$C_0(X)$-structure in C*-algebras, Multiplier Algebras and Tensor Products

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David McConnell, 29th January 2015

Summary

We begin in Chapter 2 with an introduction to the various notions of a bundle of C*-algebras that have appeared throughout the literature, and clarify the definitions of upper- and lower-semicontinuous C*-bundles not explicitly defined in a formal way elsewhere. The definition of $C_0(X)$-algebra, introduced by Kasparov [38], and its relation to C*-bundles is discussed in this chapter also. The purpose of this chapter is to bring together concepts that we will refer to in subsequent sections and which are described using various notations by different authors. Most of this is implicitly understood elsewhere, though Theorem 2.3.12, relating sub-modules of $C_0(X)$-modules and subbundles of C*-bundles, is a new result.

Chapter 3 concerns the study of the multiplier algebra $M(A)$ of a non-unital C*-algebra A. When A is the section algebra of a C*-bundle over a space X, we consider two approaches to representing $M(A)$ as the section algebra of a bundle over βX. Our first approach involves extending the evaluation homomorphisms from A to $M(A)$, using properties of the strict topology on $M(A)$, and is similar to the construction of Akemann, Pedersen and Tomiyama [1]. This gives $M(A)$ the desired structure of a C*-bundle which we describe in Theorem 3.1.6. Moreover, we show that lower-semicontinuity of A as a C*-bundle passes to $M(A)$ under this construction (Corollary 3.1.9).

The second approach is due to Archbold and Somerset [8]; when A is the section algebra of a C*-bundle arising from a $C_0(X)$-algebra, we describe how $M(A)$ carries naturally the structure of a $C(\beta X)$-algebra (hence an upper-semicontinuous C*-bundle over βX), where βX is the Stone-Čech compactification of X. We then investigate conditions under which these bundle representations of $M(A)$ coincide.

Central to this is the study of the relationship between the spaces Prim(A), Prim$(M(A))$, X and βX. Further, we relate the different types of bundle representations of $M(A)$ with the question of 'spectral synthesis in the multiplier algebra of a $C_0(X)$-algebra,' as studied by Archbold and Somerset in [9] and [11]. Some of the principal results of section 3.4, namely Corollaries 3.3.6, 3.4.7 and 3.4.8, and Theorem 3.4.10,were obtained independently by Archbold and Somerset, and have been published in [11]. Theorem 3.4.6 is a new result and has not appeared elsewhere.

In Chapter 4, we examine the relationship between sheaves and bundles in the category of C*-algebras. In the case of a sheaf of C*-algebras over a locally compact Hausdorff space X, this relationship is described in detail in Theorem 4.2.10. As a consequence, we establish a Stone-Čech type theorem for C*-bundles over locally compact Hausdorff spaces in Corollary 4.2.11. In section 4.3 we consider the particular case of a sheaf arising from the multiplier algebra of a $C_0(X)$-algebra A. We show in Theorem 4.3.7 that this sheaf is in fact equivalent to the 'sheaf of local sections' of the bundle defined by the $C(\beta X)$-algebra $M(A)$ considered in Chapter 3.

Chapters 5 and 6 concern the study of tensor products of C*-bundles, primarily the case of the minimal (or spatial) tensor product $A \otimes_\alpha B$ of two C*-algebras A and B. Our approach is focused on the related problem of determining the ideal structure of $A \otimes_\alpha B$ in terms of those of A and B. We give a complete description in Chapter 5 (Theorems 5.3.3 and 5.5.9) of the topological space of Glimm ideals of $A \otimes_\alpha B$ in terms of the ideal spaces of A and B, without additional assumptions on A and B, extending earlier results of Kaniuth [37]. As a consequence, we determine in Theorem 5.4.3 the centre $ZM(A \otimes_\alpha B)$ of the multiplier algebra of $A \otimes_\alpha B$ in terms of these spaces.

Under some additional assumptions on the ideal structure of $A \otimes_\alpha B$, we show in Theorem 5.6.2 that if the Dauns-Hofmann representations of A and B give rise to well-behaved C*-bundles, then the same is true of $A \otimes_\alpha B$, sharpening earlier results of Kaniuth [37] and Archbold [6]. The results of this chapter have been published in [45].

Finally, Chapter 6 is dedicated to the investigation of the stability of important well-behaved classes of C*-bundles, the quasi-standard C*-algebras and continuous $C_0(X)$-algebras, under the operation of taking tensor products. Our approach differs from that considered previously by Kirchberg and Wassermann [40], who studied analogous problems for fibrewise tensor products of C*-bundles. Indeed, we show in Theorem 6.4.4 that continuity of the tensor product bundle we study here is a strictly weaker property than continuity of the fibrewise tensor product.

Our main results of this chapter, Theorems 6.5.6, 6.6.6 and 6.6.10, show that neither the class of quasi-standard C*-algebras nor the continuous $C_0(X)$-algebras are stable under tensor products, and identify the largest subclass of each which is tensor stable. In the case of the minimal (respectively maximal) tensor product, this is the subclass of exact (respectively nuclear) C*-algebras. As a related result we give several new characterisations of exact C*-algebras in Theorem 6.6.3.

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Abstract

Our general theme concerns topological decompositions of C*-algebras and the interactions of these decompositions with multiplier algebras, tensor products and module structures. A primary focus is placed on modules known as $C_0(X)$-algebras.

Bundle structures, specifically C*-bundles, for a C*-algebra A (not usually unital) over a suitable base space X, are closely related to $C_0(X)$-algebras, and a natural consideration is to relate such structures on A to bundle representations of the multiplier algebra $M(A)$ of A over the Stone-Čech compactification βX of X. We discuss how the strict topology on $M(A)$ can be used in this context, and in the case of a $C_0(X)$-algebra A, relate it to the induced $C(\beta X)$-algebra structure on $M(A)$. Further preliminary results concern the ideal structure of $M(A)$ when A is a $C_0(X)$-algebra.

Sheaves of C*-algebras over X provide another approach which is partially equivalent to bundles when the topological space X is locally compact and Hausdorff. As a corollary, we show that a C*-bundle over a locally compact Hausdorff space X defines naturally a C*-bundle over βX in such a way that their algebras of continuous sections are naturally isomorphic. These results are applied to the particular case of the sheaf of multipliers of a $C_0(X)$-algebra A, which is shown to be canonically isomorphic to the sheaf of local sections arising from the $C(\beta X)$-structure on $M(A)$.

For our main results we consider the minimal tensor product $A \otimes_\alpha B$ of two C*-algebras A and B. Extending earlier results of Kaniuth [37], we obtain a complete description of the topological space of Glimm ideals of $A \otimes_\alpha B$ in terms of those of the factors (results published in [45]). As a consequence, we construct the Dauns-Hofmann bundle representation [21] of $A \otimes_\alpha B$ in terms of the corresponding representations of A and B, and describe the structure of the centre of the multiplier algebra of $A \otimes_\alpha B$ in this setting.

Given a $C_0(X)$-algebra A and a $C_0(Y)$-algebra B, we demonstrate how $A \otimes_\alpha B$ carries naturally the structure of a $C_0(X \times Y)$-algebra. We study the associated C*-bundle decomposition of $A \otimes_\alpha B$ over this space, and in particular we compare its structure to the fibrewise tensor product studied elsewhere. As a consequence, we obtain several new characterisations of the property of exactness in terms of the stability of certain classes of $C_0(X)$-algebras under the operation of forming tensor products.

Contents

Chapter 1

Introduction

Bundles or fields of C*-algebras are an important aspect of the topological decomposition theory of C*-algebras, and have lead to significant advances in the field. The problem of representing a C*-algebra A as the algebra of sections of a bundle of C*-algebras over a suitable base space may be regarded as that of finding a non-commutative Gelfand-Naimark Theorem, which represents a commutative C*-algebra A as the algebra of continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space. A number of non-commutative decomposition theorems have been obtained, notably by Fell [27], Dauns and Hofmann [21], Lee [44], and Archbold and Somerset [7]. Often, however, these constructions give rise to bundles of a very general type, the structure of which can be difficult to study.

The focus of this thesis is the investigation of certain structural properties of these bundle decompositions of C*-algebras, and the interaction of these properties with constructions such as forming multiplier algebras and tensor products. Central to the theory is the study of important topological spaces of ideals of the C*-algebra A under consideration, namely the space of primitive ideals Prim(A) (with its hull-kernel topology) and its complete regularisation, the space of Glimm ideals. These spaces may be regarded as non-commutative analogues of the character space of a commutative C*-algebra.

We begin in Chapter 2 with an account of these topological spaces of ideals of C*-algebras. We introduce also various notions of a bundle of C*-algebras that have appeared throughout the literature, and clarify the definitions of upper- and lower-semicontinuous C*-bundles not explicitly defined in a formal way elsewhere. The definition of $C_0(X)$-algebra (that is, a C*-algebra A which is also a Banach $C_0(X)$-module), introduced by Kasparov [38], and its relation to C*-bundles is discussed in this chapter also. The purpose of this chapter is to bring together concepts that we will refer to in subsequent sections and which are described using various notations by different authors. Most of this is implicitly understood elsewhere, though Theorem 2.3.12, relating sub-modules of $C_0(X)$-modules and subbundles of C*-bundles, is a new result.

In Chapter 3, we consider the multiplier algebra $M(A)$ of a (non-unital) C*-bundle A, and the different approaches to obtaining a bundle decomposition of $M(A)$. Our first approach involves extending the evaluation homomorphisms from A to $M(A)$, using properties of the strict topology on $M(A)$, and is similar to the construction of Akemann, Pedersen and Tomiyama [1]. This gives $M(A)$ the desired structure of a C*-bundle which we describe in Theorem 3.1.6. Moreover, we show that lower-semicontinuity of A as a C*-bundle passes to $M(A)$ under this construction (Corollary 3.1.9).

The second approach is due to Archbold and Somerset [8]; when A is the section algebra of a C*-bundle arising from a $C_0(X)$-algebra, we describe how $M(A)$ carries naturally the structure of a $C(\beta X)$-algebra (or a C*-bundle over βX), where βX is the Stone-Čech compactification of X. By contrast, this construction preserves upper-semicontinuity of the bundles under consideration. We then investigate conditions under which these bundle representations of $M(A)$ coincide.

Central to this is the study of the relationship between the spaces Prim(A), Prim$(M(A))$, X and βX. Further, we relate the different types of bundle representations of $M(A)$ with the question of 'spectral synthesis in the multiplier algebra of a $C_0(X)$-algebra,' as studied by Archbold and Somerset in [9] and [11]. We remark that some of the main results of Section 3.4 were obtained independently by Archbold and Somerset, and have been published in [11]. In particular, Corollary 3.3.6 corresponds to [11, Proposition 2.6(i) and (iii)], and Corollary 3.4.7 to [11, Corollary 4.3]. Stronger versions of Corollary 3.4.8 and Theorem 3.4.10 were also given in [11, Corollary 4.4] and [11, Theorem 4.6] respectively. Theorem 3.4.6 is a new result.

As is the case for objects studied in algebraic and differential geometry, the notion of a sheaf of C*-algebras is closely related to that of a C*-bundle. The connection between these notions has been studied by Hofmann [33] (in the case of sheaves and bundles of Banach spaces), and more recently by Ara and Mathieu [3] (in the particular case of

C*-algebras). However, in general, the theory of sheaves is less well-developed than that of bundles in the category of C*-algebras. In Chapter 4, we examine the relationship between these two notions. In the case of a sheaf of C*-algebras over a locally compact Hausdorff space X, this relationship is described in detail in Theorem 4.2.10. As a consequence, we establish a Stone-Čech type theorem for C*-bundles over locally compact Hausdorff spaces in Corollary 4.2.11. In Section 4.3 we consider the particular case of a sheaf arising from the multiplier algebra of a $C_0(X)$-algebra A. We show in Theorem 4.3.7 that this sheaf is in fact equivalent to the 'sheaf of local sections' of the bundle defined by the $C(\beta X)$-algebra $M(A)$ considered in Chapter 3.

Chapters 5 and 6 concern the study of tensor products of C*-bundles, primarily the case of the minimal (or spatial) tensor product $A \otimes_\alpha B$ of two C*-algebras A and B. Our approach is focused on the related problem of determining the ideal structure of $A \otimes_\alpha B$ in terms of those of A and B. We give a complete description in Chapter 5 (Theorems 5.3.3 and 5.5.9) of the topological space of Glimm ideals of $A \otimes_\alpha B$ in terms of the ideal spaces of A and B, without additional assumptions on A and B, extending earlier results of Kaniuth [37]. As a consequence, we determine in Theorem 5.4.3 the centre $ZM(A \otimes_\alpha B)$ of the multiplier algebra of $A \otimes_\alpha B$ in terms of these spaces.

Using these results we construct the Dauns-Hofmann C*-bundle associated with $A \otimes_\alpha B$ in terms of the corresponding bundles of A and B. Under some additional assumptions on the ideal structure of $A \otimes_\alpha B$, we show in Theorem 5.6.2 that if the Dauns-Hofmann representations of A and B give rise to well-behaved C*-bundles, then the same is true of $A \otimes_\alpha B$, sharpening earlier results of Kaniuth [37] and Archbold [6]. The results of this chapter have been published in [45].

Finally, Chapter 6 is dedicated to the investigation of the stability of important well-behaved classes of C*-bundles, the quasi-standard C*-algebras and continuous $C_0(X)$-algebras, under the operation of taking tensor products. Using similar techniques to those considered in Chapter 5, we describe how the minimal tensor product of a $C_0(X)$-algebra A and a $C_0(Y)$-algebra B carries the structure of a $C_0(X \times Y)$-algebra, and investigate the structure of the associated C*-bundle over $X \times Y$. Our approach differs from that considered previously by Kirchberg and Wassermann [40], who studied analogous problems for fibrewise tensor products of C*-bundles. Indeed, we show in Theorem 6.4.4 that continuity of the tensor product bundle we study here is a strictly weaker property than continuity of the fibrewise tensor product.

We show in Section 6.6 that for an inexact continuous $C_0(X)$-algebra A, one can always construct a continuous $C_0(Y)$-algebra B such that $A \otimes_\alpha B$ is discontinuous as a $C_0(X \times Y)$-algebra. As a consequence it is shown in Theorem 6.5.6 that stability of continuity is in fact equivalent to exactness of A. Thus our tensor product construction identifies exactness in precisely the same way as the fibrewise tensor product of Kirchberg and Wassermann [40, Theorem 4.5].

We apply these results in particular to those $C_0(X)$-algebras arising from C*-algebras A with Hausdorff primitive ideal space, and the *quasi-standard* C*-algebras introduced in [7]. Until now, it appears that there were no known examples of a pair of quasi-standard C*-algebras whose minimal tensor product fails to be quasi-standard. Our main results of this section, 6.6.6 and 6.6.10, show that neither of these classes are stable under tensor products, and identify the largest subclass of each which is tensor stable. In the case of the minimal (respectively maximal) tensor product, this is the subclass of exact (respectively nuclear) C*-algebras. As a related result we give several new characterisations of exact C*-algebras in Theorem 6.6.3, in terms of short exact sequences arising from Glimm and minimal primal ideals.

Chapter 2

Preliminaries on $C_0(X)$-algebras, C*-bundles and ideal spaces

In this section we introduce elements of topological decompositions of C*-algebras, in particular, the decomposition of a given C*-algebra A into the algebra of (continuous) sections of a bundle of C*-algebras. Central to these decomposition theories is the structure of certain topological spaces of ideals, the primitive and Glimm ideal spaces, of the algebra under consideration, which we introduce in Section 2.1.

There are numerous different definitions of C*-bundles appearing throughout the literature. In many cases these definitions turn out to be essentially equivalent, though the reasons for this are by no means obvious. Thus it will be convenient for us to give an account of these equivalences, in order to conveniently reference results regarding C*-bundles published elsewhere. Most of this chapter is therefore expository in nature.

In Section 2.2, we relate two of the main definitions of C*-bundles. The first, a (H)-C*-bundle by our terminology (Definition 2.2.1), is the more traditional in some sense; we consider a (locally compact Hausdorff) base space X, a total space $\mathcal{A}$ consisting of a collection of C*-algebras $\{A_x : x \in X\}$, and algebras of *continuous* sections $\Gamma(X, \mathcal{A})$. The second, which we call simply a C*-bundle (Definition 2.2.7), is closer to that of an 'algebra of operator fields,' and frequently offers a simpler framework in which to work. Results of Fell and Hofmann show that, under the assumption that the norm functions of sections are upper-semicontinuous on X, these constructions are in fact equivalent.

A more recent bundle-type decomposition is that of $C_0(X)$-algebras (Definition 2.3.1), that is, a C*-algebra A which is an essential $C_0(X)$-Banach module for some locally compact Hausdorff space X. We introduce $C_0(X)$-algebras in Section 2.3, and give an account of how $C_0(X)$-algebras are equivalent to the C*-bundles in the previous section. The advantage of $C_0(X)$-algebras is that many more technical results concerning C*-bundles have convenient formulations in the language of $C_0(X)$-algebras.

Most of the results of this section are known, or at least implicitly understood, elsewhere in the literature. Theorem 2.3.12, relating sub-$C_0(X)$-modules of a $C_0(X)$-algebra with sub-bundles of the associated C*-bundle, is new.

For a topological space X, $C(X)$ will denote the continuous complex valued functions on X, and $C^b(X)$ will denote those functions in $C(X)$ that are bounded, which is a C*-algebra with pointwise operations and supremum norm. Similarly, $C_0(X)$ will denote the C*-subalgebra of $C^b(X)$ consisting of those functions that vanish at infinity on X. If in addition X is compact, then we shall frequently identify $C_0(X)$ and $C^b(X)$ with $C(X)$, since all three are equal in this case.

If X is locally compact, non-compact and Hausdorff, then $\hat{X}$ will denote the one-point compactification $X \cup \{\infty\}$ of X, and βX the Stone-Čech compactification of X.

For a Hilbert space H, $B(H)$ will denote the C*-algebra of bounded linear operators on H, and $K(H)$ the ideal of compact operators. For C*-algebra A, by a *representation* of A we mean a $*$-homomorphism $\pi : A \to B(H)$. A representation π is said to be *irreducible* if H contains no nontrivial proper closed subspaces invariant under $\pi(A)$, and *factorial* if the closure of $\pi(A)$ in the weak operator topology on $B(H)$ has trivial centre.

For a locally compact Hausdorff space X and a C*-algebra A, $C^b(X, A)$ will denote the bounded norm-continuous A-valued functions on X, which is a C*-algebra with pointwise operations and supremum norm. We define $C_0(X, \mathcal{A})$ and $C(X, \mathcal{A})$ analogously. When $X = \mathbb{N}$, we shall often write $\ell^\infty(A)$ and $c_0(A)$ to denote $C^b(\mathbb{N}, A)$ and $C_0(\mathbb{N}, A)$ respectively.

A *state* on a C*-algebra A is a linear functional $\phi : A \to \mathbb{C}$ which is positive (in the sense that $\phi(a^*a) \geq 0$ for all $a \in A$), and satisfies $\|\phi\| = 1$. Other definitions will be introduced as they arise.

2.1 The primitive and Glimm ideal spaces of a C*-algebra

In this section we describe certain topological spaces of ideals of a C*-algebra A, which will be of interest to us in subsequent sections.

By a *primitive ideal* of a C*-algebra A we mean $\ker(\pi)$, where $\pi : A \to B(H)$ is an irreducible representation of A on some Hilbert space H. We will denote by $\mathrm{Prim}(A)$ the set of primitive ideals of the C*-algebra A.

We now describe how $\mathrm{Prim}(A)$ is regarded as a topological space with the *hull-kernel* (or *Jacobson*) topology. For $T \subseteq \mathrm{Prim}(A)$, we define the kernel of T, $k(T)$ via

$$k(T) = \bigcap \{P : P \in T\}$$

For an ideal J of A, we define its hull, $\mathrm{hull}(J)$ as

$$\mathrm{hull}(J) = \{P \in \mathrm{Prim}(A) : P \supseteq J\}.$$

Definition 2.1.1. Let A be a C*-algebra. Then the *hull-kernel* topology on $\mathrm{Prim}(A)$ is defined by the closure operation $\mathrm{cl}(T) = \mathrm{hull}(k(T))$ for all $T \subseteq \mathrm{Prim}(A)$.

The following proposition lists some well-known properties of the topological space $\mathrm{Prim}(A)$ in the hull-kernel topology, which shall be used frequently in the sequel.

Proposition 2.1.2. *Let A be a C*-algebra. Then the topological space* $\mathrm{Prim}(A)$ *has the following properties:*

(i) Every open subset $U \subseteq \mathrm{Prim}(A)$ is of the form

$$U = \{P \in \mathrm{Prim}(A) : P \not\supseteq I\}$$

for some ideal I of A. [24, Proposition 3.1.2]

(ii) For every ideal I of A, the map $P \mapsto P \cap I$ (resp. $P \mapsto P/I$) is a homeomorphism of the open subset $\{P \in \mathrm{Prim}(A) : P \not\supseteq I\}$ (resp. the closed subset $\{P \in \mathrm{Prim}(A) : P \supseteq I\}$) onto $\mathrm{Prim}(I)$ *(resp.* $\mathrm{Prim}(A/I)$*). [24, Proposition 3.2.1]*

(iii) $\mathrm{Prim}(A)$ *is a T_0-space. [24, Proposition 3.1.1]*

(iv) $\mathrm{Prim}(A)$ *is locally compact in the sense that every $P \in \mathrm{Prim}(A)$ admits a compact neighbourhood. [24, Corollaire 3.3.8]*

(v) If A is unital then $\mathrm{Prim}(A)$ *is compact. [24, Proposition 3.1.8]*

Elementary examples show that we cannot expect $\mathrm{Prim}(A)$ to admit stronger separation properties than T_0 in general. Indeed, it follows from the definition of the hull-kernel topology that a one-point subset $\{P\} \subseteq \mathrm{Prim}(A)$ is closed if and only if P is a maximal ideal of A. Note that when $\mathrm{Prim}(A)$ is non-Hausdorff, it is not necessarily true that compact subsets of $\mathrm{Prim}(A)$ are closed.

When A is a commutative C*-algebra, the Gelfand-Naimark Theorem asserts that $A \equiv C_0(X)$ for some locally compact Hausdorff space X. In this case, the irreducible representations of A are precisely the characters on A, and thus we may identify $\mathrm{Prim}(A)$ with X, where $x \in X$ is identified with the maximal ideal $\{f \in C_0(X) : f(x) = 0\}$ of A. Moreover, the hull-kernel topology on $\mathrm{Prim}(A)$ corresponds to the usual topology on X under this identification. In particular, $\mathrm{Prim}(A)$ is Hausdorff whenever A is a commutative C*-algebra.

Often when working with non-Hausdorff spaces such as $\mathrm{Prim}(A)$, it is convenient to construct a related space satisfying stronger separation axioms but preserving certain information about the original space. The *complete regularisation* of a topological space X is a procedure for associating a completely regular space ρX to X in such a way that their algebras of (bounded) continuous functions are isomorphic. We describe this construction below for general spaces X, and then return to the particular case of the complete regularisation of $\mathrm{Prim}(A)$.

Definition 2.1.3. A topological space X is said to be *completely regular* (or a *Tychonoff space*) if it is a Hausdorff space, and given any closed subset $F \subseteq X$ and a point $x \in X \backslash F$, there is a continuous function $f : X \to \mathbb{R}$ with $f(x) = 1$ and $f(F) = \{0\}$.

Theorem 2.1.4. *[29, Theorem 3.9] Let X be a topological space. Then there exist a completely regular space ρX and a continuous surjection $\rho_X : X \to \rho X$ with the property that the map $f \mapsto f \circ \rho_X$, where $f \in C^b(\rho X)$, is a $*$-isomorphism of $C^b(\rho X)$ onto $C^b(X)$.*

We give the details of the construction of ρX and ρ_X below, since we will refer to it in subsequent sections.

For $f \in C(X)$, let $\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$, the *cozero set* of f. Replacing f with $\min(|f|, 1)$, we may assume that any cozero set in X is the cozero set of some continuous function $f : X \to [0, 1]$.

Define an equivalence relation on X as follows: for $x_1, x_2 \in X$ write $x_1 \approx x_2$ if $f(x_1) = f(x_2)$ for all $f \in C^b(X)$. Let $\rho X = X/\approx$ and let $\rho_X : X \to \rho X$ be the quotient map. Each $f \in C^b(X)$ defines a function f^ρ on ρX by setting $f^\rho([x]) = f(x)$, where $[x]$ denotes the $\approx$-equivalence class of x. Denote by τ_{cr} the weak topology on ρX induced by the functions $\{f^\rho : f \in C^b(X)\}$. Then the space ρX with the topology τ_{cr} has the required properties.

Definition 2.1.5. The triple $(\rho X, \tau_{cr}, \rho_X)$ constructed in Theorem 2.1.4 is called the *complete regularisation* of the space X.

We shall often refer to ρX as the complete regularisation of X when ρ_X and τ_{cr} are understood.

An alternative construction of ρX is as follows: let I denote the closed unit interval and I^X the set of all continuous maps $f : X \to I$. Let $P(X) = \prod\{I_f : f \in I^X\}$, where $I_f = I$ for all $f \in I^X$, and let $\tau : X \to P(X)$ be defined via $\tau(x) = (f(x))_{f \in I^X}$.

Defining $I^{\rho X}$, $P(\rho X)$ and $\tau' : \rho X \to P(\rho X)$ analogously, it is clear that $I^{\rho X} = \{f^\rho : f \in I^X\}$ and hence $P(\rho X) = P(X)$. The definition of f^ρ, where $f \in I^X$, ensures that $\tau' \circ \rho_X = \tau$, hence $\tau(X)$ is equal to $\tau'(\rho X)$. On the other hand, it is well-known that τ' is a homeomorphism onto its image [55, Lemma 1.5], so that $\tau(X)$ is homeomorphic to ρX.

Now let X and Y be topological spaces and $\phi : X \to Y$ a continuous map. Then setting $\phi^\rho(\rho_X(x)) = (\rho_Y \circ \phi)(x)$ gives a map $\phi^\rho : \rho X \to \rho Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Y \\ {\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_Y} \\ \rho X & \xrightarrow[\ \phi^\rho\ ]{} & \rho Y \end{array}$$

commutes. To see that ϕ^ρ is continuous, let $U = \mathrm{coz}(f)$ be a cozero set in ρY. Then $(\rho_Y \circ \phi)^{-1}(U)$ is precisely the cozero set in X of the continuous function $f \circ \rho_Y \circ \phi$. It follows that $(\phi^\rho)^{-1}(U) = \mathrm{coz}(f \circ \rho_Y \circ \phi)^\rho$ is open in ρX by the definition of τ_{cr}.

Thus the assignment of ρX to X and ϕ^ρ to ϕ defines a covariant functor from the category of topological spaces to the subcategory of completely regular spaces, called the *Tychonoff functor*. The term Tychonoff functor was first used by Morita in [47, p. 32].

Now let A be a C*-algebra and Prim(A) the space of primitive ideals of A with the hull-kernel topology. We define Glimm(A) as the complete regularisation ρPrim(A) of Prim(A), and denote by $\rho_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ the complete regularisation map.

Let $p \in \mathrm{Glimm}(A)$ and choose $P \in \mathrm{Prim}(A)$ with $\rho_A(P) = p$. We associate to the point p the norm closed two sided ideal G_p of A given by

$$G_p = \bigcap\{Q \in \mathrm{Prim}(A) : Q \approx P\} = \bigcap\{Q \in \mathrm{Prim}(A) : \rho_A(Q) = p\} = k\left([P]\right).$$

Note that since $[P]$ is closed in Prim(A) and $G_p = k([P])$, each equivalence class in $\mathrm{Prim}(A)/\approx$ is of the form

$$[P] = \mathrm{hull}\left(k([P])\right) = \mathrm{hull}(G_p),$$

by the definition of the hull-kernel topology.

Definition 2.1.6. The ideals $\{G_p : p \in \mathrm{Glimm}(A)\}$ are known as the *Glimm ideals* of A. The space Glimm(A), with the topology τ_{cr} is called the *Glimm space* of the C*-algebra A.

Since the assignment $p \mapsto G_p$ is injective, we will regard elements of Glimm(A) as either points of a topological space or as ideals of A, depending on the context.

Example 2.1.7. *[7, p. 353] Let $\ell^\infty(M_2(\mathbb{C}))$ denote the C*-algebra of bounded sequences of 2×2 matrices over $\mathbb{C}$ (with pointwise operations and supremum norm). Let $A \subset \ell^\infty(M_2(\mathbb{C}))$ consist of those sequences $x = (x_n)$ with the property that, as $n \to \infty$, the subsequences (x_{2n}) and (x_{2n+1}) satisfy*

$$x_{2n} \to \mathrm{diag}(\lambda_1(x), \lambda_2(x)), \;\; x_{2n+1} \to \mathrm{diag}(\lambda_2(x), \lambda_3(x))$$

for some complex scalars $\lambda_i(x)$ $(1 \leq i \leq 3)$. Then A is a C-subalgebra of $\ell^\infty(M_2(\mathbb{C}))$.*

The irreducible representations of A consist of the evaluation maps $\varepsilon_n : A \to M_2(\mathbb{C})$, where $\varepsilon_n(x) = x_n$ for all $x \in A, n \in \mathbb{N}$, together with $\lambda_i : A \to \mathbb{C}$ for $i = 1, 2, 3$. Hence

$$\mathrm{Prim}(A) = \{\ker(\varepsilon_n) : n \in \mathbb{N}\} \cup \{\ker(\lambda_i) : 1 \leq i \leq 3\}.$$

The topology on Prim(A) *is as follows:*

(i) the points $\ker(\varepsilon_n)$ *are isolated for all* $n \in \mathbb{N}$,

(ii) *the point* $\ker(\lambda_1)$ *(respectively* $\ker(\lambda_3)$*) has a neighbourhood basis consisting of sets of the form* $\{\ker(\varepsilon_{2n}) : n \geq n_0\} \cup \{\ker(\lambda_1)\}$ *(respectively* $\{\ker(\varepsilon_{2n+1}) : n \geq n_0\} \cup \{\ker(\lambda_3)\}$*) for some* $n_0 \in \mathbb{N}$.

(iii) *the point* $\ker(\lambda_2)$ *has a neighbourhood basis consisting of sets of the form* $\{\ker(\varepsilon_n) : n \geq n_0\} \cup \{\ker(\lambda_2)\}$ *for some* $n_0 \in \mathbb{N}$.

It is easily seen that the equivalence classes of the relation $\approx$ *on* $\mathrm{Prim}(A)$ *satisfy* $[\ker(\varepsilon_n)] = \{\ker(\varepsilon_n)\}$ *for* $n \in \mathbb{N}$, *while* $\ker(\lambda_1) \approx \ker(\lambda_2) \approx \ker(\lambda_3)$. *Hence*

$$\mathrm{Glimm}(A) = \{\ker(\varepsilon_n) : n \in \mathbb{N}\} \cup \{\bigcap_{i=1}^{3} \ker(\lambda_i)\}$$

with topology homeomorphic to $\hat{\mathbb{N}}$.

2.2 C*-bundles

There are many different definitions of bundles or fields of C*-algebras appearing throughout the literature, although it is often the case that they are in fact equivalent. In this section we give two definitions of C*-bundles that will appear in subsequent sections, and give a short description of the relations between the two.

In the most general sense, a bundle is simply a surjective mapping $p : E \to X$, where X is called the *base space* and E as the *total space*. For each $x \in X$ we regard $p^{-1}(x)$ as the fibre of E over X, and thus think of E as the disjoint union of fibres $E = \coprod_{x \in X} p^{-1}(x)$. In our case, we will require that E carries a topology and algebraic operations in such a way that each fibre $p^{-1}(x)$ is a C*-algebra.

Definition 2.2.1 below is essentially due to Dauns and Hofmann [21], generalising an earlier definition of Fell [28]. Our terminology '(H)-C*-bundle' was used first by Dupré and Gillette [26].

Definition 2.2.1. Let $\mathcal{A}$ be a topological space and X a locally compact Hausdorff space. An *upper semicontinuous (H)-C*-bundle over* X is a triple $(\mathcal{A}, X, p)$ consisting of a continuous, open surjection $p : \mathcal{A} \to X$ together with $*$-algebraic operations and norms making each fibre $A_x := p^{-1}(\{x\})$ into a C*-algebra such that the following conditions are satisfied:

(i) The map $a \mapsto \|a\|_{A_x}$ is upper semicontinuous $\mathcal{A} \to \mathbb{R}$,

(ii) Involution $a \mapsto a^*$ is continuous $\mathcal{A} \to \mathcal{A}$,

(iii) The maps $(a, b) \mapsto ab$ and $(a, b) \mapsto a + b$ are continuous from $\mathcal{A}_p := \{(a, b) \in \mathcal{A} \times \mathcal{A} : p(a) = p(b)\}$ to $\mathcal{A}$,

(iv) For each $\lambda \in \mathbb{C}$, the map $a \mapsto \lambda a$ is continuous $\mathcal{A} \to \mathcal{A}$,

(v) If $\{a_i\}$ is a net in $\mathcal{A}$ such that $\|a_i\| \to 0$ and $p(a_i) \to x$ in X, then $a_i \to 0_x$, the zero element of A_x.

If we replace 'upper semicontinuous' with 'continuous' in (i) then $\mathcal{A}$ is called a *continuous (H)-C*-bundle over X*.

Remark 2.2.2. Fell's definition [28, p. 10] of a C*-bundle required that the norm functions in (i) be continuous. Our definition of a continuous (H)-C*-bundle is called an *(F)-C*-bundle* by Dupré and Gillette [26, p. 8], who reserve the term (H)-C*-bundle for upper-semicontinuous bundles.

For an open subset $U \subseteq X$, a (local) section γ of $(\mathcal{A}, X, p)$ is a map $\gamma : U \to \mathcal{A}$ such that $p(\gamma(y)) = y$ for all $y \in U$, i.e. $\gamma(y) \in A_y$ for all $y \in U$. The set of all bounded continuous sections $U \to \mathcal{A}$ is denoted by $\Gamma^b(U, \mathcal{A})$. The set of those $\gamma \in \Gamma^b(U, \mathcal{A})$ that vanish at infinity on U, in the sense that $\{y \in U : \|\gamma(y)\| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$, is denoted by $\Gamma_0(U, \mathcal{A})$. Then $\Gamma^b(U, \mathcal{A})$ and $\Gamma_0(U, \mathcal{A})$ are C*-algebras with pointwise operations and supremum norm, see e.g. [58, Proposition C.23] or [3, Lemma 5.2].

