.2) are $a_1 \approx -\frac{3\ln 2}{\pi}$ and $k_1 = 0$. We shall also argue that in general string α' corrections will never produce higher than $\log S$ corrections to the energy.

We shall then consider (in sect. 6.2) the excited oscillator part of the 1-loop string spectrum (i.e. a generalization of the oscillator spectrum in the $S = 0$ case [2]) by considering special limits in the (S, J) parameter space. It would be interesting to see if the anomalous dimensions of gauge-theory operators with large spin and large R-charge have similar scaling with S and J as we find on the string-theory side.

Appendix A contains a derivation of the general expression for the correction to the energy of classical rotating string solution in terms of the 2-d Hamiltonian for quadratic fluctuation fields using the conformal-gauge constraints.

2. Superstring action in $AdS_5 \times S^5$

Our starting point will be the GS superstring action in $AdS_5 \times S^5$ written in the following general form

$$I = -\frac{\sqrt{\lambda}}{2\pi} \int d^2\xi \ \left[L_B(x,y) + L_F(x,y,\theta)\right] , \qquad \sqrt{\lambda} \ \equiv \frac{R^2}{\alpha'} \tag{2.1}$$

$$L_B = \frac{1}{2}\sqrt{-g} g^{ab} \big[G^{(AdS_5)}_{mn}(x) \partial_a x^m \partial_b x^n + G^{(S^5)}_{m'n'}(y) \partial_a y^{m'} \partial_b y^{n'} \big] \ . \tag{2.2}$$

Here $\xi^a = (\tau, \sigma)$, $\sigma \equiv \sigma + 2\pi$. We shall use Minkowski signature in both target space and world sheet, so that in conformal gauge $\sqrt{-g}g^{ab} = \eta^{ab}$ =diag(-1,1). The fermionic part is

$$L_F = i(\sqrt{-g}g^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})\bar{\theta}^I \varrho_a D_b \theta^J + O(\theta^4) \ , \tag{2.3}$$

where $I, J = 1, 2$, s^{IJ} =diag(1,-1), ϱ_a are projections of the 10-d Dirac matrices,

$$\varrho_a \equiv \Gamma_A E^A_M \partial_a X^M = (\Gamma_p E^p_M + \Gamma_{p'} E^{p'}_M)\partial_a X^M \ , \tag{2.4}$$

and E^A_M is the vielbein of the 10-d target space metric (see [5,8] for details). Here $X^M = (x^m, y^{m'})$ are the coordinates and p, p' are the tangent space indices of AdS_5 and S^5. The covariant derivative D_a is the projection $\partial_a X^M D_M$ of the 10-d derivative $D^{IJ}_M = (\partial_M + \frac{1}{4}\omega^{AB}_M \Gamma_{AB})\delta^{IJ} - \frac{1}{8\cdot 5!} F_{A_1 \ldots A_5}\Gamma^{A_1 \ldots A_5}\Gamma_M \epsilon^{IJ}$. Since θ^I are 10-d MW spinors of the same chirality and since $F_5 = \epsilon_5 + *\epsilon_5$ for the $AdS_5 \times S^5$ background, it can be put into the following form

$$D_a \theta^I = (\delta^{IJ} \mathrm{D}_a - \frac{i}{2}\epsilon^{IJ}\Gamma_* \varrho_a)\theta^J \ , \qquad \Gamma_* \equiv i\Gamma_{01234} \ , \quad \Gamma_*^2 = 1 \ , \tag{2.5}$$

where $\mathrm{D}_a = \partial_a + \frac{1}{4}\partial_a X^M \omega^{AB}_M \Gamma_{AB}$ and the "mass term" originates from the R-R coupling (cf. [5,6]).[1]

In the leading large $\sqrt{\lambda} = \frac{R^2}{\alpha'}$ approximation near a classical bosonic solution we will be discussing we will need to know only the quadratic fermionic term.[2] Our approach will be similar to the one in [6,7,8] where 1-loop string correction to the Wilson loop factor (i.e. long open string configuration) [9] was discussed. Here we will follow [2,3] and will need to use global coordinates in which the space-time energy of a string state is directly related to the dimension of the corresponding state in dual gauge theory [14].

We shall use the following explicit parametrization of the (unit-radius) metrics of AdS_5 and S^5:

$$(ds^2)_{AdS_5} = G^{(AdS_5)}_{mn}(x)dx^m dx^n = -\cosh^2\rho \ dt^2 + d\rho^2 + \sinh^2\rho \ d\Omega_3 \ , \tag{2.6}$$

$$d\Omega_3 = d\beta_1^2 + \cos^2\beta_1(\ d\beta_2^2 + \cos^2\beta_2 \ d\beta_3^2) \ , \qquad \beta_3 \equiv \phi \ .$$

$$(ds^2)_{S^5} = G^{(S^5)}_{m'n'}(y)dy^{m'}dy^{n'} = d\psi_1^2 + \cos^2\psi_1(\ d\psi_2^2 + \cos^2\psi_2 \ d\Omega_3') \ , \tag{2.7}$$

$$d\Omega_3' = d\psi_3^2 + \cos^2\psi_3 \ (d\psi_4^2 + \cos^2\psi_4 \ d\psi_5^2) \ , \qquad \psi_5 \equiv \varphi \ .$$

[1] In the special "5+5" representation of the Γ-matrices used in [8] one can write $D_a\theta^I = (\delta^{IJ}\mathrm{D}_a - \frac{i}{2}\epsilon^{IJ}\tilde{\varrho}_a)\theta^J$, $\tilde{\varrho}_a \equiv \Gamma_p E^p_M + i\Gamma_{p'}E^{p'}_M$.

[2] The full form of the GS string action in $AdS_5 \times S^5$ was explicitly written down [5,13] in the Poincare coordinates of AdS_5 .

3. Classical solution for rotating string in $AdS_5 \times S^5$

The classical string solution we shall consider is a direct generalisation of both the particle (point-like string) moving with the speed of light along the φ-circle of S^5 [2] and the solution of [3] describing a closed string rotating in the (ρ, ϕ) plane of AdS_5 (this solution was originally found in [4], and was discussed also in [15]).[3] It is straightforward to check that the following rotating and boosted closed string configuration solves the classical string equations in $AdS_5 \times S^5$ space

$$t = \kappa\tau \ , \quad \phi = \omega\tau \ , \quad \varphi = \nu\tau \ , \quad \kappa, \omega, \nu = \text{const} \ ,$$

$$\rho = \rho(\sigma) = \rho(\sigma + 2\pi) \ , \quad \beta_i = 0 \ \ (i = 1, 2) \ , \quad \psi_s = 0 \ \ (s = 1, 2, 3, 4) \ , \tag{3.1}$$

where ρ is subject to the corresponding second-order equation (prime denotes derivative over σ)

$$\rho'' = (\kappa^2 - \omega^2) \sinh\rho \ \cosh\rho \ . \tag{3.2}$$

The first of the conformal gauge constraints

$$G^{(AdS_5)}_{mn}(x)(\partial_0 x^m \partial_0 x^n + \partial_1 x^m \partial_1 x^n) + G^{(S^5)}_{m'n'}(y)(\partial_0 y^{m'} \partial_0 y^{n'} + \partial_1 y^{m'} \partial_1 y^{n'}) = 0 \ , \tag{3.3}$$

$$G^{(AdS_5)}_{mn}(x)\partial_0 x^m \partial_1 x^n + G^{(S^5)}_{m'n'}(y)\partial_0 y^{m'} \partial_1 y^{n'} = 0 \ , \tag{3.4}$$

then says that $\rho(\sigma)$ must satisfy the following 1-st order equation (implying (3.2) for $\rho' \neq 0$)[4]

$$\rho'^2 = \kappa^2 \cosh^2\rho - \omega^2 \sinh^2\rho - \nu^2 \ . \tag{3.5}$$

The periodicity condition on $\rho(\sigma)$ is satisfied by considering a folded string configuration. In the simplest ("one-fold") case considered in [3] [5] the interval $0 \leq \sigma < 2\pi$ is split into 4 segments: for $0 < \sigma < \pi/2$ the function $\rho(\sigma)$ increases from 0 to its maximal value ρ_0 $(\rho'(\pi/2) = 0)$

$$(\kappa^2 - \nu^2)\cosh^2\rho_0 - (\omega^2 - \nu^2)\sinh^2\rho_0 = 0 \ , \tag{3.6}$$

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[3] Other oscillating circular string solutions in AdS space were studied in [16].

[4] In terms of the coordinate $r = \tanh\frac{\rho}{2}$ this equation can be interpreted as describing periodic motion in the central part of the inverted double-well potential.

[5] Multifold configurations correspond to states with lower spin for given energy.

then for $\pi/2 < \sigma < \pi$ decreases to zero, etc.[6] The periodicity thus implies an extra condition on the parameters:

$$2\pi = \int_0^{2\pi} d\sigma = 4 \int_0^{\rho_0} \frac{d\rho}{\sqrt{(\kappa^2 - \nu^2)\cosh^2\rho - (\omega^2 - \nu^2)\sinh^2\rho}} \ . \tag{3.7}$$

The corresponding induced 2-d metric on the (τ, σ) cylinder is conformally-flat

$$g_{ab} = G_{MN}(X)\partial_a X^M \partial_b X^N = \rho'^2(\sigma)\eta_{ab} \ . \tag{3.8}$$

Its curvature

$$R^{(2)} = -\rho'^{-2}\partial_\sigma^2 \ln \rho'^2 = -2 + \frac{2(\kappa^2 - \nu^2)(\omega^2 - \nu^2)}{\rho'^4} \tag{3.9}$$

has singularities on the lines $\sigma = \pi/2$ and $\sigma = 3\pi/2$, i.e. at the turning points where $\rho' = 0$.[7]

When $\nu = 0$, i.e. when the rotation is only in AdS_5 space, this is the case considered in [3,4]. If instead we choose $\omega = 0$, then to satisfy the periodicity condition we must set $\rho = \rho_0 =$ const, but then (3.2) implies that one should put $\rho_0 = 0$. [8] Thus we get a string shrunk to a point, placed at the center of AdS_5 and rotating along the circle in S^5 with the speed of light [2,3].

Another special case is when $\kappa = \pm\omega$: this leads to $\rho'' = 0$ and $\rho'^2 = \kappa^2 - \nu^2$=const. To have a continuous periodic $\rho(\sigma)$ we need to demand $\kappa = \pm\nu$, i.e. $\rho = \rho_0$=const. This case is equivalent to $\rho = 0$ and $\omega = 0$, $\kappa = \pm\nu$ by a coordinate transformation in AdS_5 (a combination of $SO(1,1)$ and $SO(2)$ rotation).

[6] In flat space $ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + ...$ the corresponding rotating string solution (with $\nu = 0$) is described by $t = \kappa\tau$, $\phi = \omega\tau$, $\rho = \rho_0|\sin\omega\sigma|$, $\rho_0 = \frac{\kappa}{\omega}$, so that $\rho(\sigma) = \rho(\sigma + 2\pi)$ implies that ω must be an integer (=1 in the one-fold case). The modulus accounts for the fact that ρ must be positive. An apparent singularity at $\sigma = 0$ (i.e. $\rho = 0$) is simply a coordinate one: the solution has the standard regular form when written in the cartesian coordinates in the 2-plane: $x_1 \equiv \rho\cos\phi = \rho_0 \sin\omega\sigma \cos\omega\tau$, $x_2 \equiv \rho\sin\phi = \rho_0 \sin\omega\sigma \sin\omega\tau$.

[7] The induced metric written in terms of ρ and τ as the coordinates is $ds^2 = -f(\rho)d\tau^2 + d\rho^2$ where $f = \rho'^2 = \kappa^2\cosh^2\rho - \omega^2\sinh^2\rho - \nu^2$. The turning points are thus singularities coinciding with the horizons. For a general discussion of folded classical string solutions in curved $D \geq 2$ space and related references see [17].

[8] A possible alternative is not to impose periodicity in σ and consider instead an open string stretched all the way to the boundary of AdS_5 .

Since the solution depends only on the two combinations $\kappa^2 - \nu^2$ and $\omega^2 - \nu^2$, the analysis of its properties is very similar to the one in [3]. Introducing the parameter η

$$\coth^2 \rho_0 = \frac{\omega^2 - \nu^2}{\kappa^2 - \nu^2} \equiv 1 + \eta \ , \qquad \eta > 0 \ , \tag{3.10}$$

it is not difficult to see that solutions with finite energy exist if (we choose κ, ω, ν to be positive)

$$\kappa > \nu \ , \qquad \omega > \nu \ . \tag{3.11}$$

As in [3] the three conserved momenta conjugate to t, ϕ, φ are the space-time energy E and the two angular momenta S and J:

$$E = P_t = \sqrt{\lambda}\, \kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \cosh^2 \rho \equiv \sqrt{\lambda}\, \mathcal{E} \ , \tag{3.12}$$

$$S = P_\phi = \sqrt{\lambda}\, \omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \sinh^2 \rho \equiv \sqrt{\lambda}\, \mathcal{S} \ , \tag{3.13}$$

$$J = P_\varphi = \sqrt{\lambda}\, \nu \int_0^{2\pi} \frac{d\sigma}{2\pi} \ = \sqrt{\lambda}\, \nu \ . \tag{3.14}$$

Note the relation

$$\mathcal{E} = \kappa + \frac{\kappa}{\omega} \mathcal{S} \ , \tag{3.15}$$

which together with (3.7) may be used to determine the dependence of E on S and J.