Lemma 2.2.3. *Let $(\mathcal{A}, X, p)$ be an upper-semicontinuous (H)-C*-bundle and $K \subseteq X$ an open, compact subset. Then every continuous section $\gamma : K \to \mathcal{A}$ is bounded.*

Proof. Since the map $\mathcal{A} \to \mathbb{R}_+$ sending $a \mapsto \|a\|_{A_x}$, where $a \in A_x$, is upper-semicontinuous on $\mathcal{A}$, it follows that the map $K \to \mathbb{R}_+$, $x \mapsto \|\gamma(x)\|$ is upper-semicontinuous on K. But then since K is compact, it follows that $x \mapsto \|\gamma(x)\|$ is bounded, i.e. γ is bounded. □

Example 2.2.4. *(i) Let X be a locally compact Hausdorff space and B a C*-algebra. Setting $\mathcal{A} = X \times B$ and $p : \mathcal{A} \to X$ the projection onto the first factor, the triple $(\mathcal{A}, X, p)$ is a continuous (H)-C*-bundle over X, with constant fibre $A_x = B$. A bundle of this type is called the trivial bundle over X with fibre B.*

It is easily verified that the section algebras in this case are canonically $*$-isomorphic $\Gamma_0(X, \mathcal{A}) \equiv C_0(X, B)$ and $\Gamma^b(X, \mathcal{A}) \equiv C^b(X, B)$ (where the latter are considered as C-algebras with pointwise operations and the supremum norm).*

(ii) *With $\mathcal{A}$ as in (i), fix $x_0 \in X$ and a C*-subalgebra $B_0 \subseteq B$. Let $\mathcal{A}_0$ be the subspace of $\mathcal{A}$ given by*

$$\mathcal{A}_0 = \{(x, b) \in \mathcal{A} : b \in B_0 \text{ if } x = x_0\}.$$

Setting $p_0 = p|_{\mathcal{A}_0}$, the triple $(\mathcal{A}_0, X, p_0)$ is again a continuous C-bundle over X, whose section algebras satisfy*

$$\Gamma_0(X, \mathcal{A}_0) \equiv \{f \in C_0(X, B) : f(x_0) \in B_0\},$$

and similarly for $\Gamma^b(X, \mathcal{A}_0)$.

Beyond trivial cases, it is not immediately obvious how C*-bundles arise. Suppose that we are given a locally compact Hausdorff space X and a collection of C*-algebras $\{A_x : x \in X\}$. Setting $\mathcal{A} = \coprod\{A_x : x \in X\}$ gives a well-defined surjection $p : \mathcal{A} \to X$ via $p(a) = x$ whenever $a \in A_x$. We wish to determine under what conditions does this construction equip the triple $(\mathcal{A}, X, p)$ with the structure of an upper-semicontinuous (H)-C*-bundle.

Theorem 2.2.5 below asserts that given a sufficiently large collection Γ of sections $X \to \mathcal{A}$ with suitable properties, there is a *unique* topology on $\mathcal{A}$ such that $(\mathcal{A}, X, p)$ is an upper-semicontinuous (H)-C*-bundle, such that every section in Γ becomes a continuous section $X \to \mathcal{A}$. The original result, for continuous (H)-C*-bundles, is due to Fell [28, Proposition 1.6]. The fact that the analogous result holds in the upper-semicontinuous case was observed by Dupré and Gillette [26, Proposition 1.9], though no proof was given. A self-contained proof of the general (i.e. upper-semicontinuous) case may be found in [58, Theorem C.25].

Theorem 2.2.5. (Fell) *Let X be a locally compact Hausdorff space and $\mathcal{A} = \{A_x : x \in X\}$ a collection of C*-algebras. Define $p : \mathcal{A} \to X$ via $p(a) = x$ for $a \in A_x$. Let Γ be a $*$-algebra of sections $X \to \mathcal{A}$ (with pointwise operations), such that the following conditions hold:*

(i) *the map $x \mapsto \|\gamma(x)\|$ is upper semicontinuous on X for all $\gamma \in \Gamma$,*

(ii) *for each $x_0 \in X$, the set $\{\gamma(x_0) : \gamma \in \Gamma\}$ is dense in A_{x_0}.*

Then there is a unique topology on $\mathcal{A}$ such that $(\mathcal{A}, X, p)$ is an upper-semicontinuous (H)-C-bundle over X with $\Gamma \subseteq \Gamma^b(X, \mathcal{A})$. If we replace 'upper-semicontinuous' with 'continuous' in (i), then $(\mathcal{A}, X, p)$ is a continuous (H)-C*-bundle.*

In fact, under some additional conditions on the family Γ of sections, Proposition 2.2.6 shows that we may conclude $\Gamma = \Gamma_0(\mathcal{A})$. Again, this result was shown originally by Fell [28, Proposition 1.7] in the continuous case, the upper-semicontinuous analogue may be found in [58, Proposition C.24].

Proposition 2.2.6. *Suppose that $(\mathcal{A}, X, p)$ is an upper-semicontinuous (H)-C*-bundle over X, and that $\Gamma \subseteq \Gamma_0(X, \mathcal{A})$ is a subspace such that*

(i) $\gamma \in \Gamma$ and $f \in C_0(X)$ implies that $f \cdot \gamma : x \mapsto f(x)\gamma(x)$ belongs to Γ, and

(ii) for each $x_0 \in X$, $\{\gamma(x_0) : \gamma \in \Gamma\}$ is dense in A_{x_0}.

Then Γ is dense in $\Gamma_0(X, \mathcal{A})$.

As a consequence of Theorem 2.2.5 and Proposition 2.2.6, we see that it is possible in certain cases to construct a (H)-C*-bundle by specifying the base space X, fibre algebras A_x and a suitable C*-algebra of sections A. In particular, it is possible to avoid making reference to the topology on the total space (since this is uniquely determined by the section algebra by Theorem 2.2.5). We shall favour this approach throughout much of this work for simplicity.

Motivated by these considerations, we make the following definition of a C*-bundle:

Definition 2.2.7. A C*-bundle is a triple $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ where X is a locally compact Hausdorff space, A a C*-algebra, and $\pi_x : A \to A_x$ surjective *-homomorphisms for all $x \in X$ satisfying

(i) the family $\{\pi_x : x \in X\}$ is faithful, i.e., $\bigcap_{x \in X} \ker(\pi_x) = \{0\}$, and

(ii) for each $f \in C_0(X)$ and $a \in A$ there is an element $f \cdot a \in A$ with the property that
$$\pi_x(f \cdot a) = f(x)\pi_x(a) \text{ for all } x \in X.$$

If in addition the functions $N(a) : X \to \mathbb{R}_+$, $x \mapsto \|\pi_x(a)\|$ where $a \in A$, belong to $C_0(X)$ for all $a \in A$ then we say that $\mathscr{A}$ is a *continuous C*-bundle* over X. If for all $a \in A$ the functions $N(a)$ are upper-semicontinuous (resp. lower-semicontinuous) on X, and if for each $\varepsilon > 0$ the set $\{x \in X : N(a) \geq \varepsilon\}$ has compact closure in X, then we say that $\mathscr{A}$ is an *upper-semicontinuous C*-bundle* (resp. lower-semicontinuous C*-bundle).

Remark 2.2.8. A continuous C*-bundle in the sense of Definition 2.2.7 was called a 'maximal full algebra of operator fields' by Fell [27, §1.1]. Similarly, our notion of a semicontinuous C*-bundle corresponds to Rieffel's 'semicontinuous fields of C*-algebras' [52, Definition 1.1]. The terminology 'bundle' has mostly replaced 'field' in more recent literature, e.g. [40], [48], [6].

The following Theorem makes precise the equivalence between upper-semicontinuous (H)-C*-bundles and C*-bundles.

Theorem 2.2.9. *Let X be a locally compact Hausdorff space.*

(i) Let $(\mathcal{A}, X, p)$ be an upper-semicontinuous (H)-C-bundle over X. Then setting $A = \Gamma_0(X, \mathcal{A})$, $A_x = p^{-1}(x)$ and $\pi_x : A \to A_x$ the evaluation mapping for all $x \in X$, the triple $(X, A, \pi_x : A \to A_x)$ is an upper-semicontinuous C*-bundle over X. Moreover if $(\mathcal{A}, X, p)$ is continuous then so is $(X, A, \pi_x : A \to A_x)$.*

(ii) Let $(X, A, \pi_x : A \to A_x)$ be an upper-semicontinuous C-bundle over X. Set $\mathcal{A} = \coprod\{A_x : x \in X\}$, $p : \mathcal{A} \to X$ the surjection sending each $a \in A_x$ to x, and for each $a \in A$ define a section $\hat{a} : X \to \mathcal{A}$ via $\hat{a}(x) = \pi_x(a)$. Then there is a unique topology on $\mathcal{A}$ such that $(\mathcal{A}, X, p)$ is an upper-semicontinuous (H)-C*-bundle over X and such that the map $a \mapsto \hat{a}$ is a $*$-isomorphism of A onto $\Gamma_0(X, \mathcal{A})$. If $(X, A, \pi_x : A \to A_x)$ is continuous then so is $(\mathcal{A}, X, p)$.*

Proof. (i): Since the norm on $\Gamma_0(X, \mathcal{A})$ is the supremum norm, it is clear that condition (i) of Definition 2.2.7 is satisfied. For all $\gamma \in \Gamma_0(X, \mathcal{A})$ and $f \in C_0(X)$, the section $f \cdot \gamma : x \mapsto f(x)\gamma(x)$ belongs to $\Gamma_0(X, \mathcal{A})$ by [58, Lemma C.22]. It follows from the definition of $f \cdot \gamma$ that $\pi_x(f \cdot \gamma) = f(x)\gamma(x) = f(x)\pi_x(\gamma)$ for all $x \in X$, hence condition (ii) of Definition 2.2.7 is satisfied also.

For each $\gamma \in \Gamma_0(X, \mathcal{A})$, the norm function $N(\gamma)$ is the composition $x \mapsto \gamma(x) \mapsto \|\gamma(x)\|$ of a continuous function and an upper-semicontinuous function, hence is upper-semicontinuous on X. Moreover, $N(\gamma)$ vanishes at infinity on X by definition of $\Gamma_0(X, \mathcal{A})$.

The final assertion follows from the fact that in a continuous (H)-C*-bundle, the map $\gamma(x) \mapsto \|\gamma(x)\|$ is continuous.

(ii): By Theorem 2.2.5, there is a unique topology on $\mathcal{A}$ such that $(\mathcal{A}, X, p)$ is an upper-semicontinuous (H)-C*-bundle over X with $\{\hat{a} : a \in A\} \subseteq \Gamma^b(X, \mathcal{A})$. Moreover, by assumption the former is in fact contained in $\Gamma_0(X, \mathcal{A})$.

Using condition (ii) of Defintion 2.2.7, we have

$$(f \cdot a)^\wedge(x) = \pi_x(f \cdot a) = f(x)\pi_x(a) = f(x)\hat{a}(x)$$

for all $a \in A$ and $f \in C_0(X)$. In particular, the hypotheses of Proposition 2.2.6 are satisfied, so that $\{\hat{a} : a \in A\}$ is dense in $\Gamma_0(X, \mathcal{A})$. Being closed, we must have equality.

The continuous case follows from the continuous cases of Theorem 2.2.5 and Proposition 2.2.6. □

In subsequent sections, we shall mostly make use of C*-bundles, rather than (H)-C*-bundles. One exception is Chapter 4, where it will be necessary to consider the local section algebras $\Gamma^b(U, \mathcal{A})$ for open subsets U of X. The fact that these sections do not necessarily vanish at infinity on U means that we are unable to make use of the equivalence given in Theorem 2.2.9(ii), which only applies to $\Gamma_0(-, \mathcal{A})$.

Definition 2.2.10. Let $(X, A, \pi_x : A \to A_x)$ and $(X, B, \sigma_x : B \to B_x)$ be C*-bundles over X. Suppose that the following conditions are satisfied

(i) B is a C*-subalgebra of A,

(ii) for each $x \in X$, $\sigma_x = \pi_x|_B$.

Then $(X, B, \sigma_x : B \to B_x)$ is said to be a C*-sub-bundle of $(X, A, \pi_x : A \to A_x)$.

2.3 Equivalence of $C_0(X)$-algebras and C*-bundles

Definition 2.3.1. Let X be a locally compact Hausdorff space. A *$C_0(X)$-algebra* is a C*-algebra A together with a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ with the property that $\mu_A(C_0(X))A = A$.

It follows from the Dauns-Hofmann Theorem [21] which we will discuss below (Theorem 2.3.6), that there is a $*$-isomorphism $\theta_A : C^b(\mathrm{Prim}(A)) \to ZM(A)$ with the property that

$$\theta_A(f)a + P = f(P)(a + P), \text{ for } a \in A, f \in C^b(\mathrm{Prim}(A)), P \in \mathrm{Prim}(A). \quad (2.3.1)$$

This gives an equivalent formulation of Definition 2.3.1: a $C_0(X)$-algebra is a C*-algebra A together with a continuous map $\phi_A : \mathrm{Prim}(A) \to X$. The maps μ_A and ϕ_A are related via $\mu_A(f) = \theta_A(f \circ \phi_A)$ for all $f \in C_0(X)$ [58, Proposition C.5]. We call ϕ_A the *base map* and μ_A the *structure map*.

For clarity we will denote any $C_0(X)$-algebra A by the triple (A, X, ϕ_A) or (A, X, μ_A). For $x \in X$ we define the ideal J_x via

$$J_x = \mu_A\left(\{f \in C_0(X) : f(x) = 0\}\right) A = \bigcap \{P \in \mathrm{Prim}(A) : \phi_A(P) = x\}, \qquad (2.3.2)$$

see [48, Section 2] for example.

Remark 2.3.2. We do not require that the base map $\phi_A : \mathrm{Prim}(A) \to X$ be surjective, or even that $\phi_A(\mathrm{Prim}(A))$ be dense in X. It is shown in [8, Corollary 1.3] that ϕ_A has dense range if and only if the structure map μ_A is injective.

If $x \in X \backslash \mathrm{Im}(\phi_A)$, then we may still define the ideal J_x of A via $J_x = \mu_A\left(\{f \in C_0(X) : f(x) = 0\}\right) A$; it is shown in [8, §1] that $J_x = A$ for all such x. This is consistent with our second definition of J_x in (2.3.2), when we regard the intersection of the empty set $\{P \in \mathrm{Prim}(A) : \phi_A(P) = x\}$ of ideals of A as A itself.

The relationship between $C_0(X)$-algebras and C*-bundles is well known, see [48] or [58, Appendix C] for example. We give details in the following proposition, which will be used frequently in what follows.

Proposition 2.3.3. *Let A be a C*-algebra and X a locally compact Hausdorff space.*

(i) If (A, X, μ_A) is a $C_0(X)$-algebra, then with $A_x = A/J_x$ and $\pi_x : A \to A_x$ the quotient $*$-homomorphism, the triple $(X, A, \pi_x : A \to A_x)$ is an upper-semicontinuous C-bundle [48, Theorem 2.3].*

(ii) If $(X, A, \pi_x : A \to A_x)$ is a C-bundle, then setting $\mu_A(f)a = f \cdot a$ for $f \in C_0(X), a \in A$ defines a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ such that (A, X, μ_A) is a $C_0(X)$-algebra. Moreover, $(X, A, \pi_x : A \to A_x)$ is an upper-semicontinuous C*-bundle if and only if $\ker(\pi_x) = J_x$ for all $x \in X$ [40, Lemmas 2.1 and 2.3].*

(iii) The $C_0(X)$-algebra (A, X, μ_A) gives rise to a continuous C-bundle $(X, A, \pi_x : A \to A_x)$ if and only if the corresponding base map $\phi_A : \mathrm{Prim}(A) \to X$ is an open map [44, Theorem 5].*

As a consequence of Proposition 2.3.3, we will regard $C_0(X)$-algebras and upper-semicontinuous C*-bundles as being (essentially) equivalent. Moreover, we may unambiguously speak of a $C_0(X)$-algebra (A, X, μ_A) being *continuous* if the corresponding C*-bundle $(X, A, \pi_x : A \to A_x)$ is continuous.

For clarity, we give the (H)-C*-bundle analogue of Proposition 2.3.3 below.

Corollary 2.3.4. *Let (A, X, μ_A) be a $C_0(X)$-algebra. Then there is an upper-semicontinuous (H)-C*-bundle $(\mathcal{A}, X, p)$ over X such that:*

(i) for each $x \in X$ the corresponding fibre algebra is given by $A_x = A/J_x$,

(ii) the map $\gamma : A \to \Gamma_0(X, \mathcal{A})$, where

$$\gamma(a)(x) = a + J_x \text{ for all } a \in A, x \in X$$

is a $*$-isomorphism.

Example 2.3.5. *[24, 4.7.19] Let A be the C*-algebra of sequences $a = (a_n) \subset M_2(\mathbb{C})$ that converge as $n \to \infty$ to a diagonal matrix, denoted by $\mathrm{diag}(\lambda_1(a), \lambda_2(a))$, where $\lambda_i(a)$ are complex scalars for $i = 1, 2$. Then $Z(A)$ consists of those $a \in A$ such that a_n is a scalar multiple of the identity matrix for all $n \in \mathbb{N}$, so that $\lambda_1(a) = \lambda_2(a)$ and $Z(A)$ may be identified with $C(\hat{\mathbb{N}})$. It follows that A is a $C(\hat{\mathbb{N}})$-algebra.*

The corresponding ideals $\{J_n : n \in \hat{\mathbb{N}}\}$ of (2.3.2) are given by $J_n = \{a \in A : a_n = 0\}$ for $n \in \mathbb{N}$ and $J_\infty = \{a \in A : \lambda_1(a) = \lambda_2(a) = 0\}$. The corresponding C-bundle $(A, \hat{\mathbb{N}}, \pi_n : A \to A_n)$ has fibres $A_n = M_2(\mathbb{C})$ for $n \in \mathbb{N}$ and $A_\infty = \mathbb{C} \oplus \mathbb{C}$, with $\pi_n(a) = a_n$ for $n \in \mathbb{N}$ and $\pi_\infty(a) = \lambda_1(a) \oplus \lambda_2(a)$ for all $a \in A$. Moreover, $(A, \hat{\mathbb{N}}, \pi_n : A \to A_n)$ is a continuous C*-bundle since the sequences (a_n) converge in norm.*

Given a C*-algebra A, it is natural to ask how one may construct a space Y and fibre algebras A_y in such a way that $(Y, A, \pi_y : A \to A_y)$ is a C*-bundle. Of course, taking $Y = \{y\}$ to be a one-point space and $\pi_y : A \to A_y = A$ the identity $*$-homomorphism gives a continuous C*-bundle decomposition of A in a trivial way.

At the other extreme, if $\mathrm{Prim}(A)$ is Hausdorff, then being also locally compact, we may take the identity map on $\mathrm{Prim}(A)$ as a base map. It then follows that the triple $(A, \mathrm{Prim}(A), \theta_A|_{C_0(\mathrm{Prim}(A))})$, where θ_A is the Dauns-Hofmann $*$-isomorphism of (2.3.1), is a continuous $C_0(\mathrm{Prim}(A))$-algebra. Thus by Proposition 2.3.3 we get a continuous C*-bundle $(\mathrm{Prim}(A), A, \pi_P : A \to A/P)$ over $\mathrm{Prim}(A)$, and it is easily verified that the fibre algebras are simple C*-algebras in this case. This result was obtained originally by Fell in [27].

The most general decomposition Theorem of this type was given by Dauns and Hofmann in [21], who showed that *any* C*-algebra A may be decomposed as an upper-semicontinuous C*-bundle over its space of Glimm ideals.

Theorem 2.3.6. *(J. Dauns, K.H. Hofmann) Let A be a C*-algebra. Then A is a $C_0(Y)$-algebra, and hence the section algebra of an upper-semicontinuous C*-bundle $(Y, A, \pi_p : A \to A_p)$, where*

(i) if $\mathrm{Glimm}(A)$ *is locally compact,* $Y = \mathrm{Glimm}(A)$, $A_p = A/G_p$ *and* $\pi_p = q_p : A \to A/G_p$ *the quotient $*$-homomorphism for all* $p \in \mathrm{Glimm}(A)$,

(ii) if $\mathrm{Glimm}(A)$ *is not locally compact,* $Y = \beta\mathrm{Glimm}(A)$, *and*

- *for* $p \in \mathrm{Glimm}(A)$, $A_p = A/G_p$ *and* $\pi_p = q_p : A \to A/G_p$ *the quotient $*$-homorphism, and*
- *for* $p \in \beta\mathrm{Glimm}(A)\backslash\mathrm{Glimm}(A)$, $A_p = \{0\}$.

If $\mathrm{Prim}(A)$ is Hausdorff, then being locally compact, it is necessarily completely regular. Thus $\mathrm{Prim}(A) = \mathrm{Glimm}(A)$, both as sets of ideals and topologically. In this case the Dauns-Hofmann bundle associated with A is precisely the continuous C*-bundle over $\mathrm{Prim}(A)$ obtained by Fell [27, Theorem 2.3]. More generally, Lee's theorem [44, Theorem 4] implies that the Dauns-Hofmann bundle of a C*-algebra A is a continuous C*-bundle if and only if the complete regularisation map is open. Note that if this is the case then necessarily $\mathrm{Glimm}(A)$ is locally compact.

Example 2.3.7. *Let A be the C*-algebra of Example 2.1.7. Then since* $\mathrm{Glimm}(A)$ *is homeomorphic to $\hat{\mathbb{N}}$, A is a $C(\hat{\mathbb{N}})$-algebra by Theorem 2.3.6. The fibre algebras of the corresponding C*-bundle are given by $A_n = M_2(\mathbb{C})$ for $n \in \mathbb{N}$ and $A_\infty = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, where $\pi_n(a) = a_n$ for $n \in \mathbb{N}$ and $\pi_\infty(a) = \lambda_1(a) \oplus \lambda_2(a) \oplus \lambda_3(a)$ for all $a \in A$.*

This bundle is upper-semicontinuous but fails to be continuous. To see this, consider the sequence $a = (a_n)$ with $a_{2n} = \mathrm{diag}(1, 0)$ and $a_{2n+1} = 0$ for all $n \in \mathbb{N}$. Then clearly $a \in A$, and for all $n \in \mathbb{N}$ we have $\|\pi_{2n}(a)\| = \|\pi_\infty(a)\| = 1$, while $\|\pi_{2n+1}(a)\| = 0$ for all $n \in \mathbb{N}$. It follows that $n \mapsto \|\pi_n(a)\|$ is discontinuous at infinity.

In the following example, from [21, Example 9.2], we give a C*-algebra A for which the local compactness of $\mathrm{Prim}(A)$ does not pass to $\mathrm{Glimm}(A)$. In particular, we describe how $(A, \mathrm{Glimm}(A), \mu_A)$ fails to be a $C_0(\mathrm{Glimm}(A))$-algebra in this case.

Example 2.3.8. *Let H be a separable infinite dimensional Hilbert space, $\{e_n : n = 1, 2, \ldots\}$ a fixed orthonormal basis for H, and let $K(H)$ denote the compact operators on H. The subset $D(H)$ of all compact operators that are diagonal with respect to the basis $\{e_n\}$ is a C*-subalgebra of $K(H)$. Let*

$$A = \{F \in C([-1,1], K(H)) : F(t) \in D(H) \text{ for all } t \geq 0\}.$$

With pointwise operations and norm $\|F\| = \sup\{\|F(t)\| : t \in [-1,1]\}$, A is a C-algebra. Each element F of A is given by an infinite matrix of continuous functions $F_{i,j} : [-1,1] \to \mathbb{C}$, such that*

$$F(t)e_j = \sum \{F_{i,j}(t)e_i : i = 1, 2, \ldots\}.$$

The set $\mathrm{Prim}(A)$ consists of the ideals

$$\begin{aligned} P(t) &= \{F \in A : F(t) = 0\} \text{ for } t < 0 \\ P(t,n) &= \{F \in A : F_{n,n}(t) = 0\} \text{ for } t \geq 0, n \in \mathbb{N} \end{aligned}$$

The topology on $\mathrm{Prim}(A)$ is as follows:

- *For $t < 0$, a neighbourhood basis for $P(t)$ is given by the sets*

$$N(t,\varepsilon) := \{P(q) : q < 0, |q - t| < \varepsilon\}, 0 < \varepsilon < |t|$$

- *For $t = 0$ and $n \in \mathbb{N}$, $P(0,n)$ has a neighbourhood basis of sets*

$$M(0,n,\varepsilon) := \{P(q) : -\varepsilon < q < 0\} \cup \{P(q,n) : 0 \leq q < \varepsilon\}, 0 < \varepsilon < 1$$

- *For $t > 0$ and $n \in \mathbb{N}$, a neighbourhood basis for $P(t,n)$ is given by the sets*

$$M(t,n,\varepsilon) := \{P(q,n) : q > 0, |q - t| < \varepsilon\}, 0 < \varepsilon < t$$

For each n let $I_n = [0,1] \times \{n\}$, and for $t \in [0,1]$ let $t^{(n)} = (t,n) \in I_n$. Then

$$\mathrm{Prim}(A) \equiv [-1,0) \cup (\bigcup_{n=1}^{\infty} I_n),$$

where each point $0^{(n)}$ has a neighbourhood basis of intervals $(-\varepsilon, 0) \cup [0^{(n)}, \varepsilon^{(n)})$. It is easy to see that for a pair of points $0^{(n)}$ and $0^{(m)}$, with $n \neq m$, any open neighbourhood of $0^{(n)}$ will intersect every open neighbourhood of $0^{(m)}$ in the subset $[-1,0)$.

The complete regularisation map $\rho : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ fixes the sets $[-1,0)$ and $(0^{(n)}, 1^{(n)}]$ for all n. The set $\{0^{(n)} : n \in \mathbb{N}\}$ is mapped to a single point which we will denote by 0 for convenience. Hence

$$\mathrm{Glimm}(A) = [-1,0] \cup \bigcup_{n=1}^{\infty} (0^{(n)}, 1^{(n)}],$$

where a neighbourhood basis of 0 is given by the sets $(-\delta_0, 0] \cup \bigcup_{n=1}^{\infty} (0^{(n)}, \delta_n^{(n)})$ where $0 < \delta_j < 1$ for all $j \geq 0$. All other points of $\mathrm{Glimm}(A)$ have a neighbourhood basis consisting of intervals. Moreover, the point 0 does not have a compact neighbourhood in $\mathrm{Glimm}(A)$. For a proof of these facts, see [21, Examples 3.4 and 9.2].

Consider the $*$-homomorphism $\mu_A : C_0(\mathrm{Glimm}(A)) \to ZM(A)$ obtained from the restriction $\theta_A \circ \rho_A^*|_{C_0(\mathrm{Glimm}(A))}$, where θ_A is the Dauns-Hofmann $*$-isomorphism and $\rho_A^* : C^b(\mathrm{Glimm}(A)) \to C^b(\mathrm{Prim}(A))$ the $*$-isomorphism $f \mapsto f \circ \rho_A$ of Theorem 2.1.4.

Since the point 0 does not have a compact neighbourhood in $\mathrm{Glimm}(A)$, it follows that every $f \in C_0(\mathrm{Glimm}(A))$ satisfies $f(0) = 0$. Hence for all $a \in A$ and $P \in \mathrm{hull}(G_0)$,

$$\mu_A(f)a + P = (f \circ \rho_A)(P)\,(a + P) = 0,$$

by (2.3.1). In particular, $\mu_A(C_0(\mathrm{Glimm}(A))) \cdot A \subseteq G_0$, a proper ideal of A, so that the triple $(A, \mathrm{Glimm}(A), \mu_A)$ fails to be a $C_0(\mathrm{Glimm}(A))$-algebra.

Remark 2.3.9. Our definition of a $C_0(X)$-algebra (Definition 2.3.1) requires that X be a locally compact Hausdorff space. While the space $\mathrm{Glimm}(A)$ of Example 2.3.8 fails to be locally compact, it is necessarily true that the non-unital commutative C*-algebra $C_0(\mathrm{Glimm}(A))$ may be identified with $C_0(X)$ for some locally compact, non-compact Hausdorff space X (where X is the maximal ideal space of $C_0(\mathrm{Glimm}(A))$). It is easily seen that X is homeomorphic to the subspace $\mathrm{Glimm}(A)\backslash\{0\}$ of $\mathrm{Glimm}(A)$ in this case, but that we still have $\mu_A(C_0(X)) \cdot A \subseteq G_0$, so that A fails to be a $C_0(X)$-algebra with respect to this construction.

If A is unital, then since $\mathrm{Prim}(A)$ is compact, the same is true for $\mathrm{Glimm}(A)$, and so the pathology of Example 2.3.8 cannot occur.

Definition 2.3.10. A C*-algebra A is called *quasi-central* if $P \not\supseteq Z(A)$ for any $P \in \mathrm{Prim}(A)$.

Clearly every unital C*-algebra A is quasi-central, since the unit of A belongs to $Z(A)$. Moreover, when A is a quasi-central C*-algebra, $\mathrm{Glimm}(A)$ is necessarily locally compact. This fact is well known, we give a proof in Proposition 2.3.11 below for completeness.

Proposition 2.3.11. *Let A be a quasi-central C*-algebra, and denote by $\theta_A : C^b(\mathrm{Prim}(A)) \to ZM(A)$ the Dauns-Hofmann $*$-isomorphism and by $\rho_A^* : C^b(\mathrm{Glimm}(A)) \to C^b(\mathrm{Prim}(A))$ the $*$-isomorphism induced by the complete regularisation map $\rho_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$. Then $\mathrm{Glimm}(A)$ is locally compact, and the restriction of the mapping $\theta_A \circ \rho_A^*$ to $C_0(\mathrm{Glimm}(A))$ is a $*$-isomorphism of $C_0(\mathrm{Glimm}(A))$ onto $Z(A)$.*

Proof. Let $p \in \mathrm{Glimm}(A)$ and denote by G_p the corresponding Glimm ideal of A. Then since $Z(A) \not\subseteq G_p$ there is some $z \in Z(A)$ with $\|z + G_p\| = 1$. Since $Z(A) \subseteq ZM(A)$, there is $f_z \in C^b(\mathrm{Glimm}(A))$ such that $\theta_A \circ \rho_A^*(f_z) = z$.

For any $P \in \mathrm{Prim}(A)$ and $a \in A$, (2.3.1) ensures that

$$(z + P)(a + P) = f_z(\rho_A(P))(a + P).$$

Since $z + P \in Z(A/P)$ it follows that

$$\|z + P\| = |f_z(\rho_A(P))|$$

for all $P \in \mathrm{Prim}(A)$.

Now, since $z \in A$, the set

$$\begin{aligned} K &:= \{P \in \mathrm{Prim}(A) : \|z + P\| \geq \frac{1}{2}\} \\ &= \{P \in \mathrm{Prim}(A) : |f_z(\rho_A(P))| \geq \frac{1}{2}\} \end{aligned}$$

is compact by [24, Proposition 3.3.7]. Moreover, it is easily seen that $\rho_A(K) = \{q \in \mathrm{Glimm}(A) : |f_z(q)| \geq \frac{1}{2}\}$, which is again compact. Since

$$|f_z(p)| = \|z + G_p\| = \|z + P\| = 1,$$

and f_z is continuous on Glimm(A), it follows that $\rho_A(K)$ is a compact neighbourhood of p in Glimm(A). Hence Glimm(A) is locally compact.

It is shown in [22, Proposition 1] that for a quasi-central C*-algebra A, $\theta_A(C_0(\mathrm{Prim}(A))) = Z(A)$. We will show that ρ_A^* maps $C_0(\mathrm{Glimm}(A))$ onto $C_0(\mathrm{Prim}(A))$, from which the second assertion will follow.

Let $f \in C_0(\mathrm{Prim}(A))$, then there is $f^\rho \in C^b(\mathrm{Glimm}(A))$ with $\rho_A^*(f^\rho) = f$. For any $\varepsilon > 0$, the set

$$K_\varepsilon := \{P \in \mathrm{Prim}(A) : |f(P)| \geq \varepsilon\}$$

is compact. Moreover, its image under ρ_A is compact and is precisely the set

$$\begin{aligned} \rho_A(K_\varepsilon) &= \{p \in \mathrm{Glimm}(A) : |f(P)| \geq \varepsilon \text{ for all } P \in \mathrm{hull}(G_p)\} \\ &= \{p \in \mathrm{Glimm}(A) : |f^\rho(p)| \geq \varepsilon\}. \end{aligned}$$

It follows that $f^\rho \in C_0(\mathrm{Glimm}(A))$. Thus we have shown that $\rho_A^*(C_0(\mathrm{Glimm}(A))) \supseteq C_0(\mathrm{Prim}(A))$.

Since $Z(A)$ is an ideal of $ZM(A)$, the identifications

$$\theta_A(C_0(\mathrm{Prim}(A))) = Z(A) \subseteq (\theta_A \circ \rho_A^*)(C_0(\mathrm{Glimm}(A))) \subseteq ZM(A)$$

ensure that $(\rho_A^*)^{-1}(C_0(\mathrm{Prim}(A)))$ is a closed ideal of $C_0(\mathrm{Glimm}(A))$. If this inclusion were strict, it would follow that there exists a nonempty closed subset $F \subseteq \mathrm{Glimm}(A)$ with the property that

$$(\rho_A^*)^{-1}(C_0(\mathrm{Prim}(A))) = \{f \in C_0(\mathrm{Glimm}(A)) : f|_F \equiv 0\}.$$

Take $z \in Z(A)$ and again denote by $f_z \in C_0(\mathrm{Glimm}(A))$ the unique function with $(\theta_A \circ \rho_A^*)(f_z) = z$. Then for any $P \in \mathrm{Prim}(A)$ with $\rho_A(P) \in F$ and any $a \in A$, we would have

$$\begin{aligned} (z+P)(a+P) &= (\theta_A \circ \rho_A^*)(f_z)(a+P) \\ &= f_z(\rho_A(P))(a+P) \\ &= 0+P. \end{aligned}$$

In particular, $z \in P$ for all such P, so that $P \supseteq Z(A)$, a contradiction. □

The following Theorem identifies when a subalgebra of a $C_0(X)$-algebra can be identified with a subbundle of the associated upper-semicontinuous C*-bundle.

Theorem 2.3.12. *Let (B, X, μ_B) be a $C_0(X)$-algebra, and $\iota : A \to B$ a $*$-monomorphism with the property that $\mu_B(f)\iota(a) \in \iota(A)$ for all $a \in A$ and $f \in C_0(X)$. Then*

(i) There is a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ with the property that

$$\iota(\mu_A(f)a) = \mu_B(f)\iota(a). \tag{2.3.3}$$

Hence (A, X, μ_A) is a $C_0(X)$-algebra, and ι is a $C_0(X)$-module map,

(ii) For $x \in X$ we let $I_x = \mu_A(\{f \in C_0(X) : f(x) = 0\})A$ and $J_x = \mu_B(\{f \in C_0(X) : f(x) = 0\})B$. Then we have $\iota(I_x) = J_x \cap \iota(A)$,

(iii) If B is a continuous $C_0(X)$-algebra then so is A.

Proof. (i) Denote by C the C*-subalgebra of $M(B)$ generated by $\iota(A)$ and $\mu_B(C_0(X))$. Since $\iota(A)$ is closed under multiplication by $\mu_B(C_0(X))$ by assumption, $\iota(A)$ is a closed two-sided ideal of C. Then by [20, Proposition 3.7(i)], there is a $*$-homomorphism $\sigma : C \to M(A)$ extending ι^{-1} on $\iota(A)$. Moreover, for $f \in C_0(X)$ and $a \in A$, we have

$$(\sigma \circ \mu_B)(f)a = \sigma(\mu_B(f)\iota(a)) = \sigma(\iota(a)\mu_B(f)) = a(\sigma \circ \mu_B)(f),$$

since $\sigma \circ \iota$ is the identity on A. As in [45, (3.1)], we see that $(\sigma \circ \mu_B)(C_0(X)) \subseteq ZM(A)$. Thus we get a $*$-homomorphism $\mu_A = \sigma \circ \mu_B : C_0(X) \to ZM(A)$. To see that μ_A is non-degenerate, let (f_λ) be an approximate identity for $C_0(X)$. Then since μ_B is non-degenerate, $\mu_B(f_\lambda)b \to b$ for all $b \in B$. In particular, for all $a \in A$ we have

$$\mu_A(f_\lambda)a = \sigma(\mu_B(f_\lambda)\iota(a)) \to \sigma(\iota(a)) = a,$$

which shows that $\mu_A(C_0(X))A$ is dense in A. By the Cohen-Hewitt factorisation Theorem, $\mu_A(C_0(X))A = A$.

To see that (2.3.3) holds, note that for $f \in C_0(X)$ and $a \in A$, we have

$$\begin{aligned}\iota(\mu_A(f)a) &= \iota((\sigma \circ \mu_B)(f)a) \\ &= \iota(\sigma(\mu_B(f)\iota(a))) \\ &= \mu_B(f)\iota(a)\end{aligned}$$

Thus (A, X, μ_A) has the required properties.

(ii) The inclusion $\iota(I_x) \subseteq J_x \cap \iota(A)$ follows from (2.3.3). Now let $a \in J_x \cap \iota(A)$ and $\varepsilon > 0$. By upper-semicontinuity there is a neighbourhood U of x in X, with compact closure $\overline{U}$, such that $\|\iota(a) + J_y\| < \varepsilon$ for all $y \in U$. Choose a continuous $f : X \to [0,1]$ such that $f(x) = 1$ and $f(X \backslash U) \equiv 0$, then $f \in C_0(X)$ since $\overline{U}$ is compact. Then

$$(1 - \mu_A(f))a + I_x = (a - f(x)a) + I_x = 0 + I_x,$$

so that $(1 - \mu_A(f))a \in I_x$.

Now we have

$$\|(1 - \mu_A(f))a - a\| = \|\mu_A(f)a\| = \|\sigma(\mu_B(f)\iota(a))\| = \|\mu_B(f)\iota(a)\|,$$

since σ is injective on $\iota(A)$. Moreover,

$$\begin{aligned} \|\mu_B(f)\iota(a)\| = \sup_{y \in X} \|\mu_B(f)\iota(a) + J_y\| &= \sup_{y \in X} \|f(y)(\iota(a) + J_y)\| \\ &= \sup_{y \in U} |f(y)| \, \|\iota(a) + J_y\| \\ &\leq \sup_{y \in U} \|\iota(a) + J_y\| \leq \varepsilon. \end{aligned}$$

Combining these facts, we see that

$$\|a + I_x\| \leq \|(1 - \mu_A(f))a - a\| = \|\mu_B(f)\iota(a)\| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and I_x closed, $a \in I_x$.

(iii) By [24, Corollary 1.8.4] and by part (ii) we may identify

$$\frac{A}{I_x} = \frac{\iota(A)}{\iota(I_x)} = \frac{\iota(A)}{J_x \cap \iota(A)} \equiv \frac{\iota(A) + J_x}{J_x} \subseteq \frac{B}{J_x}.$$

Hence for $a \in A$ and $x \in X$, $\|a + I_x\| = \|\iota(a) + J_x\|$. Since norm functions of elements of B are continuous on X, the same is true for A. □

Chapter 3

Multipliers of $C_0(X)$-algebras

In this chapter, we consider the problem of obtaining a C*-bundle decomposition of the multiplier algebra $M(A)$ of a C*-bundle $(X, A, \pi_x : A \to A_x)$. Central to this is the notion of the 'strict topology' on $M(A)$, and how it relates to the embedding of Prim(A) as an open subset of Prim$(M(A))$.

We examine two different approaches to obtaining such a bundle representation of $M(A)$ over βX, the Stone-Čech compactification of X. The first approach (Theorem 3.1.6) depends on extending each $*$-homomorphisms $\pi_x : A \to A_x$ to a (strictly continuous) $*$-homomorphisms $\tilde{\pi}_x : M(A) \to M(A_x)$. Indeed, this may be interpreted as a generalisation of a theorem of Akemann, Pedersen and Tomiyama [1, Theorem 3.3].

In the case that the bundle $(X, A, \pi_x : A \to A_x)$ arises from a $C_0(X)$-algebra (A, X, μ_A), then Archbold and Somerset have shown in [8] how $M(A)$ carries naturally the structure of a $C(\beta X)$-algebra. It follows that $M(A)$ may be decomposed as an upper-semicontinuous C*-bundle over βX.

A natural question that arises from these constructions is that of determining when these bundle decompositions of $M(A)$ agree. The final two sections of this chapter are dedicated to the study of this question. We give a partial characterisation of those C*-algebras for which this occurs in Section 3.4.

We remark that some of the main results of Section 3.4 were obtained independently by Archbold and Somerset, and have been published in [11]. In particular, Corollary 3.3.6 corresponds to [11, Proposition 2.6(i) and (iii)], and Corollary 3.4.7 to [11, Corollary 4.3]. Stronger versions of Corollary 3.4.8 and Theorem 3.4.10 were also given in [11, Corollary 4.4] and [11, Theorem 4.6] respectively. The results of Theorems 3.1.6 and 3.4.6, and Corollary 3.1.9, are new.

3.1 The strict topology on $M(A)$

Definition 3.1.1. Let A be a C*-algebra. The *strict topology* on $M(A)$ is the locally convex topology generated by the seminorms

$$\{b \mapsto \|ab\|, b \mapsto \|ba\| : a \in A\}.$$

If A is unital then $M(A) = A$ and the strict topology is equal to the norm topology. In general, $M(A)$ is the strict completion of A [20, Proposition 3.5 and 3.6].

Notation 3.1.2. *For a subset $S \subseteq M(A)$ we will denote by $\tilde{S}$ its closure in the strict topology on $M(A)$.*

Proposition 3.1.3. *([20, Proposition 3.8] and [8, Proposition 1.1(i)]) Let A and B be C*-algebras and $\pi : A \to B$ a $*$-homomorphism. Then π extends uniquely to a strictly continuous $*$-homomorphism $\tilde{\pi} : M(A) \to M(\pi(A))$ defined via*

$$\tilde{\pi}(m)\pi(a) = \pi(ma) \text{ and } \pi(a)\tilde{\pi}(m) = \pi(am)$$

for all $a \in A$ and $m \in M(A)$. Moreover, the kernel of $\tilde{\pi}$ satisfies

$$\ker \tilde{\pi} = (\ker(\pi))^{\sim} = \{b \in M(A) : bA + Ab \subseteq \ker(\pi)\}.$$

Note that Proposition 3.1.3 allows us to identify the strict closure in $M(A)$ of an ideal J of A. Indeed, letting $\pi : A \to A/J$ be the quotient $*$-homomorphism, its extension $\tilde{\pi} : M(A) \to M(A/J)$ has kernel

$$\tilde{J} = \{b \in M(A) : bA + Ab \subseteq J\}. \tag{3.1.1}$$

Proposition 3.1.4. *Let A be a C*-algebra.*

(i) If $P \in \mathrm{Prim}(A)$ then $\tilde{P} \in \mathrm{Prim}(M(A))$, and $\tilde{P}$ is the unique primitive ideal of $M(A)$ whose intersection with A is P.

(ii) *The map* $\mathrm{Prim}(A) \to \mathrm{Prim}(M(A)), P \mapsto \tilde{P}$ *is a homeomorphism of* $\mathrm{Prim}(A)$ *onto the dense open subspace*

$$\{Q \in \mathrm{Prim}(M(A)) : Q \not\supseteq A\}$$

of $\mathrm{Prim}(M(A))$.

Proof. (i): By [24, Proposition 2.11.2(ii)] there is a unique $Q \in \mathrm{Prim}(M(A))$ with $Q \cap A = P$. The fact that $Q = \tilde{P}$ is shown in [20, Lemma 6.1]. (ii): Is shown in [20, Proposition 6.2]. □

Remark 3.1.5. In the commutative case, the embedding map of Proposition 3.1.4(ii) is well understood. Indeed, let B be a non-unital commutative C*-algebra. Then by the Gelfand-Naimark Theorem, $B \equiv C_0(X)$ where X is a locally compact, non-compact Hausdorff space. Then $M(B) = C^b(X) = C(\beta X)$. For $x \in X$, let $M_x = \{f \in C_0(X) : f(x) = 0\}$, and for $x \in \beta X$, let $M^x = \{f \in C(\beta X) : f(x) = 0\}$. Then $\mathrm{Prim}(B) = \{M_x : x \in X\} \equiv X$ and $\mathrm{Prim}(M(B)) = \{M^x : x \in \beta X\} \equiv \beta X$. $\mathrm{Prim}(B)$ is embedded in $\mathrm{Prim}(M(B))$ via $M_x \mapsto M^x (x \in X)$, and it is easily seen that for each $x \in X$, M^x is the ideal $\tilde{M_x}$ of Proposition 3.1.3.

Theorem 3.1.6. *Let* $(X, A, \pi_x : A \to A_x)$ *be a C*-bundle. For* $x \in X$ *set* $M_x = \tilde{\pi}_x(M(A))$, *and for* $q \in \beta X \backslash X$ *let* $M_q = \{0\}$ *and let* $\tilde{\pi}_q$ *be the zero* $*$*-homomorphism. Then the triple* $(\beta X, M(A), \tilde{\pi}_x : M(A) \to M_x)$ *is a C*-bundle. If in addition* A *is* σ*-unital, then* $M_x = M(A_x)$ *for all* $x \in X$.

Proof. We show that conditions (i) and (ii) of Definition 2.2.7 are satisfied. To see that $\bigcap_{x \in \beta X} \ker(\tilde{\pi}_x) = \{0\}$, let $m \in M(A)$ and suppose that $m \in \bigcap_{x \in \beta X} \ker(\tilde{\pi}_x)$. Then for all $x \in X$ and $a \in A$ we have

$$\tilde{\pi}_x(m)\pi_x(a) = \pi_x(ma) = 0 \text{ and } \pi_x(a)\tilde{\pi}_x(m) = \pi_x(am) = 0.$$

It follows that $ma, am \in \ker(\pi_x)$ for all $x \in X$. Since $(X, A, \pi_x : A \to A_x)$ is a C*-bundle, this implies that $ma = am = 0$ for all $a \in A$, so that $m = 0$.

To see that condition (ii) is satisfied, we first note that by Proposition 2.3.3(ii) there is a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ such that $\mu_A(f)a = f \cdot a$ for all $f \in C_0(X)$ and $a \in A$. By Proposition 3.1.3, μ_A extends to a strictly continuous $*$-homomorphism $\tilde{\mu}_A : M(C_0(X)) = C(\beta X) \to ZM(A)$. For $g \in C(\beta X)$ and $m \in M(A)$, define $g \cdot m$ to be $\tilde{\mu}_A(g)m$.

We claim that $\tilde{\pi}_q(g \cdot m) = g(q)\tilde{\pi}_q(m)$ for all $g \in C(\beta X), m \in M(A)$ and $q \in \beta X$. Note that if $q \in \beta X \backslash X$ there is nothing to prove. Let $x \in X$ and $a \in A$, and note that a factorises as $a = f \cdot b$ for some $f \in C_0(X)$ and $b \in A$ by the Cohen-Hewitt factorisation theorem [25, Theorem 16.1]. Then since

$$(g \cdot m)(f \cdot b) = \tilde{\mu}_A(g)m\mu_A(f)b = \mu_A(gf)(mb) = (gf) \cdot (mb),$$

we have

$$\begin{aligned} \tilde{\pi}_x(g \cdot m)\pi_x(a) = \tilde{\pi}_x(g \cdot m)\pi_x(f \cdot b) &= \pi_x((gf) \cdot (mb)) \\ &= (gf)(x)\pi_x(mb) \\ &= g(x)f(x)\tilde{\pi}_x(m)\pi_x(b) \\ &= g(x)\tilde{\pi}_x(m)\pi_x(f \cdot b) \\ &= g(x)\tilde{\pi}_x(m)\pi_x(a) \end{aligned}$$

for all $x \in X$. Thus for all $a \in A, x \in X$, we have

$$(\tilde{\pi}_x(g \cdot m) - g(x)\tilde{\pi}_x(m))\, \pi_x(a) = 0.$$

Since A_x is an essential ideal of M_x [8, Proposition 1.1(ii) and (v)], it follows that

$$\tilde{\pi}_x(g \cdot m) = g(x)\tilde{\pi}_x(m)$$

for all $x \in X$, so condition (ii) follows.

Finally, if A is σ-unital then the $*$-homomorphisms $\tilde{\pi}_x : M(A) \to M(A_x)$ are necessarily surjective by [49, Theorem 10]. □

Remark 3.1.7. Note that if $(X, A, \pi_x : A \to A_x)$ is a C*-bundle and $x \in X$ is such that $A_x = \{0\}$, then π_x is the zero $*$-homomorphism, and hence $M_x = \tilde{\pi}_x(M(A)) = \{0\}$ also.

Theorem 3.1.6 makes no reference to continuity of the bundles under consideration. The following example shows that when $(X, A, \pi_n : A \to A_n)$ is a continuous C*-bundle, $(\beta X, M(A), \tilde{\pi}_x : M(A) \to M_x)$ may fail to be continuous.

Example 3.1.8. *Let H be a separable infinite dimensional Hilbert space and $A = C(\hat{\mathbb{N}}, K(H))$, regarded as a C*-bundle $(\hat{\mathbb{N}}, A, \pi_n : A \to A_n)$ where $A_n = K(H)$ and $\pi_n : A \to A_n$ the evaluation mappings for all $n \in \hat{\mathbb{N}}$. Then $M(A) = C(\hat{\mathbb{N}}, B(H)_{s*})$, where $B(H)_{s*}$ denotes $B(H)$ with the strong-$*$ topology [1, Corollary 3.5].*

In the C-bundle* $(\hat{\mathbb{N}}, M(A), \tilde{\pi}_n : M(A) \to M_n)$*, we have* $M_n = B(H)$ *and the maps* $\tilde{\pi}_n$ *are again the evaluation mappings for all* $n \in \hat{\mathbb{N}}$*. To see that* $(\hat{\mathbb{N}}, M(A), \tilde{\pi}_n : M(A) \to M_n)$ *is not a continuous C*-bundle, let* $\{e_n\}$ *be an orthonormal basis for* H *and let* p_n *be the projection onto the one-dimensional subspace corresponding to* e_n *for all* $n \in \mathbb{N}$*. Setting*

$$f(n) = \begin{cases} p_n & n \in \mathbb{N} \\ 0 & n = \infty \end{cases}$$

then f *is strong-*-continuous from* $\hat{\mathbb{N}} \to B(H)$*, hence* $f \in M(A)$*. But the norm function of* f *fails to be upper-semicontinuous at* ∞*, hence* $(\hat{\mathbb{N}}, M(A), \tilde{\pi}_n : M(A) \to M(A_n))$ *is not an upper-semicontinuous C*-bundle.*

The C*-bundle $(\hat{\mathbb{N}}, M(A), \tilde{\pi}_n : M(A) \to M(A)_n)$ of Example 3.1.8 is in fact lower-semicontinuous. This is a special case of Corollary 3.1.9 below.

Corollary 3.1.9. *Let* $(X, A, \pi_x : A \to A_x)$ *be a lower-semicontinuous C*-bundle. Then the bundle* $(M(A), \beta X, \tilde{\pi}_x : M(A) \to M_x)$ *of Theorem 3.1.6 is lower-semicontinuous.*

Proof. We remark that for a C*-algebra B and $m \in M(B)$, it always holds that

$$\|m\| = \sup\{\|bm\| : b \in B, \|b\| \leq 1\} = \sup\{\|mb\| : b \in B, \|b\| \leq 1\},$$

see [20, §2].

Now let $(X, A, \pi_x : A \to A_x)$ be a lower-semicontinuous C*-bundle and take $m \in M(A)$. Denote by $N(m) : \beta X \to \mathbb{R}_+$ the function $N(m)(x) = \|\tilde{\pi}_x(m)\|$, and note that for all $a \in A$ and $x \in X$ we have $N(a)(x) = \|\pi_x(a)\|$ since $\tilde{\pi}_x$ extends π_x.

We first show that $N(m)$ is lower semicontinuous on X. Note that for all $x \in X$, $M_x \subseteq M(A_x)$ and so

$$\begin{aligned} \|\tilde{\pi}_x(m)\| &= \sup\{\|\tilde{\pi}_x(m)b\| : b \in A_x, \|b\| \leq 1\} \\ &= \sup\{\|\tilde{\pi}_x(m)\pi_x(a)\| : a \in A, \|\pi_x(a)\| \leq 1\} \\ &= \sup\{\|\pi_x(ma)\| : a \in A, \|a\| \leq 1\}, \end{aligned}$$

since π_x maps the unit ball of A onto the unit ball of A_x. It follows that the norm function $N(m)$ restricted to X is the pointwise supremum of the collection $\{N(ma) : a \in A, \|a\| \leq 1\}$, which are lower-semicontinuous on X since $ma \in A$ for all $a \in A$. It follows that $N(m)$ is lower-semicontinuous on X [19, Theorem IV.6.2.4].

If $x \in \beta X \backslash X$, then $N(m)(x) = 0$, hence $N(m)$ is trivially lower-semicontinuous at these points. The fact that the sets $\{x \in \beta X : N(m)(x) \geq \varepsilon\}$ have compact closure in βX for all $m \in M(A), \varepsilon > 0$ is immediate from the fact that βX itself is compact. $\square$

Remark 3.1.10. We now describe how Proposition 3.1.4 allows us to construct $\mathrm{Glimm}(M(A))$ in terms of $\mathrm{Glimm}(A)$ (as a topological space). By applying the Dauns-Hofmann isomorphism to $M(A)$, we get a $*$-isomorphism $\theta_{M(A)} : C(\mathrm{Prim}(M(A))) \to ZM(A)$, again satisfying

$$\theta_{M(A)}(f)m + Q = f(Q)\,(m + Q)\,,$$

for $f \in C(\mathrm{Prim}(M(A))), m \in M(A), Q \in \mathrm{Prim}(M(A))$. Thus $C^b(\mathrm{Prim}(A))$ and $C(\mathrm{Prim}(M(A)))$ are $*$-isomorphic, and it follows that their structure spaces are homeomorphic. Together with the universal properties of the complete regularisation and Stone-Čech compactification, there is a commutative diagram of $*$-isomorphisms:

$$\begin{array}{ccccc}
C^b(\mathrm{Prim}(A)) & & \xrightarrow{\theta_{M(A)}^{-1}\circ\theta_A} & & C(\mathrm{Prim}(M(A))) \\
\downarrow{\scriptstyle \rho_A^*} & \searrow^{\theta_A} & & \swarrow^{\theta_{M(A)}} & \\
C^b(\mathrm{Glimm}(A)) & & ZM(A) & & \downarrow{\scriptstyle \rho_{M(A)}^*} \\
\downarrow & & & & \\
C(\beta\mathrm{Glimm}(A)) & \dashleftarrow & \cdots & \cdots & C(\mathrm{Glimm}(M(A)))
\end{array}$$

Since $\mathrm{Prim}(M(A))$ is compact, the same is true of $\mathrm{Glimm}(M(A))$, and thus the $*$-isomorphism $C(\mathrm{Glimm}(M(A)) \to C(\beta\mathrm{Glimm}(A))$ is dual to a homeomorphism $\iota : \beta\mathrm{Glimm}(A) \to \mathrm{Glimm}(M(A))$. Moreover, it is shown in [8, Proposition 4.7] that the map ι satisfies $\rho_{M(A)}(\tilde{P}) = \iota \circ \rho_A(P)$ for all $P \in \mathrm{Prim}(A)$. In particular the diagram

$$\begin{array}{ccc}
\mathrm{Prim}(A) & \xrightarrow{\sim} & \mathrm{Prim}(M(A)) \\
\downarrow{\scriptstyle \rho_A} & & \\
\mathrm{Glimm}(A) & & \downarrow{\scriptstyle \rho_{M(A)}} \\
\downarrow & & \\
\beta\mathrm{Glimm}(A) & \xrightarrow{\iota} & \mathrm{Glimm}(M(A))
\end{array}$$

commutes. The advantage of this fact is that, as a topological space, $\mathrm{Glimm}(M(A))$ depends only on the topological space $\mathrm{Glimm}(A)$ (which in turn is defined uniquely by $\mathrm{Prim}(A)$), and may thus be constructed without knowledge of $\mathrm{Prim}(M(A))\backslash\mathrm{Prim}(A)$.

3.2 The $C(\beta X)$-algebra associated with the multiplier algebra of a $C_0(X)$-algebra

In this section we describe how the multiplier algebra of a $C_0(X)$-algebra A may be regarded as a $C(\beta X)$-algebra in a natural way. This may be regarded as the $C_0(X)$-algebraic analogue of Theorem 3.1.6, although the upper-semicontinuous C*-bundle over βX defined by the $C(\beta X)$-algebra structure on $M(A)$ over βX (via Proposition 2.3.3) does not agree in general with that of Theorem 3.1.6.

If (A, X, μ_A) is a $C_0(X)$-algebra with structure map $\phi_A : \mathrm{Prim}(A) \to X$, then as in the proof of Theorem 3.1.6 we get a $*$-homomorphism $\tilde{\mu}_A : C(\beta X) \to ZM(A)$. This construction gives $M(A)$ the structure of a $C(\beta X)$-algebra, whose properties we describe in the following proposition.

Proposition 3.2.1. *[8, Proposition 1.2] Let (A, X, μ_A) be a $C_0(X)$-algebra with base map μ_A and structure map ϕ_A. Then setting $\mu_{M(A)}(f) = \theta_A(f \circ \phi_A)$ for $f \in C(\beta X)$, the triple $(M(A), \beta X, \mu_{M(A)})$ is a $C(\beta X)$-algebra. Moreover, we have the following properties:*

(i) for $f \in C_0(X)$, $\mu_{M(A)}(f) = \mu_A(f)$,

(ii) $\mu_{M(A)}(1) = 1_{M(A)}$,

(iii) Denoting by $\phi_{M(A)} : \mathrm{Prim}(M(A)) \to \beta X$ the base map associated with $\mu_{M(A)}$, we have $\phi_{M(A)}(\tilde{P}) = \phi_A(P)$ for all $P \in \mathrm{Prim}(A)$.

Remark 3.2.2. It is not immediately evident that the $*$-homomorphism $\mu_{M(A)} : C(\beta X) \to ZM(A)$ of Proposition 3.2.1 agrees with the strictly continuous extension $\tilde{\mu}_A : C(\beta X) \to ZM(A)$ of μ_A from Proposition 3.1.3. Indeed, for $g \in C(\beta X)$, $\tilde{\mu}_A(g)$ is defined by the action

$$\tilde{\mu}_A(g)\mu_A(f) = \mu_A(gf) \text{ for all } f \in C_0(X).$$

But then for any such f, we have

$$\begin{aligned} \mu_{M(A)}(g)\mu_A(f) &= \theta_A((g|_X) \circ \phi_A)\theta_A(f \circ \phi_A) \\ &= \theta_A((gf) \circ \phi_A) \\ &= \mu_A(gf), \end{aligned}$$

so that necessarily $\mu_{M(A)}(g) = \tilde{\mu}_A(g)$ for all $g \in C(\beta X)$.

Returning to the case of a $C_0(X)$-algebra (A, X, μ_A) and the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$, we now describe the relationship between their associated (upper-semicontinuous) C*-bundles. By analogy with the ideals J_x of A defined in (2.3.2), we define for $y \in \beta X$ the ideals H_y of $M(A)$ via

$$\begin{aligned} H_y &= \mu_{M(A)}(\{f \in C(\beta X : f(y) = 0\})\, M(A) && (3.2.1) \\ &= \bigcap \{Q \in \mathrm{Prim}(M(A)) : \phi_{M(A)}(Q) = y\}. && (3.2.2) \end{aligned}$$

We remark that for $x \in X$, the quotient morphism $q_x : A \to A/J_x$ gives rise to a strictly continuous $*$-homomorphism $\tilde{q}_x : M(A) \to M(A/J_x)$, whose kernel is the ideal $\tilde{J}_x$ of (3.1.1). Thus we may identify $M(A)/\tilde{J}_x$ canonically with $\tilde{q}_x(M(A)) \subseteq M(A/J_x)$ and write $M(A)/\tilde{J}_x \subseteq M(A/J_x)$, with equality holding whenever A is σ-unital, as in Theorem 3.1.6.

One might expect that for $x \in X$ the ideals H_x and $\tilde{J}_x$ are equal; however, this is not true in general. The following proposition lists some of the known relations between the ideals J_x, H_x and $\tilde{J}_x$.

Proposition 3.2.3. *[8, Proposition 1.3] Let (A, X, μ_A) be a $C_0(X)$-algebra. Then for each $x \in X$ the ideals J_x, H_x and $\tilde{J}_x$ satisfy the following relations:*

(i) $J_x \subseteq H_x \subseteq \tilde{J}_x$,

(ii) $H_x \cap A = \tilde{J}_x \cap A = J_x$, and

(iii) $\tilde{J}_x = H_x$ if and only if H_x is strictly closed in $M(A)$.

With (A, X, μ_A) a $C_0(X)$-algebra and $(M(A), \beta X, \mu_{M(A)})$ the $C(\beta X)$-algebra of Proposition 3.2.1, we get upper-semicontinuous C*-bundles over X and βX respectively by Proposition 2.3.3(i). The following theorem describes the relationship between the two, and in particular, we compare the bundle associated with $(M(A), \beta X, \mu_{M(A)})$ and the bundle $(\beta X, M(A), \tilde{\pi}_x : M(A) \to M_x)$ of Theorem 3.1.6.

Theorem 3.2.4. *Let (A, X, μ_A) be a $C_0(X)$-algebra and $(M(A), \beta X, \mu_{M(A)})$ the $C(\beta X)$-algebra of Proposition 3.2.1. Denote by $(X, A, \pi_x : A \to A_x)$ and $(\beta X, M(A), \sigma_y : M(A) \to M(A)_y)$ the upper-semicontinuous C*-bundles of Proposition 2.3.3(i) associated with (A, X, μ_A) and $(M(A), \beta X, \mu_{M(A)})$ respectively. Then $(X, A, \pi_x : A \to A_x)$ and $(\beta X, M(A), \sigma_y : M(A) \to M(A)_y)$ satisfy the following relations:*

(i) The fibre algebras of $(\beta X, M(A), \sigma_y : M(A) \to M(A)_y)$ are given by $M(A)_y = M(A)/H_y$ and the $*$-homomorphisms $\sigma_y : M(A) \to M(A)_y$ are the quotient $*$-homomorphisms for all $y \in \beta X$.

(ii) Extend $(X, A, \pi_x : A \to A_x)$ to a bundle $(\beta X, A, \pi_y : A \to A_y)$, where for $y \in \beta X \backslash X$, we set $A_y = 0$ and let $\pi_y : A \to A_y$ be the zero $*$-homomorphism. Then $(\beta X, A, \pi_y : A \to A_y)$ is a subbundle of $(\beta X, M(A), \sigma_y : M(A) \to M(A)_y)$.

(iii) For $x \in X$, A_x is an essential ideal of $M(A)_x$ if and only if H_x is strictly closed in $M(A)$. Hence we may regard $M(A)_x \subseteq M(A_x)$ if and only if H_x is strictly closed in $M(A)$.

If in addition A is σ-unital, then assertion (iii) becomes

(iii)' for $x \in X$ we may make the identification $M(A)_x = M(A_x)$ if and only if H_x is strictly closed in $M(A)$.

Proof. Assertion (i) follows from the fact that $(M(A), \beta X, \mu_{M(A)})$ is a $C(\beta X)$-algebra, and the construction of the upper-semicontinuous C*-bundle $(\beta X, M(A), \sigma_y : M(A) \to M(A)_y)$ in Proposition 2.3.3(i).

By construction, the image of $\mu_{M(A)}$ is contained in $ZM(A)$, and thus we have $\mu_{M(A)}(C(\beta X)) \cdot A \subseteq A$. Assertion (ii) then follows from Theorem 2.3.12.

Since for each $x \in X$ we have $H_x \cap A = \tilde{J}_x \cap A = J_x$ by Proposition 3.2.3(ii), [24, Proposition 1.8.4] shows that we may identify

$$\frac{A}{J_x} \equiv \frac{A + \tilde{J}_x}{\tilde{J}_x} \lhd \frac{M(A)}{\tilde{J}_x}$$

and

$$\frac{A}{J_x} \equiv \frac{A + H_x}{H_x} \lhd \frac{M(A)}{H_x}.$$

Thus by [20, Proposition 3.7], we get $*$-homomorphisms

$$\sigma : \frac{M(A)}{H_x} \to M(\frac{A}{J_x}) \text{ and } \pi : \frac{M(A)}{\tilde{J}_x} \to M(\frac{A}{J_x}),$$

extending the identity on the (copies of) A/J_x in each case. Moreover, since by [8, Proposition 1.1(v)], $\frac{A+\tilde{J}_x}{\tilde{J}_x}$ is an essential ideal of $\frac{M(A)}{\tilde{J}_x}$, π is injective.

Since by Proposition 3.2.3 we have $H_x \subseteq \tilde{J}_x$, it follows that there is a surjective $*$-homomorphism $\tau : \frac{M(A)}{H_x} \to \frac{M(A)}{\tilde{J}_x}$. Together with the fact that π and σ extend the identity on $\frac{A}{J_x}$, we thus get a commutative diagram :

$$\begin{array}{ccc} \frac{M(A)}{H_x} & \xrightarrow{\tau} & \frac{M(A)}{\tilde{J}_x} \\ & \searrow^{\sigma} & \downarrow \pi \\ & & M(\frac{A}{J_x}) \end{array}$$

Thus σ is injective if and only if τ is, i.e. if and only if $H_x = \tilde{J}_x$. This occurs if and only if H_x is strictly closed in $M(A)$ by Proposition 3.2.3(iii). Again, using [20, Proposition 3.7], we see that σ is injective if and only if A/J_x is an essential ideal of $M(A)/H_x$

If A is σ-unital then the canonical mapping $\tilde{q}_x : M(A) \to M(A/J_x)$ is necessarily surjective [49, Theorem 10], so that (iii)' follows. □

One might expect that in well-behaved cases, the bundle decompositions of $M(A)$ defined in Theorem 3.2.4 agrees with that of Theorem 3.1.6. This is of particular interest in the case that (A, X, μ_A) is a continuous $C_0(X)$-algebra, since the former is upper-semicontinuous, and the latter is lower-semicontinuous by Corollary 3.1.9. However, we cannot expect the fibre algebras to agree at points of $\mathrm{Im}(\phi_{M(A)})\backslash\mathrm{Im}(\phi_A)$.

Indeed, when $x \in \mathrm{Im}(\phi_{M(A)})\backslash\mathrm{Im}(\phi_A)$, then H_x is a proper ideal of $M(A)$ by Remark 2.3.2. But since $x \notin \mathrm{Im}(\phi_A)$, it follows that every $Q \in \mathrm{Prim}M(A)$ with $Q \supseteq H_x$ satisfies $Q \supseteq A$. In particular, $H_x \supseteq A$, so that H_x cannot be strictly closed since A is strictly dense in $M(A)$. By contrast, the fibre algebras of Theorem 3.1.6 at these points are $\{0\}$ by definition.

The following corollary shows that for a continuous $C_0(X)$-algebra (A, X, μ_A) with surjective base map, the assumption that $\tilde{J}_x = H_x$ for all $x \in X$ is enough to ensure continuity of $(M(A), \beta X, \mu_{M(A)})$.

Corollary 3.2.5. *Let (A, X, μ_A) be a continuous $C_0(X)$-algebra and $(M(A), \beta X, \mu_{M(A)})$ the $C(\beta X)$-algebra associated with its multiplier algebra. Suppose that the base map $\phi_A : \mathrm{Prim}(A) \to X$ associated with (A, X, μ_A) is surjective, and that $\tilde{J}_x = H_x$ for all $x \in X$. Then $(M(A), \beta X, \mu_{M(A)})$ is a continuous $C(\beta X)$-algebra.*

Proof. For all $c \in M(A)$, the map $x \mapsto \left\|c + \tilde{J}_x\right\|$ is lower-semicontinuous on X by Corollary 3.1.9, and the map $x \mapsto \|c + H_x\|$ is upper-semicontinuous on βX by Proposition 2.3.3(i). Since $\tilde{J}_x = H_x$ for all $x \in \mathrm{Im}(\phi_A)$ by assumption, it follows that $x \mapsto \|c + H_x\|$ is continuous on X for all $c \in M(A)$. By [8, Proposition 2.8], it follows that

$x \mapsto \|c + H_x\|$ is continuous on βX, whence the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$ is continuous.