In the full quantum theory S and J should take quantized values. In the semiclassical approximation we shall consider we shall assume that their values are very large, i.e. that $\frac{S}{\sqrt{\lambda}}$ and $\frac{J}{\sqrt{\lambda}}$ are finite for $\sqrt{\lambda} \gg 1$, or, equivalently, that the parameters of the classical solution κ, ω, ν are fixed in the limit of large $\sqrt{\lambda}$.

Computing the integrals in (3.7),(3.12),(3.13) (using $d\sigma = \rho'^{-1} d\rho$ and (3.5)) we find

$$\sqrt{\kappa^2 - \nu^2} = \frac{1}{\sqrt{\eta}} \ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta}\right) \ , \tag{3.16}$$

$$\mathcal{E} = \frac{\kappa}{\sqrt{\kappa^2 - \nu^2}} \frac{1}{\sqrt{\eta}} \ {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta}\right) \ , \qquad \mathcal{S} = \frac{\omega}{\sqrt{\kappa^2 - \nu^2}} \frac{1}{2\eta\sqrt{\eta}} \ {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; -\frac{1}{\eta}\right) \ . \tag{3.17}$$

Let us now follow [3] and consider the limits of "short" ($\rho_0 \to 0$, i.e. $\eta \to \infty$) and "long" ($\rho_0 \to \infty$, i.e. $\eta \to 0$) strings.

3.1. **Short string**

For $\eta \gg 1$ we get from (3.16) and (3.10)

$$\kappa^2 \approx \nu^2 + \frac{1}{\eta} , \qquad \omega^2 \approx \kappa^2 + 1 \approx \nu^2 + 1 + \frac{1}{\eta} . \tag{3.18}$$

Taking into account that $\omega^2 - \kappa^2 \approx 1$, an approximate solution describing a short rotating string has $\rho(\sigma)$ satisfying

$$\sinh^2 \rho \approx \rho^2 \approx \frac{1}{\eta} \sin^2 \sigma . \tag{3.19}$$

Eq.(3.17) implies

$$\frac{1}{\eta} \approx \frac{2\mathcal{S}}{\sqrt{1+\nu^2}} \ll 1 , \quad \text{i.e.} \quad 4\mathcal{S}^2 \ll 1 + \nu^2 . \tag{3.20}$$

Eq.(3.15) then determines the dependence of the energy E on the angular momenta $J = \sqrt{\lambda}\ \nu$ and $S = \sqrt{\lambda}\ \mathcal{S}$

$$\mathcal{E} \approx \sqrt{\nu^2 + \frac{2\mathcal{S}}{\sqrt{1+\nu^2}}} + \sqrt{\frac{\nu^2 + \frac{2\mathcal{S}}{\sqrt{1+\nu^2}}}{1 + \nu^2 + \frac{2\mathcal{S}}{\sqrt{1+\nu^2}}}}\ \mathcal{S} . \tag{3.21}$$

This relation is valid for any ν and $\mathcal{S}$ satisfying (3.20). Further simplification depends on a particular value of ν. If ν is small, i.e. $\nu \ll 1$ then $\mathcal{S} \ll 1$ and (3.21) becomes

$$\mathcal{E} \approx \sqrt{\nu^2 + 2\mathcal{S} + ...} + ... , \quad \text{i.e.} \quad E^2 \approx J^2 + 2\sqrt{\lambda} S + ... \tag{3.22}$$

This expression has a simple interpretation. The limit of short strings probes small-curvature region of AdS_5 where $\rho \approx 0$, i.e. the second term in the r.h.s. of (3.15) and thus the second square root in (3.21) should be small, and the energy spectrum should be approximately the same as in flat space. Indeed, (3.22) is the correct relativistic expression for the energy of a string in flat space moving along a φ-direction circle with momentum ν and rotating in a 2-plane with spin $\mathcal{S}$. If the boost energy is smaller than the rotation one, i.e. if $\nu^2 \ll \mathcal{S}$, then we are back to the flat-space Regge trajectory [3]:

$$\mathcal{E} \approx \sqrt{2\mathcal{S}} + \frac{\nu^2}{2\sqrt{2\mathcal{S}}} . \tag{3.23}$$

A more interesting case is that of large $\nu \gg 1$. Then from (3.20) $\nu \gg 2\mathcal{S}$, and thus (3.21) becomes

$$\mathcal{E} \approx \nu + \mathcal{S} + \frac{\mathcal{S}}{2\nu^2} + ... , \quad \text{i.e.} \quad E \approx J + S + \frac{\lambda S}{2J^2} + ... \tag{3.24}$$

Remarkably, this expression can be directly related to the leading quantum term in the spectrum of strings in $AdS_5 \times S^5$ in the frame boosted to the speed of light along the φ-circle of S^5 [2]:

$$E = J + \sum_{n=-\infty}^{\infty} \sqrt{1 + \frac{\lambda n^2}{J^2}}\ N_n \ + \ O(\frac{1}{\sqrt{\lambda}}) \ . \tag{3.25}$$

Here $J = \sqrt{\lambda}\nu$ is the quantum value of the momentum and N_n stands for the oscillator occupation number of a particular state (see [11] for detailed expressions; see also section 4 below). Let us first recall that in the flat-space string theory the state with a large angular momentum on the leading Regge trajectory can be described either by expanding near a classical rotating string configuration or by expanding near a point-like string vacuum state and building up the angular momentum out of quantum oscillator modes (the semiclassical configuration with large angular momentum will then be represented by the corresponding coherent state $e^{\sqrt{S}a^\dagger_1}|0>$). In the light-cone frame where $P^- = -\frac{1}{\alpha' p^+}\sum_n |n|N_n$ the leading Regge trajectory $P^- \sim S$ will be represented by states $(a^\dagger_1)^S|0>$ for which $n = 1,\ N_1 = S$.

In exactly the same fashion, the classical energy (3.24) of rotated string in the frame boosted along the direction φ transverse to the rotation plane is correctly captured by the quantum spectrum (3.25) of string oscillations in the S^5-boosted frame: applying (3.25) to the oscillator state with $n = 1,\ N_1 = S$ $(1 \ll S \ll J)$ and expanding in large J one reproduces (3.24).

Note that the term linear in S in (3.24) comes from the second term in (3.15) and (3.21), i.e. its presence is due to the curvature of AdS_5 . Indeed, this term is reproduced by the first term ("1") under the square root in (3.25) (related to the mass term of the light-cone string coordinates in the plane wave background [10]) whose origin can be traced to the curvature of the AdS_5 background [12,2].

This observation supports the suggestion of [3] that parts of semiclassical $AdS_5 \times S^5$ string spectrum can be captured by expanding near different classical string solutions. We see that there is an overlap between the leading-order (large $\sqrt{\lambda}$) quantum spectrum obtained by expanding near S^5-boosted point-like string state with *no* rotation in AdS_5 and a classical spectrum obtained by expanding near a highly boosted *and* rotating string solution.

3.2. **Long string**

For long strings for which the maximal value ρ_0 of $\rho(\sigma)$ is large, i.e. $\eta \ll 1$ we get from (3.16),(3.10) and (3.17)

$$\kappa^2 \approx \nu^2 + \frac{1}{\pi^2} \ln^2 \frac{1}{\eta} \ , \qquad \omega^2 \approx \nu^2 + \frac{1}{\pi^2} (1+\eta) \ln^2 \frac{1}{\eta} \ , \tag{3.26}$$

$$\mathcal{S} \approx \frac{2\omega}{\eta \ln \frac{1}{\eta}} \ . \tag{3.27}$$

We learn that for long strings the spin is always large, $\mathcal{S} \gg 1$, irrespective of the value of ν. Here there is no simple relation between $\mathcal{E}$ and $\mathcal{S}$ for general ν (cf. (3.21)) so we need to consider special cases.

In the case of small ν, i.e. $\nu \ll \ln \frac{1}{\eta}$, (3.27) implies that

$$\frac{1}{\eta} \approx \frac{\pi}{2} \mathcal{S} \ , \tag{3.28}$$

and using the relation (3.15), we obtain the energy

$$\mathcal{E} \approx \mathcal{S} + \frac{1}{\pi} \ln \mathcal{S} \ + \ \frac{\pi \nu^2}{2 \ln \mathcal{S}} \ , \quad \text{i.e.} \qquad E \approx S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} \ + \ \frac{\pi J^2}{2\sqrt{\lambda} \ln \frac{S}{\sqrt{\lambda}}} \ . \tag{3.29}$$

We have kept only the first $\ln \mathcal{S}$ -correction and the leading term in ν^2. The first two terms in (3.29) are the ones obtained in [3] where ν was equal to zero.[9]

In the opposite case of large ν, i.e. $\nu \gg \ln \frac{1}{\eta}$ we find from (3.27),(3.26)

$$\eta \ln \frac{1}{\eta} \approx \frac{2\nu}{\mathcal{S}} \ . \tag{3.30}$$

Since $\eta \ll 1$, this implies that $\nu \ll \mathcal{S}$. The leading asymptotics for the solution of the equation (3.30) for η is

$$\ln \frac{1}{\eta} \approx \ln \frac{\mathcal{S}}{\nu} \ . \tag{3.31}$$

Using (3.15), we then find that the energy in this case of $\ln \frac{\mathcal{S}}{\nu} \ll \nu \ll \mathcal{S}$ is given by

$$\mathcal{E} \approx \mathcal{S} + \nu + \frac{1}{2\pi^2 \nu} \ln^2 \frac{\mathcal{S}}{\nu} \ , \quad \text{i.e.} \qquad E \approx S + J + \frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J} \ . \tag{3.32}$$

[9] Note that using the relation (3.15) to find the energy as a function of spin it is sufficient to determine only the leading $\mathcal{S}$-dependence of η.

In contrast to the large ν limit of the short string case (3.24) here the third correction term is not related to the "plane-wave" spectrum (3.25). This is not surprising since there one considered the limit of large boost and small string oscillations, while in the long-string case the spin is always larger than the boost parameter ν.

Compared to the small boost ν case (3.29) the energy $\mathcal{E}$ in (3.32) no longer has a characteristic large-spin $\ln \mathcal{S}$ term found in [3]. In general, $\mathcal{E}(\mathcal{S},\nu)$ contains $k(\nu)\ln \mathcal{S}$ term with coefficient function such that $k(\nu \ll 1) \to \frac{1}{\pi}$ (cf. (3.29)) and $k(\nu \gg 1) \to \frac{\ln \nu}{\pi^2 \nu} \to 0$ (cf. (3.32)). Eq. (3.32) predicts the presence of the $\frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J}$ term in the strong-coupling asymptotics of the anomalous dimension of the gauge-theory operators with large spin and large R-charge [2]. It would be very interesting to see if the $\ln S$ term [3] in the anomalous dimensions of the corresponding $N=4$ SYM operators with large R-charge is suppressed also at weak 't Hooft coupling.

4. 1-loop approximation: boost in S^5

Our aim is to compute the leading quantum corrections to the energy spectrum by expanding the action to quadratic order in fluctuations near the classical solution. In flat space, where the action is gaussian in the conformal gauge, this would effectively account for the full string spectrum of states, irrespective of a particular classical solution one starts with. In the present non-linear sigma model case this will be a good approximation to the string spectrum only up to $\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}$ corrections.[10]

In this section we shall consider the case of rotation in S^5 only, i.e. the solution (3.1) with $\omega = 0,\ \nu = \kappa,\ \rho = 0$, reproducing the "plane-wave" string oscillator spectrum of [2] by an explicit version of the argument suggested in [3]. We shall use the $AdS_5 \times S^5$ string sigma model action in the conformal gauge. Given the solution

$$t = \varphi = \nu\tau\ , \quad \rho = 0\ , \qquad \beta_l = 0 \quad (l = 1,2,3)\ , \qquad \psi_s = 0 \quad (s = 1,2,3,4)\ , \qquad (4.1)$$

we will expand the action and the constraints up to quadratic order in fluctuations.

Since the point $\rho = 0$ is special with regard to small fluctuations along S^3 directions in AdS_5 (the corresponding tangent-space metric is degenerate)[11] it is useful to replace

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[10] As usual, the semiclassical expansion will be well-defined if the parameters of the classical solution $(\kappa, \omega, \nu, ...)$ are fixed in the limit $\frac{1}{\sqrt{\lambda}} \to 0$.

[11] This is the same as starting with the model $L = (\partial\rho)^2 + \rho^2(\partial\varphi)^2$ and trying to do semiclassical expansion near the point $\rho = 0$.