□

Remark 3.2.6. Archbold and Somerset observed in [11] that for a σ-unital continuous $C_0(X)$-algebra (A, X, μ_A) with surjective base map, the condition $\tilde{J}_x = H_x$ for all $x \in X$ implies continuity of $(M(A), \beta X, \mu_{M(A)})$. Moreover, if A is separable, then the converse holds. This observation follows from comparing the results of [8, Theorem 3.8] with those of [11, Corollary 4.7]. However, we believe the argument given in Corollary 3.2.5 sheds more light on the relation between these two notions, in particular regarding the different bundle representations of $M(A)$ considered in Corollary 3.1.9 and Theorem 3.2.4.

3.3 Strict closures of the ideals H_x

In this section we examine a natural question that arises as a consequence of Theorem 3.2.4(ii), that is, when are the ideals $\tilde{J}_x$ and H_x equal. We begin with some general results regarding the connection between strictly closed ideals of $M(A)$ and the hull-kernel topology on $\mathrm{Prim}(M(A))$.

Notation 3.3.1. *To allow for the identification of* $\mathrm{Prim}(A)$ *as an open subset of* $\mathrm{Prim}(M(A))$*, we fix some notation in order to avoid ambiguity regarding hulls and kernels. We shall write* $\mathrm{hull}_{\mathrm{Prim}(A)}(I)$ *to denote the hull of an ideal I of A in* $\mathrm{Prim}(A)$*, and $k_A(S)$ the kernel of a subset $S \subseteq$* $\mathrm{Prim}(A)$*. Analogously we define* $\mathrm{hull}_{\mathrm{Prim}M(A)}(J)$ *(J an ideal of $M(A)$) and $k_{M(A)}(S)$ (S a subset of* $\mathrm{Prim}M(A)$*).*

Theorem 3.3.2. *Let A be a C^*-algebra and let J be a proper norm-closed two-sided ideal of $M(A)$. The following are equivalent:*

(i) J is strictly closed in $M(A)$

(ii) $J = (J \cap A)^{\sim}$

(iii) $\mathrm{hull}_{\mathrm{Prim}M(A)}(J) \cap \mathrm{Prim}(A)$ *is dense in* $\mathrm{hull}_{\mathrm{Prim}(M(A))}(J)$

Proof. If A is unital then the strict topology on $M(A) = A$ is the norm topology and (i),(ii) and (iii) are trivially satisfied for any norm closed two sided ideal J of A. So we may assume that A is non-unital.

Note that (i) to (iii) are satisfied if $J = \{0\}$. On the other hand, no proper ideal J of $M(A)$ with $J \supseteq A$ can be satisfy conditions (i),(ii) or (iii). Indeed, since A is strictly dense in $M(A)$, if $J \supseteq A$ then clearly (i) and (ii) are impossible. Moreover, identifying $\mathrm{Prim}(A) = \{Q \in \mathrm{Prim}(M(A)) : Q \not\supseteq A\}$, $\mathrm{hull}_{\mathrm{Prim}(M(A))}(J) \cap \mathrm{Prim}(A) = \emptyset$ in this case, so (iii) cannot hold either. So we may assume without loss of generality that $J \cap A$ is a nonzero proper ideal of A.

$(ii) \Rightarrow (i)$ is obvious. To see that $(i) \Rightarrow (ii)$, note that by (3.1.1) it always holds that $J \subseteq (J \cap A)^{\sim}$, since if $x \in J$ then for any $a \in A$, $ax, xa \in J \cap A$, so that $x \in (J \cap A)^{\sim}$. Suppose this inclusion were strict. Then there would be a net (a_λ) in $J \cap A$ strictly converging to $x \in (J \cap A)^{\sim} \backslash J$. Since this net is contained in J, J cannot be strictly closed.

To see the equivalence of (ii) and (iii), let $F = \mathrm{hull}_{\mathrm{Prim}(M(A))}(J)$ and $F_0 = F \cap \mathrm{Prim}(A)$, note that $k_{M(A)}(F) = J$. For $P \in \mathrm{Prim}(A)$, it is clear that $\tilde{P} \supseteq J$ if and only if $\tilde{P} \supseteq \tilde{J}$, which occurs if and only if $P \supseteq J \cap A$ by (3.1.1). Hence the kernel $k_{M(A)}$ of F_0, regarded as a subset of $\mathrm{Prim}(M(A))$, satisfies

$$\begin{aligned} k_{M(A)}(F_0) &= \bigcap \{\tilde{P} : P \in \mathrm{Prim}(A), \tilde{P} \supseteq J\} \\ &= \bigcap \{\tilde{P} : P \in \mathrm{Prim}(A), P \supseteq J \cap A\} \\ &= (J \cap A)^{\sim}, \end{aligned}$$

where the final equality follows from [8, Proposition 1.1(iv)].

Thus if $J = (J \cap A)^{\sim}$, then

$$\mathrm{cl}_{\ \mathrm{Prim}(M(A))}(F_0) = \mathrm{hull}_{\mathrm{Prim}(M(A))}(k_{M(A)}(F_0)) = \mathrm{hull}_{\mathrm{Prim}(M(A))}(J) = F,$$

so (ii) implies (iii). Conversely, if $\mathrm{cl}_{\ \mathrm{Prim}(M(A))}(F_0) = F$, then

$$J = k_{M(A)}(F) = k_{M(A)}(\mathrm{cl}_{\ \mathrm{Prim} M(A)}(F_0)) = k_{M(A)}(F_0) = (J \cap A)^{\sim},$$

so (iii) implies (ii). □

Example 3.3.3. *If $A = C_0(X)$ is commutative, then any ideal J of $M(A) = C(\beta X)$ is of the form*

$$J = \{f \in C(\beta X) : f|_F \equiv 0\}$$

for some closed subset $F \subseteq \beta X$. Thus by Theorem 3.3.2, J is strictly closed if and only if $F \cap X$ is dense in F.

Corollary 3.3.4. *Let (A, X, ϕ_A) be a $C_0(X)$-algebra and denote by $\phi_{M(A)}$: $\mathrm{Prim}(M(A)) \to \beta X$ the unique extension of $\phi_A : \mathrm{Prim}(A) \to X$, where we regard $\mathrm{Prim}(A)$ as a dense open subset of $\mathrm{Prim}(M(A))$. Then for each $x \in \mathrm{Im}(\phi_A)$, H_x is strictly closed in $M(A)$ if and only if $\phi_A^{-1}(\{x\})$ is dense in $\phi_{M(A)}^{-1}(\{x\})$.*

Proof. By Theorem 3.3.2, H_x is strictly closed in $M(A)$ if and only if $\mathrm{hull}_{\mathrm{Prim}(M(A))}(H_x) \cap \mathrm{Prim}(A)$ is dense in $\mathrm{hull}_{\mathrm{Prim}(M(A))}(H_x)$. But then

$$\mathrm{hull}_{\mathrm{Prim}(M(A))}(H_x) = \phi_{M(A)}^{-1}(\{x\}),$$

and since $\phi_{M(A)}$ extends ϕ_A,

$$\mathrm{hull}_{\mathrm{Prim}(M(A))}(H_x) \cap \mathrm{Prim}(A) = \phi_{M(A)}^{-1}(\{x\}) \cap \mathrm{Prim}(A) = \phi_A^{-1}(\{x\}),$$

from which the conclusion follows. □

Let I be a norm-closed two-sided ideal of a C*-algebra A. Then I is said to be *modular* if the quotient C*-algebra A/I is unital (obviously if A is unital this is true for every proper ideal). The relationship between modular ideals and the topology on $\mathrm{Prim}(A)$ has been studied in [23, Section 3], where a result similar to Lemma 3.3.5 below was obtained for the primitive ideal space of the minimal unitisation A_1 of a C*-algebra A.

Lemma 3.3.5. *Let A be a C*-algebra and I a norm closed, two-sided modular ideal of A. Then A/I is isomorphic to $M(A)/\tilde{I}$, and we can identify*

$$\mathrm{hull}_{\mathrm{Prim}(M(A))}(\tilde{I}) = \{\tilde{P} : P \in \mathrm{hull}_{\mathrm{Prim}(A)}(I)\}.$$

In particular, no $Q \in \mathrm{Prim}M(A)$ with $Q \supseteq A$ has $Q \supseteq \tilde{I}$.

Proof. By [24, Theorem 1.8.4], $A/I \equiv (A + \tilde{I})/\tilde{I}$, and by [8, Proposition 1.1(v)], the latter is an essential ideal of $M(A)/\tilde{I}$. Being unital, they must be equal.

Since $\mathrm{Prim}(A/I) = \mathrm{hull}_{\mathrm{Prim}(A)}(I)$ and $\mathrm{Prim}(M(A)/\tilde{I}) = \mathrm{hull}_{\mathrm{Prim}(M(A))}(\tilde{I})$, the $*$-isomorphism $A/I \equiv M(A)/\tilde{I}$ gives rise to a commutative diagram of homeomorphisms

$$\begin{array}{ccc} \mathrm{Prim}(\frac{A}{I}) & \xrightarrow{P/I \mapsto \tilde{P}/\tilde{I}} & \mathrm{Prim}(\frac{M(A)}{\tilde{I}}) \\ {\scriptstyle P \mapsto P/I} \uparrow & & \uparrow {\scriptstyle Q \mapsto Q/\tilde{I}} \\ \mathrm{hull}_{\mathrm{Prim}(A)}(I) & \overset{P \mapsto \tilde{P}}{\dashrightarrow} & \mathrm{hull}_{\mathrm{Prim}(M(A))}(\tilde{I}) \end{array}$$

Thus every $Q \in \text{hull}_{\text{Prim}(M(A))}(\tilde{I})$ satisfies $Q = \tilde{P}$ for some $P \in \text{hull}_{\text{Prim}(A)}(I)$. In particular we may regard $\text{hull}_{\text{Prim}(M(A))}(\tilde{I}) \subseteq \text{Prim}(A)$, so that there is no $Q \in \text{Prim}(M(A))\backslash\text{Prim}(A)$ with $Q \supseteq \tilde{I}$. □

Part (i) of Corollary 3.3.6 below was shown in [8, Proposition 2.2].

Corollary 3.3.6. *Let (A, X, μ_A) be a $C_0(X)$-algebra with base map ϕ_A and structure map μ_A, and let $x \in \text{Im}\phi$.*

(i) If $\mu_A(C_0(X)) \cap A \not\subseteq J_x$, then J_x is a modular ideal of A and H_x is strictly closed in $M(A)$.

(ii) If $\mu_A(C_0(X)) \cap A \subseteq J_x$ and J_x is a modular ideal of A, then H_x is not strictly closed in $M(A)$.

Proof. As in the remark preceding the corollary, part (i) was shown in [8, Proposition 2.2].

To prove (ii), note that since J_x is modular, $\phi_A^{-1}(x)$ is closed in Prim$M(A)$ by Lemma 3.3.5. Then together with Corollary 3.3.4, we see that H_x is strictly closed in $M(A)$ iff $\phi_{M(A)}^{-1}(x) = \phi_A^{-1}(x)$. Since $\phi_A^{-1}(x)$ is contained in Prim(A), it follows that H_x will be strictly closed if and only if no $R \in \text{Prim}(M(A))$ with $R \supseteq A$ has $\phi_{M(A)}(R) = x$. But then [8, Lemma 2.1] shows that since $\mu_A(C_0(X)) \cap A \subseteq J_x$, there is some $R \in \text{Prim}(M(A))$ such that $R \supseteq A$ and $\phi_{M(A)}(R) = x$. Thus H_x is not strictly closed in $M(A)$.

□

Corollary 3.3.7. *Let A be a quasi-central C^*-algebra and denote by $(A, \text{Glimm}(A), \mu_A)$ the $C_0(\text{Glimm}(A))$-algebra associated with the Dauns-Hofmann representation of A. Then H_p is strictly closed in $M(A)$ for all $p \in \text{Glimm}(A)$.*

Proof. Since A is quasi-central, Glimm(A) is locally compact and $\mu_A(C_0(\text{Glimm}(A))) = Z(A)$ by Proposition 2.3.11. Since no $P \in \text{Prim}(A)$ contains $Z(A)$, it follows that $Z(A) \not\subseteq G_p$ for all $p \in \text{Glimm}(A)$. In particular, $\{p \in \text{Glimm}(A) : \mu_A(C_0(\text{Glimm}(A))) \cap A \not\subseteq G_p\} = \text{Glimm}(A)$, so that H_p is strictly closed in $M(A)$ for all $p \in \text{Glimm}(A)$ by Corollary 3.3.6(i). □

Example 3.3.8. *Let $A \subset C^b(\mathbb{N}, M_2(\mathbb{C}))$ be the C^*-subalgebra of sequences $x = (x_n)$ such that $x_n \to \text{diag}(\lambda(x), 0)$ for some complex scalar $\lambda(x)$. Then A is a continuous*

$C(\hat{\mathbb{N}})$-algebra, where for $f \in C(\hat{\mathbb{N}})$, $\mu_A(f)(n) = f(n)1_{2\times 2}$ for all $n \in \hat{\mathbb{N}}$. For all $n \in \hat{\mathbb{N}}$, the ideals J_n satisfy $J_n = \{x = (x_m) \in A : x_n = 0\}$.

$M(A)$ may be identified with those sequences $y = (y_n) \in C^b(\mathbb{N}, M_2(\mathbb{C}))$ such that

(i) $y_n^{(1,2)} \to 0$ and $y_n^{(2,1)} \to 0$,

(ii) $y_n^{(1,1)} \to \lambda_1(y)$ and

(iii) the sequence $(y_n^{(2,2)})_{n\geq n_0}$ is bounded,

see e.g. [4, Example 4.5].

Note that for all $n \in \hat{\mathbb{N}}$, A/J_n is unital, and the set

$$\{n \in \hat{\mathbb{N}} : \mu_A(C(\hat{\mathbb{N}})) \cap A \not\subseteq J_n\}$$

is identified with $\mathbb{N}$. In particular we have $\tilde{J}_n = H_n$ for all $n \in \mathbb{N}$, while $\tilde{J}_\infty \supsetneq H_\infty$ by Corollary 3.3.6.

In fact, it is straightforward to verify that we have $\tilde{J}_n = \{y \in M(A) : y_n = 0\}$ for all $n \in \hat{\mathbb{N}}$, while $H_\infty = \{y \in M(A) : y_n \to 0\}$. Thus $M(A)/\tilde{J}_\infty \equiv \mathbb{C}$ while $M(A)/H_\infty \equiv \mathbb{C} \oplus \ell^\infty/c_0$.

We remark that the C*-algebra A of Example 3.3.8 shows that the question of whether or not H_x is strictly closed in $M(A)$ is not determined by the base map $\phi_A : \mathrm{Prim}(A) \to X$ alone in general. To see this, let $B = C(\hat{\mathbb{N}})$, regarded as a $C(\hat{\mathbb{N}})$-algebra in the obvious way. Then $\mathrm{Prim}(A)$ is homeomorphic to $\mathrm{Prim}(B) = \hat{\mathbb{N}}$, and thus we may identify the base maps ϕ_A and ϕ_B, being the identity map in each case. Since $M(B) = B$, it follows that $H_n = \tilde{J}_n = J_n$ for all $n \in \hat{\mathbb{N}}$ (these are the maximal ideals of $C(\hat{\mathbb{N}})$), and in particular H_∞ is strictly closed in $M(B)$.

3.4 Strict closures and spectral synthesis

We now give an account of how the question 'when is $H_x = \tilde{J}_x$' may interpreted as one of 'spectral synthesis' for the multiplier algebra of a $C_0(X)$-algebra. The main ideas and techniques in this section were developed by Archbold and Somerset in [9] and [11].

For a topological space X denote by $C_R(X)$ the ring of continuous real valued functions on X, and for $f \in C_R(X)$ let $Z(f) = \{x \in X : f(x) = 0\}$. We say that a subset $Z \subseteq X$ is a zero set if $Z = Z(f)$ for some $f \in C_R(X)$.

Definition 3.4.1. Let X be a topological space and denote by $Z(X)$ the collection of all zero sets of continuous functions on X. A nonempty subfamily $\mathcal{F}$ of $Z(X)$ is called a *z-filter on* X if

(i) $\emptyset \notin \mathcal{F}$,

(ii) If $Z_1, Z_2 \in \mathcal{F}$ then $Z_1 \cap Z_2 \in \mathcal{F}$, and

(iii) If $Z' \in Z(X)$ such that $Z' \supseteq Z$ for some $Z \in \mathcal{F}$ then $Z' \in \mathcal{F}$.

Let I be an (algebraic) ideal in $C_R(X)$. Then the collection

$$Z[I] = \{Z(f) : f \in I\}$$

defines a z-filter on X. We say that I is a *z-ideal* if $Z(f) \in Z[I]$ implies $f \in I$.

Conversely if $\mathcal{F}$ is a z-filter on X then the set

$$I[\mathcal{F}] = \{f \in C_R(X) : Z(f) \in \mathcal{F}\}$$

is an (algebraic) ideal in $C_R(X)$. Moreover, there is a bijective correspondence between the set of z-ideals of $C_R(X)$ and z-filters on X given by $I \mapsto Z[I]$ (so that $I = I[Z[I]]$). For a proof of these facts see [29, Section 2.2 to 2.7].

Now let A be a $C_0(X)$-algebra with base map ϕ_A. For $b \in M(A)$, define

$$Z(b) = \{x \in \mathrm{Im}\phi_A : b \in \tilde{J}_x\} = \{x \in \mathrm{Im}\phi_A : \|b + \tilde{J}_x\| = 0\},$$

and note that $Z(b)$ is not a zero-set (that is, of a continuous function) on $\mathrm{Im}\phi_A$ in general, since the norm function of b need not be continuous.

In what follows let $X_\phi = \mathrm{Im}\phi_A$. To each z-filter $\mathcal{F}$ on X_ϕ (hence to each z-ideal $I[\mathcal{F}]$ in $C_R(X_\phi)$) we define an algebraic ideal of $M(A)$ given by

$$L_{\mathcal{F}}^{\mathrm{alg}} = \{b \in M(A) : Z(b) \supseteq Z \text{ for some } Z \in \mathcal{F}\}, \tag{3.4.1}$$

and let $L_{\mathcal{F}}$ be its norm closure. In [9, Theorem 3.2] it is shown that the map $\mathcal{F} \mapsto L_{\mathcal{F}}$ is an injection whenever J_x is non-modular for all $x \in X_\phi$.

With $M_p = \{f \in C_R(X_\phi) : f(p) = 0\}$, let $Z[M_p]$ be the collection of zero sets of functions in M_p, i.e.

$$Z[M_p] = \{Z \in Z(X_\phi) : p \in Z\}.$$

Let O_p be the (not necessarily closed) ideal of $f \in C_R(X_\phi)$ such that f vanishes in a neighbourhood of p. Then

$$Z[O_p] = \{Z \in Z(X_\phi) : p \in \text{int}(Z)\}$$

When $M_p = O_p$, p is called a P-point in X_ϕ. If $q \in X_\phi$ is a P-point in X then q is a P-point in X_ϕ [29, 4L].

Our interest in the ideals M_p and O_p is as follows:

Theorem 3.4.2. *[9, Theorem 4.3] Let (A, X, ϕ_A) be a σ-unital $C_0(X)$-algebra with base map ϕ_A and let $X_\phi = \text{Im}\phi_A$. Then the map $\mathcal{F} \mapsto L_{\mathcal{F}}$ of (3.4.1) has the following properties:*

(i) for $x \in X_\phi$, $L_{Z[M_x]} = \tilde{J}_x$,

(ii) for $p \in \text{cl}_{\beta X}(X_\phi)$, $L_{Z[O_p]} = H_p$.

The following example exhibits a non-σ-unital $C_0(X)$-algebra (A, X, μ_A) and a P-point $x \in X_\phi$ (so that $M_x = O_x$) with the property that $\tilde{J}_x \neq H_x$. In particular, this shows that the conclusion of Theorem 3.4.2 may fail when the assumption that A is σ-unital is dropped.

Example 3.4.3. *Let ω_1 be the first uncountable ordinal and $X = [0, \omega_1]$ with the order topology. Let A be the C^*-subalgebra of $C^b(X, M_2(\mathbb{C}))$ consisting of those $f \in C^b(X, M_2(\mathbb{C}))$ such that $f(\omega_1)$ is a matrix with zeros everywhere except in the $(1,1)$-entry.*

Then $\text{Prim}(A)$ is easily seen to be homeomorphic to X, and thus (A, X, ϕ_A) is a continuous $C(X)$-algebra where $\phi_A : \text{Prim}(A) \to X$ is the canonical homeomorphism.

Now, the centre of A may be identified with $C_0([0, \omega_1))$, acting as scalar multiples of the 2×2 identity matrix. Moreover, the structure map $\mu_A : C(X) \to ZM(A)$ is precisely the Dauns-Hofmann isomorphism of (2.3.1), and so

$$\mu_A(C(X)) \cap A = ZM(A) \cap A = Z(A) \equiv C_0([0, \omega_1)).$$

Thus for $x \in X$, we have $J_x \supseteq Z(A)$ if and only if $x = \omega_1$. Since $A/J_{\omega_1} \equiv \mathbb{C}$, J_{ω_1} is a modular ideal of A, and so H_{ω_1} is not strictly closed by Corollary 3.3.6.

On the other hand, ω_1 is a P-point in X by [29, 5O], and so the z-ideals M_{ω_1} and O_{ω_1} of $C_R(X)$ are equal. Since $\tilde{J}_{\omega_1} \neq H_{\omega_1}$, this shows that either $L_{Z[M_{\omega_1}]} \neq \tilde{J}_{\omega_1}$ or $L_{Z[O_{\omega_1}]} \neq H_{\omega_1}$.

Returning to the case of a σ-unital $C_0(X)$-algebra (A, X, ϕ_A), Theorem 3.4.2 shows that the question of identifying those points $x \in X_\phi$ for which $\tilde{J}_x = H_x$ may be interpreted as one of 'spectral synthesis.' More precisely, an element $b \in M(A)$ belongs to H_x if and only if b may be approximated in norm by elements vanishing on a neighbourhood of x. Since b is zero as a multiplier of A/J_x if and only if $b \in \tilde{J}_x$, we will regard $M(A)$ as admitting 'spectral synthesis at x' if and only if $\tilde{J}_x = H_x$.

In the case that the ideals J_x are not modular for any $x \in X_\phi$ then injectivity of the map $\mathcal{F} \mapsto L_\mathcal{F}$ [9, Theorem 3.2] shows that the ideals J_x and H_x $(x \in X_\phi)$ are equal if and only if x is a P-point in X_ϕ. While Theorem 3.4.2 does not require this assumption, without it, the map $\mathcal{F} \mapsto L_\mathcal{F}$ need not be injective. In particular, for $x \in X_\phi$, it is possible that $M_x \neq O_x$ (x not a P-point in X_ϕ) but $L_{Z[M_x]} = L_{Z[O_x]}$.

We will show that for certain $C_0(X)$-algebras and open subsets $E \subseteq X_\phi$, the existence of a dense subset of E consisting of points x with J_x non-modular is enough to show $M_p \neq O_p$ implies $L_{Z[M_p]} \neq L_{Z[O_p]}$ for $p \in E$.

The following theorem relates zero sets of bounded continuous functions on X_ϕ with elements of $M(A)$. It is the main technical tool used in [10] to prove injectivity of the embedding map.

Theorem 3.4.4. *[10, Theorem 3.2] Let (A, X, ϕ_A) be a σ-unital $C_0(X)$-algebra with base map ϕ_A and let $X_\phi = \mathrm{Im}\phi_A$. Let $u \in A$ be strictly positive with $\|u\| = 1$. Let $f \in C^b(X_\phi)$ with $0 \leq f \leq 1$, and Z be the zero set of f. Then there exists $b \in M(A)$ with $0 \leq b \leq 1$ such that*

(i) $b \in A + \tilde{J}_x$ for all $x \in X_\phi \backslash Z$,

(ii) $1 - b \in \tilde{J}_x$ for all $x \in Z$

Moreover, denoting by $\mathrm{sp}(u + J_x)$ the spectrum of $u + J_x$ in A/J_x and by $V = \{x \in X_\phi \backslash Z : 2\min \mathrm{sp}(u + J_x) \leq f(x)\}$, then H_x is not strictly closed in $M(A)$ for all $x \in \mathrm{cl}_{X_\phi}(V) \cap Z$.

We will need a slight modification of [10, Corollary 3.1]:

Lemma 3.4.5. *Let (A, X, ϕ_A) be a σ-unital $C_0(X)$-algebra with $X_\phi = \mathrm{Im}\phi_A$. Then for each zero set Z in X_ϕ there exists an element $b^Z \in M(A)$ with $0 \leq b^Z \leq 1$ such that*

(i) $\|b^Z + \tilde{J}_x\| = 0$ for all $x \in Z$,

(ii) $\|b^Z + \tilde{J}_x\| = 1$ *whenever* $x \in X_\phi \backslash Z$ *and* J_x *is not a modular ideal of* A.

Proof. Apply Theorem 3.4.4 to get an element $b \in M(A)$, $0 \leq b \leq 1$ satisfying conditions (i) and (ii) of Theorem 3.4.4. Set $b^Z = 1 - b$. Then by (ii), $\|b^Z + \tilde{J}_x\| = 0$ for all $x \in Z$.

Recall that $A/J_x \equiv (A + \tilde{J}_x)/\tilde{J}_x$ is an essential ideal of the unital C*-algebra $M(A)/\tilde{J}_x$ [8, Proposition 1.1(v)]. In particular, if J_x is not modular then necessarily $A + \tilde{J}_x \neq M(A)$.

By (i) of Theorem 3.4.4, $b = 1 - b^Z \in A + \tilde{J}_x$ for all $x \in X_\phi \backslash Z$. If J_x is not modular, then this implies that $\|b^Z + (A + \tilde{J}_x)\| = 1$, and hence

$$1 = \|b^Z + (A + \tilde{J}_x)\| \leq \|b^Z + \tilde{J}_x\| \leq \|b^Z\| \leq 1,$$

which completes the proof. □

Theorem 3.4.6. *Let* (A, X, ϕ_A) *be a* σ*-unital* $C_0(X)$*-algebra with base map* ϕ_A *and let* $X_\phi = \mathrm{Im}\phi_A$. *Suppose that* $E \subseteq X_\phi$ *is a relatively open subset of* X_ϕ *such that the set* $\{y \in E : J_y$ *is not modular* $\}$ *is dense in* E. *Then for any* $x \in E$, $\tilde{J}_x = H_x$ *if and only if* x *is a P-point in* X_ϕ.

Proof. If x is a P-point in X_ϕ then $\tilde{J}_x = H_x$ by [10, Theorem 4.3].

Suppose x is not a P-point in X_ϕ, so there exists a zero set Z of X_ϕ with $x \in Z$ but $x \notin \mathrm{int}_{X_\phi}(Z)$. Then $E \backslash Z$ is open in E, hence open in X_ϕ. Let $c \in L^{\mathrm{alg}}_{Z[O_x]}$. Then there is a zero set $W \in Z[O_x]$ such that

$$Z(c) := \{y \in X_\phi : \|c + \tilde{J}_y\| = 0\} \supseteq W.$$

Let $\mathcal{O}$ be an open neighbourhood of x in X_ϕ with $\mathcal{O}$ contained in W. Then $\mathcal{O} \not\subseteq Z$ (since otherwise $x \in \mathrm{int}_{X_\phi}(Z)$). Hence $(E \backslash Z) \cap \mathcal{O}$ is open in X_ϕ, nonempty and contained in E. Hence there is a point $y \in (E \backslash Z) \cap \mathcal{O}$ such that J_y is not a modular ideal of A. For this y, the element b^Z of $M(A)$ of Lemma 3.4.5 satisfies $\|b^Z + \tilde{J}_y\| = 1$, while $\|c + \tilde{J}_y\| = 0$. In particular,

$$\|b^Z - c\| \geq \|(b^Z - c) + \tilde{J}_y\| = 1,$$

and since our choice of $c \in L^{\mathrm{alg}}_{Z[O_x]}$ was arbitrary, $b^Z \notin H_x$. On the other hand, $b^Z \in \tilde{J}_x$ by Lemma 3.4.5(i). It follows that H_x is not strictly closed in $M(A)$. □

We investigate the conditions that ensure that the hypothesis of Theorem 3.4.6 are satisfied by the set $E = X_\phi \backslash \overline{U}$, where

$$U = \{x \in X : \mu_A(C_0(X)) \cap A \not\subseteq J_x\}. \tag{3.4.2}$$

Note that since $J_x = A$ whenever $x \in X \backslash X_\phi$, the set U is contained in X_ϕ. We have seen already in Corollary 3.3.6 that for all $x \in U$, H_x is strictly closed in $M(A)$.

Under the additional assumption that the $C_0(X)$-algebra A is continuous, Theorem 3.4.6 characterises the set of points $x \in X_\phi \backslash \overline{U}$ such that H_x is strictly closed in $M(A)$:

Corollary 3.4.7. *Let (A, X, ϕ_A) be a σ-unital continuous $C_0(X)$-algebra with base map ϕ_A, let $X_\phi = \mathrm{Im}(\phi_A)$ and*

$$U = \{x \in X : \mu(C_0(X)) \cap A \not\subseteq J_x\}.$$

Then for $x \in E = X_\phi \backslash \overline{U}$, H_x is strictly closed in $M(A)$ if and only if x is a P-point in X_ϕ.

Proof. By [8, Lemma 3.6], the set of $x \in \mathrm{Im}\phi$ such that J_x is not a modular ideal of A is dense in $X_\phi \backslash \overline{U}$. Hence the conclusion follows from Theorem 3.4.6. □

A straightforward application of Theorem 3.4.6 is the case where (A, X, ϕ_A) is a σ-unital $C_0(X)$-algebra with $Z(A) = \{0\}$. Indeed, in that case we have $\mu_A(C_0(X)) \cap A \subseteq ZM(A) \cap A = Z(A) = \{0\}$, so that $U = \emptyset$.

As we have seen, if in addition (A, X, μ_A) is continuous, then the set of $x \in X_\phi$ with J_x not modular is dense in X_ϕ. Hence for $x \in X_\phi$, the equality $\tilde{J}_x = H_x$ holds if and only if x is a P-point in X_ϕ by Corollary 3.4.7.

There are however other conditions on a $C_0(X)$-algebra (A, X, ϕ_A) with $Z(A) = \{0\}$ which ensure that this characterisation is still valid. In [23], it is shown that the existence of a set of modular primitive ideals with nonempty interior in $\mathrm{Prim}(A)$ is sufficient in certain cases to imply $Z(A) \neq \{0\}$.

Let T be a T_0 space. A point $x \in T$ is said to be *separated* if given any $y \in T$ with $y \notin \mathrm{cl}_T(\{x\})$, x and y have disjoint neighbourhoods. For $F \subseteq T$, let $S(F)$ be the set of separated points of F. If for every closed nonempty $F \subseteq T$, the interior of $S(F)$ is nonempty, then T is said to be *quasi-separated* [23, Definition 7].

Corollary 3.4.8. *Let (A, X, ϕ_A) be a σ-unital $C_0(X)$-algebra with base map ϕ_A, $Z(A) = \{0\}$, and suppose that at least one of the following conditions hold:*

(i) A is a continuous $C_0(X)$-algebra with respect to the base map ϕ_A,

(ii) A is liminal, or

(iii) A is separable and Prim(A) *is quasi-separated.*

Then $\tilde{J}_x = H_x$ if and only if x is a P-point in $X_\phi := \mathrm{Im}\phi_A$.

Proof. In case (i) $U = \emptyset$ and so Corollary 3.4.7 gives the result. For cases (ii) and (iii), by [23, Propositions 11 and 12], $Z(A) \neq \{0\}$ if and only if the set of modular primitive ideals has nonempty interior. So if $Z(A) = \{0\}$, the set of non-modular primitive ideals is dense in Prim(A).

Take $x \in X_\phi$ such that J_x is a modular ideal of A, and suppose $P \in \mathrm{Prim}(A)$ with $\phi(P) = x$. Then by [23, Lemme 4 and Proposition 9], P must be a modular ideal of A (since otherwise $P \in \mathrm{Prim}(I)$ for every modular ideal I of A, while we know that $P \notin \mathrm{Prim}(J_x)$).

Now suppose V is an open subset of X_ϕ such that J_x is modular for all $x \in V$. Then $\phi_A^{-1}(V)$ would be an open subset of Prim(A) consisting of modular ideals, which is impossible. Hence the set of $x \in X_\phi$ such that J_x is non-modular is dense in X_ϕ. The conclusion thus follows from Theorem 3.4.6. □

Let (A, X, ϕ_A) be a σ-unital, continuous $C_0(X)$-algebra with base map ϕ_A and let $X_\phi = \mathrm{Im}\phi_A$. Then Corollaries 3.3.6 and 3.4.7 characterise the set of x in U and $X_\phi \backslash \overline{U}$ (where $\overline{U}$ denotes the closure of U in X_ϕ) such that H_x is strictly closed in $M(A)$. All that remain are points on the boundary of $\overline{U}$. We have been unable to characterise those points x on the boundary of $\overline{U}$ such that H_x is strictly closed in general. However, when $M(A)$ is a continuous $C(\beta X)$-algebra, the following Lemma allows us to give a full characterisation.

Lemma 3.4.9. *[8, Lemma 3.3] Let (A, X, ϕ_A) be a σ-unital $C_0(X)$-algebra, u a strictly positive element of A with $\|u\| = 1$, and suppose that $M(A)$ is a continuous $C(\beta X)$-algebra. Define $f : X \to [0, 1]$ as*

$$f(x) = (1 - \|(1-u) + H_x\|)^{\frac{1}{2}}.$$

Then

(i) f *is continuous on* X*,*

(ii) U *is the cozero set of* f*,*

(iii) If $x \in X$ *and* $0 < f(x) \leq \frac{1}{2}$*, then* $2\min \operatorname{sp}(u + J_x) \leq f(x)$ *(hence* x *belongs to the set* V *of Theorem 3.4.4).*

The following Theorem characterises completely the set of points $x \in X_\phi$ such that H_x is strictly closed in $M(A)$ when $M(A)$ is a continuous $C(\beta X)$-algebra:

Theorem 3.4.10. *Let* (A, X, ϕ_A) *be a* σ*-unital* $C_0(X)$*-algebra with base map* ϕ_A *such that* $M(A)$ *is a continuous* $C(\beta X)$*-algebra, let* $X_\phi = \operatorname{Im}\phi_A$ *and*

$$U = \{x \in X_\phi : \mu(C_0(X)) \cap A \not\subseteq J_x\}.$$

Then for any $x \in X_\phi$*,* H_x *is strictly closed in* $M(A)$ *if and only if either*

(i) $x \in U$*, or*

(ii) $x \in X_\phi \backslash \overline{U}$ *and* x *is a P-point in* X*,*

where $\overline{U} = \operatorname{cl}_{X_\phi}(U)$*.*

Proof. If $x \in U$ then H_x is strictly closed by Corollary 3.3.6.

If (ii) holds then, since A is necessarily a continuous $C_0(X)$-algebra, H_x is strictly closed by Corollary 3.4.7.

Now suppose neither (i) nor (ii) hold. Then $x \in \overline{U}\backslash U$. Let f be the function given in Lemma 3.4.9, and note that $f(x) = 0$. Then any neighbourhood of x contains a neighbourhood W of x with $f(y) \leq \frac{1}{2}$ for all $y \in W$, hence $2\min \operatorname{sp}(u + J_q) \leq f(q)$ for all $q \in W \cap U$ by Lemma 3.4.9 (iii). In particular, $x \in \operatorname{cl}_X(V)$, so H_x cannot be strictly closed by Theorem 3.4.4. □

We conclude this section with two examples illustrating the relationship between the strict topology on the multiplier algebra of a $C_0(X)$-algebra (A, X, μ_A) and the ideal structure of the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$.

Example 3.4.11. *Let* $A = C(\hat{\mathbb{N}}, c_0) \equiv C(\hat{\mathbb{N}}) \otimes c_0 \equiv C_0(\hat{\mathbb{N}} \times \mathbb{N})$ *be the trivial continuous* $C(\hat{\mathbb{N}})$*-algebra with fibre* c_0*. Then* $\operatorname{Prim}(A)$ *is canonically isomorphic to* $\hat{\mathbb{N}} \times \mathbb{N}$*, and the base map* $\phi_A : \operatorname{Prim}(A) \to \hat{\mathbb{N}}$ *is the projection onto the first coordinate. Since*

$M(c_0) = \ell^\infty = C(\beta\mathbb{N})$, it follows from [1, Corollary 3.4] that $M(A) = C(\hat{\mathbb{N}}, \ell^\infty_\beta)$, where ℓ^∞_β denotes ℓ^∞ with the strict topology induced by regarding ℓ^∞ as the multiplier algebra of c_0.

By Theorem 3.1.6, $M(A)$ may be regarded as a C^*-bundle over $\hat{\mathbb{N}}(=\beta\hat{\mathbb{N}})$, with fibre algebras $M(A)_n = \ell^\infty$ for all $n \in \hat{\mathbb{N}}$. On the other hand, by Proposition 3.2.1, $M(A)$ is canonically a $C(\hat{\mathbb{N}})$-algebra with base map $\phi_{M(A)} : \mathrm{Prim}(M(A)) \to \hat{\mathbb{N}}$ the unique continuous extension of ϕ_A to $\mathrm{Prim}(M(A))$. Regarding A as $C_0(\hat{\mathbb{N}} \times \mathbb{N})$, it is clear that $\mathrm{Prim}(M(A)) = \beta(\hat{\mathbb{N}} \times \mathbb{N})$, and that $\phi_{M(A)}$ is the Stone-Čech extension of ϕ_A [29, Theorem 6.5(I)].

As in Theorem 3.2.4, the $C(\hat{\mathbb{N}})$-algebra $(M(A), \hat{\mathbb{N}}, \phi_{M(A)})$ gives rise to an upper-semicontinuous C^*-bundle $(\hat{\mathbb{N}}, M(A), \sigma_n : M(A) \to M(A)_n)$, with fibre algebras satisfying $M(A)_n = M(A)/H_n$ for all $n \in \hat{\mathbb{N}}$. Moreover, by [9, Theorem 4.9], we may make the identifications $M(A)_n = M(A)/\tilde{J}_n = M(A_n) \equiv \ell^\infty$ for all $n \in \mathbb{N}$, while $M(A)_\infty = M(A)/H_\infty$, with $H_\infty \subsetneq \tilde{J}_\infty$. Moreover, there are uncountably many distinct norm-closed ideals J of $M(A)$ satisfying $H_\infty \subsetneq J \subsetneq \tilde{J}_\infty$.

We make some remarks about the structure of the fibre algebra $M(A)_\infty$:

(i) as a commutative, unital C^*-algebra, there is a compact space K with $M(A)_\infty \equiv C(K)$. Moreover, K may be identified with the subset $\mathrm{hull}(H_\infty)$ of $\mathrm{Prim}(M(A))$, so that $K = \phi^{-1}_{M(A)}(\{\infty\})$.

(ii) Since $H_\infty \subsetneq \tilde{J}_\infty$, it follows that $M(A)/\tilde{J}_\infty \equiv \ell^\infty$ is a nontrivial subquotient of $M(A)_\infty$. In particular, K contains a copy of $\beta\mathbb{N}$ as a proper closed subset.

(iii) For each ideal J of $M(A)$ with $H_\infty \subsetneq J \subsetneq \tilde{J}_\infty$ there is a corresponding closed subset $F_J \subset K$ with $J/H_\infty = \{f \in C(K) : f|_{F_J} \equiv 0\}$. Thus we get uncountably many such subsets of K, with $K \backslash F_J$ nonempty in each case.

Example 3.4.12 below should be compared with the (non-σ-unital) $C_0(X)$-algebra of Example 3.4.3.

Example 3.4.12. *Let ω_1 be the first uncountable ordinal and consider the space $X = [0, \omega_1]$ with the order topology. Then for $\alpha \in X$, we have*

(i) If α is a successor ordinal, then α is isolated in X,

(ii) If $\alpha \neq \omega_1$ is a limit ordinal, then α is a non-P-point of X, and

(iii) $\alpha = \omega_1$ *is a (non-isolated) P-point of* X *[29, 5O].*

If B *is any separable, non-unital* C^**-algebra then the trivial continuous* $C(X)$*-algebra* $A = C(X, B)$ *has* $M(A) = C^b(X, M(B)_\beta)$ *by [1, Corollary 3.4]. Moreover, since* X *is compact and* B *separable,* A *is* σ*-unital. Since* $\mu_A(f) = f \cdot 1_{M(B)}$ *for all* $f \in C(X)$*, it is clear that the set* U *of (3.4.2) satisfies* $U = \emptyset$*. Thus by Corollary 3.4.7, we have* $\tilde{J}_\alpha = H_\alpha$ *if and only if either* α *is a successor ordinal or* $\alpha = \omega_1$*.*

Chapter 4

Sheaves of C*-algebras

This chapter concerns the interplay between sheaves of C*-algebras and C*-bundles, with a particular focus on sheaves arising from $C_0(X)$-algebras.

Our definition of a sheaf of C*-algebras is due to Ara and Mathieu [3], and is closely related to the sheaves of Banach spaces studied by Hofmann [33]. When working in the category of C*-algebras, it is natural to restrict the 'gluing property' (Definition 4.1.1(iv)) of a sheaf to *bounded* families of local sections. Thus our notion of a sheaf differs from the 'classical' sheaves studied in algebraic and differential geometry.

One advantage of the sheaf theoretic approach is that sheaves may be conveniently defined over non-Hausdorff base spaces. Indeed, sheaves distinguish between points of a topological space having distinct open neighbourhoods, i.e., T_0-spaces such as $\mathrm{Prim}(A)$. By contrast, the theory of C*-bundles is not well developed in this setting.

We may regard sheaves of C*-algebras as a generalisation of C*-bundles. Indeed, given an upper-semicontinuous (H)-C*-bundle $(\mathcal{A}, X, p)$ over a locally compact Hausdorff space X, we obtain naturally the sheaf of local sections of the bundle. More precisely, we assign to each open subset $U \subseteq X$ the C*-algebra $\Gamma^b(U, \mathcal{A})$, and for a smaller open subset $V \subseteq U$, restriction from U to V corresponds to the usual notion of restriction of a continuous section.

In the other direction, starting from a sheaf $\mathfrak{A}$ of C*-algebras, one must first construct the 'stalks' of the sheaf at each point of the base space. This construction gives rise to a bundle of a very general type, with the property that sections of the sheaf

$\mathfrak{A}$ define continuous sections of the resulting bundle. This construction is due to Ara and Mathieu [3, Section 5], although the main ideas are similar to those used by Hofmann [33]. We will not discuss this approach in its utmost generality, however, we give some examples (Examples 4.1.6 and 4.1.7) to illustrate the difficulties that arise in the case of non-Hausdorff base spaces.

Even in the case of a sheaf $\mathfrak{A}$ over a locally compact Hausdorff space X, it is unknown in general whether or not $\mathfrak{A}$ is isomorphic to the local section sheaf of the associated bundle. We obtain partial results in this direction in Section 4.2. As a consequence, we give an interesting 'Stone-Čech compactification'-type theorem for upper-semicontinuous (H)-C*-bundles over locally compact Hausdorff spaces (Corollary 4.2.11).

The final section of this chapter concerns the study of the so-called 'multiplier sheaf' $\mathfrak{M}_X$ associated with a $C_0(X)$-algebra (A, X, μ_A). We show that, with no additional restrictions on A or X, $\mathfrak{M}_X$ is canonically isomorphic to the local section sheaf of the associated bundle (Theorem 3.1.6). Moreover, this bundle may be identified with that arising from the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$ of Proposition 3.2.1 in a natural way.

4.1 Sheaves of C*-algebras

Let X be a topological space (not assumed to be Hausdorff) and $\mathcal{O}(X)$ the collection of open subsets of X. Then $\mathcal{O}(X)$ is a category whose morphisms are inclusions, that is, for $V, U \in \mathcal{O}(X)$, $V \to U$ if and only if $V \subseteq U$.

Definition 4.1.1. [3], [33] A *presheaf* of C*-algebras is a contravariant functor $\mathfrak{A} : \mathcal{O}(X) \to \mathcal{C}^*$; that is, for $U, V \in \mathcal{O}(X)$ with $V \subseteq U$ we have a $*$-homomorphism $\Phi_{V,U} : \mathfrak{A}(U) \to \mathfrak{A}(V)$, such that the maps $\Phi_{V,U}$ satisfy the following properties:

(i) For all $U \in \mathcal{O}(X)$, $\Phi_{U,U}$ is the identity $*$-homomorphism, and

(ii) For $W, V, U \in \mathcal{O}(X)$ with $W \subseteq V \subseteq U$, the composition relation $\Phi_{W,U} = \Phi_{W,V} \circ \Phi_{V,U}$ holds.

A *sheaf* of C*-algebras $\mathfrak{A}$ is a presheaf satisfying the additional conditions:

(iii) $\mathfrak{A}(\emptyset) = \{0\}$, and

(iv) For every collection $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, and $\{s_i : i \in I\}$ with $s_i \in \mathfrak{A}(U_i)$ for all i, such that $\Phi_{U_i \cap U_j, U_i}(s_i) = \Phi_{U_i \cap U_j, U_j}(s_j)$ and $\sup\{\|s_i\| : i \in I\} < \infty$, setting $U = \bigcup_{i \in I} U_i$, there is a unique $s \in \mathfrak{A}(U)$ with $\Phi_{U_i,U}(s) = s_i$ for all $i \in I$.

We begin with some commutative examples of presheaves and sheaves, in particular to illustrate condition (iv) of Definition 4.1.1.

Example 4.1.2. *(i) Let X be a locally compact, non-compact Hausdorff space and let $A = C_0(X)$. For $F \subseteq X$ closed defined the closed ideal $I(F) = \{f \in A : f|_F \equiv 0\}$, and for $U \in \mathcal{O}(X)$ set $\mathfrak{A}(X)(U) = A/I(\overline{U})$. Then it is easily verified that $\mathfrak{A}(X)(U) \equiv C_0(\overline{U})$ for each $U \in \mathcal{O}(X)$. For $V \subseteq U \in \mathcal{O}(X)$, we have $I(\overline{U}) \subseteq I(\overline{V})$, and hence there is a $*$-homomorphism*

$$\Phi_{V,U} : \mathfrak{A}(X)(U) = C_0(\overline{U}) \to \mathfrak{A}(X)(V) = C_0(\overline{V}).$$

For $f \in C_0(\overline{U})$, $\Phi_{V,U}(f)$ is precisely its usual restriction $f|_{\overline{V}}$. Then it is clear that $\mathfrak{A}(X)$, together with these restriction mappings, is a presheaf of C-algebras over X.*

To see that $\mathfrak{A}(X)$ is not a sheaf, for each $x \in X$ choose an open neighbourhood U_x of x such that $\overline{U}_x$ is compact. Then $\{U_x : x \in X\}$ is an open cover of X, and $\mathfrak{A}(X)(U)$ is a unital C-algebra for all $x \in X$. If we define $f_x = 1_{\overline{U}_x} \in \mathfrak{A}(X)(U_x)$ for all $x \in X$, then clearly we have the relations $f_x|_{\overline{U}_x \cap \overline{U}_y} = f_y|_{\overline{U}_x \cap \overline{U}_y}$ for all $x, y \in X$. But there is no $f \in \mathfrak{A}(X)(X)$ for which $f|_{\overline{U}_x} = 1_{\overline{U}_x}$ for all $x \in X$, since $C_0(X)$ has no unit.*

(ii) With X as in (i), setting $\mathfrak{C}(X)(U) = C^b(U)$ for all $U \in \mathcal{O}(X)$, together with the usual restriction mappings, defines a sheaf of C-algebras over X.*

(iii) When X is compact, $\overline{U}$ is compact for all $U \in \mathcal{O}(X)$ and hence with $\mathfrak{A}(X)$ as in (i), $\mathfrak{A}(X)(U)$ is unital for all such U. However, it is still possible that the presheaf $\mathfrak{A}(X)$ fails to be a sheaf. Indeed, take $X = \hat{\mathbb{N}}$ and consider the open subset $U = \mathbb{N}$, and the open covering $\{U_n = \{n\} : n \in \mathbb{N}\}$ of U. Then $\mathfrak{A}(X)(U) = C(\overline{U}) = C(\hat{\mathbb{N}})$, and $\mathfrak{A}(X)(U_n) = \mathbb{C}$ for all n. For each n define $f_n \in \mathfrak{A}(U_n)$ via

$$f_n(n) = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Then there is no $f \in C(\hat{\mathbb{N}})$ with $f|_{U_n} = f_n(n)$ for all n, hence $\mathfrak{A}(X)$ is not a sheaf.

(iv) *Suppose now that X is an extremally disconnected compact Hausdorff space. Then the presheaf $\mathfrak{A}(X)$ of part (i) is a sheaf (and in fact equals $\mathfrak{C}(X)$ of part (ii)).*

Indeed, let $U \in \mathcal{O}(X)$ and $\{U_i : i \in I\}$ an open cover of U. Note that $\overline{U}$ is open in X, hence $\overline{U}$ is extremally disconnected [29, 1H5]. It follows that each $\overline{U}_i$ is open in $\overline{U}$, hence $U_0 = \bigcup_{i \in I} \overline{U}_i$ is an open subset of $\overline{U}$.

Consider a bounded collection $\{f_i : i \in I\}$ with $f_i \in \mathfrak{A}(X)(U_i)$ for all i, such that $f_i|_{\overline{U}_i \cap \overline{U}_j} = f_j|_{\overline{U}_i \cap \overline{U}_j}$ for all $i, j \in I$. First we define $f : U_0 \to \mathbb{C}$ via $f(x) = f_i(x)$ for $x \in \overline{U}_i$, then f is well-defined and bounded. Moreover, since $\overline{U}_i$ is open for all i, it follows that for each $x \in X$, f agrees with the continuous function f_i on some neighbourhood $\overline{U}_i$ of x, hence f is continuous on U_0.

Finally, since U_0 is an open subset of $\overline{U}$, there is $\overline{f} \in C^b(\overline{U}) = C(\overline{U})$ extending f [29, 1H6]. Hence $\overline{f}|_{\overline{U}_i} = f_i$ for all $i \in I$, which shows that $\mathfrak{A}(X)$ is a sheaf.

The fact that $\mathfrak{A}(X)(U) = \mathfrak{C}(X)(U)$ for all $U \in \mathcal{O}(X)$ is evident from [29, 1H6] also.

Consider now a C*-algebra A and open subsets U, V of $\mathrm{Prim}(A)$ with $V \subseteq U$. Then U and V correspond to closed two sided ideals $A(U)$ and $A(V)$ respectively, such that $A(U) = k(\mathrm{Prim}(A) \backslash U)$ and $A(V) = k(\mathrm{Prim}(A) \backslash V)$.

Since $V \subseteq U$, $\mathrm{Prim}(A) \backslash U \subseteq \mathrm{Prim}(A) \backslash V$, and so

$$A(V) = k\left(\mathrm{Prim}(A) \backslash V\right) \subseteq k\left(\mathrm{Prim}(A) \backslash U\right) = A(U).$$

In particular, $A(V)$ is a closed two-sided ideal of $A(U)$.

This construction gives non-commutative generalisations of the presheaf $\mathfrak{A}(X)$ and the sheaf $\mathfrak{C}(X)$ of Example 4.1.2. Indeed, for open subsets $U, V \in \mathcal{O}(\mathrm{Prim}(A)$ with $V \subseteq U$, we have $\mathrm{Prim}(A) \backslash \overline{U} \subseteq \mathrm{Prim}(A) \backslash \overline{V}$, and so $A(\mathrm{Prim}(A) \backslash \overline{U}) = k(U)$ is an ideal of $A(\mathrm{Prim}(A) \backslash \overline{V}) = k(V)$. Thus $k(V)/k(U)$ is an ideal of $A/k(U)$ and we get a $*$-homomorphism

$$\frac{A}{k(U)} \to \frac{A/k(U)}{k(V)/k(U)} \equiv \frac{A}{k(V)}.$$

Setting $\mathfrak{A}(U) = A/k(U)$ and $\Phi_{V,U} : A/k(U) \to A/k(V)$ the quotient $*$-homomorphisms described above for all $U, V \in \mathcal{O}(X)$ with $V \subseteq U$, gives a presheaf $\mathfrak{A}$

of C*-algebras over Prim(A). We remark that when $A = C_0(X)$ is commutative, $\mathfrak{A}$ is precisely the presheaf $\mathfrak{A}(X)$ of Example 4.1.2(i).

Let A be a C*-algebra and I a norm-closed two sided ideal of A. Then I is also an ideal of $M(A)$, and hence there is a unique $*$-homomorphism $\pi_I : M(A) \to M(I)$ extending the identity on I [20, Proposition 3.7].

Now if we first associate $A(U)$ to $U \in \mathcal{O}(\mathrm{Prim}(A))$, then for any $V \in \mathcal{O}(\mathrm{Prim}(A))$ with $V \subseteq U$, $A(V)$ is a closed two-sided ideal of $A(U)$. Thus by [20, Proposition 3.7], we get a $*$-homomorphism $\Phi_{V,U} : M(A(U)) \to M(A(V))$ extending the identity on $A(V)$.

Definition 4.1.3. For a C*-algebra A, we define the *multiplier sheaf* $\mathfrak{M}_A$ *of* A as the sheaf over Prim(A) with $\mathfrak{M}_A(U) = M(A(U))$ and the restriction mappings $\Phi_{V,U} : \mathfrak{M}_A(U) \to \mathfrak{M}_A(V)$ defined as above [3, Propostion 3.4].

It clear that when $A = C_0(X)$ is a commutative C*-algebra, then for all $U \in \mathcal{O}(X)$ we have $\mathfrak{M}_A(U) = M(C_0(U)) = C^b(U)$, and that the restriction $*$-homomorphisms $\Phi_{V,U}$ are the usual restriction mappings for all $U, V \in \mathcal{O}(X)$ with $V \subseteq U$. Thus the multiplier sheaf generalises the sheaf $\mathfrak{C}(X)$ of Example 4.1.2(ii).

Definition 4.1.4. Let X and Y be topological spaces, $\psi : X \to Y$ a continuous map and $\mathfrak{A}$ a presheaf of C*-algebras over X. Then setting $(\psi_*\mathfrak{A})(U) = \mathfrak{A}(\psi^{-1}(U))$ for all $U \in \mathcal{O}(Y)$, and $\psi_*\Phi_{V,U} = \Phi_{\psi^{-1}(V),\psi^{-1}(U)}$ whenever $V, U \in \mathcal{O}(Y)$ with $V \subseteq U$, defines a presheaf $\psi_*\mathfrak{A}$ of C*-algebras over Y. The presheaf $\psi_*\mathfrak{A}$ is called the *direct image* of the presheaf $\mathfrak{A}$ under the map ψ.

We remark that if the presheaf $\mathfrak{A}$ in Definition 4.1.4 is a sheaf, then the same is true for $\psi_*\mathfrak{A}$.

Let I be a directed set and suppose that $\{A_i\}_{i\in I}$ is a family of C*-algebras indexed by I. Suppose that for all $i, j \in I$ with $i \leq j$ there is a $*$-homomorphism $\Psi_{j,i} : A_i \to A_j$, such that whenever $i \leq j \leq k$ we have $\Psi_{k,j} \circ \Psi_{j,i} = \Psi_{k,i}$. Then the family $(A_i, \Psi_{j,i})$ is called a *directed system of C*-algebras.*

Denote by

$$A' = \left\{ (a_i)_{i\in I} \in \prod_{i\in I} A_i : \exists\, i_0 \in I \text{ with } a_j = \Psi_{j,i_0}(a_{i_0})\ \forall\, j \geq i_0 \right\}, \tag{4.1.1}$$

then A' is a $*$-algebra in the obvious way. Let $p : A' \to \mathbb{R}$ be the seminorm $p\left((a_j)\right) = \inf_{j\geq i_0} \|a_j\|$, where $i_0 \in I$ is such that $a_j = \Psi_{j,i_0}(a_{i_0})$ whenever $j \geq i_0$. Observe that

p has the C*-property $p(a^*a) = p(a)^2$, and hence induces a C*-norm on the $*$-algebra $A'/\ker(p)$.

We define the *direct limit* $\varinjlim_{i\in I} A_i$ of the directed system $(A_i, \Psi_{j,i})$ as the C*-completion of $A'/\ker p$.

Definition 4.1.5. Suppose that X is a topological space and $\mathfrak{A}$ a presheaf of C*-algebras over X. For $x_0 \in X$, let $\{U_i : i \in I\}$ be the downwards-directed set of open neighbourhoods of x_0 in X. Then we get a directed system of C*-algebras $(\mathfrak{A}(U_i), \Phi_{U_j,U_i})$, and we define

$$\mathfrak{A}_{x_0} = \varinjlim_{i\in I} \mathfrak{A}(U_i),$$

the *stalk of* $\mathfrak{A}$ at x_0.

We remark that for each $U \in \mathcal{O}(X)$ and $x_0 \in U$ we get a $*$-homomorphism

$$\Phi_{x_0,U} : \mathfrak{A}(U) \to \mathfrak{A}_{x_0} \tag{4.1.2}$$

which we regard as the evaluation map at x_0. To see this, let $\mathfrak{A}'_{x_0}$ be defined as in (4.1.1), and let $p_{x_0} : \mathfrak{A}'_{x_0} \to \mathbb{R}$ be the corresponding seminorm. Then each $a \in \mathfrak{A}(U)$ defines $(a_i) \in \mathfrak{A}'_{x_0}$ via

$$a_i = \begin{cases} 0 & U_i \not\subseteq U \\ \Phi_{U_i,U}(a) & U_i \subseteq U \end{cases}$$

and the map $a \mapsto (a_i)$ is clearly injective. Thus we may regard the quotient mapping $\mathfrak{A}'_{x_0}/\ker(p_{x_0})$, when restricted to the image of $\mathfrak{A}(U)$, as a $*$-homomorphism $\Phi_{x_0,U} : \mathfrak{A}(U) \to \mathfrak{A}_{x_0}$ for all $x_0 \in U$.

The stalks of a sheaf of C*-algebras are in general difficult to compute. In the case of the multiplier sheaf $\mathfrak{M}$ of a C*-algebra A over Prim(A), one might expect that $(\mathfrak{M}_A)_P = M(A/P)$ for all $P \in \mathrm{Prim}(A)$. However, the following two examples show that this need not be the case.

Example 4.1.6. *Let A be the C*-algebra of sequences $x = (x_n) \subset M_2(\mathbb{C})$ such that $x_n \to \mathrm{diag}(\lambda_1(x), \lambda_2(x))$. Then the irreducible representations of A are given by $\varepsilon_n : A \to M_2(\mathbb{C})$, $\varepsilon_n(x) = x_n$ for $n \in \mathbb{N}$, and $\lambda_i : A \to \mathbb{C}$ for $i = 1, 2$. Thus*

$$\mathrm{Prim}(A) = \{\ker(\varepsilon_n) : n \in \mathbb{N}\} \cup \{\ker(\lambda_i) : i = 1, 2\},$$

where the points $\ker(\varepsilon_n)$ *are isolated for all* $n \in \mathbb{N}$*, and each* $\ker(\lambda_i)$ *has a neighbourhood basis consisting of sets of the form*

$$U_{i,n_0} := \{\ker(\varepsilon_n) : n \geq n_0\} \cup \{\ker \lambda_i\}$$

for some $n_0 \in \mathbb{N}, i = 1, 2$.

For each $n_0 \in \mathbb{N}$*, the multiplier algebra* $M(A(U_{1,n_0}))$ *of the ideal* $A(U_{1,n_0})$ *corresponding to a neighbourhood* U_{1,n_0} *of* $\ker(\lambda_1)$ *is given by those sequences* $y = (y_n) \in C^b(\mathbb{N}, M_2(\mathbb{C}))$ *such that*

(i) $y_n^{(1,2)} \to 0$ *and* $y_n^{(2,1)} \to 0$,

(ii) $y_n^{(1,1)} \to \lambda_1(y)$ *and*

(iii) the sequence $(y_n^{(2,2)})_{n \geq n_0}$ *is bounded,*

see e.g. [4, Example 4.5]. It follows that the stalk of $\mathfrak{M}$ *at* $\ker(\lambda_1)$ *is given by*

$$(\mathfrak{M}_A)_{\ker(\lambda_1)} = \varinjlim_{n_0 \to \infty} M(A(U_{1,n_0})) \equiv \mathbb{C} \oplus \ell^\infty / c_0 \equiv \mathbb{C} \oplus C(\beta\mathbb{N} \backslash \mathbb{N}).$$

Similarly, $(\mathfrak{M}_A)_{\ker(\lambda_2)} \equiv C(\beta\mathbb{N}\backslash\mathbb{N}) \oplus \mathbb{C}$.

On the other hand, the corresponding primitive quotients $A/\ker(\lambda_i)$ *are one-dimensional for* $i = 1, 2$.

Example 4.1.7. *Let* H *be a separable infinite dimensional Hilbert space and* $A = C(\hat{\mathbb{N}}, B(H))$*, where* $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ *is the one-point compactification of* $\mathbb{N}$. *Then* $\mathrm{Prim}(A) = \hat{\mathbb{N}} \times \{\{0\}, K(H)\}$. *We will compute the stalk of the multiplier sheaf of* A *over* $\mathrm{Prim}(A)$ *at each point of* $\mathrm{Prim}(A)$.

We first describe the neighbourhood bases of the points in $\mathrm{Prim}(A)$ *and the ideals of* A *to which each open neighbourhood corresponds.*

(i) For each $n \in \mathbb{N}$*, the one-point sets* $\{(n, \{0\})\}$ *are open and have closure*

$$\overline{\{(n, \{0\})\}} = \{(n, \{0\}), (n, K(H))\}.$$

It is easily seen that the corresponding ideal $A((n, \{0\}))$ *is identified with those* $f \in A$ *with* $f(m) = 0$ *for* $m \neq n$ *and* $f(n) \in K(H)$. *Thus* $A((n, \{0\}))$ *is canonically identified with* $K(H)$.

(ii) The points $(n, K(H))$ are closed, and have a minimal open neighbourhood O_n given by

$$O_n = \{(n, \{0\}), (n, K(H))\}.$$

$A(O_n)$ is then identified with those $f \in A$ such that $f(m) = 0$ whenever $m \neq n$, hence $A(O_n) \equiv B(H)$.

(iii) The point $(\infty, \{0\})$ has a neighbourhood basis consisting of sets of the form

$$U_{n_0} = \{(n, \{0\}) : n \geq n_0\}$$

for some $n_0 \in \mathbb{N}$. Thus

$$A(U_{n_0}) = \{f \in A : f(n) = 0 \text{ for } n < n_0, f(n) \in K(H) \text{ for } n \geq n_0\},$$

and $A(U_{n_0})$ is identified with $C^b(\hat{\mathbb{N}}, K(H))$.

(iv) The point $(\infty, K(H))$ has a neighbourhood basis consisting of sets of the form

$$V_{n_0} = \{(n, \{0\}), (n, K(H)) : n \geq n_0\}$$

for some $n_0 \in \mathbb{N}$. Hence

$$A(V_{n_0}) = \{f \in A : f(n) = 0 \text{ for } n < n_0\},$$

which is $*$-isomorphic to A.

It is clear from (i) and (ii) that for each $n \in \mathbb{N}$ the stalks of $\mathfrak{M}_A$ satisfy $(\mathfrak{M}_A)_{(n,\{0\})} = (\mathfrak{M}_A)_{(n,K(H))} = B(H)$.

Considering the point $(\infty, \{0\})$, we have $\mathfrak{M}(U_{n_0}) \equiv C^b(\hat{\mathbb{N}}, B(H)_{s*})$ for all $n_0 \in \mathbb{N}$, where $C^b(\hat{\mathbb{N}}, B(H)_{s*})$ denotes the C^*-algebra of norm-bounded stong-$*$ continuous $B(H)$-valued functions on $\hat{\mathbb{N}}$, see [1, Corollary 3.4] or [51, Proposition 2.57]. Let $\{U_i : i \in I\}$ be the set of neighbourhoods of $(\infty, \{0\})$ in $\mathrm{Prim}(A)$ and $(f_i)_{i\in I} \in \mathfrak{M}'_{(\infty,\{0\})}$. Then there is $n_0 \in \mathbb{N}$ and $f \in \mathfrak{M}(U_{n_0})$ such that $f_i = f|_{U_i}$ for all i with $U_i \subseteq U_{n_0}$.

With the identification $\mathfrak{M}(U_{n_0}) = C^b(\hat{\mathbb{N}}, B(H)_{s*})$, and denoting by $p : \mathfrak{M}'_{(\infty,\{0\})} \to \mathbb{R}_+$ the canonical seminorm, it is clear that $p(f) = \limsup \|f(n)\| = 0$ if and only if $\|f(n)\| \to 0$. It follows that $\mathfrak{M}_{(\infty,\{0\})}$ is $*$-isomorphic to

$$\frac{C^b(\hat{\mathbb{N}}, B(H)_{s*})}{C_0(\mathbb{N}, B(H))}.$$

The fact that this quotient C-algebra is not isomorphic to $B(H)$, the multiplier algebra of the primitive quotient of A at $(\infty, \{0\})$, may be deduced from [9, Theorem 4.9].*

For the point $(\infty, K(H))$, it is clear that for each $n_0 \in \mathbb{N}$, $A(V_{n_0})$ is unital and hence equals $M(A(V_{n_0}))$. Thus the stalk $\mathfrak{M}_{(\infty,K(H))}$ is the quotient of A modulo those sequences converging to zero in norm, which is easily seen to be $*$-isomorphic to $B(H)$. Again, this differs from the multiplier algebra of the primitive quotient of A at this point, which is identified with the Calkin algebra $B(H)/K(H)$.

4.2 Sheaves and C*-bundles

In this section we examine the relationship between (pre-) sheaves of C*-algebras and C*-bundles. In [33], Hofmann showed that there is a natural equivalence between sheaves and bundles of certain classes of Banach spaces over hereditarily paracompact Hausdorff spaces. In the context of C*-algebras, this problem was considered by Ara and Mathieu in [3], with particular reference to the study of local multiplier algebras.

We shall only consider bundles and sheaves over locally compact Hausdorff spaces. Many of the constructions in this section are still valid over more general classes of spaces, although our definition of (H)-C*-bundle (Definition 2.2.1) requires additional assumptions in these cases, in order to ensure that the (local) section algebra is non-trivial, for example [33, Definition 3.3] and [3, Definition 5.1]. As a consequence, some of the results from [33] and [3] quoted in this section are not given in their utmost generality. In particular, Theorems 4.2.1 and 4.2.3 are both valid over non-Hausdorff spaces (provided that [3, Definition 5.1] is taken as the definition of a (H)-C*-bundle).

The following Theorem describes how the local section algebras of an upper-semicontinuous (H)-C*-bundle give rise to a sheaf of C*-algebras over the same space.

Theorem 4.2.1. *[3, Theorem 5.3] Let $(\mathcal{A}, X, p)$ be an upper-semicontinuous (H)-C*-bundle over a locally compact Hausdorff space X. Then for each $U \in \mathcal{O}(X)$, set $\mathfrak{A}(U) = \Gamma^b(U, \mathcal{A})$, and define $\Phi_{V,U} : \Gamma^b(U, \mathcal{A}) \to \Gamma^b(V, \mathcal{A})$ via*

$$\Phi_{V,U}(\gamma)(x) = \gamma(x) \text{ for all } x \in V,$$

whenever $V, U \in \mathcal{O}(X)$ with $V \subseteq U$. Then the assignment $U \mapsto \Gamma^b(U, \mathcal{A})$, together with these restriction mappings $\Phi_{V,U}$, defines a sheaf $\Gamma^b(-, \mathcal{A})$ of C-algebras over X*

Definition 4.2.2. Given an upper-semicontinuous (H)-C*-bundle $(\mathcal{A}, X, p)$ over a locally compact Hausdorff space X, the sheaf $\Gamma^b(-, \mathcal{A})$ defined in Theorem 4.2.1 is called the *local section sheaf* of the bundle $(\mathcal{A}, X, p)$.