(ρ, β_i) by 4 cartesian coordinates η_k $(k = 1,2,3,4)$ and then consider their fluctuations. Equivalently, we may start with the AdS_5 metric (2.6) written in the coordinates

$$ds^2 = -\frac{(1+r^2)^2}{(1-r^2)^2}dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\Omega_3) = -\frac{(1+\frac{1}{4}\eta^2)^2}{(1-\frac{1}{4}\eta^2)^2}dt^2 + \frac{d\eta_k d\eta_k}{(1-\frac{1}{4}\eta^2)^2} \ , \quad (4.2)$$

and then expand (2.1) near $\eta_k = 0$, i.e.

$$t = \nu\tau + \tfrac{1}{\lambda^{1/4}}\tilde{t} \ , \qquad \eta_k = \tfrac{1}{\lambda^{1/4}}\tilde{\eta}_k \ , \qquad \varphi = \nu\tau + \tfrac{1}{\lambda^{1/4}}\tilde{\varphi} \ , \qquad \psi_s = \tfrac{1}{\lambda^{1/4}}\tilde{\psi}_s \ . \qquad (4.3)$$

This leads to the following bosonic action for the quadratic fluctuations:

$$I_B^{(2)} = -\frac{1}{4\pi}\int d^2\xi \ \big[-\partial_a\tilde{t}\partial^a\tilde{t} + \partial_a\tilde{\varphi}\partial^a\tilde{\varphi} + \nu^2(\tilde{\eta}_k^2 + \tilde{\psi}_s^2) + \partial_a\tilde{\eta}_k\partial^a\tilde{\eta}_k + \partial_a\tilde{\psi}_s\partial^a\tilde{\psi}_s\big] \ . \quad (4.4)$$

This is the same action that is found by expanding the string action in the plane wave background of [12]

$$I_B^{(pw)} = -\frac{1}{4\pi}\int d^2\xi \ \big[\partial_a x^+\partial^a x^- - \tfrac{1}{4}(\eta_k^2 + \psi_s^2)\partial^a x^+\partial_a x^+ + \partial_a\eta_k\partial^a\eta_k + \partial_a\psi_s\partial^a\psi_s\big] \ , \ (4.5)$$

to quadratic order near the following classical solution

$$x^+ = p^+\tau \ , \qquad x^- = 0 \ , \qquad \eta_k = 0 \ , \qquad \psi_s = 0 \ , \qquad x^\pm \equiv \varphi \pm t \ , \qquad p^+ \equiv 2\nu \ . \qquad (4.6)$$

Starting with the action (4.5) (supplemented with fermions (2.3) to preserve conformal invariance at the quantum level) one may then find the corresponding spectrum by imposing the quantum light-cone gauge condition

$$\tilde{x}^+ = 0 \ , \qquad \text{i.e.} \qquad \tilde{\varphi} + \tilde{t} = 0 \ . \qquad (4.7)$$

The same is effectively possible (to the leading order in $\frac{1}{\sqrt{\lambda}}$ expansion) also in the present case, even though the full non-linear action (2.1) does not allow one to fix the light-cone gauge in its standard form. The resulting light-cone gauge action is then simply that of 8 massive 2-d fields as in the case of (4.5) in the light-cone gauge [10]

$$I_B^{(2)} = -\frac{1}{4\pi}\int d^2\xi \ \big[\partial_a\tilde{\eta}_k\partial^a\tilde{\eta}_k + \partial_a\tilde{\psi}_s\partial^a\tilde{\psi}_s + \nu^2(\tilde{\eta}_k^2 + \tilde{\psi}_s^2)\big] \ . \qquad (4.8)$$

The relevant quadratic part of the fermionic action (2.3) also takes a simple form (the only non-vanishing $\partial_a X^M$ factors in (2.4) are $\partial_0 t = \partial_0\varphi = \nu$) and becomes the same as in [10] if we choose the light-cone kappa-symmetry gauge $\Gamma^+\theta^I = 0$ (see [11] and section

5.3 below). As a result, the quadratic fluctuation part of the superstring action expanded near the solution (4.1) takes indeed the same form as the full GS action in the maximally supersymmetric plane wave background. Therefore, its spectrum (to the leading order in $\frac{1}{\sqrt{\lambda}}$) is the same as found in [11,2].

In more detail, the action (4.4) should be supplemented by the conformal gauge constraints (3.3),(3.4) which, expanded to quadratic order in the fluctuations, take the form

$$2\lambda^{1/4}\nu\partial_0\tilde{x}^- - \nu^2(\tilde{\eta}_k^2 + \tilde{\psi}_s^2) + \partial_0\tilde{\eta}_k\partial_0\tilde{\eta}_k + \partial_1\tilde{\eta}_k\partial_1\tilde{\eta}_k + \partial_0\tilde{\psi}_s\partial_0\tilde{\psi}_s + \partial_1\tilde{\psi}_s\partial_1\tilde{\psi}_s$$

$$+ \partial_0\tilde{x}^+\partial_0\tilde{x}^- + \partial_1\tilde{x}^+\partial_1\tilde{x}^- + O(\tfrac{1}{\sqrt{\lambda}}) + ... = 0 \ , \tag{4.9}$$

$$2\lambda^{1/4}\nu\partial_1\tilde{x}^- + \partial_0\tilde{\eta}_k\partial_1\tilde{\eta}_k + \partial_0\tilde{\psi}_s\partial_1\tilde{\psi}_s + \frac{1}{2}\partial_0\tilde{x}^+\partial_1\tilde{x}^- + \frac{1}{2}\partial_1\tilde{x}^+\partial_0\tilde{x}^- + O(\tfrac{1}{\sqrt{\lambda}}) + ... = 0 \ , \tag{4.10}$$

where dots stand for the fermionic contributions. As usual, one may either impose the constraints on states "on average" or impose a proper gauge condition (light-cone gauge in the present case), directly solve constraints and quantise the remaining degrees of freedom. Both approaches lead to the same expression for the physical spectrum, but the second is more direct in the present case. Here we should solve constraints perturbatively in large λ, so that we learn that $\partial_a\tilde{x}^- \approx 0$; that means that quadratic terms in $\partial\tilde{x}^\pm$ in (4.9),(4.10) and in the action can be omitted to the leading order. Since $\tilde{x}^-$ is "conjugate" to $\tilde{x}^+$ in the action (4.4), this is essentially equivalent to imposing the quantum light-cone gauge (4.7). We thus find the expressions for $\partial_0\tilde{x}^-$ and $\partial_1\tilde{x}^-$ in terms of 8 fluctuations $\tilde{\eta}_k, \tilde{\psi}_s$.

To find the correction to the space-time energy conjugate to the time coordinate t we note that to second order in fluctuations the relations for P_t and P_φ are (cf. (3.12) and (4.2),(2.6),(2.7))

$$E = P_t = \int_0^{2\pi} \frac{d\sigma}{2\pi} \ (\sqrt{\lambda}\ \nu + \lambda^{1/4}\partial_0\tilde{t} + \nu\tilde{\eta}_k^2 + ...) \ , \tag{4.11}$$

$$J = P_\varphi = \int_0^{2\pi} \frac{d\sigma}{2\pi} \ (\sqrt{\lambda}\ \nu + \lambda^{1/4}\partial_0\tilde{\varphi} - \nu\tilde{\psi}_s^2 + ...) \ . \tag{4.12}$$

Dots stand for fermionic and higher order terms. Taking their difference and using the constraint (4.9) to eliminate the $\partial_0(\tilde{t}-\tilde{\varphi})$ in terms of the transverse oscillators we find that the extra terms in (4.11),(4.12) effectively change the sign of the $\tilde{\eta}_k$ and $\tilde{\psi}_s$ mass terms in the constraint (4.9) written in the form $\int d\sigma\partial_0\tilde{x}^- = ...$. The end result is that $E - J$ is given by the expectation value of the transverse Hamiltonian (see also Appendix). This is the same expression as found by directly applying the quantum light-cone gauge (4.7), giving the expectation value of $-P^- = E-J$ in terms of the eigen-values of the "light-cone " Hamiltonian, i.e. equivalent (once the fermions are included) to the one (3.25) in [2,11]

$$E-J = \frac{1}{\nu}\sum_{n=-\infty}^{\infty} \sqrt{n^2+\nu^2}\ N_n \ + \ O(\frac{1}{\sqrt{\lambda}}) = \sum_{n=-\infty}^{\infty} \sqrt{1+\frac{\lambda n^2}{J^2}}\ N_n \ + \ O(\frac{1}{\sqrt{\lambda}}) \ . \quad (4.13)$$

Here N_n stands for the occupation number of the 8 sets of the bosonic and fermionic oscillators in the corresponding oscillator string state. Note that this expression is valid only to the leading order in expansion in $\frac{1}{\sqrt{\lambda}}$; that is why one is able to replace the tree-level value of the momentum ν in the first equality in (4.13) by the exact expression $\frac{J}{\sqrt{\lambda}}$.

To clarify the meaning of (4.13) let us first note that we are considering a sector of string states with given (quantized) value of the expectation value of the quantum momentum operator $\hat{J}$, i.e. for any state $|\Psi\rangle$ in this sector

$$\langle\Psi|\hat{J}|\Psi\rangle \equiv J = J_0 \equiv \sqrt{\lambda}\ \nu \ . \quad (4.14)$$

This should be true, in particular, for the ground state $|0\rangle$ containing no transverse oscillations. An apparent contradiction with the presence of corrections in (4.12) is resolved by noting that $|0\rangle$ will, in general, be different from the transverse oscillator vacuum state by terms of order $O(\frac{1}{\sqrt{\lambda}})$, i.e. $|0\rangle = |0\rangle_0 + \frac{1}{\sqrt{\lambda}}|0\rangle_1 + ...$, so that ${}_0\langle 0|\hat{J}|0\rangle_0 = J_0 + \frac{c_1}{\sqrt{\lambda}} + ...$. Then eq. (4.13) gives the energies of the string states with $J = J_0$ which are "close" to the "ground state" with $E = J$. The energy of the ground state (which is a BPS state) does not receive a 1-loop correction: the vacuum energy contributions of the bosonic and fermionic oscillators cancel out because of the effective 2-d supersymmetry [10,2,11].

To compute the $O(\frac{1}{\sqrt{\lambda}})$ term in (4.13) one is to go beyond the leading quadratic approximation. This gives a non-linear action containing quartic and higher terms in $\tilde{t}, \tilde{\varphi}, \tilde{\psi}_s, \tilde{\eta}_k$ (its explicit form follows from the metric of S^5 (2.6) and the metric of AdS_5 (4.2)). To quartic order in the fields

$$I_B^{(4)} = -\frac{1}{4\pi}\int d^2\xi \ \Big[-(1+\tfrac{1}{\sqrt{\lambda}}\tilde{\eta}_k^2)\partial_a\tilde{t}\partial^a\tilde{t} + \nu^2[\tilde{\eta}_k^2 + \tfrac{1}{2}\tfrac{1}{\sqrt{\lambda}}(\tilde{\eta}_k^2)^2]$$
$$+ \ (1+\tfrac{1}{\sqrt{\lambda}}\tilde{\psi}_s^2)\partial_a\tilde{\varphi}\partial^a\tilde{\varphi} + \nu^2[\tilde{\psi}_s^2 + \tfrac{1}{\sqrt{\lambda}}(\tfrac{1}{3}\tilde{\psi}_s^4 + \sum_{1=s<s'}^{4}\tilde{\psi}_s^2\tilde{\psi}_{s'}^2)]$$
$$+ \ (1+\tfrac{1}{2}\tfrac{1}{\sqrt{\lambda}}\tilde{\eta}_k^2)\partial_a\tilde{\eta}_n\partial^a\tilde{\eta}_n + \sum_{i=1}^{5}(1-\tfrac{1}{\sqrt{\lambda}}\sum_{k=1}^{i-1}\tilde{\psi}_k^2)\partial_a\tilde{\psi}_i\partial^a\tilde{\psi}_i\Big] \ . \quad (4.15)$$

Here we may again proceed with solving the corresponding conformal-gauge constraints perturbatively in $\frac{1}{\sqrt{\lambda}}$. This will be effectively equivalent to imposing the light-cone gauge at lower order in λ, i.e. to "deforming" the gauge condition $x^+ = p^+\tau + O(\frac{1}{\sqrt{\lambda}})$. One would also need to take into account the quartic fermionic terms in (2.3).

It is possible to determine the general structure of higher-order string sigma model loop ($\alpha' \sim \frac{1}{\sqrt{\lambda}}$) corrections to the energy relation (4.13).[12] Using that the sigma-model action (2.1) expanded near the classical solution (4.1) should contain only two derivatives or two powers of ν in all interaction terms (cf. (4.15)) and should lead to UV finite quantum 2-d effective action one concludes that l-loop correction to the space-time energy of a particular oscillator string state with quantum numbers n_i should have the structure $(\Delta E)_l = \frac{1}{\nu}(\Delta E_{2-\mathrm{d}})_l = (\frac{1}{\sqrt{\lambda}\,\nu})^{l-1} F_l(\frac{1}{\nu^2}; n_i)$. The function F_l should vanish for $n_i = 0$ (the ground state energy should not receive corrections on the basis of supersymmetry) and should have a regular expansion in positive powers of $\frac{1}{\nu^2}$, approaching a constant for $\nu \to \infty$. Expressed in terms of $J = \sqrt{\lambda}\ \nu$ this gives: $(\Delta E)_l = \frac{1}{J^{l-1}} F_l(\frac{\lambda}{J^2}; n_i)$, with F_l having power series expansion in $\frac{\lambda}{J^2}$ (cf. (4.13) for $l = 1$). Thus, in the large J sector one is able to re-interpret the string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ expansion as expansion in positive powers of $\frac{\lambda}{J^2}$, opening an interesting possibility of direct comparison with perturbative computations of anomalous dimensions of corresponding operators with large R-change on the SYM side.