Conversely, in [3, Theorem 5.6], it is shown how to associate an upper-semicontinuous (H)-C*-bundle to a presheaf $\mathfrak{A}$:

Theorem 4.2.3. *[3, Theorem 5.6] Let X be a locally compact Hausdorff space and $\mathfrak{A}$ a presheaf of C*-algebras over X. Then there is a canonically associated upper semicontinuous (H)-C*-bundle $(\mathcal{A}, X, p)$ over X such that*

(i) For each $x \in X$ the fibre A_x of $\mathcal{A}$ at x is given by the stalk $\mathfrak{A}_x$ of $\mathfrak{A}$ at x,

(ii) For each $U \in \mathcal{O}(X)$ we get an injective $*$-homomorphism $\nu_U : \mathfrak{A}(U) \to \Gamma^b(U, \mathcal{A})$ defined via

$$\nu_U(s)(x) = \Phi_{x,U}(s), \quad \textit{where } x \in U, s \in \mathfrak{A}(U).$$

(iii) For each $x_0 \in X$, $a \in A_{x_0}$, and $\varepsilon > 0$, there is an open neighbourhood U of x_0 and $s \in \mathfrak{A}(U)$ such that $\|\nu_U(s)(x_0) - a\| < \varepsilon$.

Suppose the sheaf $\mathfrak{A}$ of Theorem 4.2.3 is the local section sheaf $\Gamma^b(-, \mathcal{B})$ of some upper-semicontinuous (H)-C*-bundle $(\mathcal{B}, X, q)$ over X. Then the upper-semicontinuous (H)-C*-bundle $(\mathcal{A}, X, p)$ induced by $\mathfrak{A}$ is canonically isomorphic to $(\mathcal{B}, X, q)$; that is, there is a homeomorphism of $\mathcal{A}$ onto $\mathcal{B}$ whose restriction to each fibre A_x is a $*$-isomorphism of A_x onto B_x. In particular, the mapping $\nu_U : \mathfrak{A} \to \Gamma^b(U, \mathcal{A})$ of Theorem 4.2.3(ii) is necessarily a $*$-isomorphism for all $U \in \mathcal{O}(X)$. This is discussed in [3, §5].

Given a presheaf $\mathfrak{A}$ and $U, V \in \mathcal{O}(X)$ with $V \subseteq U$, it is clear from the definition of the $*$-homomorphisms $\Phi_{x_0,U}$ and $\Phi_{x_0,V}$ that $\Phi_{x_0,U} = \Phi_{x_0,V} \circ \Phi_{V,U}$ for all $x_0 \in V$. It follows that for all $a \in \mathfrak{A}(U)$ and $x_0 \in V$, we have $\nu_U(a)(x_0) = (\nu_V \circ \Phi_{V,U})(a)(x_0)$. In particular, the diagram

$$\begin{array}{ccc} \mathfrak{A}(U) & \xrightarrow{\nu_U} & \Gamma^b(U, \mathcal{A}) \\ {\scriptstyle \Phi_{V,U}} \downarrow & & \downarrow {\scriptstyle \gamma \mapsto \gamma|_V} \\ \mathfrak{A}(V) & \xrightarrow{\nu_V} & \Gamma^b(V, \mathcal{A}) \end{array}$$

commutes for all $U, V \in \mathcal{O}(X)$ with $V \subseteq U$.

A natural question which arises from this construction is as follows; starting from a sheaf $\mathfrak{A}$ of a C*-algebras over X, under what conditions may we make the identification $\mathfrak{A}(U) \equiv \Gamma^b(U, \mathcal{A})$ for all $U \in \mathcal{O}(X)$, where $(\mathcal{A}, X, p)$ is the associated (H)-C*-bundle over X? Evidently, this will occur precisely when the maps $\nu_U : \mathfrak{A}(U) \to \Gamma^b(U, \mathcal{A})$ are surjective for all $U \in \mathcal{O}(X)$.

Definition 4.2.4. Let X be a locally compact Hausdorff space and $\mathfrak{C}(X)$ the sheaf $\mathfrak{C}(X)(U) = C^b(U)$ for all $U \in \mathcal{O}(X)$ together with the usual restriction mappings. A sheaf $\mathfrak{A}$ of C*-algebras over X is said to be a $\mathfrak{C}(X)$*-sheaf* if for every nonempty $U \in \mathcal{O}(X)$, $\mathfrak{A}(U)$ is an essential $C^b(U)$ Banach module ($1 \cdot a = a$ for all $a \in \mathfrak{A}(U)$[1]), such that the restriction mappings commute in the following sense: for $V, U \in \mathcal{O}(X)$ with $V \subseteq U$, $a \in \mathfrak{A}(U)$ and $f \in C^b(U)$,

$$\Phi_{V,U}(f \cdot a) = (f|_V) \cdot \Phi_{V,U}(a).$$

Most of the sheaves of interest to us are $\mathfrak{C}(X)$-sheaves. The following example shows that one may easily construct sheaves that are not $\mathfrak{C}(X)$-sheaves in general.

Example 4.2.5. *Let X and Y be locally compact Hausdorff spaces, where Y contains at least two points, $p : X \times Y \to X$ the (open) projection mapping, and $\mathfrak{A}$ a sheaf of C*-algebras over X, with $\mathfrak{A}(X) \neq \{0\}$. For $U \in \mathcal{O}(X \times Y)$, set $(p^*\mathfrak{A})(U) = \mathfrak{A}(p(U))$ and let $p^*\Phi_{V,U} = \Phi_{p(V),p(U)} : (p^*\mathfrak{A})(U) \to (p^*\mathfrak{A})(V)$ whenever $U, V \in \mathcal{O}(X \times Y)$ with $V \subseteq U$. Then $p^*\mathfrak{A}$ is a sheaf[2] of C*-algebras over $X \times Y$.*

We claim that $p^*\mathfrak{A}$ is not a $\mathfrak{C}(X \times Y)$-sheaf. Indeed, let V_1, V_2 be disjoint open subsets of Y, with disjoint compact closures in Y, and take $f \in C^b(X \times Y)$ with $f|_{X \times V_1} \equiv 0$ and $f|_{X \times V_2} \equiv 1$. Then $(p^*\mathfrak{A})(X \times V_1) = (p^*\mathfrak{A})(X \times V_2) = \mathfrak{A}(X)$.

Let a be a nonzero-element of $\mathfrak{A}(X)$, then since

$$p^*\Phi_{X \times V_1, X \times Y} = p^*\Phi_{X \times V_2, X \times Y} = \Phi_{X,X} = \mathrm{id}_{\mathfrak{A}(X)},$$

it follows that $p^*\Phi_{X \times V_i, X \times Y}(a) \neq 0$ for $i = 1, 2$. Thus, if $p^*\mathfrak{A}$ were a $\mathfrak{C}(X \times Y)$-sheaf, it would follow that there was an element $f \cdot a \in \mathfrak{A}(X) = p^*\mathfrak{A}(X \times Y)$. Since the restriction

[1]Equivalently, $C^b(U) \cdot \mathfrak{A}(U) = \mathfrak{A}(U)$ by [25, Corollary 15.3]

[2]The sheaf $p^*\mathfrak{A}$ is the *inverse image sheaf* of $\mathfrak{A}$ under the map p.

mappings must commute in the sense of Definition 4.2.4, we would then have

$$\begin{aligned} f\cdot a = \Phi_{X,X}(f\cdot a) &= p^*\Phi_{X\times V_1,X\times Y}(f\cdot a) \\ &= (f|_{X\times V_1})\cdot(p^*\Phi_{X\times V_1,X\times Y}(a)) = 0 \end{aligned}$$

while also

$$\begin{aligned} f\cdot a = \Phi_{X,X}(f\cdot a) &= p^*\Phi_{X\times V_2,X\times Y}(f\cdot a) \\ &= (f|_{X\times V_2})\cdot(p^*\Phi_{X\times V_2,X\times Y}(a)) \\ &= (1_{X\times V_2})\cdot a = a \neq 0, \end{aligned}$$

which is clearly impossible.

The following proposition shows that the local section sheaf of an upper-semicontinuous (H)-C*-bundle is a $\mathfrak{C}(X)$-sheaf in a natural way. This fact was observed in [3, §5], we include a proof for completeness.

Proposition 4.2.6. *Let $(\mathcal{A}, X, p)$ be an upper-semicontinuous (H)-C*-bundle over a locally compact Hausdorff space X. Then for each $U \in \mathcal{O}(X)$, $\gamma \in \Gamma^b(U,\mathcal{A})$ and $f \in C^b(U)$, the section $f\cdot\gamma : U \to \mathcal{A}$, where*

$$(f\cdot\gamma)(x) = f(x)\gamma(x) \text{ for all } x \in U,$$

belongs to $\Gamma^b(U,\mathcal{A})$. Moreover, the local section sheaf $\Gamma^b(-,\mathcal{A})$ is a $\mathfrak{C}(X)$-sheaf with respect to this action of $C^b(U)$ on $\Gamma^b(U,\mathcal{A})$.

Proof. With U,γ and f as in the statement of the proposition, then the fact that $f\cdot\gamma$ belongs to $\Gamma^b(U,\mathcal{A})$ is shown in [58, Lemma C.22]. Moreover, it is clear from the definition of $f\cdot\gamma$ that

$$\|f\cdot\gamma\| = \sup_{x\in U}\|f(x)\gamma(x)\| = \sup_{x\in U}|f(x)|\cdot\|\gamma(x)\| \leq \|f\|\,\|\gamma\|,$$

and that $1\cdot\gamma = \gamma$ for all $\gamma \in \Gamma^b(U,\mathcal{A})$. It follows that $\Gamma^b(U,\mathcal{A})$ is an essential $C^b(U)$-Banach module.

Given $U, V \in \mathcal{O}(X)$ with $V \subseteq U$, then for all $f \in C^b(U), \gamma \in \Gamma^b(U,\mathcal{A})$ and $x \in V$

$$((f\cdot\gamma)|_V)(x) = (f\cdot\gamma)(x) = f(x)\gamma(x) = ((f|_V)\cdot(\gamma|_V))(x)$$

whence $(f\cdot\gamma)|_V = (f|_V)\cdot(\gamma|_V)$. Hence $\Gamma^b(-,\mathcal{A})$ is a $\mathfrak{C}(X)$-sheaf as required. □

As a consequence of Proposition 4.2.6, starting from a sheaf $\mathfrak{A}$, a necessary condition for the maps ν_U of Theorem 4.2.3(ii) to be surjective is that $\mathfrak{A}$ be a $\mathfrak{C}(X)$-sheaf. The following Lemma shows that for any $\mathfrak{C}(X)$ sheaf $\mathfrak{A}$, the action of $C^b(U)$ on $\mathfrak{A}(U)$ corresponds precisely to pointwise multiplication by scalar valued functions when we identify $\mathfrak{A}(U)$ with $\nu_U(\mathfrak{A}(U)) \subseteq \Gamma^b(U, \mathcal{A})$. Again, this fact was observed for unital $\mathfrak{C}(X)$-sheaves in [3, §5].

Lemma 4.2.7. *Let $\mathfrak{A}$ be a $\mathfrak{C}(X)$-sheaf. Then for $s \in \mathfrak{A}(U)$ and $f \in C^b(U)$, we have*

$$\nu_U(f \cdot s)(x_0) = \Phi_{x_0,U}(f \cdot s) = f(x_0)(\nu_U(s))(x_0)$$

for all $x_0 \in U$.

Proof. Let $\varepsilon > 0$ be given, then there is an open neighbourhood W of x_0 contained in U such that $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$ for all $x \in W$. In particular, $\|(f|_W) - f(x_0)\|_W < \varepsilon$, and so for any $s \in \mathfrak{A}(U)$,

$$\begin{aligned} \|\Phi_{W,U}(f \cdot s - f(x_0)s)\| &= \|\Phi_{W,U}(f \cdot s) - f(x_0)\Phi_{W,U}(s)\| \\ &= \|((f|_W) - f(x_0)) \cdot \Phi_{W,U}(s)\| \\ &\leq \|(f|_W) - f(x_0)\|_W \|\Phi_{W,U}(s)\| \\ &< \varepsilon\|\Phi_{W,U}(s)\| \leq \varepsilon \|s\| . \end{aligned}$$

It follows that $\Phi_{x_0,U}(f \cdot s - f(x_0)s) = 0$, and so $\nu_U(f \cdot s)(x_0) = f(x_0)\nu_U(s)(x_0)$. □

If $\mathfrak{A}$ is a $\mathfrak{C}(X)$-sheaf, then $\mathfrak{A}(U)$ is a $C(\beta U)$-algebra for all $U \in \mathcal{O}(X)$. In particular, $\mathfrak{A}(X)$ is a $C(\beta X)$-algebra, and so $\mathfrak{A}(X)$ is canonically isomorphic to the section algebra of an upper-semicontinuous (H)-C*-bundle over βX. We wish to compare this bundle to that defined by the sheaf $\mathfrak{A}$ in Theorem 4.2.3. First we describe how a sheaf A over X may be identified with a sheaf over βX via the direct image functor.

If we denote by $\iota : X \hookrightarrow \beta X$ the canonical inclusion, then the direct image sheaf $\iota_*\mathfrak{A}$ over βX is defined via

$$\iota_*\mathfrak{A}(V) = \mathfrak{A}(\iota^{-1}(V)) = \mathfrak{A}(V \cap X), \text{ where } V \in \mathcal{O}(\beta X),$$

with restriction mappings $(\iota_*\Phi)_{V,U} : (\iota_*\mathfrak{A})(U) \to (\iota_*\mathfrak{A})(V)$ for $U, V \in \mathcal{O}(\beta X)$ with $V \subseteq U$ defined as

$$(\iota_*\Phi)_{V,U} = \Phi_{V\cap X, U\cap X}.$$

We will denote by $(\iota_*\mathcal{A}, \beta X, \iota_* p)$ the bundle over βX defined by $\iota_*\mathfrak{A}$, and by $(\iota_*\nu)_U : (\iota_*\mathfrak{A})(U) \to \Gamma^b(U, \iota_*\mathfrak{A})$ the injective $*$-homomorphisms of Theorem 4.2.3(ii) for all $U \in \mathcal{O}(\beta X)$.

The following Lemma shows that when $\mathfrak{A}$ is a $\mathfrak{C}(X)$-sheaf, then the stalks of $\mathfrak{A}$ and $\iota_*\mathfrak{A}$ agree at points of X. Moreover, we show that every element of a stalk is the image of a global section of the sheaf.

Lemma 4.2.8. *Let X be a locally compact Hausdorff space, $\mathfrak{A}$ a $\mathfrak{C}(X)$-sheaf of C^*-algebras over X, and $\iota_*\mathfrak{A}$ the direct image sheaf corresponding to the inclusion mapping $\iota : X \hookrightarrow \beta X$.*

(i) $\iota_*\mathfrak{A}$ is a $\mathfrak{C}(\beta X)$-sheaf, where for all $U \in \mathcal{O}(\beta X)$ the action of $C^b(U)$ on $(\iota_*\mathfrak{A})(U)$ is given by

$$f \cdot a = (f|_{U\cap X}) \cdot a, \text{ for all } a \in (\iota_*\mathfrak{A})(U), f \in C^b(U).$$

(ii) For each $x_0 \in X$, the stalks $\mathfrak{A}_{x_0}$ and $(\iota_*\mathfrak{A})_{x_0}$ are isomorphic,

(iii) For each $x_0 \in X$ (respectively βX) and $a \in \mathfrak{A}_{x_0}$ there is a global section $\gamma \in \nu_X(\mathfrak{A}(X)) \subseteq \Gamma^b(X, \mathcal{A})$ (respectively $\gamma \in (\iota_*\nu)_{\beta X}(\iota_*\mathfrak{A})(\beta X))$ with $\gamma(x_0) = a$.

Proof. (i) For every nonempty $U \in \mathcal{O}(\beta X)$ we have $(\iota_*\mathfrak{A})(U) = \mathfrak{A}(U \cap X)$ by definition. Since $\mathfrak{A}(U \cap X)$ is an essential $C^b(U \cap X)$-module, and $C^b(U \cap X)$ is a $C^b(U)$-module with respect to the usual restriction mappings, it is clear that $(\iota_*\mathfrak{A})(U)$ is an essential $C^b(U)$-module. Moreover, for $V, U \in \mathcal{O}(\beta X)$ with $V \subseteq U$, the restriction mappings satisfy

$$\begin{aligned}(\iota_*\Phi)_{V,U}(f \cdot a) &= \Phi_{V\cap X, U\cap X}((f|_{U\cap X}) \cdot a) \\ &= (f|_{V\cap X}) \cdot \Phi_{V\cap X, U\cap X}(a) \\ &= (f|_V) \cdot ((\iota_*\Phi)_{V,U}(a)).\end{aligned}$$

for all $a \in \iota_*\mathfrak{A}(U)$ and $f \in C^b(U)$.

(ii) For $x_0 \in X$ denote by $\{V_i : i \in I\} \subseteq \mathcal{O}(\beta X)$ the directed set of open neighbourhoods of x_0 in βX. Then we have $\iota_*\mathfrak{A}(V_i) = \mathfrak{A}(V_i \cap X)$ for all $i \in I$, and since X is open in βX there is an index $i_0 \in I$ with $V_{i_0} = X$, so that $V_i \subseteq X$

whenever $i \geq i_0$. In particular, $\{V_i : i \geq i_0\} \subseteq \mathcal{O}(X)$ is the directed set of open neighbourhoods of x_0 in X. Thus

$$\begin{aligned}(\iota_*\mathfrak{A})_{x_0} := \varinjlim_{i\in I} \iota_*\mathfrak{A}(V_i) &= \varinjlim_{i\geq i_0} \iota_*\mathfrak{A}(V_i) \\ &= \varinjlim_{i\geq i_0} \mathfrak{A}(V_i) \\ &= \mathfrak{A}_{x_0}.\end{aligned}$$

(iii) For $\varepsilon > 0$, there is $U \in \mathcal{O}(X)$ and $s \in \mathfrak{A}(U)$ such that $\|\nu_U(s)(x_0) - a\| < \varepsilon$ by Theorem 4.2.3(iii). Let $f \in C^b(X)$ such that $f(x_0) = 1$ and with f supported on a compact neighbourhood K of x_0 contained in U, and set $V = X \backslash K$. Then for any $x \in U \cap V$, we have

$$\nu_{(U\cap V)}\left(\Phi_{(U\cap V),U}\left((f|_U)\cdot s\right)\right)(x) = f(x)\nu_{U\cap V}(s) = 0.$$

Since $\nu_{(U\cap V)}$ is injective, this implies that $\Phi_{(U\cap V),U}\left((f|_U)\cdot s\right) = 0$.

Now we have open sets U, V with $U \cup V = X$ and a pair of sections $0 \in \mathfrak{A}(V)$ and $(f|_U)\cdot s \in \mathfrak{A}(U)$ whose restrictions to $U \cap V$ agree. Since $\mathfrak{A}$ is a sheaf, there is $s' \in \mathfrak{A}(X)$ with $\Phi_{U,X}(s') = (f|_U)\cdot s$. In particular, $\|\nu_X(s')(x_0) - a\| < \varepsilon$, so $\nu_X(\mathfrak{A}(X))(x_0)$ is dense in $\mathfrak{A}_{x_0}$. Since $\nu_X(\mathfrak{A}(X))(x_0)$ is closed, it must in fact equal $\mathfrak{A}_{x_0}$.

The case of βX follows from applying this result to the $C(\beta X)$-sheaf $\iota_*\mathfrak{A}$.

□

If $\mathfrak{A}$ is a $\mathfrak{C}(X)$-sheaf, then in particular $\mathfrak{A}(X)$ is a $C(\beta X)$-algebra. Denote by $A = \mathfrak{A}(X)$ and for $x_0 \in X$ let $J_{x_0} = \{f \in C(\beta X) : f(x_0) = 0\} \cdot A$, which is closed by the Cohen factorisation Theorem [25, Theorem 16.1].

Proposition 4.2.9. *Let $\mathfrak{A}$ be a $\mathfrak{C}(X)$-sheaf of C*-algebras over a locally compact Hausdorff space X. Then with the above notation, for any $x_0 \in X$, the map $\Phi_{x_0,X} : A \to \mathfrak{A}_{x_0}$ is surjective with kernel J_{x_0}, so that we have a natural identification $\mathfrak{A}_{x_0} = A/J_{x_0}$, such that for each $a \in A$ we have $\nu_X(a)(x_0) = a + J_{x_0}$.*

Proof. We first show that $\ker \Phi_{x_0,X} = J_{x_0}$, so that $a \mapsto a + J_{x_0}$ defines an embedding of A/J_{x_0} into $\mathfrak{A}_{x_0}$. Let $a \in A$ with $\|a\| = 1$ and $f \in C(\beta X)$ with $f(x_0) = 0$, so that

$f \cdot a \in J_{x_0}$. If $\varepsilon > 0$ is given, there is an open neighbourhood W of x_0 such that $\|f(x)\| < \frac{\varepsilon}{2}$ for any $x \in W$. In particular, $\|f|_W\| \leq \frac{\varepsilon}{2}$. Then

$$\begin{aligned} \|\Phi_{W,X}(f \cdot a)\| &= \|(f|_W) \cdot \Phi_{W,X}(a)\| \\ &\leq \|f|_W\|\|\Phi_{W,X}(a)\| \\ &\leq \frac{\varepsilon}{2}\|a\| < \varepsilon. \end{aligned}$$

Hence $f \cdot a \in \ker \Phi_{x_0,X}$, so that $J_{x_0} \subseteq \ker \Phi_{x_0,X}$.

On the other hand, suppose $a \in \ker \Phi_{x_0,X}$ and take $\varepsilon > 0$. Then there is an open neighbourhood W of x_0 such that $\|\Phi_{W,X}(a)\| < \frac{\varepsilon}{2}$. In particular, for any $x \in W$ we have

$$\|(\nu_W \circ \Phi_{W,X})(a)(x)\| = \|\nu_X(a)(x)\| < \frac{\varepsilon}{2}.$$

Let $f : \beta X \to [0,1]$ be continuous with $f(x_0) = 1$ and $f|_{X \setminus W} = 0$. Since ν_X is isometric,

$$\begin{aligned} \|a - (1-f) \cdot a\| &= \sup_{x \in \beta X} \|\nu_X(a - (1-f) \cdot a)(x)\| \\ &= \sup_{x \in \beta X} \|\nu_X(a)(x) - (1-f)(x)\nu_X(a)(x)\| \\ &= \sup_{x \in X} \|f(x)\nu_X(a)(x)\| \leq \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Since $(1-f) \cdot a \in J_{x_0}$ and J_{x_0} is closed, $a \in J_{x_0}$.

In particular, we now have $A/J_{x_0} = A/\ker \Phi_{x_0,X} \subseteq \mathfrak{A}_{x_0}$. Equality then follows from Lemma 4.2.8(iii). □

We remark that, with $\mathfrak{A}$ as in Proposition 4.2.9, a similar identification may be obtained for the stalks of the $\mathfrak{C}(\beta X)$-sheaf $\iota_*\mathfrak{A}$. Indeed, let $A = (\iota_*\mathfrak{A})(\beta X) = \mathfrak{A}(X)$ and $J_{x_0} = \{f \in C(\beta X) : f(x_0) = 0\} \cdot A$ for $x_0 \in \beta X$. Then we see that $(\iota_*\mathfrak{A})_{x_0} = A/J_{x_0}$ for all $x_0 \in \beta X$ also, and that for all $a \in A$ we have the identification

$$(\iota_*\Phi)_{x_0,\beta X}(a) = (\iota_*\nu)(\beta X)(a)(x_0) = a + J_{x_0}.$$

Theorem 4.2.10. *Let $\mathfrak{A}$ be a $\mathfrak{C}(X)$-sheaf of C^*-algebras over a locally compact Hausdorff space X, and $(\mathcal{A}, X, p)$ (respectively $(\iota_*\mathcal{A}, \beta X, \iota_*p)$) the upper-semicontinuous (H)-C^*-bundle over X (respectively βX) associated with $\mathfrak{A}$ (respectively $\iota_*\mathfrak{A}$).*

(i) *The map* $(\iota_*\nu)_{\beta X} : (\iota_*\mathfrak{A})(\beta X) \to \Gamma^b(\beta X, \iota_*\mathcal{A})$ *is a* $*$*-isomorphism. Moreover, denoting by* $A = \mathfrak{A}(X) = (\iota_*\mathfrak{A})(\beta X)$ *and by* $J_x = \{f \in C(\beta X) : f(x) = 0\} \cdot A$, *where* $x \in \beta X$, *this isomorphism satisfies*

$$(\iota_*\nu)_{\beta X}(a)(x) = a + J_x, \text{ for all } a \in A, x \in \beta X.$$

(ii) *The bundles* $(\mathcal{A}, X, p)$ *and* $(\iota_*\mathcal{A}, \beta X, \iota_* p)$ *satisfy the relations*

$$\mathcal{A} = (\iota_* p)^{-1}(X) \text{ and } p = (\iota_* p)|_{\mathcal{A}},$$

so that the total space $\mathcal{A}$ *of* $(\mathcal{A}, X, p)$ *is precisely the subspace* $(\iota_* p)^{-1}(X)$ *of* $\iota_*\mathcal{A}$ *with the relative topology. In particular, for each* $U \in \mathcal{O}(X)$ *we may regard* $\nu_U(\mathfrak{A}(U)) \subseteq \Gamma^b(U, \iota_*\mathcal{A})$.

Proof. (i): We remark that since βX is compact, every continuous section $\gamma : \beta X \to \iota_*\mathcal{A}$ is bounded by Lemma 2.2.3, and so we have $\Gamma(\beta X, \iota_*\mathcal{A}) = \Gamma^b(\beta X, \iota_*\mathcal{A}) = \Gamma_0(\beta X, \iota_*\mathcal{A})$. Note also that $(\iota_*\nu)_{\beta X}$ is an injective $*$-homomorphism by Theorem 4.2.3. We show that the algebra of sections $(\iota_*\nu)_{\beta X}(A)$ of the bundle $(\iota_*\mathcal{A}, \beta X, \iota_* p)$ satisfies conditions (i) and (ii) of Proposition 2.2.6, whence it will follow that $(\iota_*\nu)_{\beta X}$ is surjective.

Take $\gamma = (\iota_*\nu)_{\beta X}(a)$, where $a \in A$, and $f \in C(\beta X)$. Then by Lemma 4.2.7, for each $x \in \beta X$ we have

$$(\iota_*\nu)_{\beta X}(f \cdot a)(x) = f(x)(\iota_*\nu)_{\beta X}(a)(x) = f(x)\gamma(x),$$

so that in particular the section $f \cdot \gamma$, where $(f \cdot \gamma)(x) = f(x)\gamma(x)$, belongs to $(\iota_*\nu)_{\beta X}(A)$.

For $x \in \beta X$ and $c \in (\iota_*\mathfrak{A})_x = (\iota_* p)^{-1}(x)$, Lemma 4.2.8(iii) gives an element $a \in A$ with $(\iota_*\nu)_{\beta X}(a)(x) = c$. Thus we may apply Proposition 2.2.6 to the subalgebra $(\iota_*\nu)_{\beta X}(A)$ of $\Gamma(\beta X, \iota_*\mathcal{A}) = \Gamma_0(\beta X, \iota_*\mathcal{A})$, and conclude that $(\iota_*\nu)_{\beta X}(A)$ is dense in $\Gamma(\beta X, \iota_*\mathcal{A})$. Being also closed, equality follows, hence $(\iota_*\nu)_{\beta X}$ is an isomorphism.

The second assertion is immediate from Proposition 4.2.9 applied to the $\mathfrak{C}(\beta X)$-sheaf $\iota_*\mathfrak{A}$.

(ii): By Lemma 4.2.8(ii), for each $x_0 \in X$ we have $\mathfrak{A}_{x_0} = (\iota_*\mathfrak{A})_{x_0}$, and so $\mathcal{A} = (\iota_* p)^{-1}(X)$, as a collection of fibre algebras. It is clear from this fact and the definitions of $(\mathcal{A}, X, p)$ and $(\iota_*\mathcal{A}, \beta X, \iota_* p)$ that $p = (\iota_* p)|_{\mathcal{A}}$.

By Proposition 4.2.9, applied separately to the $\mathfrak{C}(X)$-sheaf A and the $\mathfrak{C}(\beta X)$-sheaf $\iota_*\mathfrak{A}$ we have

$$(\iota_*\nu)_{\beta X}(a)(x_0) = \nu_X(a)(x_0) = a + J_{x_0},$$

for all $a \in A$ and $x_0 \in X$. In particular, given $\gamma \in (\iota_*\nu)_{\beta X}(\iota_*\mathfrak{A}(\beta X))$ and $a \in A$ with $\gamma = (\iota_*\nu)_{\beta X}(a)$, the restriction of γ to X agrees with the continuous section $\nu_X(a) \in \Gamma^b(X, \mathcal{A})$.

Denote by Γ the collection of sections $X \to \coprod_{x\in X} \mathfrak{A}_x$ given by

$$\Gamma = \{\gamma|_X : \gamma \in (\iota_*\nu)_{\beta X}((\iota_*\mathfrak{A})(\beta x))\}.$$

Then we have shown that each $\gamma \in \Gamma$ is continuous as a section $X \to \mathcal{A}$. Moreover, each $\gamma \in \Gamma$ is necessarily continuous as a section $X \to \iota_*\mathcal{A}$ by definition of $(\iota_*\nu)_{\beta X}$ (being the restriction to X of an element of $\Gamma(\beta X, \iota_*\mathcal{A})$).

It is clear that Γ satisfies the hypotheses of Theorem 2.2.5, and so the (H)-C*-bundle topology on $\coprod_{x\in X} \mathfrak{A}_x$ for which each $\gamma \in \Gamma$ is continuous is unique. Since both $\mathcal{A}$ and $(\iota_*p)^{-1}(X)$, have this property, it follows that $\mathcal{A} = (\iota_*p)^{-1}(X)$.

Finally, for $U \in \mathcal{O}(X)$, $\nu_U(\mathfrak{A}(U)) \subseteq \Gamma^b(U, \mathcal{A})$ by definition of ν_U. But then since $\mathcal{A} = (\iota_*p)^{-1}(X)$, we have $\Gamma^b(U, \mathcal{A}) = \Gamma^b(U, \iota_*\mathcal{A})$.

□

In the particular case that the sheaf $\mathfrak{A}$ in Theorem 4.2.10 is the local section sheaf $\Gamma^b(-, \mathcal{A})$ of an upper-semicontinuous (H)-C*-bundle $(\mathcal{A}, X, p)$, the direct image sheaf $\iota_*\mathfrak{A}$ gives rise to a bundle over βX containing $(\mathcal{A}, X, p)$ as a sub-bundle. We show in Corollary 4.2.11 below that in fact this bundle has certain Stone-Čech compactification-type properties. While we will not make use of this result in subsequent sections, we believe that it is interesting in its own right.

Corollary 4.2.11. *Let X be a locally compact Hausdorff space and $(\mathcal{A}, X, p)$ an upper-semicontinuous (H)-C*-bundle over X. Then there is an upper-semicontinuous (H)-C*-bundle $(\mathcal{A}^\beta, \beta X, p^\beta)$ over βX with $(p^\beta)^{-1}(X) = \mathcal{A}$ and $p^\beta|_X = p$, such that every $\gamma \in \Gamma^b(X, \mathcal{A})$ has a unique extension to a continuous section $\gamma^\beta \in \Gamma(\beta X, \mathcal{A}^\beta)$.*

Proof. Let $\mathfrak{A}$ be the bounded section sheaf $\mathfrak{A}(U) = \Gamma^b(U, \mathcal{A})$, $U \in \mathcal{O}(X)$, associated with the bundle $(\mathcal{A}, X, p)$, which is a $\mathfrak{C}(X)$-sheaf by Proposition 4.2.6. It is well known that the upper-semicontinuous (H)-C*-bundle over X induced by $\mathfrak{A}$ is precisely $(\mathcal{A}, X, p)$, and that ν_U is the identity $*$-isomorphism for all $U \in \mathcal{O}(X)$, see the proof of [3, Theorem 5.6] for example.

The direct image sheaf $\iota_*\mathfrak{A}$ gives rise to an upper-semicontinuous (H)-C*-bundle $(\iota_*\mathcal{A}, \beta X, \iota_*p)$, which we take as $(\mathcal{A}^\beta, \beta X, p^\beta)$. Then it follows from Theorem 4.2.10(ii) that $(p^\beta)^{-1}(X) = \mathcal{A}$ and $p^\beta|_X = p$.

For $\gamma \in \Gamma^b(X, \mathcal{A}) = (\iota_*\mathfrak{A})$, define $\gamma^\beta \in \Gamma(\beta X, \mathcal{A}^\beta)$ via $\gamma^\beta = (\iota_*\nu)_{\beta X}(\gamma)$. Then with $A = \mathfrak{A}(X) = (\iota_*\mathfrak{A})(\beta X)$ and $J_x = \{f \in C(\beta X) : f(x) = 0\} \cdot A$ for $x \in X$, we have

$$\gamma^\beta(x) = \gamma + J_x = \gamma(x)$$

by Theorem 4.2.10(i). Hence γ^β is a continuous extension of γ to a section $\beta X \to \mathcal{A}^\beta$, and we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{A}(X) = \Gamma^b(X, \mathcal{A}) & & \\ {\scriptstyle \iota_*}\downarrow & \searrow^{\gamma \mapsto \gamma^\beta} & \\ (\iota_*\mathfrak{A})(\beta X) = \iota_*\Gamma^b(X, \mathcal{A}) & \xrightarrow[(\iota_*\nu)_{\beta X}]{} & \Gamma(\beta X, \mathcal{A}^\beta). \end{array}$$

By Theorem 4.2.10(i), $(\iota_*\nu)_{\beta X}$ is a $*$-isomorphism, and it follows that $\gamma \mapsto \gamma^\beta$ is a $*$-isomorphism of $\Gamma^b(X, \mathcal{A})$ onto $\Gamma(X, \mathcal{A}^\beta)$. In particular γ^β is unique. □

4.3 The multiplier sheaf of a $C_0(X)$-algebra

In this section we consider a $C_0(X)$-algebra (A, X, ϕ_A) and the associated sheaf of C*-algebras defined by the direct image $\phi_{A*}\mathfrak{M}_A$, where $\mathfrak{M}_A$ is the multiplier sheaf of A over $\mathrm{Prim}(A)$ of Definition 4.1.3. Our main result states that

(i) the upper-semicontinuous (H)-C*-bundle over X induced by $\phi_{A*}\mathfrak{M}_A$ agrees with that defined by the $C(\beta X)$-algebra $(M(A), \beta X, \phi_{M(A)})$ (restricted to X), and

(ii) $\mathfrak{M}_X$ is canonically $*$-isomorphic to the local section sheaf $\Gamma^b(-, \mathcal{M})$ of this bundle.

We begin with the study of the ideals of A corresponding to open subsets of X.