5. 1-loop approximation: rotation in AdS_5 and boost in S^5

We would like now to repeat the analysis of the previous section in the case of a more general solution (3.1), including rotation in AdS_5 . The goal is to determine the quantum string spectrum at the leading order in large $\sqrt{\lambda}$. We shall expand the string action near the background (3.1) as in (4.3):

$$t = \kappa\tau + \frac{1}{\lambda^{1/4}}\tilde{t}\ , \qquad \rho = \rho(\sigma) + \frac{1}{\lambda^{1/4}}\tilde{\rho}\ , \qquad \phi = \omega\tau + \frac{1}{\lambda^{1/4}}\tilde{\phi}\ , \qquad \varphi = \nu\tau + \frac{1}{\lambda^{1/4}}\tilde{\varphi}\ ,$$

$$\beta_i = \frac{1}{\lambda^{1/4}}\tilde{\beta}_i \quad (i = 1,2)\ , \qquad \psi_s = \frac{1}{\lambda^{1/4}}\tilde{\psi}_s \quad (s = 1,2,3,4)\ . \tag{5.1}$$

As in the previous section, one possible strategy is to use the conformal gauge and expand the action and the constraints to the second order in the fluctuations, solve the constraints perturbatively in $\frac{1}{\sqrt{\lambda}}$, express the energy E in terms of the quantum fields and then compute the leading-order quantum term in the spectrum.

[12] We are grateful to D. Berenstein and I. Klebanov for important discussions of this issue.

As we shall see below, this conformal gauge approach turns out to be more complicated than in the pure S^5 boost case: in contrast to the discussion in the previous section where the two fluctuation fields $(\tilde{t}, \tilde{\varphi})$ effectively decoupled (cf. (4.4)), here $\tilde{t}$ will mix non-trivially with $\tilde{\phi}$ and $\tilde{\rho}$. Also, the masses of the fields will depend on σ, making it hard to find the spectrum exactly. A study of the resulting 1-loop correction to the energy spectrum will be the subject of section 6.

An a priori possible alternative to the conformal gauge approach is to start with the Nambu action in the static gauge, i.e. to consider only fluctuations normal to the embedded 2-d surface. However, this approach turns out to be problematic in the present case where the induced metric (3.8) has singularities. While the resulting action for small fluctuations is somewhat simpler than the conformal gauge action, one of the mass terms contains a 2-d curvature part and thus is singular at the turning points. In addition, the resulting bosonic+fermionic action does not appear to be finite: like in the case of the 1-loop correction to the Wilson loop [7,8] the 2-d logarithmic divergences do not manifestly cancel out, implying a subtlety [8] in the definition of the quantum measure in the approach based directly on the Nambu action. As we shall see, problem can be avoided [8] in the Polyakov approach to GS string in the conformal gauge: since the conformal anomaly cancels out in the flat 10-d space, one may trade the conformally-flat induced metric for a flat background metric on the cylinder (τ, σ). Then the 2-d ghosts decouple, all masses are regular for all values of σ and the finiteness of the 2-d quantum theory defined by the quadratic fluctuation action becomes manifest.[13]

Below we shall start with computing the bosonic part of the action, both in the static gauge and in the conformal gauge, and then determine the corresponding quadratic fermionic term.

5.1. Bosons: static gauge

Let us start with the Nambu action and impose the static gauge on t, ρ, i.e. set their fluctuations to zero:

$$\tilde{t} = 0 \ , \qquad \tilde{\rho} = 0 \ . \tag{5.2}$$

The form of the resulting action for small fluctuations is then a special case of the general expression for a semiclassical expansion of the string action in $AdS_5 \times S^5$ (see [18,7,8]).

[13] This is a consequence of a mass-squared sum rule which itself is a consequence of the effective 2-d supersymmetry (spontaneously broken by the classical background).

For $\nu = 0$ one finds one "transverse" scalar field fluctuation $\bar{\phi}$ with a non-trivial 2-d curvature dependent mass term, 2 massive fluctuations in the two other directions of S^3 and 5 massless S^5 fluctuations [14]

$$I_B^{(2)} = -\frac{1}{4\pi}\int d^2\xi\sqrt{-g}\big[g^{ab}\partial_a\bar{\phi}\partial_b\bar{\phi} + (4+R^{(2)})\bar{\phi}^2 + g^{ab}\partial_a\bar{\beta}_i\partial_b\bar{\beta}_i + 2\bar{\beta}_i^2$$

$$+ g^{ab}\partial_a\tilde{\varphi}\partial_b\tilde{\varphi} + g^{ab}\partial_a\tilde{\psi}_s\partial_b\tilde{\psi}_s\big] , \tag{5.3}$$

where g_{ab} is the induced metric (3.8) and $R^{(2)}$ is its curvature (3.9). The fields $\bar{\phi}$ and $\bar{\beta}_i$ are related to $\tilde{\phi}$ and $\tilde{\beta}_i$ in (5.1) by a ρ-dependent rescaling needed to put the kinetic terms in the canonical form. In more detail, expanding the square root of the Nambu action we get the following term depending on $\tilde{\phi}$:

$$L(\tilde{\phi}) = -\rho'^2 + \frac{1}{2}\sinh^2\rho\ (1+\frac{\omega^2\sinh^2\rho}{\rho'^2})[(\partial_0\tilde{\phi})^2 - (\partial_1\tilde{\phi})^2] + O(\tilde{\phi}^4) , \tag{5.4}$$

where ρ' is given by (3.5). After the field redefinition that puts the kinetic term in the canonical form

$$\bar{\phi} \equiv \frac{\sinh\rho}{\sqrt{1-\frac{\omega^2}{\kappa^2}\tanh^2\rho}}\tilde{\phi} , \tag{5.5}$$

we get the $\bar{\phi}$ part of (5.3). Explicitly, using that the induced metric is conformally flat (5.3) can be written as

$$I_B^{(2)} = -\frac{1}{4\pi}\int d^2\xi\big[\partial_a\bar{\phi}\partial^a\bar{\phi} + m_\phi^2\bar{\phi}^2 + \partial_a\bar{\beta}_i\partial^a\bar{\beta}_i + m_\beta^2\bar{\beta}_i^2 + \partial_a\tilde{\varphi}\partial^a\tilde{\varphi} + \partial_a\tilde{\psi}_s\partial^a\tilde{\psi}_s\big] , \tag{5.6}$$

where (using (3.9))

$$m_\phi^2 = \sqrt{-g}(4+R^{(2)}) = 2\rho'^2 + \frac{2\kappa^2\omega^2}{\rho'^2} , \tag{5.7}$$

$$m_\beta^2 = 2\rho'^2 = 2[(\kappa^2-\omega^2)\cosh^2\rho + \omega^2] . \tag{5.8}$$

As we shall see below in section 5.3, for $\nu = 0$ the contribution of the GS fermions can be effectively represented by 8 copies of 2-d Majorana fermions with masses ± 1 defined on the 2-d surface with induced metric, i.e. with the same squared Dirac operator $-\hat{\nabla}^2 + \frac{1}{4}R^{(2)} + 1$ as in [8,7] ($\hat{\nabla}$ is the covariant derivative in the induced metric). As in the Wilson loop case

[14] The same expression for the quadratic-fluctuation action is found by expanding not in $\tilde{\phi}$, but in the direction $\zeta = \rho'^{-1}(\omega\tanh\rho\ \tilde{\theta} + \kappa\coth\tilde{\phi})$ normal to the embedded 2-d surface.

considered in [7], the logarithmic divergences coming from the fermionic part of the action will not then cancel against the divergences corresponding to the 2-d bosonic theory (5.6)

$$b_2 = \sum_k \sqrt{-g}(\frac{R^{(2)}}{6} d_k - m_k^2) = \sqrt{-g}\big[\frac{R^{(2)}}{6}(1+2+5) - (4 + R^{(2)} + 2\times 2)\big] \ . \qquad (5.9)$$

As discussed in [8], the remaining divergence $\sim R^{(2)}$ should be cancelled out by a measure contribution needed to make the Nambu string partition function equivalent to the partition function in the Polyakov approach (with the induced metric used as the background one).

There is also another apparent problem related with the presence of that $R^{(2)}$ term in the mass (5.7): this term blows up at the turning points, while all other masses go there to zero.[15] Both problems can be avoided by starting with the Polyakov action in the conformal gauge where (due to cancellation of conformal anomaly) one is able to choose the background metric on the cylinder to be flat, making all masses regular at the folding points.

Let us mention that for non-zero ν the action (5.3) becomes more complicated: the fluctuations $\bar{\phi}$ and $\tilde{\varphi}$ mix in a non-trivial way. This mixing will be absent in the conformal gauge where the AdS_5 and S^5 parts of the bosonic action are decoupled (but it will reappear if one would try to solve the constraints and use their solution in the action).

5.2. Bosons: conformal gauge

Let us now present the expression for the quadratic fluctuation action in the conformal gauge. Expanding the sigma model as in (4.3),(5.1) gives the following bosonic action for the quadratic fluctuations (cf. (4.4))

$$I_B^{(2)} = -\frac{1}{4\pi}\int d^2\xi\ \big[-\cosh^2\rho\ \partial_a\tilde{t}\partial^a\tilde{t} + \sinh^2\rho\ \partial_a\tilde{\phi}\partial^a\tilde{\phi} + \ 2\sinh 2\rho\ \tilde{\rho}(\kappa\partial_0\tilde{t} - \omega\partial_0\tilde{\phi})$$
$$+ \partial_a\tilde{\rho}\partial^a\tilde{\rho} + (\kappa^2 - \omega^2)\cosh 2\rho\ \tilde{\rho}^2 + \ \sinh^2\rho\ (\partial_a\tilde{\beta}_i\partial^a\tilde{\beta}_i + \omega^2\tilde{\beta}_i^2)$$
$$+ \partial_a\tilde{\varphi}\partial^a\tilde{\varphi} + \partial_a\tilde{\psi}_s\partial^a\tilde{\psi}_s + \nu^2\tilde{\psi}_s^2\big] \ . \qquad (5.10)$$

The linearised (leading-order in $\frac{1}{\sqrt{\lambda}}$) part of the conformal-gauge constraints (3.3),(3.4) corresponding to the action (5.10) has the form (cf. (4.9),(4.10))

$$-\kappa\cosh^2\rho\,\partial_0\tilde{t} + \omega\sinh^2\rho\,\partial_0\tilde{\phi} + \nu\partial_0\tilde{\varphi} + \rho'\partial_1\tilde{\rho} + \frac{1}{2}(\omega^2 - \kappa^2)\sinh 2\rho\,\tilde{\rho} \approx 0 \ , \qquad (5.11)$$

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[15] The vanishing of masses at the turning points is an expected behaviour as the fold points should move with velocity of light.

$$-\kappa \cosh^2 \rho\, \partial_1 \tilde{t} + \omega \sinh^2 \rho\, \partial_1 \tilde{\phi} + \nu \partial_1 \tilde{\varphi} + \rho' \partial_0 \tilde{\rho} \approx 0 \ . \tag{5.12}$$

Using these constraints to eliminate one of the fluctuation fields, e.g., $\tilde{t}$ (cf. $\tilde{x}^-$ in (4.9),(4.10)) from the action (5.10), one should discover that some linear combination of the remaining fields (cf. $\tilde{x}^+$ in (4.4)) then decouples from the action (it should then be formally gauge-fixed to zero to avoid degeneracy of the path integral). The resulting action for the 8 "transverse" bosonic fluctuation fields should be (classically) equivalent to the static-gauge action (5.6).

Redefining the fields to put the kinetic terms in the standard form

$$\bar{t} = \cosh \rho\ \tilde{t}\ , \quad \bar{\phi} = \sinh \rho\ \tilde{\phi}\ , \quad \bar{\beta}_i = \sinh \rho\ \tilde{\beta}_i\ , \quad \bar{\rho} \equiv \tilde{\rho}\ , \tag{5.13}$$

transforms (5.10) into

$$I_B^{(2)} \equiv I_1(\bar{t}, \bar{\phi}, \bar{\rho}) + I_2(\bar{\beta}_i, \tilde{\varphi}, \tilde{\psi}_s)\ ,$$

$$I_1 = -\frac{1}{4\pi} \int d^2\xi\ \big[- \partial_a \bar{t} \partial^a \bar{t} - \mu_t^2 \bar{t}^2 + \partial_a \bar{\phi} \partial^a \bar{\phi} + \mu_\phi^2 \bar{\phi}^2$$
$$+\ 4\bar{\rho}(\kappa \sinh \rho\ \partial_0 \bar{t} - \omega \cosh \rho\ \partial_0 \bar{\phi}) + \partial_a \bar{\rho} \partial^a \bar{\rho} + \mu_\rho^2 \bar{\rho}^2 \big]\ , \tag{5.14}$$

$$I_2 = -\frac{1}{4\pi} \int d^2\xi\ \big[\partial_a \bar{\beta}_i \partial^a \bar{\beta}_i + m_{_\beta}^2 \bar{\beta}_i^2 + \partial_a \tilde{\varphi} \partial^a \tilde{\varphi} + \partial_a \tilde{\psi}_s \partial^a \tilde{\psi}_s + \nu^2 \tilde{\psi}_s^2 \big]\ , \tag{5.15}$$

where (using (3.5) and (3.2))

$$\mu_t^2 = \rho'^2 + \tanh \rho\ \rho'' = 2\rho'^2 - \kappa^2 + \nu^2\ , \qquad \mu_\phi^2 = \rho'^2 + \coth \rho\ \rho'' = 2\rho'^2 - \omega^2 + \nu^2\ , \tag{5.16}$$

$$\mu_\rho^2 = (\kappa^2 - \omega^2) \cosh 2\rho = 2\rho'^2 - \kappa^2 - \omega^2 + 2\nu^2\ \ , \qquad m_{_\beta}^2 = 2\rho'^2 + \nu^2\ . \tag{5.17}$$

The non-constant part of all masses is thus the same and is equal to $2\rho'^2$ (cf. (5.6)).

While in the case of the pure boost in S^5 the small fluctuation operators had constant coefficients (cf. (4.4)), here we have a non-trivial dependence on σ through the classical solution $\rho(\sigma)$. This is very similar to the case of the 1-loop expansion [6,7,8] near the classical string profile appearing in the Wilson loop [9] calculations. The system splits into three non-trivially coupled fluctuations of the effective AdS_3 sector, two massive fluctuations in S^3 and one massless and 4 massive fluctuations in S^5.

The action (5.14) can be obtained in a more geometrical way as a special case of the general semiclassical expansion of the $AdS_5 \times S^5$ sigma model in terms of tangent-space fluctuations [8]. Expanding the string sigma model action (2.1) about a classical solution

$x^M \to x_0^M + z^M$, and introducing the tangent-space components of the fluctuation fields $z^A = E^A_M(x_0) z^M$, we get the following action for the quadratic fluctuations

$$I_B^{(2)} = -\frac{1}{4\pi} \int d^2\xi \sqrt{-g} \left(\eta_{AB} g^{ab} \mathrm{D}_a z^A \mathrm{D}_b z^B + X_{AB} z^A z^B \right) , \tag{5.18}$$

$$X_{AB} = -g^{ab} e_a^C e_b^D R_{ABCD} , \qquad e_a^A \equiv \partial_a x_0^M E^a_M(x_0) . \tag{5.19}$$

D_a is the covariant derivative containing the projection of the target space spin connection,

$$\mathrm{D}_a z^A = \partial_a z^A + w_a^{AB} z^B , \qquad\qquad w_a^{AB} = \partial_a x_0^M \omega_M^{AB} , \tag{5.20}$$

where ω_M^{AB} is the spin connection of the target space $AdS_5 \times S^5$. The S^5 part of (5.18) is the same as in (5.15), while for the AdS_5 part with the curvature $R_{ACBD} = -\eta_{AB}\eta_{CD} + \eta_{AD}\eta_{CB}$ we get

$$X^{AB} = \eta_{CD} g^{ab} e_a^C e_b^D \eta^{AB} - g^{ab} e_a^A e_b^B . \tag{5.21}$$

Since the induced metric g_{ab} is a conformally flat (3.8) it can be replaced by η_{ab}. Explicitly, in the global coordinates in (2.6) we find by expanding near the solution (3.1): $z^A = (\bar{t}, \bar{\rho}, \bar{\phi}, \tilde{\beta}_i)$, $\quad E^0_0 = \cosh\rho$, $\ E^1_1 = 1$, $E^2_2 = \sinh\rho$, $\quad E^i_i = \sinh\rho$,

$$w_0^{01} = \kappa \sinh\rho , \qquad w_0^{21} = \omega \cosh\rho , \qquad w_1^{AB} = 0 , \tag{5.22}$$

$$e_0^0 = \kappa\cosh\rho , \qquad e_0^2 = \omega\sinh\rho , \qquad e_1^1 = \rho' , \qquad \eta_{CD}\eta^{ab} e_a^C e_b^D = 2\rho'^2 + \nu^2 .$$

Note that the 2-d $SO(1,2)$ "gauge potential" w_a^{AB} depends only on $\rho(\sigma)$ and has a non-vanishing field strength. The non-trivial "AdS_3" part of the quadratic action (5.14) is then

$$\begin{aligned} I_1 = -\frac{1}{4\pi} \int d^2\xi \ \Big[&(\partial_0 \bar{t} + \kappa\sinh\rho\ \bar{\rho})^2 - (\partial_1 \bar{t})^2 - \ (\partial_0\bar{\phi} + \omega\cosh\rho\ \bar{\rho})^2 + (\partial_1\bar{\phi})^2 + \\ &- \ (\partial_0\bar{\rho} + \kappa\sinh\rho\ \bar{t} - \omega\cosh\rho\ \bar{\phi})^2 + (\rho'^2 + \nu^2)\bar{\rho}^2 + (\partial_1\bar{\rho})^2 \\ &+ \ (2\rho'^2 + \nu^2)(-\bar{t}^2 + \bar{\phi}^2) + (\kappa\cosh\rho\ \bar{t} - \omega\sinh\rho\ \bar{\phi})^2 \Big] . \end{aligned} \tag{5.23}$$

This is easily seen to be equivalent to (5.14). Note also that in the case of no rotation in AdS_5 , i.e. $\rho = 0, \ \omega = 0, \ \kappa = \nu$ the mass of $\bar{t}$ vanishes while the masses of $\bar{\phi}$ and $\bar{\rho}$ become equal to ν, in agreement with (4.4) ($\tilde{\eta}_k$ in (4.4) corresponds to $\bar{\phi}, \bar{\rho}, \tilde{\beta}_i$ here).

In general, the action in conformal gauge should be supplemented by the 2-d ghost action [8] ($\alpha = 0, 1$):

$$I_{gh} = -\frac{1}{4\pi} \int d^2\xi \ \sqrt{-g} \ \ (\nabla^a \zeta^\alpha \nabla_a \zeta_\alpha - \frac{1}{2} R^{(2)} \zeta^\alpha \zeta_\alpha) \ , \tag{5.24}$$

where $\nabla_a \zeta^\alpha = \partial_a \zeta^\alpha + \omega_a^{\alpha\beta} \zeta_\beta$ is the 2-d covariant derivative with respect to the background 2-d metric. In the case when this metric is identified with the induced (conformally flat) metric (3.8) we have: $g_{ab} = \mathrm{e}_a^\alpha \mathrm{e}_b^\beta \eta_{\alpha\beta}, \ \mathrm{e}_a^\alpha = \rho' \delta_a^\alpha, \ \zeta^\alpha = \mathrm{e}_a^\alpha \zeta^q$ and $\omega_0^{01} = \frac{\rho''}{\rho'}$, $\omega_1^{01} = 0$. Since the total conformal anomaly must cancel after the fermions are included (see discussion in [8]), we may choose the background metric on the 2-cylinder (which is used to define the norms and operators in the conformal gauge) to be not the induced but simply the flat metric. This is quite natural, given the singularity of the induced metric. Then the ghost contribution becomes trivial and can be ignored, and we are left with a system of fields on a flat cylinder with non-constant (but everywhere regular) masses.

The resulting contribution of the fields in (5.23) to the 2-d logarithmic divergences is then proportional to the trace of their mass matrix (cf. (5.9))

$$\rho'^2 + \nu^2 + 2 \times (2\rho'^2 + \nu^2) - \kappa^2 \cosh^2 \rho + \omega^2 \sinh^2 \rho = 2(2\rho'^2 + \nu^2) \ .$$

The total contribution of the bosonic fluctuations $\bar{t}, \bar{\phi}, \bar{\rho}, \tilde{\beta}_i, \varphi, \psi_k$

$$b_2 = -\eta^{AB} X_{AB} = -2(2\rho'^2 + \nu^2) - 2 \times (2\rho'^2 + \nu^2) - 4\nu^2 = -8(\rho'^2 + \nu^2) \tag{5.25}$$

will be exactly cancelled by the contribution of the fermions, checking the conformal invariance of the theory. [16]

Let us comment on some special cases. When $\nu = 0$ and $\omega \gg \kappa$ the range of variation of ρ is small and we should be back to the flat space case, i.e. the fluctuations should diagonalize and become massless. For $\rho \to 0, \ \omega \gg \kappa$ (5.14) can be written as

$$I_1 \approx -\frac{1}{4\pi} \int d^2\xi \ \big(- \partial_a \bar{t} \partial^a \bar{t} + \partial_a \bar{\phi} \partial^a \bar{\phi} - \omega^2 \bar{\phi}^2 - 4\omega \bar{\rho} \ \partial_0 \bar{\phi} + \partial_a \bar{\rho} \partial^a \bar{\rho} - \omega^2 \bar{\rho}^2 \big) \ . \tag{5.26}$$

[16] For g_{ab} chosen to be the induced metric, the contributions of the two ghosts to the partition function should effectively cancel the contributions of the two "longitudinal" fluctuations (combinations of $\bar{t}$ and $\bar{\rho}$) leading to the same final result as in the static gauge. In particular, the extra ghost contribution to the mass^2 term in the divergences $\Delta b_2 = \sqrt{-g}(-2 \times \frac{1}{2} R^{(2)})$ reproduces the $R^{(2)}$ term in (5.9) (see [8]). Since the ghost contribution can be ignored in the case of the flat choice of the fiducial metric, this suggests that the correct prescription for quantisation in the static gauge should be to omit the $R^{(2)}$ term in the mass of $\bar{\phi}$ in (5.7) and define the norms of the fields using flat metric on the cylinder.

This should be same as the action obtained by starting with the metric $ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2$ and repeating the expansion near $t = \kappa\tau,\ \phi = \omega\tau,\ \rho'^2 = \kappa^2 - \omega^2\rho^2$ solution. Indeed, after the redefinition[17] $\eta_1 = \cos\omega\tau\ \bar\rho - \rho\sin\omega\tau\ \tilde\phi = \cos\omega\tau\ \bar\rho - \sin\omega\tau\ \bar\phi$, $\eta_2 = \sin\omega\tau\ \bar\rho + \rho\cos\omega\tau\ \tilde\phi = \sin\omega\tau\ \bar\rho + \cos\omega\tau\ \bar\phi$, we get the massless Lagrangian $\sim (\partial\eta_i)^2$.

Next, let us check that the expansion near the point-like solution with $\rho' = 0$, $\rho = \rho_0$, $\kappa = \omega = \nu$ (particle located at radius ρ_0 and rotating in both S^5 and S^3) is indeed equivalent to the expansion near the solution with $\kappa = \nu,\ \omega = 0,\ \rho = 0$. Here (5.14) becomes

$$I_1(\bar t, \bar\phi, \bar\rho) = -\frac{1}{4\pi}\int d^2\xi\ \big[-\partial_a\bar t\partial^a\bar t + \partial_a\bar\phi\partial^a\bar\phi - 4\kappa\partial_0\bar\rho(\sinh\rho\ \bar t - \cosh\rho\ \bar\phi) + \partial_a\bar\rho\partial^a\bar\rho\big]\ . \quad (5.27)$$

Applying first the $SO(1,1)$ rotation $\hat\phi = \cosh\rho\ \bar\phi - \sinh\rho\ \bar t,\quad \hat t = \cosh\rho\ \bar t - \sinh\rho\ \bar\phi$, and then the above $SO(2)$ rotation in the $\eta_i = (\bar\rho, \hat\phi)$ plane we end up with the Lagrangian $-\partial_a\hat t\partial^a\hat t + \partial_a\eta_i\partial^a\eta_i + \nu^2\eta_i^2$, which is in agreement with (4.8).

5.3. Fermions

The derivation of the quadratic fermionic action from (2.3) is similar to the one in [8]. The 2-d projections of Γ-matrices that enter the fermionic action are (the indices $A = 0,1,2,9$ are used to label the t,ρ,ϕ,φ directions in the tangent space):[18] $\varrho_\alpha = \mathrm{e}^a_\alpha E^A_M(x_0)\partial_a x^M_0\Gamma_A = t^A_\alpha\Gamma_A,\ \ \mathrm{e}^a_\alpha = \rho'^{-1}\delta^a_\alpha$, or explicitly ($\alpha = \hat 0, \hat 1$)

$$\varrho_{\hat 0} = \rho'^{-1}(\kappa\cosh\rho\ \Gamma_0 + \omega\sinh\rho\ \Gamma_2 + \nu\Gamma_9)\ , \qquad \varrho_{\hat 1} = \Gamma_1\ , \qquad \varrho_{(\alpha}\varrho_{\beta)} = \eta_{\alpha\beta}\ . \qquad (5.28)$$

Let us first consider the more transparent case of $\nu = 0$. As in [8] one can make a local $SO(1,9)$ rotation which transforms the set of σ-dependent 10-d Dirac matrices into 10 constant Dirac matrices, $\varrho_\alpha(\sigma) = t^A_\alpha\Gamma_A = S(\sigma)\Gamma_\alpha S^{-1}(\sigma),\ \ \varrho_u(\sigma) = n^A_u\Gamma_A = S(\sigma)\Gamma_u S^{-1}(\sigma)$. One is then able to write the quadratic part of the GS action (2.3) as an action for a set

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[17] This effectively amounts to going back to cartesian coordinates $x_1 = \rho\cos\phi,\ x_2 = \rho\sin\phi$ and then expanding near classical background.