Proposition 4.3.1. *Let (A, X, ϕ_A) be a $C_0(X)$-algebra with base map ϕ_A and structure map μ_A. For $x \in X$, denote by $J_x = \bigcap\{P \in \mathrm{Prim}(A) : \phi_A(P) = x\}$ and for $U \in \mathcal{O}_X$ let $A(U) = \bigcap\{J_x : x \in X \backslash U\}$.*

(i) $\mathrm{Prim}(A(U))$ is canonically homeomorphic to the open subset $\phi_A^{-1}(U)$ of $\mathrm{Prim}(A)$.

(ii) $(A(U), U, \phi_{A(U)})$ is a $C_0(U)$ algebra with base map $\phi_{A(U)}$ given by the restriction of ϕ_A to $\phi_A^{-1}(U)$,

(iii) For each $x \in U$, we have $A(U) + J_x = A$, and the ideal $J(U)_x$ of $A(U)$ given by

$$J(U)_x = \bigcap\{P \in \mathrm{Prim}(A(U)) : \phi_{A(U)}(P) = x\}$$

satisfies $J(U)_x = A(U) \cap J_x$, so that $A(U)/J(U)_x \equiv A/J_x$.

(iv) Denote by $(\mathcal{A}, X, p)$ the upper-semicontinuous (H)-C-bundle over X defined by (A, X, ϕ_A), and by $\gamma_A : A \to \Gamma_0(X, \mathcal{A})$ the $*$-isomorphism of Corollary 2.3.4. Then the restriction of γ_A to $A(U)$ is a $*$-isomorphism of $A(U)$ onto $\Gamma_0(U, \mathcal{A})$.*

Proof. (i): Note that for $P \in \mathrm{Prim}(A)$ we have $P \supseteq A(U)$ if and only if $\phi_A(P) \notin U$. Thus we may identify

$$\phi_A^{-1}(U) = \{P \in \mathrm{Prim}(A) : P \not\supseteq A(U)\},$$

and so the map $P \mapsto P \cap A(U)$ is a homeomorphism of $\phi_A^{-1}(U)$ onto $\mathrm{Prim}(A(U))$ by [24, Proposition 3.2.1].

(ii): Note that U is locally compact since it is an open subset of X. (ii) is then immediate from the fact that $\mathrm{Prim}(A(U))$ is homeomorphic to $\phi_A^{-1}(U)$.

(iii): By [24, Proposition 2.11.5] we have

$$\begin{aligned} J(U)_x &:= \bigcap\{P \in \mathrm{Prim}(A(U)) : \phi_{A(U)}(P) = x\} \\ &= \bigcap\{Q \cap A(U) : Q \in \phi_A^{-1}(U), \phi_A(Q) = x\} \\ &= \bigcap\{Q \in \mathrm{Prim}(A) : \phi_A(Q) = x\} \cap A(U) \\ &= A(U) \cap J_x. \end{aligned}$$

By [24, Corollary 1.8.4], we identify $A(U)/A(U) \cap J_x = (A(U) + J_x)/J_x$, which is an ideal of A/J_x. Now take $b \in A/J_x$ and $a \in A$ such that $a + J_x = b$. Choose $f \in C_0(X)$ with $f(x) = 1$ and $f|_{X \setminus U} \equiv 0$. For any $P \in \mathrm{Prim}(A)$ such that $P \supseteq A(U)$ we have $\phi_A(P) \notin U$ and so

$$\mu_A(f)a + P = \theta_A(f \circ \phi_A)a + P = f(\phi_A(P))\,(a + P) = 0 + P.$$

In particular $\mu_A(f)a \in P$ whenever $P \supseteq A(U)$, hence $\mu_A(f)a \in A(U)$. Moreover,

$$\mu_A(f)a + J_x = f(x)\,(a + J_x) = a + J_x = b,$$

which shows that $b \in (A(U) + J_x)/J_x$, and so $A(U)/(A(U) \cap J_x) \equiv A/J_x$.

(iv): Let $(\mathcal{A}(\mathcal{U}), U, q)$ denote the upper-semicontinuous (H)-C*-bundle over U defined by $(A(U), U, \phi_{A(U)})$, and $\gamma_{A(U)} : A(U) \to \Gamma_0(U, \mathcal{A}(\mathcal{U}))$ the $*$-isomorphism of Corollary 2.3.4. By (ii), for each $x \in U$, the fibres $p^{-1}(x)$ and $q^{-1}(x)$ of $\mathcal{A}$ and $\mathcal{A}(\mathcal{U})$ respectively are $*$-isomorphic. Moreover, the definitions of γ_A and $\gamma_{A(U)}$ ensure that for all $a \in A(U)$ and $x \in U$ we have $\gamma_A(a)(x) = \gamma_{A(U)}(a)(x)$ (via the identification $A/J_x \equiv A(U)/J(U)_x$ of (ii)). In particular, for all $a \in A(U)$, $\gamma_{A(U)}(a)$ agrees with the restriction to U of continuous local section $\gamma_A(a)$ of $(\mathcal{A}, X, p)$.

It follows from these fact that $\gamma_{A(U)}(A(U))$ $(= \Gamma_0(U, \mathcal{A}(U)))$ may be regarded as a C*-subalgebra of $\Gamma^b(U, \mathcal{A})$. Moreover, by the definitions of γ_A and $A(U)$ it is evident that $\gamma_A(a)(x) = 0$ for each $a \in A(U)$ and $x \in X \backslash U$. Since $\gamma_{A(U)}(a)$ vanishes at infinity on U, for each $\varepsilon > 0$ the set

$$\{x \in X : \|\gamma_A(a)(x)\| \geq \varepsilon\} = \{x \in U : \left\|\gamma_{A(U)}(a)(x)\right\| \geq \varepsilon\}$$

is compact, so that in fact $\gamma_A(A(U)) \subseteq \Gamma_0(U, \mathcal{A})$. By (ii) $\gamma_A(A(U))$ is closed under pointwise multiplication by $C_0(U)$, so that $\gamma_A(A(U)) = \Gamma_0(U, \mathcal{A})$ by Proposition 2.2.6.

□

Now suppose that (A, X, ϕ_A) is a $C_0(X)$-algebra, and consider open subsets U, V of X with $V \subseteq U$. Then $\mathrm{Prim}(A) \backslash \phi_A^{-1}(U) \subseteq \mathrm{Prim}(A) \backslash \phi_A^{-1}(V)$, so that with the notation of Proposition 4.3.1,

$$A(V) = k\left(\mathrm{Prim}(A) \backslash \phi_A^{-1}(V)\right) \subseteq k\left(\mathrm{Prim}(A) \backslash \phi_A^{-1}(U)\right) = A(U).$$

In particular, $A(V)$ is a closed two-sided ideal of $A(U)$. Thus we get a $*$-homomorphism $\Phi_{V,U} : M(A(U)) \to M(A(V))$ extending the identity on $A(V)$ by [20, Proposition 3.7].

Definition 4.3.2. The *multiplier sheaf of A over X*, which we denote by $\mathfrak{M}_X$ is defined by assigning to each open subset U of X the C*-algebra $M(A(U))$, with the restriction mappings $\Phi_{V,U}$, whenever $V \subseteq U$, defined as above.

If we denote by $\mathfrak{M}_A$ the multiplier sheaf of A over $\mathrm{Prim}(A)$, then it is clear that $\mathfrak{M}_X$ is precisely the direct image sheaf $\phi_{A*}\mathfrak{M}_A$ defined via $(\phi_{A*}\mathfrak{M}_A)(U) = \mathfrak{M}_A(\phi_A^{-1}(U))$ for all $U \in \mathcal{O}(X)$. It is evident from this fact that $\mathfrak{M}_X$ satisfies the gluing condition of a sheaf.

We have seen already in Proposition 3.2.1 that the multiplier algebra of a $C_0(X)$-algebra (A, X, ϕ_A) defines a $C(\beta X)$-algebra $(M(A), \beta X, \phi_{M(A)})$. In particular, $M(A)$

is an essential $C^b(X)$-module. Applying this fact together with Proposition 4.3.1 to each of the algebras $M(A(U))$ for $U \in \mathcal{O}_X$, we show that the multiplier sheaf is in fact a $\mathfrak{C}(X)$-sheaf.

Lemma 4.3.3. *Let A be a $C_0(X)$-algebra. Then the multiplier sheaf $\mathfrak{M}_X$ of A over X is a $\mathfrak{C}(X)$-sheaf.*

Proof. Let $U \in \mathcal{O}(X)$, then $A(U)$ is a $C_0(U)$-algebra by Proposition 4.3.1(i). It then follows from Theorem 3.2.4 that $\mathfrak{M}_X(U) = M(A(U))$ is a $C(\beta U)$-algebra, i.e. an essential $C^b(U)$-module. For $V, U \in \mathcal{O}(X)$ with $V \subseteq U$, we claim that

$$\Phi_{V,U}(\mu_{M(A(U))}(f)) = \mu_{M(A(V))}(f|_V)$$

for all $f \in C^b(U)$. Note that it is sufficient to show that

$$\left(\mu_{M(A(V))}(f|_V) - \Phi_{V,U}(\mu_{M(A(U))}(f))\right) a \in P \cap A(V)$$

for all $f \in C^b(U), a \in A(V)$ and $P \in \{Q \in \mathrm{Prim}(A(U)) : Q \not\supseteq A(V)\}$.

Note that since $A(V)$ is an ideal of $M(A(U))$, and since $\Phi_{V,U}$ is the identity on $A(V)$, we have

$$\Phi_{V,U}(\mu_{M(A(U))}(f))a + P \cap A(V) = \mu_{M(A(U))}(f)a + P \cap A(V).$$

Denoting by $\phi_{A(U)} : \mathrm{Prim}(A(U)) \to U$ and $\phi_{A(V)} : \mathrm{Prim}(A(V)) \to V$ the relevant base maps, it is clear that $\phi_{A(U)}\big|_{\mathrm{Prim}(A(V))} = \phi_{A(V)}$. Regarding $A(V)/(P \cap A(V)) \subseteq A(U)/P$, we may thus identify

$$\begin{aligned} \mu_{M(A(U))}(f)a + P \cap A(V) &= \mu_{M(A(U))}(f)a + P \\ &= f(\phi_{A(U)}(P))(a + P). \end{aligned}$$

Thus it is clear that

$$\begin{aligned} \Phi_{V,U}(\mu_{M(A(U))}(f))a + P \cap A(V) &= f(\Phi_{A(U)}(P))\,(a + P \cap A(V)) \\ &= (f|_V)(\phi_{A(V)}(P \cap A(V)))\,(a + P \cap A(V)) \\ &= \mu_{M(A(V))}(f|_V)a + P \cap A(V), \end{aligned}$$

as required. □

We have seen in Theorem 3.2.4 that for a $C_0(X)$-algebra (A, X, μ_A), the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$ gives rise to an upper-semicontinuous (H)-C*-bundle over βX with global section algebra canonically isomorphic to $M(A)$. Considering the direct image sheaf $\iota_*\mathfrak{M}_X$, where $\mathfrak{M}_X$ is the multiplier sheaf of A over X, we get an upper-semicontinuous (H)-C*-bundle over βX by Theorem 4.2.3. Via Theorem 4.2.10, we now show that both bundles are in fact equivalent.

Theorem 4.3.4. *Let (A, X, μ_A) be a $C_0(X)$-algebra, $(M(A), \beta X, \mu_{M(A)})$ the $C(\beta X)$-algebra defined by its multiplier algebra, and $\mathfrak{M}_X$ the multiplier sheaf of A over X. Denoting by $\iota : X \hookrightarrow \beta X$ the inclusion mapping, then the upper-semicontinuous (H)-C*-bundle over βX associated with $\iota_*\mathfrak{M}_X$ may be naturally identified with that associated with the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$.*

Proof. Denote by $(\mathcal{M}, \beta X, p)$ the bundle defined by the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$ in Corollary 2.3.4, and by $(\mathcal{M}', \beta X, p')$ that defined by the sheaf $\iota_*\mathfrak{M}_X$ in Theorem 4.2.10, with total spaces $\mathcal{M} = \{M_p : p \in \beta X\}$ and $\mathcal{M}' = \{M'_p : p \in \beta X\}$. For each $p \in \beta X$, let H_p denote the ideal of $M(A)$ defined in (3.2.1).

By Theorem 3.2.4(i) we have $M_p \equiv M(A)/H_p$ for all $p \in \beta X$, and by Corollary 2.3.4 the mapping $\gamma_{M(A)} : M(A) \to \Gamma(\beta X, \mathcal{M})$, where $\gamma_{M(A)}(c)(p) = c + H_p$ for all $c \in M(A)$ and $p \in \beta X$, is a $*$-isomorphism.

Since $\mathfrak{M}_X$ is a $\mathfrak{C}(X)$-sheaf by Lemma 4.3.3, $\iota_*\mathfrak{M}_X$ is a $C(\beta X)$-sheaf by Lemma 4.2.8. Moreover, since $\mathfrak{M}_X(X) = \iota_*\mathfrak{M}_X(\beta X) = M(A)$ and $H_p = \{f \in C(\beta X) : f(p) = 0\} \cdot M(A)$ for all $p \in \beta X$, it is clear from Proposition 4.2.9 that $M'_p \equiv M(A)/H_p$ for all $p \in \beta X$. The mapping $(\iota_*\nu)_{\beta X} : M(A) \to \Gamma(\beta X, \mathcal{M}')$ is a $*$-isomorphism by Theorem 4.2.10(i), and also satisfies $(\iota_*\nu)_{\beta X}(c)(p) = c + H_p$ for all $c \in M(A)$ and $p \in \beta X$.

It follows that we may identify $M_p = M'_p = M(A)/H_p$ for all $p \in \beta X$, and thus $\mathcal{M}$ and $\mathcal{M}'$ consist of the same collection of fibre algebras. Moreover, the global section algebras $\Gamma(\beta X, \mathcal{M})$ and $\Gamma(\beta X, \mathcal{M}')$ are both isomorphic to $M(A)$ via the identification of $c \in M(A)$ with the section $p \mapsto c + H_p$. Since the (H)-C*-bundle topology on $\coprod_{p \in \beta X} M(A)/H_p$, for which every $c \in M(A)$ defines a continuous section in this manner, is unique by Proposition 2.2.6, it follows that $\mathcal{M} = \mathcal{M}'$ as topological spaces. □

Lemma 4.3.5. *Let (A, X, ϕ_A) be a $C_0(X)$-algebra and $U \in \mathcal{O}_X$. For $c \in M(A)$ denote* *by* $c|_U := \Phi_{U,X}(c)$ *its restriction to an element of* $M(A(U))$. *Then*

$$\|c|_U\| = \sup_{y \in U} \|c + \tilde{J}_y\|,$$

where J_y *is the ideal of* A *of (2.3.2), and* $\tilde{J}_y$ *its strict closure in* $M(A)$ *as in (3.1.1).*

Proof. As a consequence of Proposition 4.3.1(ii), we will regard A/J_y and $A(U)/J(U)_y$ as being equal for all $y \in U$. Denote by $\pi_y : A \to A/J_y$ and $\pi_{U,y} : A(U) \to A/J_y$ the quotient mappings, and by $\tilde{\pi}_y : M(A) \to M(A/J_y)$ and $\tilde{\pi}_{U,y} : M(A(U)) \to M(A/J_y)$ their extensions to $M(A)$ and $M(A(U))$ respectively. We first claim that for all $c \in M(A)$ we have $\tilde{\pi}_y(c) = \tilde{\pi}_{U,y}(c|_U)$.

Note that since $\Phi_{U,X}$ is the identity on $A(U)$, for all $c \in M(A)$ and $b \in A(U)$ we have

$$(c|_U)b = \Phi_{U,X}(c)\Phi_{U,X}(b) = \Phi_{U,X}(cb) = cb.$$

It follows that for all $c \in M(A), b \in A(U)$,

$$\tilde{\pi}_{U,y}(c|_U)\pi_{U,y}(b) = \pi_{U,y}((c|_U)b) = \pi_y(cb) = \tilde{\pi}_y(c)\pi_y(b)$$

and similarly $\pi_{U,y}(b)\tilde{\pi}_{U,y}(c|_U) = \pi_y(b)\tilde{\pi}_y(c)$. Now as each $a \in A/J_y$ may be written as $a = \pi_{U,y}(b) = \pi_y(b)$ for some $b \in A(U)$, it follows that $\tilde{\pi}_{U,y}(c|_U) = \tilde{\pi}_y(c)$ for all $c \in M(A)$.

Since $\bigcap\{\ker(\tilde{\pi}_{U,y}) : y \in U\} = \{0\}$ by Theorem 3.1.6, it follows that for any $c \in M(A)$ we have

$$\begin{aligned} \|c|_U\| &= \sup_{y \in U} \|\tilde{\pi}_{U,y}(c|_U)\| \\ &= \sup_{y \in U} \|\tilde{\pi}_y(c)\| \\ &= \sup_{y \in U} \left\|c + \tilde{J}_y\right\|. \end{aligned}$$

□

Proposition 4.3.6. *Let* (A, X, ϕ_A) *be a* $C_0(X)$*-algebra,* $(M(A), \beta X, \phi_{M(A)})$ *the* $C(\beta X)$*-algebra associated with* $M(A)$*, and* $(\mathcal{A}, X, p)$ *and* $(\mathcal{M}, \beta X, q)$ *the upper-semicontinuous (H)-C*-bundles associated with* (A, X, ϕ_A) *and* $(M(A), \beta X, \phi_{M(A)})$ *respectively. Denoting by* $\gamma_A : A \to \Gamma_0(X, \mathcal{A})$ *and* $\gamma_{M(A)} : M(A) \to \Gamma(\beta X, \mathcal{M})$ *the* $*$*-isomorphisms of Corollary 2.3.4, the bundles* $\mathcal{A}$ *and* $\mathcal{M}$ *satisfy the following relations:*

(i) For each $x \in X$, denoting by $A_x = p^{-1}(x)$ and $M_x = q^{-1}(x)$ the fibre algebras, then A_x may be identified with an ideal of M_x in such a way that $\gamma_A(a)(x) = \gamma_{M(A)}(a)(x)$ for all $a \in A$.

(ii) The map $\iota : \mathcal{A} \to \mathcal{M}$, where for all $x \in X$, $\iota|_{p^{-1}(x)} : A_x \to M_x$ is the identification of part (i), is a homeomorphism onto its image.

(iii) For all $U \in \mathcal{O}_X$, $\Gamma_0(U, \mathcal{A})$ is an essential ideal of $\Gamma^b(U, \mathcal{M})$, when we identify $\mathcal{A}$ with its image $\iota(\mathcal{A}) \subseteq \mathcal{M}$.

Proof. (i): For each $x \in X$, let J_x and H_x be as defined in (2.3.2) and (3.2.1) respectively. Then since $J_x = H_x \cap A$ by Proposition 3.2.3(ii), we may identify $A/J_x = A/(H_x \cap A) = (A + H_x)/H_x$ by [24, Corollary 1.8.4], which is a closed two sided ideal of $M(A)/H_x$. Under this identification, the image of $a + J_x$ of $a \in A$ is identified with $a + H_x$ (regarding a as an element of $M(A)$).

By Corollary 2.3.4, the $*$-isomorphisms γ_A and $\gamma_{M(A)}$ satisfy $\gamma_A(a)(x) = a + J_x$ and $\gamma_{M(A)}(a)(x) = a + H_x$ for all $a \in A$ and $x \in X$, so that $\gamma_A(a)(x) = \gamma_{M(A)}(a)(x)$ via the identification above.

(ii): It is clear that $\iota(\mathcal{A})$ with the subspace topology inherited from $\mathcal{M}$, satisfies the conditions of Definition 2.2.1. Moreover, since $\gamma_A(a) = \gamma_{M(A)}(a)$ for all $a \in A$, it follows that $\gamma_A(A) \subseteq \Gamma^b(X, \iota(\mathcal{A}))$. The family of sections $\gamma_A(A)$ satisfy conditions (i) and (ii) Theorem 2.2.5, hence the subspace topology on $\iota(\mathcal{A})$ is the unique topology for which these sections are continuous. It follows that ι is a homeomorphism onto its image.

(iii): We first show that $\Gamma_0(U, \mathcal{A})$ is a closed two-sided ideal of $\Gamma^b(U, \mathcal{M})$. Let $s \in \Gamma^b(U, \mathcal{M})$ and $a \in \Gamma_0(U, \mathcal{A})$. Since for each $x \in U$, A_x is an ideal of M_x, we have $s(x)a(x), a(x)s(x) \in A_x$ for all such x. Thus $sa, as \in \Gamma^b(U, \mathcal{A})$.

To see that the norm functions of sa and as vanish at infinity on U, note that for each x we have $\|(sa)(x)\| \leq \|s(x)\| \cdot \|a(x)\| \leq \|s\| \cdot \|a(x)\|$. It follows (if $s \neq 0$) that for each $\varepsilon > 0$,

$$\begin{aligned} \{x \in U : \|(sa)(x)\| \geq \varepsilon\} &\subseteq \{x \in U : \|s\| \cdot \|a(x)\| \geq \varepsilon\} \\ &= \{x \in U : \|a(x)\| \geq \frac{\varepsilon}{\|s\|}\}, \end{aligned}$$

and the latter is compact since $x \mapsto \|a(x)\|$ vanishes at infinity on U. Hence the set of $\{x \in U : \|(sa)(x)\| \geq \varepsilon\}$, being closed by upper-semicontinuity of $x \mapsto \|(sa)(x)\|$, is compact for each $\varepsilon > 0$. Applying a similar argument to the norm function of as shows that $sa, as \in \Gamma_0(U, \mathcal{A})$.

To see that $\Gamma_0(U, \mathcal{A})$ is essential in $\Gamma^b(U, \mathcal{M})$, let $s \in \Gamma^b(U, \mathcal{M})$ such that s that annihilates $\Gamma_0(U, \mathcal{A})$. We claim that $s = 0$.

For each $x_0 \in U$, there is $m \in M(A)$ such that $\gamma_{M(A)}(m)(x_0) = s(x_0)$ (that is, $s(x_0) = m + H_{x_0}$). We will show that $\gamma_{M(A)}(m)(x_0) = 0$, from which the claim will follow. Denote by $m' = \gamma_{M(A)}(m)\big|_U = (\nu_U \circ \Phi_{U,X})(m)$, and let $\varepsilon > 0$ be given. Since $(m' - s)(x_0) = 0$ and $x \mapsto \|(m' - s)(x)\|$ is upper-semicontinuous on U, there is an open neighbourhood W of x_0 contained in U with $\|(m' - s)(x)\| < \frac{\varepsilon}{2}$ for all $x \in W$. It follows that

$$\|(m' - s)\big|_W\| = \sup_{x \in W} \|(m' - s)(x)\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Now take $a \in \Gamma_0(W, \mathcal{A})$ with $\|a\| = 1$. Since $W \subseteq U$, we have $A(W) \subseteq A(U)$ and so $\Gamma_0(W, \mathcal{A}) \subseteq \Gamma_0(U, \mathcal{A})$ by Proposition 4.3.1(iv). In particular this implies that $(s|_W)a = 0$. Then

$$\begin{aligned} \|(m'|_W)a\| &= \| \left((m' - s + s)\big|_W\right) a\| \\ &\leq \|(m' - s)\big|_W\| \cdot \|a\| + \|(s|_W)a\| \\ &= \|(m' - s)\big|_W\| \leq \frac{\varepsilon}{2} < \varepsilon. \end{aligned} \tag{4.3.1}$$

Since $\nu_W : M(A(W)) \to \Gamma^b(W, \mathcal{M})$ is injective, it is isometric, and since restriction is transitive,

$$m'|_W = \left(\gamma_{M(A)}(m)\big|_U\right)\Big|_W = \gamma_{M(A)}(m)\big|_W = \nu_W \circ \Phi_{W,X}(m),$$

so that in particular $\|m'|_W\| = \|\Phi_{W,X}(m)\|$. By [20, §2], the norm of $(\Phi_{W,X})(m)$ is given by

$$\|(\Phi_{W,X})(m)\| = \sup\{\|(\Phi_{W,X})(m)b\| : b \in A(W), \|b\| = 1\}.$$

Now ν_W restricted to $A(W)$ is a $*$-isomorphism of $A(W)$ onto $\Gamma_0(W, \mathcal{A})$ by Proposition 4.3.1(iv), so for any $a \in \Gamma_0(W, \mathcal{A})$ with $\|a\| = 1$ there is $b \in A(W)$ with $\nu_W(b) = \gamma_{A(W)}(b) = a$ and $\|b\| = 1$. Thus

$$\begin{aligned} \|(\Phi_{W,X})(m)\| &= \sup\{\| \left((\nu_W \circ \Phi_{W,X})(m)\right) \nu_W(b)\| : b \in A(W), \|b\| = 1\} \\ &= \sup\{\|(m'|_W)\nu_W(b)\| : b \in A(W), \|b\| = 1\} \\ &= \sup\{\|(m'|_W)a\| : a \in \Gamma_0(W, \mathcal{A}), \|a\| = 1\} < \varepsilon, \end{aligned}$$

where the inequality on the last line follows from (4.3.1).

By Lemma 4.3.5, we have

$$\|(\Phi_{W,X})(m)\| = \sup_{x \in W} \|b + \tilde{J}_x\|,$$

while $\|\gamma_{M(A)}(m)(x_0)\| = \|m + H_{x_0}\|$. But then by [8, Lemma 1.5] this gives

$$\begin{aligned} \|\gamma_{M(A)}(m)(x_0)\| = \|m + H_{x_0}\| &= \inf_{x_0 \in V} \left(\sup_{y \in V} \|m + \tilde{J}_y\| \right) \\ &= \inf_{x_0 \in V} \|(\Phi_{W,X})(m)\| < \varepsilon, \end{aligned}$$

the infimum being taken over all open neighbourhoods V of x_0 in X. Since $\varepsilon > 0$ was arbitrary, this gives $\|\gamma_{M(A)}(m)(x_0)\| = 0$, and so $s(x_0) = 0$.

We conclude that $sa = 0$ for all $a \in \Gamma_0(U, \mathcal{A})$ implies $s = 0$, so that $\Gamma_0(U, \mathcal{A})$ is an essential ideal of $\Gamma^b(U, \mathcal{M})$.

□

Theorem 4.3.7. *Let (A, X, μ_A) be a $C_0(X)$-algebra, $\mathfrak{M}_X$ the associated multiplier sheaf of A over X (Definition 4.1.3), and $(\mathcal{M}, \beta X, p)$ the upper-semicontinuous (H)-C*-bundle over βX associated with $\iota_* \mathfrak{M}_X$ by Theorem 4.2.10. Then for each open $U \subseteq X$, the map $\nu_U : \mathfrak{M}_X(U) \to \Gamma^b(U, \mathcal{M})$ of Theorem 4.2.3 is a $*$-isomorphism. In particular $\mathfrak{M}_X$ may be identified with the local section sheaf $U \mapsto \Gamma^b(U, \mathcal{M})$.*

Proof. The restriction of ν_U to $A(U)$ is a $*$-isomorphism by Proposition 4.3.1. By Proposition 4.3.6(iii), $\Gamma_0(U, \mathcal{A})$ is an essential ideal of $\Gamma^b(U, \mathcal{M})$, so there is a $*$-isomorphism $\eta : \Gamma^b(U, \mathcal{M}) \to M(\Gamma_0(U, \mathcal{A}))$ extending the identity on $\Gamma_0(U, \mathcal{A})$. Thus we get a commutative diagram

$$\begin{array}{ccccc} M(A(U)) & \xhookrightarrow{\nu_U} & \Gamma^b(U, \mathcal{M}) & \xrightarrow{\eta} & M(\Gamma_0(U, \mathcal{A})) \\ \uparrow & & \uparrow & \nearrow & \\ A(U) & \xrightarrow[\nu_U]{} & \Gamma_0(U, \mathcal{A}) & & \end{array}$$

Note that since $\eta \circ \nu_U$ extends the $*$-isomorphism $\nu_U|_{A(U)}$, it must be surjective. But then since η is injective, ν_U must be surjective. □

In certain cases Theorem 4.3.7 allows us to construct the multiplier sheaf of a $C_0(X)$-algebra as sections of a bundle depending only on A, as the following corollary shows.

Corollary 4.3.8. *Let (A, X, μ_A) be a σ-unital $C_0(X)$-algebra and denote by $(\mathcal{A}, X, p)$ the associated upper-semicontinuous (H)-C*-bundle over X. If for all $x \in X$ we have $\mu_A(C_0(X)) \cap A \not\subseteq J_x$, then with the identification of Theorem 4.3.7, the multiplier sheaf of A over X satisfies $\mathfrak{M}_X(U) = \Gamma^b(U, \mathcal{A})$ for all $U \in \mathcal{O}(X)$.*

Proof. Since $\mu_A(C_0(X)) \cap A \not\subseteq J_x$ for all $x \in X$, A/J_x is unital for all $x \in X$ by [8, Proposition 2.2], and thus the ideal H_x of (3.2.1) is strictly closed in $M(A)$ by Corollary 3.3.6. It then follows from Theorem 3.2.4(ii)' that $M(A)/H_x = M(A/J_x) = A/J_x$ for all x.

Together with Proposition 4.3.6(ii), this last fact shows that $(\mathcal{A}, X, p) = (\mathcal{M}, X, q)$, and so $\mathfrak{M}_X(U) = \Gamma^b(U, \mathcal{A})$ by Theorem 4.3.7.

□

Recall that by Proposition 2.3.11 any quasi-central C*-algebra A is a $C_0(\text{Prim}(Z(A)))$-algebra, where the structure map is the inclusion mapping $C_0(\text{Prim}(Z(A))) \equiv Z(A) \hookrightarrow ZM(A)$.

Corollary 4.3.9. *Let A be a quasi-central C*-algebra, $(A, \text{Prim}(Z(A)), \mu_A)$ its canonical representation as a $C_0(\text{Prim}(Z(A)))$-algebra, and $(\mathcal{A}, \text{Prim}(Z(A)), p)$ the associated upper-semicontinuous (H)-C*-bundle over $\text{Prim}(Z(A))$. Then, with the identification of Theorem 4.3.7, the multiplier sheaf of A over $\text{Prim}(Z(A))$ satisfies $\mathfrak{M}_{\text{Prim}(Z(A))}(U) = \Gamma^b(U, \mathcal{A})$ for all $U \in \mathcal{O}(\text{Prim}(Z(A)))$.*

Proof. Note that $\mu_A(C_0(\text{Prim}(Z(A))) = Z(A)$ in this case, and since $P \not\supseteq Z(A)$ for any $P \in \text{Prim}(A)$, it follows that $J_x \not\supseteq Z(A)$ for any $x \in \text{Prim}(Z(A))$. Thus Corollary 4.3.8 applies. □

Chapter 5

The Glimm space of the minimal tensor product

In this section we study the space of Glimm ideals of the minimal tensor product $A \otimes_\alpha B$ of two C*-algebras A and B. We show that there is a natural open bijection of $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ onto $\mathrm{Glimm}(A \otimes_\alpha B)$, and identify a large class of C*-algebras A for which this map is a homeomorphism for all B. In the case that the map fails to be a homeomorphism, we show that the topology on $\mathrm{Glimm}(A \otimes_\alpha B)$ depends only on that of the product space $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$. In particular, $\mathrm{Glimm}(A \otimes_\alpha B)$ may be constructed in all cases without knowledge of those ideals of $A \otimes_\alpha B$ which do not arise from a pair of ideals of A and B.

This result, together with the Dauns-Hofmann Theorem, allows us to construct the centre $ZM(A \otimes_\alpha B)$ of the multiplier algebra of $A \otimes_\alpha B$ in terms of these spaces. We obtain a precise characterisation of those C*-algebras A and B for which $ZM(A \otimes_\alpha B) = ZM(A) \otimes ZM(B)$, and show that this may occur even when $M(A) \otimes_\alpha M(B)$ is strictly smaller than $M(A \otimes_\alpha B)$.

When the Glimm spaces are regarded as sets of ideals, we show that the map sending

$$(G, H) \mapsto G \otimes_\alpha B + A \otimes_\alpha H$$

implements the above bijection. This extends an earlier result of Kaniuth [37], who obtained a similar result under the assumption that $A \otimes_\alpha B$ satisfies the property (F) of Tomiyama [54]. Our approach is similar to that of Kaniuth, in that we use an alternative definition of the Glimm space based on the complete regularisation of the space of factorial, rather than primitive, ideals. The reasons for this are discussed in Section 5.5.

Finally, we apply these results to the study of the Dauns-Hofmann representation of $A \otimes_\alpha B$ over Glimm$(A \otimes_\alpha B)$. We discuss conditions under which the resulting C*-bundle depends only on the corresponding bundles associated with the Dauns-Hofmann representations of A and B. Moreover, we investigate the stability of the properties of continuity and quasi-standardness of these bundles under the operation of taking minimal tensor products.

Sections 5.1 to 5.7 have appeared in the article [45].

5.1 Introduction

The focus of our work is the relationship between a C^*-algebra A and its collection of primitive ideals Prim(A), a topological space in the hull kernel topology, the associated complete regularisation space Glimm(A) of Prim(A), and in particular how Glimm$(A \otimes_\alpha B)$ relates to Glimm(A) and Glimm(B). Here $A \otimes_\alpha B$ is the minimal (or spatial) tensor product of two C^*-algebras A and B (Definition 5.3.1) and our work is motivated by a desire to extend earlier work such as [37], [6], [42].

As described in Theorem 2.3.6, Dauns and Hofmann showed in [21] that any C*-algebra A may be represented as the section algebra of an upper semicontinuous C*-bundle over Glimm(A) if this space is locally compact, or over its Stone-Čech compactification otherwise. Under this representation the fibre algebras are given by the Glimm quotients of A. Thus in the case of the minimal tensor product of C*-algebras A and B, a natural question that arises is to determine Glimm$(A \otimes_\alpha B)$ in terms of Glimm(A) and Glimm(B), both topologically and as a collection of ideals of $A \otimes_\alpha B$. A related problem (over more general base spaces) was studied by Kirchberg and Wassermann in [40], and later by Archbold in [6], by considering the fibrewise tensor product of the corresponding bundles of A and B.

We will denote by Id$'(A)$ the set of all proper norm-closed two sided ideals of A. By Fac(A) we mean the space of kernels of factor representations of A, which is a topological space in the hull-kernel topology (Definition 5.5.1).