[18] Considering the (t,ρ,ϕ,φ) subspace we may introduce the two tangent $t^M_\alpha = \mathrm{e}^a_\alpha\partial_a x^M_0$ and 2 normal vectors n^M_u to the embedded world sheet, $G_{MN}t^M_\alpha t^N_\beta = \eta_{\alpha\beta},\ G_{MN}t^M_\alpha n^N_u = 0,\ G_{MN}n^M_u n^N_v = \delta_{uv}$ ($M = 0,1,2,9$ counts the directions t,ρ,ϕ,φ). Then we find that $t^M_{\hat 0} = \rho'^{-1}(\kappa,0,\omega,\nu)$, $t^M_{\hat 1} = (0,1,0,0)$, and $n^M_2 = \gamma^{-1}(\omega\tanh\rho, 0, \kappa\coth\rho, 0)$, $n^M_9 = \rho'^{-1}(\gamma^{-1}\nu\kappa, 0, \gamma^{-1}\nu\omega, \gamma)$, $\gamma^2 \equiv \rho'^2 + \nu^2$.

of 2-d Dirac fermions coupled to curved induced 2-d metric. Explicitly, in the present case we observe that (see (3.5))

$$\varrho_{\hat{0}} = \cosh\alpha\ \Gamma_0 + \sinh\alpha\ \Gamma_2\ , \tag{5.29}$$

$$\cosh\alpha\ = \frac{\kappa\cosh\rho}{\sqrt{\kappa^2\cosh^2\rho - \omega^2\sinh^2\rho}}\ , \qquad \frac{d\alpha}{d\sigma} = \frac{\kappa\omega}{\rho'} = \sqrt{\omega^2\cosh^2\alpha - \kappa^2\sinh^2\alpha}\ , \tag{5.30}$$

so that the required transformation S is a boost in the (t,ϕ) plane:

$$S = \exp(\frac{1}{2}\alpha\Gamma_0\Gamma_2)\ . \tag{5.31}$$

After the field redefinition $\theta^I \to \Psi^I \equiv S^{-1}\theta^I$, we then find that (2.3),(2.5) becomes

$$L_F = i\big(\sqrt{-g}g^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ}\big)\big(\bar{\Psi}^I\tau_a\hat{\nabla}_b\Psi^J - \frac{1}{2}i\epsilon^{JK}\bar{\Psi}^I\tau_a\Gamma_*\tau_b\Psi^K\big)\ , \tag{5.32}$$

where τ_a play the role of the curved space 2-d Dirac matrices

$$\tau_a = e_a^\alpha\Gamma_\alpha = \rho'\delta_a^A\Gamma_A\ , \qquad \tau_0 = S^{-1}\varrho_0 S = \rho'\Gamma_0\ , \qquad \tau_1 = S^{-1}\varrho_1 S = \rho'\Gamma_1\ , \tag{5.33}$$

and $\hat{\nabla}_a$ is the 2-d curved space spinor covariant derivative for the induced metric $g_{ab} = \rho'^2\eta_{ab}$,

$$\hat{\nabla}_a = \partial_a + \tfrac{1}{4}\omega_a^{\alpha\beta}\Gamma_{\alpha\beta}\ , \qquad \hat{\nabla}_0 = \partial_0 + \frac{1}{2}\frac{\rho''}{\rho'}\Gamma_{01}\ , \qquad \hat{\nabla}_1 = \partial_1\ . \tag{5.34}$$

In more detail, the result of applying the rotation (5.31) to the covariant derivative $\mathrm{D}_a \equiv \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$ in (2.3), i.e.

$$\mathrm{D}_0 = \partial_0 - \frac{1}{2}\Gamma_1(\kappa\sinh\rho\ \Gamma_0 + \omega\cosh\rho\ \Gamma_2)\ , \qquad \mathrm{D}_1 = \partial_1\ , \tag{5.35}$$

is $\tilde{\mathrm{D}}_a = S^{-1}\mathrm{D}_a S = \hat{\nabla}_a + B_a\ ,\ B_a \equiv \frac{\kappa\omega}{2\rho'}\Gamma_2\ (\Gamma_1, -\Gamma_0)$, but since $\eta^{\alpha\beta}\Gamma_\alpha B_\beta = 0,\ \epsilon^{\alpha\beta}\Gamma_\alpha B_\beta = 0$, the extra connection term cancels out in the rotated fermionic action (2.3) and we are left with the same result (5.32) as in [8]. One can then fix the kappa-symmetry gauge by imposing, e.g., $\Psi^1 = \Psi^2$, thus ending up with a similar action as in [8]

$$L_F = 2i\sqrt{-g}\left(\bar{\Psi}\tau^a\hat{\nabla}_a\Psi + i\bar{\Psi}M\Psi\right)\ , \tag{5.36}$$

$$M \equiv \tfrac{\epsilon^{ab}}{2\sqrt{-g}}\tau_a\Gamma_*\tau_b = \tau_3\Gamma_* = i\Gamma_{234}\ , \qquad \tau_3 \equiv \tfrac{\epsilon^{ab}}{2\sqrt{-g}}\tau_a\tau_b = \Gamma_0\Gamma_1\ , \qquad M^2 = 1\ . \tag{5.37}$$

Choosing a representation for Γ_A such that $\Gamma_{0,1}$ are essentially 2-d Dirac matrices times a unit 8×8 matrix and τ_* is diagonal, we end up with 8 species of massive 2-d Majorana fermions on a 2-d surface with a curved metric equal to the induced metric with a "σ_3" mass term. The square of the resulting fermionic operator $\Delta_F = -\hat{\nabla}^2 + \frac{1}{4}R^{(2)} + 1$ is the same as for a 2-d fermion with unit mass (cf. (5.3)). Furthermore, since the induced 2-d metric is conformally flat, and the total conformal anomaly must cancel out [8], one is able to rescale the fermions by $(-g)^{1/4} = \rho'^{1/2}$, transforming their kinetic term into a flat-space Dirac form. We then get 4+4 flat-space 2-d Majorana fermions with σ-dependent masses $m_F = \pm\rho'$ (cf. (5.17)).

Let us now include the ν-dependence. First, in the case of no rotation in AdS_5 , i.e. the background (4.1) ($\rho = 0,\ \kappa = \nu$) we get in (2.3),(2.4): $\varrho_0 = \nu(\Gamma_0 + \Gamma_9),\ \varrho_1 = 0$. It is then natural to impose the kappa-symmetry gauge $\Gamma^+\theta^I = 0,\ \Gamma^\pm = \mp\Gamma_0 + \Gamma_9$. We finish with the following quadratic term in the fermionic part of the action

$$L_F = i\nu\left(\bar\theta^1\Gamma^-\partial_+\theta^1 + \bar\theta^2\Gamma^-\partial_-\theta^2 - 2\nu\bar\theta^1\Gamma^-\Pi\theta^2\right) , \qquad (5.38)$$

$$\Pi \equiv i\Gamma_*\Gamma_0 = \Gamma_{1234} , \qquad \Pi^2 = 1 .$$

This is exactly the same as the full fermionic light-cone GS action [10,11] in the plane wave background of [12], which describes 4+4 2-d Majorana fermions with mass $m_F = \pm\nu$.

When both ν and ω are non-vanishing, we may again transform ρ_α into Γ_α by applying the two $SO(1,2)$ rotations: a boost in (02) plane (5.29), and a similar boost in the (09) plane (cf. (5.30))

$$S = \exp(\frac{1}{2}\alpha_1\Gamma_0\Gamma_2)\ \exp(\frac{1}{2}\alpha_2\Gamma_0\Gamma_9) , \qquad (5.39)$$

$$\cosh\alpha_1 \ = \frac{\kappa\cosh\rho}{\sqrt{\rho'^2+\nu^2}} , \qquad \sinh\alpha_2 \ = \frac{\nu}{\rho'} .$$

Here ρ' is given by (3.5), so that α_1 is as in (5.30) and $\frac{d\alpha_1}{d\sigma} = \frac{\kappa\omega\rho'}{\rho'^2+\nu^2}$, $\frac{d\alpha_2}{d\sigma} = -\frac{\nu\rho''}{\rho'\sqrt{\rho'^2+\nu^2}}$. After the first boost $\varrho_{\hat 0} = \sqrt{1+\frac{\nu^2}{\rho'^2}}\Gamma_0 + \frac{\nu}{\rho'}\Gamma_9$, while Γ_* in (2.5) unchanged; after the second boost $\varrho_\alpha = (\Gamma_0, \Gamma_1)$ while

$$\Gamma_* \to \tilde\Gamma_* = S^{-1}\Gamma_* S = i(\cosh\alpha_2\Gamma_0 - \sinh\alpha_2\Gamma_9)\Gamma_{1234} .$$

Repeating the same steps as in the discussion of the $\nu = 0$ case above, i.e. choosing the $\Psi^1 = \Psi^2$ gauge ($\Psi^I = S^{-1}\theta^I$) and Weyl-rescaling the fermions ($\Psi \to \rho'^{-1/2}\Psi$) to put the kinetic term in the flat-space form we get (cf. (5.32),(5.36),(5.37))

$$L_F = 2i\left(\bar\Psi\tau^a\partial_a\Psi + i\rho'\bar\Psi\mathcal{M}\Psi\right) , \qquad (5.40)$$

$$\mathcal{M} \equiv \frac{1}{2}\epsilon^{\alpha\beta}\varrho_\alpha\tilde{\Gamma}_*\varrho_\beta = \cosh\alpha_2 \ M = i\rho'^{-1}\sqrt{\rho'^2+\nu^2}\ \Gamma_{234} \ . \tag{5.41}$$

This action[19] describes again a system of 4+4 2-d Majorana fermions with masses

$$m_F = \pm\sqrt{\rho'^2+\nu^2} \ , \tag{5.42}$$

incorporating consistently the two special cases $\nu = 0$ and $\rho' = 0$ discussed above. The contribution of fermions to the 2-d logarithmic divergences thus cancels out the one of the conformal-gauge bosonic fluctuations in (5.25), as required by the conformal invariance of the $AdS_5 \times S^5$ string sigma model [5,8].

6. Quantum correction to energy spectrum of rotating string

Our aim is to compute the leading (1-loop) corrections to the energy of string excitations above the rotating string ground state (3.1). As usual in semiclassical soliton quantization, we shall consider a sector of string states for which the expectation values for the spin $\hat{S}$ and the φ-momentum $\hat{J}$ operators are fixed and equal to their classical values S_0 and J_0 in (3.13),(4.12), i.e. as in (4.14) we assume

$$\langle\Psi|\hat{J}|\Psi\rangle = J = J_0 \equiv \sqrt{\lambda}\ \nu \ , \qquad \langle\Psi|\hat{S}|\Psi\rangle = S = S_0 \equiv \sqrt{\lambda}\ \mathcal{S} \ . \tag{6.1}$$

The expectation value for the ground state energy is shifted from the classical value E_0 in (3.12) by the expectation value of the 2-d Hamiltonian for quadratic fluctuations (see (A.8))

$$E_{\rm vac} = E_0 + \frac{1}{\kappa}\Delta E_{\rm 2d} \ , \qquad \Delta E_{\rm 2d} = \langle 0|\hat{H}_{\rm 2d}|0\rangle \ . \tag{6.2}$$

In contrast to what happened for the BPS state with $E = J$ ($N_n|0\rangle = N_n|0\rangle_0 + O(\frac{1}{\sqrt{\lambda}}) = O(\frac{1}{\sqrt{\lambda}})$ in (4.13)) here the 1-loop quantum correction ΔE will no longer vanish. We shall estimate ΔE for large spin $\mathcal{S}$ and $J = 0$ to the leading order in $\frac{1}{\sqrt{\lambda}}$ and will show that, like the classical contribution (3.29), it scales as $\ln S$, i.e. there are no stronger $\ln^k S$ corrections at the first quantum order. This provides support to the conjecture of [3] that the $\ln S$ behaviour seen at the string level can be interpolated to weak coupling, i.e. that for large $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ and $J = 0$ one has

$$E - S \approx f(\lambda)\ln\frac{S}{\sqrt{\lambda}} \ , \tag{6.3}$$

[19] Note that after the rescaling of the fermions this action corresponding to the $\theta^1 = \theta^2$ gauge has a regular limit $\rho' \to 0$ that is equivalent to the $\Gamma^+\theta^I = 0$ action (5.38).

where in the string sigma model loop expansion (the value of a_1 will be computed below)

$$f(\lambda)_{\lambda\gg 1} = a_0\sqrt{\lambda} \ + a_1 + a_2\frac{1}{\sqrt{\lambda}} + \dots \ , \tag{6.4}$$

$$a_0 = \frac{1}{\pi} \ , \qquad\qquad a_1 \approx -\frac{3\ln 2}{\pi} \ , \tag{6.5}$$

while $f(\lambda)_{\lambda\ll 1} = b_1\lambda + b_2\lambda^2 + \dots$ should follow from the SYM perturbation theory for the anomalous dimension of the corresponding operator (see [3] and refs. there). It is crucial for consistency of the proposal of [3] that both string-theory and gauge-theory perturbative expansions should not contain terms with higher than first powers of $\ln\frac{S}{\sqrt{\lambda}}$ (which would dominate over the leading-order $\ln\frac{S}{\sqrt{\lambda}}$ result).

As in the pure-boost case discussed in section 4, starting with ground state one can then build up a tower of string oscillator states on top of it. All of them will have the same S and J but different oscillator occupation numbers and thus different quantum energies. There will be several sub-sectors of states created by oscillators of different fluctuation fields in the quadratic action. Considering all possible cases will produce a generalization of the quantum spectrum (4.13) of small string oscillations near the boosted $E = J$ state to the case of a non-zero spin.

In section 5 we have determined the Lagrangian for small fluctuations near the solution (3.1): it is the sum of the bosonic part given in the conformal gauge by (5.14) (or (5.23)) and (5.15), and the fermionic part given by (5.40). It describes a collection of fields with σ-dependent mass terms. One should diagonalise the corresponding second-order differential operators defined on the cylinder (τ, σ), quantise the "oscillator" modes and impose the constraints (3.3),(3.4), determining the expression for the energy and the analog of the level matching condition. While this was straightforward to do in the case of the $S = 0, J \neq 0$ solution discussed in section 4 where the bosonic (4.4) and fermionic (5.38) parts of the action had constant coefficients, this is a complicated task in the AdS_5 rotating case, unless one makes certain approximations. Below we shall consider the same "short string" and "long string" limiting cases [3] of the rotating solution as in sections 3.1 and 3.2. We shall also note that since all the fields in the quadratic fluctuation action (5.14),(5.15),(5.40) have similar masses, to capture some qualitative features of the string oscillator spectrum it is enough to consider the contributions to the energy coming from excitations of one scalar field $\bar{\beta}_i$ in (5.15) (describing fluctuations in AdS_5 3-sphere directions transverse to the rotation plane).

6.1. **Ground state energy shift**

Let us start with estimating the energy of the rotating solution in the "long string" limit. For simplicity, we shall set J (i.e. the boost parameter ν) to zero and to avoid dealing with "longitudinal" modes consider the bosonic fluctuation action as it follows from the Nambu action in the static gauge (5.3). We shall use that in the "long string" limit ($\eta \ll 1$, $S \gg 1$, see (3.28)) ρ' is approximately constant and very large (ρ'' in (3.2) is small since $\kappa \approx \omega$, see (3.26)). In this case the induced 2-d curvature term $R^{(2)}$ in (3.9) is approximately zero everywhere except a very small interval of σ near the turning points $\sigma = \pi/2$ and $\sigma = 3\pi/2$ where ρ' rapidly approaches zero. As we have discussed in section 5, to have manifest conformal invariance (and to maintain equivalence with the Polyakov string quantization) one should actually omit this problematic $R^{(2)}$ term in (5.7) altogether. Then we finish with the following system of 2-d fields defined on a flat 2-d cylinder and having approximately constant masses (see (5.6),(5.40)): 1 scalar with mass squared $m^2_\phi = 4\rho'^2$, 2 scalars with $m^2_\beta = 2\rho'^2$, 5 massless scalars, 8 Majorana fermions with $m^2_F = \rho'^2$. Using (6.2)[20] we then finish with the following estimate for the leading quantum correction to the classical energy (3.12): [21]

$$\Delta E \approx \frac{1}{\kappa} \sum_{n=1}^{\infty} \left[\sqrt{n^2 + 4\kappa^2} + 2\sqrt{n^2 + 2\kappa^2} + 5\sqrt{n^2} - 8\sqrt{n^2 + \kappa^2} \ \right] + O(\frac{1}{\sqrt{\lambda}}) \ , \qquad (6.6)$$

where we dropped the $n = 0$ contribution in the 2-d vacuum energy $\frac{1}{2}\sum_{n=-\infty}^{\infty} |\omega_n|$ which is subleading at large κ and used that (see (3.5),(5.8),(3.26))

$$\rho' \approx \kappa \ , \qquad \kappa \approx \ \frac{1}{\pi} \ln \mathcal{S} \ \gg 1 \ . \qquad (6.7)$$

[20] Note that in the static gauge (5.2) ($t = \kappa\tau$) the space-time and the 2-d energy are equal, up to κ-factor.

[21] Though the masses do change near the turning points, one is able to argue that since the change occurs only during a very short interval of σ, this does not, for large $\mathcal{S}$, significantly influence the spectrum of the Laplace operators. Indeed, in the long string limit the σ-dependent part of the mass $m^2 \sim \rho'^2$ can be approximated by the sum of two delta-functions located at the turning points. The corresponding Schrodinger equation can be solved explicitly, and one finds that the leading asymptotics at large κ can be estimated by ignoring σ-dependent terms in the masses. For constant masses the expression for the quantum correction to the energy becomes essentially the same as in section 6.5 in [8]. Here we consider the theory on a flat cylinder (τ, σ), so the individual field contributions to the 2-d vacuum energy are the same as in the theory with the kinetic term $-\partial^2 + m^2$. This is an analog of the 2-d vacuum energy $(D-2)\sum_{n=1}^{\infty} n = (D-2)\zeta(-1) = -\frac{D-2}{12}$ in the closed bosonic string case: it vanishes in the standard flat-space GS string case when all masses are set to zero.

In contrast to the S^5-boost case (or the "plane-wave" spectrum (4.13)) where the 2-d vacuum energies cancelled out between 2-d massive bosons and fermions, here the effective 2-d supersymmetry is spontaneously broken by the classical solution ($\rho' \neq 0$) so that the vacuum energy is finite (due to mass-squared sum rule) but is non-zero.

To obtain the leading term in ΔE at large κ we observe that it can be found by replacing the convergent sum in (6.6) by the integral $\sum_{n=1}^{\infty} f(n) \approx \int_1^\infty dx\, f(x)$, where $f(x) = \sqrt{x^2+4\kappa^2} + 2\sqrt{x^2+2\kappa^2} + 5\sqrt{x^2} - 8\sqrt{x^2+\kappa^2}$. Computing the integral, we find that its asymptotics at large κ is given by

$$\int_1^\infty dx\, f(x) \ \approx\ -3\ln 2\ \kappa^2 + O(\kappa)\ . \tag{6.8}$$

Taking into account the factor of $\frac{1}{\kappa}$ in (6.6), we finish with

$$\Delta E \approx -\frac{3\ln 2}{\pi} \ln \mathcal{S}\ + O(\frac{1}{\sqrt{\lambda}})\ , \tag{6.9}$$

i.e. with the result for the coefficient a_1 quoted in (6.5). The most important conclusion is the absence of higher powers of $\ln \mathcal{S}$ at the 1-loop order.

Let us now argue that the same should be true also at higher orders of string inverse tension ($\frac{1}{\sqrt{\lambda}}$) expansion. Again, the space-time energy shift will be related to the 2-d energy by the $\frac{1}{\kappa}$ factor (see (6.2)), so we need to show that $\Delta E_{\rm 2d}$ scales at most as κ^2 for large $\kappa \sim \ln \mathcal{S}$. The 2-d quantum field theory in question contains a collection of massive fields with approximately constant (for large $\mathcal{S}$, i.e. in the long string case) masses and proportional to κ with interaction terms containing two derivatives or two powers of κ (cf. (4.15)). It may be modelled by a theory $L = \sqrt{\lambda}[(1+n_1\phi^2+...)(\partial\phi)^2 + m^2(\phi^2+k_1\phi^4+...)]\ , \quad m \sim \kappa$ supplemented by fermionic terms that should cancel all UV divergences. The theory is defined on the cylinder $0 < \tau < T,\ 0 \le \sigma < 2\pi L$, where $T \to \infty$ and L is fixed (above we had $L = 1$). Since this theory must be UV finite, dimensional considerations imply that the n-loop contribution to the corresponding 2-d effective action vacuum energy should scale as $\Gamma_l = T(\Delta E_{\rm 2d})_l = (\frac{1}{\sqrt{\lambda}})^{l-1} V m^2 f_l(mL)$, where $V = TL$ is the 2-d volume and f_l is a finite function of dimensionless ratio of the two IR scales.[22] Since in the infinite volume limit $L \to \infty$ the function f_l should approach a finite constant c_l, the same should be true also for fixed L but for large m, i.e. $f_l(mL)_{m\gg 1} \to c_l$. As a result, for large $m \sim \kappa$ we get $(\Delta E_{\rm 2d})_l \approx b_l \kappa^2$. This implies that string corrections to the energy of the rotating string solution should produce a non-trivial function $f(\lambda)$ in (1.1) but should keep $h(\lambda) = 0$. Combined with the expected absence of the higher $\ln^k S$ terms on the gauge theory side (see [3] and refs. there) this provides a non-trivial check of the gauge theory – string theory correspondence.

[22] In our case of several fields with different masses f will also contain finite ratios of the masses, but these will stay constant in the limit of large κ.

6.2. **Massive scalar contribution to the quantum spectrum**

To find the spectrum of excited string states near the rotating and boosted string ground state on should study different sectors of states created by oscillators corresponding to different fluctuation fields in the quadratic action. While in pure-boost case all the "transverse" fluctuations were decoupled and had the same constant mass ν the situation for $\mathcal{S} \neq 0$ is, in general, more involved. However, since all the fields in the quadratic fluctuation action (5.14),(5.15),(5.40) have similar masses, we may get a qualitative picture of the spectrum by looking at few simplest cases. One is a sub-sector of states carrying excitations of the scalar field $\bar{\beta}_i$ in (5.15) with the mass given in (5.17), i.e.

$$m^2_{_\beta} = 2\rho'^2 + \nu^2 = 2(\kappa^2 - \omega^2)\sinh^2\rho + 2\kappa^2 - \nu^2 \ . \tag{6.10}$$

We will estimate the contribution of this field to the "excited" part of the energy spectrum in the two special limits in the parameter space of the classical solution. As explained in Appendix A, the correction to the classical energy is essentially determined by the 2-d Hamiltonian associated with the fluctuation Lagrangian.

Short string case

Here $\omega^2 - \kappa^2 \approx 1$, $\eta \gg 1$ and an approximate solution for $\rho(\sigma)$ is given by (3.19). Using (3.18) and (3.19) we can rewrite (6.10) as

$$m^2_{_\beta} \approx -2(\kappa^2 - \nu^2)\sin^2\sigma + 2\kappa^2 - \nu^2 \approx \nu^2 + \frac{2}{\eta}\cos^2\sigma \ . \tag{6.11}$$

To find the spectrum of scalar modes we expand $\bar{\beta}(\tau,\sigma) = \sum_n b_n(\tau)\Phi_n(\sigma)$ where

$$-\Phi''_n(\sigma) + m^2_{_\beta}(\sigma)\Phi_n(\sigma) = l_n^2\Phi_n(\sigma) \ , \qquad \Phi_n(\sigma) = \Phi_n(\sigma + 2\pi) \ . \tag{6.12}$$

For large η the mass (6.11) is approximately constant and we are back to the case of the standard massive oscillator spectrum. Computing the first correction in perturbation theory in $1/\eta$, we get ($\langle\cos^2\sigma\rangle = \frac{1}{2}$)

$$l_n^2 \approx n^2 + \nu^2 + \frac{1}{\eta} \approx n^2 + \kappa^2 \ . \tag{6.13}$$

Using the proportionality relation between the space-time and the 2-d energy (cf. (6.2),(A.8)) we find that the scalar $\bar{\beta}$ contribution to the energy of an excited string state is given by (cf. (4.13))

$$E \approx E_{\rm vac} + \frac{1}{\kappa}\sum_{n=-\infty}^{\infty} \sqrt{n^2+\kappa^2}\ N_n + O(\frac{1}{\sqrt{\lambda}}) \ , \tag{6.14}$$

where $E_{\rm vac}$ is the "ground state" energy ($\approx E_0$ in (3.12) in the short string limit), and N_n are the occupation numbers for the oscillator states of $\bar{\beta}$.

If ν is small as in (3.22), $\nu^2 \ll \mathcal{S} \ll 1$, i.e. $\kappa^2 \approx 2\mathcal{S}$ (see (3.18),(3.20)) we find to the leading order ($n \neq 0$)

$$E \approx \sqrt{\lambda}\ (\sqrt{2\mathcal{S}} + \frac{\nu^2}{2\sqrt{2\mathcal{S}}}) + \frac{1}{\sqrt{2\mathcal{S}}} \sum_{n=-\infty}^{\infty} (1 - \frac{\nu^2}{4\mathcal{S}} + \frac{\mathcal{S}}{n^2})\ |n| N_n + ...\ , \qquad (6.15)$$

where we replaced $E_{\rm vac}$ by E_0 in (3.22). Since in the short string case $\mathcal{S} \ll 1$ (see (3.20)), the excited states can be very heavy.

In the opposite case of large $\nu \gg 1$ when $\kappa^2 \approx \nu^2 + \frac{2\mathcal{S}}{\nu}$, the energy (6.14) of states with $n \ll \nu$ becomes (cf. (3.24),(3.25))

$$E \approx \sqrt{\lambda}\ (\nu + \mathcal{S} + \frac{\mathcal{S}}{2\nu^2}) + \sum_{n=-\infty}^{\infty} (1 + \frac{n^2}{2\nu^2} - \frac{n^2\mathcal{S}}{\nu^5})\ N_n + ...\ . \qquad (6.16)$$

The first two terms in the second bracket are the same as in the spectrum (3.25) corresponding to the case of $\mathcal{S} = 0$.

Long string case

To approximate the expression for the mass (6.10) in the limit of large ρ_0 or small η let us first describe the form of the solution for $\rho(\sigma)$ in this case. Introducing the function

$$y(\sigma) = \eta\ \sinh^2 \rho(\sigma)\ , \qquad (6.17)$$

we observe that it takes values from 0 to 1, and satisfies, according to (3.5),(3.10)

$$\frac{1}{2\sqrt{\kappa^2 - \nu^2}} \int_0^{y(\sigma)} \frac{dx}{\sqrt{x(x+\eta)(1-x)}} = \sigma\ , \qquad y(\frac{\pi}{2}) = \eta \sinh^2 \rho_0 = 1\ . \qquad (6.18)$$

Computing the integral and solving the resulting equation at small values of η, we get $y(\sigma)$ for $0 \leq \sigma < \frac{\pi}{2}$ that satisfies $y(0) = 0, \quad y(\frac{\pi}{2}) = 1 + O(\eta^2)$:

$$y(\sigma) \approx \frac{1}{4(1+f)^2} \left[-\eta(1+f) + 8f + 4f\sqrt{4 + \eta + \eta f} \right]\ , \qquad f(\sigma) \equiv \left(\frac{\eta}{16}\right)^{1 - \frac{2\sigma}{\pi}}\ . \qquad (6.19)$$

In the limit $\eta \to 0$ the solution $y(\sigma)$ goes to zero as $\eta^{\frac{\pi - 2\sigma}{\pi}}$ at all points in $0 \leq \sigma < \frac{\pi}{2}$ and is equal to 1 at $\sigma = \frac{\pi}{2}$. This also means that $y(\sigma) \ln^2 \eta$ goes to 0 at all points $0 \leq \sigma < \frac{\pi}{2}$.

Substituting the solution (6.19) into (6.10), and taking into account (3.10),(3.26), we get

$$m_{\beta}^2 \approx \nu^2 + \frac{2}{\pi^2}(1 - y(\sigma)) \ln^2 \eta \ \approx \nu^2 + \frac{2}{\pi^2} \ln^2 \eta \ . \tag{6.20}$$

Thus, we see that the fluctuations $\bar{\beta}_i$ around the long string have σ-independent and very heavy mass. The energy of the excited states is then given by (cf. (6.6))

$$E \approx E_{\rm vac} + \frac{1}{\kappa} \sum_{n=-\infty}^{\infty} \sqrt{n^2 + \nu^2 + \frac{2}{\pi^2} \ln^2 \eta} \ N_n \ . \tag{6.21}$$

In the case of small ν, i.e. (see (3.28)) $\nu^2 \ll \ln^2 \eta \approx \ln^2 \mathcal{S}$ we get (using (3.26))

$$E \approx E_{\rm vac} + \frac{\pi^2}{2\sqrt{2} \ln^2 \mathcal{S}} \sum_{n=-\infty}^{\infty} (n^2 - \nu^2) \ N_n \ . \tag{6.22}$$

Note that we do not get $O(\ln S)$ terms in the oscillator part of the spectrum.

In the opposite case when $\nu^2 \gg \ln^2 \eta \approx \ln^2 \frac{\mathcal{S}}{\nu}$

$$E \approx E_{\rm vac} + \frac{1}{2\nu^2} \sum_{n=-\infty}^{\infty} \ (n^2 + \frac{1}{\pi^2} \ln^2 \frac{\mathcal{S}}{\nu}) \ N_n \ . \tag{6.23}$$

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Appendix A. Expression for the energy in conformal gauge

To compute the 1-loop correction to the space-time energy of a given classical string configuration we need the expression for the energy in terms of the fluctuation fields, governed by the quadratic fluctuation action like (5.14),(5.15). As described on the example of the boosted solution in section 4, one is to solve the conformal gauge constraints, expressing the energy in terms of the quantum values of space-time momenta or conserved charges and a term quadratic in the fluctuation fields (which turned out to be the 2-d Hamiltonian of the transverse fluctuation fields, see (4.9)–(4.13)). Quantizing the fluctuation fields using the quadratic part of the string action in the conformal gauge, one may then be able to compute the expectation value of the energy in a given quantum state.

Here we will derive the expression for the energy for the case of the expansion near the rotating solution (3.1). In general, the space-time energy is the zero mode of the momentum conjugated to the time variable t

$$E = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} \ G_{tt} \ \partial_0 t \ , \tag{A.1}$$

where $G_{tt} \equiv -G_{00}$, i.e. in the present AdS_5 case (2.6) $G_{tt} = \cosh^2 \rho$. As usual, the l.h.s. of the first of the two conformal gauge constraints (3.3) is proportional to the density of the full 2-d Hamiltonian

$$\mathcal{H}(t, \phi, \varphi, \rho, ...) = \frac{1}{2} G_{MN}(X) \left(\partial_0 X^M \partial_0 X^N + \partial_1 X^M \partial_1 X^N \right)$$

$$= -\frac{1}{2} G_{tt} \left(\partial_0 t \partial_0 t + \partial_1 t \partial_1 t \right) + ... \ . \tag{A.2}$$

Splitting the field t as $\kappa\tau + \tilde{t}$, and taking into account that the metric is invariant under the time translations, we get from the constraint $\mathcal{H} = 0$

$$G_{tt} \partial_0 t = \frac{1}{2} \kappa G_{tt} + \frac{1}{\kappa} \mathcal{H}(\tilde{t}, \phi, \varphi, \rho, ...) \ , \tag{A.3}$$

where $\mathcal{H}(\tilde{t}, \phi, \varphi, \rho, ...)$ is given by (A.2) with the replacement $t \to \tilde{t}$. In addition to the energy, our background has two other conserved charges related to the invariance of the metric under translations along ϕ and φ directions (cf. (3.13),(4.12)):

$$S = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} G_{\phi\phi} \ \partial_0 \phi \ , \qquad J = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} G_{\varphi\varphi} \ \partial_0 \varphi \ , \tag{A.4}$$

where $G_{\phi\phi} = \sinh^2 \rho \ \cos^2 \beta_1 \cos^2 \beta_2$ and $G_{\varphi\varphi} = \cos^2 \psi_1 \cos^2 \psi_2 \cos^2 \psi_3 \cos^2 \psi_4$. Since the conformal-gauge string action and thus the Hamiltonian depend quadratically on ϕ and φ, we may repeat the same splitting procedure with ϕ and φ in (A.3): setting $\phi = \omega\tau + \tilde{\phi}$, $\varphi = \nu\tau + \tilde{\varphi}$, using the definitions (A.4) and integrating (A.3) we arrive at the following expression for the energy (A.1):

$$E = \frac{\omega}{\kappa} S + \frac{\nu}{\kappa} J + \frac{\sqrt{\lambda}}{\kappa} \int \frac{d\sigma}{2\pi} \left[\frac{1}{2} \kappa^2 G_{tt} - \frac{1}{2} \omega^2 G_{\phi\phi} - \frac{1}{2} \nu^2 G_{\varphi\varphi} + \mathcal{H}(\tilde{t}, \tilde{\phi}, \tilde{\varphi}, \rho, \ldots) \right] \ , \tag{A.5}$$

where $\mathcal{H}(\tilde{t}, \tilde{\phi}, \tilde{\varphi}, \rho, \ldots)$ is again given by the l.h.s. of (A.2) with the replacement $t \to \tilde{t}, \ \phi \to \tilde{\phi}, \ \varphi \to \tilde{\varphi}$.

Next, let us expand the remaining fields near their classical values as in (5.1): $\rho(\tau,\sigma) = \rho(\sigma) + \frac{1}{\lambda^{1/4}}\tilde{\rho},\ \beta_i = \frac{1}{\lambda^{1/4}}\tilde{\beta}_i,\ \psi_s = \frac{1}{\lambda^{1/4}}\tilde{\psi}_s$ $(i=1,2;\ s=1,2,3,4)$, rescaling also $\tilde{t},\tilde{\phi},\tilde{\varphi}$ by $\frac{1}{\lambda^{1/4}}$. To quadratic order in the fluctuations we then get

$$\begin{aligned} E =& E_0 + \frac{\omega}{\kappa}(S-S_0) + \frac{\nu}{\kappa}(J-J_0) + \frac{1}{\kappa}\int \frac{d\sigma}{2\pi}\left[\frac{1}{2}(\kappa^2-\omega^2)\cosh 2\rho\ \tilde{\rho}^2\right. \\ &\left. + \frac{1}{2}\omega^2\sinh^2\rho\ \tilde{\beta}_i^2 + \frac{1}{2}\nu^2\tilde{\psi}_s^2 + \mathcal{H}^{(2)}(\tilde{t},\tilde{\phi},\tilde{\varphi},\tilde{\rho},\tilde{\beta}_i,\tilde{\psi}_s)\right] , \end{aligned} \tag{A.6}$$

where $E_0,\ S_0,\ J_0$ are the "classical" values of energy and charges computed on the rotating string background, i.e. given by (3.12),(3.13),(4.12). Comparing this to the action (5.14),(5.15) or, equivalently, (5.23), it is easy to check that the integrand in (A.6) coincides with the 2-d Hamiltonian $\mathcal{H}_{2\mathrm{d}}$ corresponding to the quadratic fluctuation action (5.14),(5.15) or, equivalently, (5.23). The explicit expression for $\mathcal{H}_{2\mathrm{d}}$ in terms of fields and their derivatives can be obtained from the (minus) Lagrangian by dropping the terms linear in time derivative (i.e. mixed terms in (5.14)), and by reversing the signs of the terms quadratic in time derivative. In terms of redefined fields in (5.14),(5.15)

$$\begin{aligned} \mathcal{H}_{2\mathrm{d}} = \frac{1}{2}\big[&- \partial_0\bar{t}\partial_0\bar{t} - \partial_1\bar{t}\partial_1\bar{t} - \mu_t^2\bar{t}^2 + \partial_0\bar{\phi}\partial_0\bar{\phi} + \partial_1\bar{\phi}\partial_1\bar{\phi} + \mu_\phi^2\bar{\phi}^2 \\ &+ \partial_0\bar{\rho}\partial_0\bar{\rho} + \partial_1\bar{\rho}\partial_1\bar{\rho} + \mu_\rho^2\bar{\rho}^2 + \partial_0\bar{\beta}_i\partial_0\bar{\beta}_i + \partial_1\bar{\beta}_i\partial_1\bar{\beta}_i + m_\beta^2\bar{\beta}_i^2 \\ &+ \partial_0\tilde{\varphi}\partial_0\tilde{\varphi} + \partial_1\tilde{\varphi}\partial_1\tilde{\varphi} + \partial_0\tilde{\psi}_s\partial_0\tilde{\psi}_s + \partial_0\tilde{\psi}_s\partial_0\tilde{\psi}_s + \nu^2\tilde{\psi}_s^2\big] . \end{aligned} \tag{A.7}$$

The final form of (A.6) is thus

$$E = E_0 + \frac{\omega}{\kappa}(S-S_0) + \frac{\nu}{\kappa}(J-J_0) + \frac{1}{\kappa}\int \frac{d\sigma}{2\pi}\,\mathcal{H}_{2\mathrm{d}}(\tilde{t},\tilde{\phi},\tilde{\varphi},\tilde{\rho},\tilde{\beta}_i,\tilde{\psi}_s) . \tag{A.8}$$

This expression generalises the one in the pure boost case, which, upon quantisation led to eq.(4.13) (there $\omega=0,\ \kappa=\nu,\ S=S_0=0,\ E_0=J_0$). In general, assuming one can diagonalise the second-order differential operators appearing in (5.14),(5.15), one can then choose a specific quantum state (subject to the "level matching" condition following from the second constraint (3.4)) with given quantized values of J and S equal to their classical values (see (6.1)) and compute the average value of E in (A.8):

$$E = \langle\Psi|\hat{E}|\Psi\rangle = E_0 + \frac{1}{\kappa}\langle\Psi|\hat{H}_{2\mathrm{d}}|\Psi\rangle , \tag{A.9}$$

$$H_{2\mathrm{d}} \equiv \int \frac{d\sigma}{2\pi}\,\mathcal{H}_{2\mathrm{d}}(\tilde{t},\tilde{\phi},\tilde{\varphi},\tilde{\rho},\tilde{\beta}_i,\tilde{\psi}_s) .$$

Finally, one may express the classical parameters ν,ω,κ in terms of S and J in the leading quantum correction (i.e. up to $O(\frac{1}{\sqrt{\lambda}})$ terms as in (4.13)) using the classical relations of section 3.

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