There is a natural embedding of Id$'(A) \times$ Id$'(B)$ into Id$'(A \otimes_\alpha B)$ sending $(I, J) \mapsto$

$\ker(q_I \otimes q_J)$, where q_I and q_J are the quotient maps. The restrictions of this map to the spaces of primitive and factorial ideals are known to be homeomorphisms onto dense subspaces of $\mathrm{Prim}(A \otimes_\alpha B)$ and $\mathrm{Fac}(A \otimes_\alpha B)$ respectively, see [59], [42]. Recently A.J. Lazar has shown in [42] that any continuous map $f : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to Y$, where Y is a T_1 space has a continuous 'extension' to $\mathrm{Prim}(A \otimes_\alpha B)$, where we identify $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ with its image under the above embedding.

We begin in Section 5.2 with considerations of the complete regularisation of a product $X \times Y$ of topological spaces, gathering together and extending results on this theme from the literature. Central to this is the theory of w-compact spaces, introduced by Ishii in [35]. We establish (Proposition 5.2.9) conditions on a C*-algebra A that ensure that the complete regularisation of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ is homeomorphic to the product space $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ for any C*-algebra B. In the presence of a countable approximate unit for A, a necessary and sufficient condition for this to occur is that $\mathrm{Glimm}(A)$ be locally compact. In the general case, sufficient conditions include compactness of $\mathrm{Prim}(A)$ (e.g. if A is unital), or that the complete regularisation map of $\mathrm{Prim}(A)$ is open (e.g. if A is quasi-standard, see [7]). Local compactness of $\mathrm{Glimm}(A)$ is always a necessary condition.

Using the extension result of Lazar together with the universal property of the complete regularisation of a topological space described in Section 5.2, we show (Theorem 5.3.3) that as a topological space $\mathrm{Glimm}(A \otimes_\alpha B)$ is the same as (or can be identified in a natural way with) the complete regularisation of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$. We investigate conditions under which the latter coincides with the cartesian product space $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, while showing that the underlying sets always agree. In Corollary 5.3.4 we give some rather general conditions on A or on B for this coincidence but show an example in Section 5.7 where it fails.

An equivalent formulation of the Dauns-Hofmann isomorphism (2.3.1) identifies $ZM(A)$ with the C*-algebra of bounded continuous functions on $\mathrm{Glimm}(A)$. We show in Theorem 5.4.3 that for any C*-algebras A and B, $ZM(A \otimes_\alpha B)$ can be identified with the bounded continuous functions on (the complete regularisation of) $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$, and give necessary and sufficient conditions (Theorem 5.4.6) for $ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B)$.

In Section 5.5, we determine the set of Glimm ideals of $A \otimes_\alpha B$ in terms of the Glimm ideals of A and B. In order to do this, we use an alternative construction of Glimm(A) based on the complete regularisation of Fac(A) (rather than Prim(A)) first considered by Kaniuth in [37]. The reason for this is the fact that there exists a continuous surjection $\mathrm{Fac}(A \otimes_\alpha B) \to \mathrm{Fac}(A) \times \mathrm{Fac}(B)$, while it is not known if the restriction of this map to $\mathrm{Prim}(A \otimes_\alpha B)$ has range $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$.

We show in Theorem 5.5.9 that the map $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B) \to \mathrm{Glimm}(A \otimes_\alpha B)$ sending $(G, H) \mapsto G \otimes_\alpha B + A \otimes_\alpha H$ determines the homeomorphism of these spaces when considered as sets of ideals. This extends Kaniuth's result [37, Theorem 2.3], which was proved under the assumption that $A \otimes_\alpha B$ satisfies Tomiyama's property (F), defined below.

In Section 5.6 we consider the problem of determining conditions for which the canonical upper semicontinuous bundle representation of $A \otimes_\alpha B$ over $\mathrm{Glimm}(A \otimes_\alpha B)$ of [21] is in fact continuous (i.e. $A \otimes_\alpha B$ defines a maximal full algebra of operator fields in the sense of Fell [27]). The main result of this section (Theorem 5.6.2) shows that, under the assumption that $\ker(q_G \otimes q_H) = G \otimes_\alpha B + A \otimes_\alpha H$ for all pairs of Glimm ideals (G, H) of A and B, this representation of $A \otimes_\alpha B$ is continuous precisely when the corresponding bundle representations of A and B (over Glimm(A) and Glimm(B) respectively) are continuous. We also show that, under a different assumption that does not require that these ideals be equal, continuity of A and B is a necessary condition for continuity of $A \otimes_\alpha B$ (Proposition 5.6.4).

Let α and β be states of A and B respectively. Then the product state $\alpha \otimes \beta$ of $A \otimes_\alpha B$ is defined via $(\alpha \otimes \beta)(a \otimes b) = \alpha(a)\beta(b)$ on elementary tensors $a \otimes b$ and extended to $A \otimes_\alpha B$ by linearity and continuity. If $I, J \in \mathrm{Id}'(A \otimes_\alpha B)$ with $I \not\subseteq J$, then we say that a state γ of $A \otimes_\alpha B$ separates I and J if $\gamma(J) = \{0\}$ and there exists $c \in I \backslash J$ with $\gamma(c) = 1$. The minimal tensor product $A \otimes_\alpha B$ is said to satisfy Tomiyama's property (F) if given any pair $I, J \in \mathrm{Id}'(A \otimes_\alpha B)$ with $I \neq J$, there is a product state of $A \otimes_\alpha B$ separating I and J. There are many equivalent characterisations of property (F), see [42, Proposition 5.1] for example.

If $\alpha : X \to Y$ is a continuous map between topological spaces we will denote by $\alpha^* : C(Y) \to C(X)$ the unique $*$-homomorphism given by $\alpha^*(f) = f \circ \alpha$ for $f \in C(Y)$.

5.2 The complete regularisation of a product of topological spaces

In this section we relate the complete regularisation $\rho(X \times Y)$ of the product of two topological spaces X and Y to the product space $\rho X \times \rho Y$.

Lemma 5.2.1. *Let X and Y be topological spaces. Then there is an open bijection*

$$\rho(X) \times \rho(Y) \to \rho(X \times Y)$$

sending $(\rho_X(x), \rho_Y(y)) \mapsto \rho_{X\times Y}(x, y)$.

Proof. We first show that $(X/\approx) \times (Y/\approx) = (X \times Y)/\approx$ as sets; specifically that $(x_1, y_1) \approx (x_2, y_2)$ if and only if $x_1 \approx x_2$ and $y_1 \approx y_2$. Indeed, if $(x_1, y_1) \approx (x_2, y_2)$ and $f \in C^b(X)$ then $f \circ \pi_X \in C^b(X \times Y)$, hence $f(x_1) = f \circ \pi_X(x_1, y_1) = f \circ \pi_X(x_2, y_2) = f(x_2)$, and $x_1 \approx x_2$.

On the other hand if both $x_1 \approx x_2$ and $y_1 \approx y_2$, take $g \in C^b(X\times Y)$; then $g(x_1, y_1) = g(x_2, y_1) = g(x_2, y_2)$. It follows that the mapping $(\rho_X(x), \rho_Y(y)) \mapsto \rho_{X\times Y}(x, y)$ is a well-defined bijection.

In order to show that the above map is open we use the fact that in a completely regular space, the cozero sets of continuous functions form a base for the topology [29, 3.4]. Consider a basic open set $\mathrm{coz}(f^\rho) \times \mathrm{coz}(g^\rho)$ in $\rho X \times \rho Y$, where $f \in C^b(X), g \in C^b(Y)$. Then $h(x, y) = (f^\rho \circ \rho_X)(x)(g^\rho \circ \rho_Y)(y)$ defines an element of $C^b(X \times Y)$, and hence gives $h^\rho \in C^b(\rho(X \times Y))$ with $h^\rho \circ \rho_{X\times Y} = h$. Then $\mathrm{coz}(h^\rho) = \mathrm{coz}(f^\rho) \times \mathrm{coz}(g^\rho)$, so that $\mathrm{coz}(f^\rho) \times \mathrm{coz}(g^\rho)$ is an open subset of $\rho(X \times Y)$. □

In view of Lemma 5.2.1, for any topological spaces X and Y we identify $\rho(X)\times\rho(Y)$ and $\rho(X\times Y)$ as sets. This canonical map is not a homeomorphism in general, however (see Example 5.7.1). Thus in what follows, we will denote by $\rho X \times \rho Y$ this product space with the (possibly weaker) product topology τ_p, and by $\rho(X \times Y)$ the space with the topology τ_{cr} induced by the functions in $C^b(X \times Y)$.

The following result was obtained originally in [34]:

Theorem 5.2.2. (Hoshina and Morita) *[36, Theorem 2.4]*

Let X and Y be topological spaces. The following are equivalent:

(i) $\rho X \times \rho Y = \rho(X \times Y)$,

(ii) For any cozero set G of $X \times Y$ and any point $(x, y) \in G$ there are cozero sets U and V of X and Y respectively with $(x, y) \in U \times V \subseteq G$.

Definition 5.2.3. Let $(X, \mathcal{T})$ be a topological space. For a subspace $Y \subseteq X$ denote by τ_Y the topology on Y generated by $\{\mathrm{coz}(f) : f \in C(Y)\}$.

For a topological space $(X, \mathcal{T})$ and a subset $U \subseteq X$ we denote by $\mathcal{T} \restriction_U$ the subspace topology on U inherited from $\mathcal{T}$. For two topologies $\mathcal{T}_1, \mathcal{T}_2$ on X we write $\mathcal{T}_1 \leq \mathcal{T}_2$ to say that $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$.

For any subset $U \subseteq X$, we denote by $\overline{U}$ the $\mathcal{T}$-closure of U and by $\mathrm{cl}_{\tau_X}(U)$ the τ_X-closure of U. The following lemma establishes some basic properties of the τ-topologies on subspaces of a topological space $(X, \mathcal{T})$:

Lemma 5.2.4. *Let $(X, \mathcal{T})$ be a topological space and let $U \subseteq X$. Then:*

(i) $\tau_U \leq \mathcal{T} \restriction_U$,

(ii) U is τ_X open if and only if $U = \rho_X^{-1}(W)$ for some open subset $W \subseteq \rho X$,

(iii) If U is τ_X-open then U is saturated with respect to the relation $\approx$ on X (hence $X \backslash U$ is saturated also),

(iv) $\mathrm{cl}_{\tau_X}(U) = \rho_X^{-1}\left(\overline{\rho_X(U)}\right)$, where $\overline{\rho_X(U)}$ is the τ_{cr}-closure of $\rho_X(U)$ in ρX,

(v) If $U \subseteq V \subseteq X$ then $\tau_V \restriction_U \leq \tau_U$.

Proof. (i) Every basic τ_U-open set is of the form $\{x \in U : f(x) \neq 0\}$ with $f : U \to [0, 1]$ continuous, hence is open in $\mathcal{T} \restriction_U$.

(ii) Note that for every continuous $f : X \to [0, 1]$, $\mathrm{coz}(f) = \rho_X^{-1}(\mathrm{coz}(f^\rho))$ by the construction of f^ρ. Hence $U \subseteq X$ is τ_X-open if and only if there exist continuous functions $f_i : X \to [0, 1]$ for all i in some index set I such that

$$U = \bigcup_{i \in I} \mathrm{coz}(f_i) = \bigcup_{i \in I} \rho_X^{-1}\left(\mathrm{coz}(f_i^\rho)\right) = \rho_X^{-1}\left(\bigcup_{i \in I} \mathrm{coz}(f_i^\rho)\right).$$

Since the τ_{cr}-open subsets of ρX are unions of cozero sets, the conclusion follows.

(iii) Suppose U is τ_X-open and $x \in U$. Then there is a cozero set neighbourhood $\mathrm{coz}(f)$ of x contained in U, where $f : X \to [0, 1]$ is continuous. Thus for any $y \in X \backslash U$, $f(y) = 0$ and $f(x) \neq 0$. Hence $x \not\approx y$ for any such y, and so $[x] \subseteq U$.

(iv) By (ii) $\rho_X^{-1}\left(\overline{\rho_X(U)}\right)$ is τ_X closed. Now suppose $F \subseteq X$ is τ_X closed and $U \subseteq F$. Then $\rho_X(F)$ is closed in ρX by (ii), and contains $\rho_X(U)$, hence contains $\overline{\rho_X(U)}$. By (iii), this gives $F = \rho_X^{-1}(\rho_X(F)) \supseteq \rho_X^{-1}\left(\overline{\rho_X(U)}\right)$, as required.

(v) If $U \subseteq V \subseteq X$, then every $f \in C(V)$ has $f\restriction_U \in C(U)$. Hence $\mathrm{coz}(f) \cap U = \mathrm{coz}(f\restriction_U)$ is a cozero set of U, so that the subspace topology $\tau_V\restriction_U$ is weaker than τ_U. □

Definition 5.2.5. A topological space $(X, \mathcal{T})$ is said to be *w-compact* if given any $\mathcal{T}$-open covering $\{U_\alpha\}_{\alpha \in A}$ of X, then there exist $\alpha_1, \ldots \alpha_n \in A$ such that $X = \mathrm{cl}_{\tau_X}(U_{\alpha_1} \cup \ldots \cup U_{\alpha_n})$.

It is shown in [36, Proposition 3.3] that X is w-compact if and only if any family $\{Q_\alpha\}$ of τ_X-open subsets of X with the finite intersection property has $\bigcap \overline{Q_\alpha} \neq \emptyset$.

The class of w-compact spaces was introduced by Ishii in [35] to characterise the topological spaces X for which $\rho(X \times Y) = \rho X \times \rho Y$, for any topological space Y:

Theorem 5.2.6. *[36, Theorem 4.1] For a topological space X the following are equivalent:*

(i) $\rho(X \times Y) = \rho X \times \rho Y$ for any space Y,

(ii) For each $x \in X$ there is a cozero set neighbourhood U of x such that $\overline{U}$ is w-compact.

We will show in Proposition 5.2.9 that condition (ii) of Theorem 5.2.6 is satisfied by Prim(A) for a large class of C*-algebras A. The following Lemma gives a sufficient condition for a point x in a general topological space X to have a cozero set neighbourhood with w-compact closure.

Lemma 5.2.7. *Let $(X, \mathcal{T})$ be a topological space and suppose that $\rho_X(x) \in \rho X$ has a compact neighbourhood K such that there is a compact $C \subseteq X$ with $\rho_X(C) = K$. Then x has a cozero set neighbourhood U in X with $\overline{U}$ w-compact.*

Proof. Choose $f \in C(\rho X)$ with $f(\rho_X(x)) = 1$ and $f(\rho X \backslash \mathrm{int}K) = \{0\}$. Let $U = \rho_X^{-1}(\mathrm{coz}(f)) = \mathrm{coz}(f \circ \rho_X)$, a cozero set neighbourhood of x in X. We claim that $U \cap C$ is τ_U-dense in U.

Let V be a cozero set of U, then V is also a cozero set of X by [36, Lemma 3.9]. Choose $g \in C(\rho X)$ such that $V = \mathrm{coz}(g \circ \rho_X)$. Note that for any $v \in \rho_X(V) = \mathrm{coz}(g)$

there is $y \in C$ such that $\rho_X(y) = v$, hence $g \circ \rho_X(y) = g(v) \neq 0$. So $V \cap C$ is non-empty, and since every τ_U open subset of U is a union of cozero sets such as V, $U \cap C$ is τ_U-dense.

If $Q \subseteq \overline{U}$ is a nonempty $\tau_{\overline{U}}$-open subset, then it is relatively open (in the subspace topology $\mathcal{T} \restriction_{\overline{U}}$) by Lemma 5.2.4 (i). In particular, $Q \cap U$ is nonempty, and moreover is τ_U open by Lemma 5.2.4 (v).

Take a collection $\{Q_\alpha\}$ of $\tau_{\overline{U}}$-open subsets of $\overline{U}$ with the finite intersection property. Then for every finite subcollection $\{Q_{\alpha_j}\}_{j=1}^n$, the intersection $\bigcap_{j=1}^n (Q_{\alpha_j} \cap U) = \left(\bigcap_{j=1}^n Q_{\alpha_j}\right) \cap U$ is nonempty. It follows that $\{Q_\alpha \cap U\}$ is a collection of τ_U-open subsets of U with the finite intersection property. Since $U \cap C$ is τ_U-dense,

$$\bigcap_{j=1}^n (Q_{\alpha_j} \cap U \cap C) = \left(\bigcap_{j=1}^n Q_{\alpha_j} \cap U\right) \cap C$$

is nonempty for every such subcollection. Thus $\{Q_\alpha \cap U \cap C\}$ is a collection of subsets of C with the finite intersection property. Since C is compact, $\bigcap \overline{(Q_\alpha \cap U \cap C)} \cap C \neq \emptyset$. As

$$\overline{(Q_\alpha \cap U \cap C)} \cap C \subseteq \overline{Q_\alpha}$$

for each α, this implies that $\bigcap \overline{Q_\alpha} \neq \emptyset$. Hence $\overline{U}$ is w-compact. □

Lemma 5.2.8. *Let X be a topological space and suppose that every $x \in X$ has a cozero set neighbourhood with w-compact closure. Then ρX is locally compact.*

Proof. If $A \subseteq X$ is w-compact, then $\rho_X(A)$ is w-compact by [36, Proposition 3.10]. But then since $\rho_X(A)$ is completely regular it is homeomorphic to its complete regularisation $\rho(\rho_X(A))$, hence is compact by [36, Proposition 3.4].

For each point $x \in X$, let U_x be a cozero set neighbourhood of x with $\overline{U_x}$ w-compact. Then $\rho_X(\overline{U_x})$ is compact, and is a neighbourhood of $\rho_X(x)$ since $\rho_X(U_x)$ is open by Lemma 5.2.4(iv). □

Proposition 5.2.9. *Let A be a C^*-algebra such that one of the following conditions hold:*

(i) $\mathrm{Prim}(A)$ *is compact,*

(ii) the complete regularisation map $\rho_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ *is open, or*

(iii) A is σ-unital and $\mathrm{Glimm}(A)$ *is locally compact.*

Then every $P \in \mathrm{Prim}(A)$ has a cozero set neighbourhood with w-compact closure. Hence for any C-algebra B, the complete regularisation $\rho(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$ of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ is homeomorphic to the product space $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$.*

Conversely, if $\mathrm{Glimm}(A)$ is not locally compact then there is $P \in \mathrm{Prim}(A)$ that does not have a cozero set neighbourhood with w-compact closure.

Proof. Note that if (i) holds then $\mathrm{Glimm}(A)$, being the continuous image of the compact space $\mathrm{Prim}(A)$, is compact. The proposition is then immediate by Lemma 5.2.7 with $C = \mathrm{Prim}(A)$ and $K = \mathrm{Glimm}(A)$.

In cases (ii) and (iii), take $P \in \mathrm{Prim}(A)$ with $\rho_A(P) = x$ and let K' be a compact neighbourhood of x in $\mathrm{Glimm}(A)$. By [43, Theorem 2.1 and Proposition 2.5], K' is contained in a compact subset of $\mathrm{Glimm}(A)$ of the form

$$K := \{G \in \mathrm{Glimm}(A) : \|a + G\| \geq \alpha\} = \rho_A(\{P \in \mathrm{Prim}(A) : \|a + P\| \geq \alpha\}),$$

for some $a \in A$ and $\alpha > 0$, and the set $\{P \in \mathrm{Prim}(A) : \|a+P\| \geq \alpha\}$ is compact by [24, Proposition 3.3.7]. Then K is a compact neighbourhood of x, and the conclusion thus follows from Lemma 5.2.7.

It then follows from Theorem 5.2.6 that if any of the conditions (i) to (iii) hold, $\rho(\mathrm{Prim}(A)\times Y) = \rho(\mathrm{Prim}(A))\times\rho(Y)$ for any space Y. In particular, if B is a C*-algebra then we have

$$\rho\left(\mathrm{Prim}(A) \times \mathrm{Prim}(B)\right) = \rho\left(\mathrm{Prim}(A)\right) \times \rho\left(\mathrm{Prim}(B)\right) = \mathrm{Glimm}(A) \times \mathrm{Glimm}(B).$$

On the other hand if $\mathrm{Glimm}(A)$ is not locally compact, then by Lemma 5.2.8 there is $P \in \mathrm{Prim}(A)$ for which no cozero set neighbourhood of P has w-compact closure.

□

Remark 5.2.10. Suppose that A is a C*-algebra such that $\mathrm{Prim}(A)$ does not satisfy condition (ii) of Theorem 5.2.6. Then there is a topological space Y for which $\rho(\mathrm{Prim}(A) \times Y) \neq \mathrm{Glimm}(A) \times \rho(Y)$. It is not immediately evident whether this space Y can be chosen as $\mathrm{Prim}(B)$ for some C*-algebra B. Thus the partial converse in Proposition 5.2.9 does not preclude the possibility that $\rho\left(\mathrm{Prim}(A) \times \mathrm{Prim}(B)\right) = \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ for all C*-algebras A and B.

We will show in Example 5.7.1 however that $\rho(X \times Y) \neq \rho(X) \times \rho(Y)$ is indeed possible when X and Y are primitive ideal spaces of C*-algebras. Specifically, we construct a C*-algebra A for which $\rho(\mathrm{Prim}(A) \times \mathrm{Prim}(A)) \neq \mathrm{Glimm}(A) \times \mathrm{Glimm}(A)$

Remark 5.2.11. Another natural topology on the complete regularisation ρX of a space X is the quotient topology τ_q induced by the complete regularisation map ρ_X; that is, the strongest topology on ρX for which ρ_X is continuous. Since ρ_X is continuous as a map into $(\rho X, \tau_{cr})$, it always holds that $\tau_{cr} \leq \tau_q$. However, there is an example due to D.W.B. Somerset of a space X for which $\tau_{cr} \neq \tau_q$ on ρX, and a C*-algebra A with $\mathrm{Prim}(A)$ homeomorphic to X [41, Appendix].

It follows from [7, p. 351] and [41, Theorem 2.6] that if A is a C*-algebra satisfying one of the conditions (i) to (iii) of Proposition 5.2.9, then necessarily $\tau_{cr} = \tau_q$ on $\mathrm{Glimm}(A)$.

5.3 The Glimm space of the minimal tensor product of C*-algebras

In this section we show that, as a topological space $\mathrm{Glimm}(A \otimes_\alpha B)$ can be naturally identified with $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, when the latter space is considered as the complete regularisation of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$. We first recall the definition of $A \otimes_\alpha B$ and discuss the canonical embedding of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ in $\mathrm{Prim}(A \otimes_\alpha B)$.

Definition 5.3.1. Let A and B be C*-algebras and let $A \odot B$ denote their $*$-algebraic tensor product. For representations $\pi : A \to B(H)$ and $\sigma : B \to B(K)$ of A and B on Hilbert spaces H and K respectively, there is a unique (algebraic) $*$-homomorphism

$$\pi \odot \sigma : A \odot B \to B(H) \odot B(K) \subseteq B(H \hat{\otimes} K),$$

where $H \hat{\otimes} K$ denotes the Hilbert space tensor product of H and K.

Setting

$$\|c\|_\alpha = \sup \|(\pi \odot \sigma)(c)\|, c \in A \odot B$$

where π and σ range over all representations of A and B respectively, defines a C*-norm $\|\cdot\|_\alpha$ on $A \odot B$. The completion $A \otimes_\alpha B$ of $A \odot B$ in this norm is called the *minimal* or *spatial tensor product* of A and B.

Let $\pi : A \to A'$ and $\sigma : B \to B'$ be $*$-homomorphisms of C*-algebras. Then there is a unique $*$-homomorphism $\pi \otimes \sigma : A \otimes_\alpha B \to A' \otimes_\alpha B'$, such that $(\pi \otimes \sigma)(a \otimes b) = \pi(a) \otimes \sigma(b)$ for all elementary tensors $a \otimes b \in A \otimes B$. In particular let $(I, J) \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$ and denote by $q_I : A \to A/I, q_J : B \to B/J$ the quotient homomorphisms. Then we have a $*$-homomorphism $q_I \otimes q_J : A \otimes_\alpha B \to (A/I) \otimes_\alpha (B/J)$.

For more details on the constructions of $A \otimes_\alpha B$, $H \hat{\otimes} K$ and $\pi \otimes \sigma$ we refer the reader to [31] and [53][Chapter IV].

We now define two natural maps $\Phi, \Delta : \mathrm{Id}'(A) \times \mathrm{Id}'(B) \to \mathrm{Id}'(A \otimes_\alpha B)$ via

$$\Phi(I, J) = \ker(q_I \otimes q_J) \tag{5.3.1}$$

$$\Delta(I, J) = I \otimes_\alpha B + A \otimes_\alpha J. \tag{5.3.2}$$

The following Proposition lists some known properties of the map Φ.

Proposition 5.3.2. *Let A and B be C*-algebras and $A \otimes_\alpha B$ their minimal C*-tensor product. Then the map Φ defined by (5.3.1) has the following properties:*

(i) *If $I, K \in \mathrm{Id}'(A)$ and $J, L \in \mathrm{Id}'(B)$ are such that $I \supseteq K$ and $J \supseteq L$ then $\Phi(I, J) \supseteq \Phi(K, L)$* [42, Lemma 2.2],

(ii) *The restriction of Φ to $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ is a homeomorphism onto its image which is dense in $\mathrm{Prim}(A \otimes_\alpha B)$* [59, lemme 16],

(iii) *The restriction of Φ to $\mathrm{Fac}(A) \times \mathrm{Fac}(B)$ is a homeomorphism onto its image which is dense in $\mathrm{Fac}(A \otimes_\alpha B)$* [42, Corollary 2.7],

(iv) *For $I, J \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$, $\Phi\left(\mathrm{hull}(I) \times \mathrm{hull}(J)\right)$ is dense in $\mathrm{hull}(\Phi(I, J))$* [42, Corollary 2.3],

(v) *For $I, J \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$, we have*

$$\Phi(I, J) = \bigcap \{\Phi(P, Q) : (P, Q) \in \mathrm{hull}(I) \times \mathrm{hull}(J)\}$$

[42, Remark 2.4]

Theorem 5.3.3 below identifies the complete regularisation of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ with that of $\mathrm{Prim}(A \otimes_\alpha B)$. As discussed in the remarks following Lemma 5.2.1, we need to take into account the appropriate topology on the former space. Thus we will refer to $\rho\,(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$ as $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})$, and $\rho\,(\mathrm{Prim}(A)) \times \rho\,(\mathrm{Prim}(B))$ as $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_p)$

Theorem 5.3.3. *Let A and B be C^*-algebras, let $A \otimes_\alpha B$ be their minimal C^*-tensor product and denote by ρ_A, ρ_B and ρ_α the complete regularisation maps of* $\mathrm{Prim}(A), \mathrm{Prim}(B)$ *and* $\mathrm{Prim}(A \otimes_\alpha B)$ *respectively. Then there is a homeomorphism* $\psi : \mathrm{Glimm}(A \otimes_\alpha B) \to (\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})$ *given by*

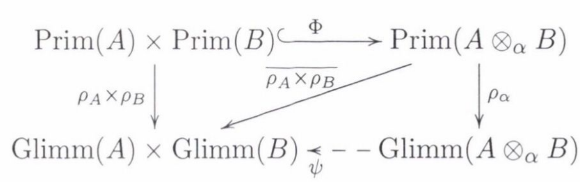
$$(\psi \circ \rho_\alpha)(\Phi(P, Q)) = (\rho_A(P), \rho_B(Q)).$$

It follows that ψ defines a continuous bijection $\mathrm{Glimm}(A \otimes_\alpha B) \to (\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_p)$.

Proof. The map $\rho_A \times \rho_B : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to (\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})$ is the complete regularisation map of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ by Lemma 5.2.1. For the remainder of the proof we will consider $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ with this topology (from which the second assertion will follow since τ_p is weaker).

By [42, Theorem 3.2], the map $(\rho_A \times \rho_B) \circ \Phi^{-1} : \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B)) \to \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ extends uniquely to a continuous map $\overline{(\rho_A \times \rho_B)} : \mathrm{Prim}(A \otimes_\alpha B) \to \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$.

Since $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ is completely regular, $\overline{(\rho_A \times \rho_B)}$ induces a continuous (surjective) map $\psi : \mathrm{Glimm}(A \otimes_\alpha B) \to \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ with the property that $\psi \circ \rho_\alpha = \overline{(\rho_A \times \rho_B)}$ [55, Corollary 1.8].



To show that ψ is in fact a homeomorphism, it suffices to show that the $*$-homomorphism $\psi^* : C^b(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)) \to C^b(\mathrm{Glimm}(A \otimes_\alpha B))$, $\psi^*(f) = f \circ \psi$ is surjective [29, Theorem 10.3 (b)].

To this end, let $f \in C^b(\mathrm{Glimm}(A \otimes_\alpha B))$, so that $f \circ \rho_\alpha \in C^b(\mathrm{Prim}(A \otimes_\alpha B))$ and hence $f \circ \rho_\alpha \circ \Phi \in C^b(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$. Denote by $g \in C^b(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B))$ the unique function such that $g \circ (\rho_A \times \rho_B) = f \circ \rho_\alpha \circ \Phi$. Then $f \circ \rho_\alpha$ and $g \circ \overline{(\rho_A \times \rho_B)}$ are both continuous extensions of $g \circ (\rho_A \times \rho_B) \circ \Phi^{-1}$ to $\mathrm{Prim}(A \otimes_\alpha B)$, hence must agree by [42, Theorem 3.2].

Take $m \in \text{Glimm}(A \otimes_\alpha B)$ and $M \in \text{Prim}(A \otimes_\alpha B)$ such that $\rho_\alpha(M) = m$. Then

$$\begin{aligned} \psi^*(g)(m) = (g \circ \psi)(m) &= (g \circ \psi \circ \rho_\alpha)(M) \\ &= (g \circ \overline{(\rho_A \times \rho_B)})(M) \\ &= (f \circ \rho_\alpha)(M) \\ &= f(m) \end{aligned}$$

It follows that $\psi^*(g) = f$, hence ψ^* is surjective. □

Note that Theorem 5.3.3 shows that $\rho_\alpha \circ \Phi$ is surjective. In particular, given any $M \in \text{Prim}(A \otimes_\alpha B)$ there exist $(P, Q) \in \text{Prim}(A) \times \text{Prim}(B)$ such that $M \approx \Phi(P, Q)$.

Corollary 5.3.4. *Let A and B be C*-algebras such that either A or B satisfies one of the conditions (i)-(iii) of Proposition 5.2.9. Then $\tau_{cr} = \tau_p$ on* $\text{Glimm}(A) \times \text{Glimm}(B)$, *and hence* $\text{Glimm}(A \otimes_\alpha B)$ *is homeomorphic to* $(\text{Glimm}(A) \times \text{Glimm}(B), \tau_p)$ *via the map ψ of Theorem 5.3.3.*

Proof. Immediate from Proposition 5.2.9 and Theorem 5.3.3. □

5.4 The central multipliers of $A \otimes_\alpha B$

In this section we apply Theorem 5.3.3 to determine the centre of the multiplier algebra of $A \otimes_\alpha B$ in terms of the topological space $(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})$. We show in Theorem 5.4.3 that $ZM(A \otimes_\alpha B)$ is $*$-isomorphic to the C*-algebra of continuous functions on the Stone-Čech compactification of $(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})$. Further in Theorem 5.4.6 we give necessary and sufficient conditions on this space for which $ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B)$.

The embedding of $M(A) \otimes_\alpha M(B) \subseteq M(A \otimes_\alpha B)$ is discussed in [1], we include a proof in Lemma 5.4.1 below for completeness. It is shown in [32, Corollary 1] that for C*-algebras C and D we have $Z(C \otimes_\alpha D) = Z(C) \otimes Z(D)$ (where $Z(C) \otimes Z(D)$ is the unique C*-completion of the algebraic tensor product $Z(C) \odot Z(D)$ by nuclearity). In particular it follows that for any C*-algebras A and B we may identify $Z(M(A) \otimes_\alpha M(B)) = ZM(A) \otimes ZM(B)$. Thus in this section we are concerned with relating the centre of the larger algebra $M(A \otimes_\alpha B)$ with that of $M(A) \otimes_\alpha M(B)$.

Recall that an ideal I of a C*-algebra C is said to be *essential* in C if given any nonzero ideal J of C, $J \cap I \neq \{0\}$. Equivalently for any $c \in C$, $cI = Ic = \{0\}$ implies $c = 0$.

Suppose that C is a C*-algebra and $z \in M(C)$ such that $zc = cz$ for all $c \in C$. Then it is easily verified that $(zm - mz)C = C(zm - mz) = \{0\}$ for all $m \in M(C)$. Since C is an essential ideal of $M(C)$, we conclude that

$$ZM(C) = \{z \in M(C) : zc = cz \text{ for all } c \in C\}. \tag{5.4.1}$$

Lemma 5.4.1. *There is a canonical embedding $\Theta : M(A) \otimes_\alpha M(B) \to M(A \otimes_\alpha B)$ such that*

$$(\Theta(x \otimes y))(a \otimes b) = xa \otimes yb \text{ and } (a \otimes b)(\Theta(x \otimes y)) = ax \otimes by$$

for all $a \in A, b \in B, x \in M(A), y \in M(B)$. Moreover, $\Theta(ZM(A) \otimes ZM(B)) \subseteq ZM(A \otimes_\alpha B)$.

Proof. Clearly $M(A) \otimes_\alpha M(B)$ contains $A \otimes_\alpha B$ as a two-sided ideal. Suppose J is a nonzero ideal of $M(A) \otimes_\alpha M(B)$. Then by [2, Proposition 4.5], J contains a nonzero elementary tensor $x \otimes y$ where $x \in M(A), y \in M(B)$. Since A is essential in $M(A)$, there is $a \in A$ with either $ax \neq 0$ or $xa \neq 0$. Suppose w.l.o.g. that $xa \neq 0$, so that $\|(xa)^*xa\| = \|xa\|^2 \neq 0$. Setting $a' = (xa)^*$, we then have an element $a' \in A$ with $a'xa \neq 0$. Similarly there are $b, b' \in B$ with $b'yb \neq 0$. It follows that

$$a'xa \otimes b'yb = (a' \otimes b')(x \otimes y)(a \otimes b)$$

is a nonzero element of $J \cap (A \otimes_\alpha B)$. Hence $A \otimes_\alpha B$ is essential in $M(A) \otimes_\alpha M(B)$.

By [20, Proposition 3.7 (i) and (ii)], there is a unique $*$-homomorphism $\Theta : M(A) \otimes_\alpha M(B) \to M(A \otimes_\alpha B)$ extending the canonical inclusion of $A \otimes_\alpha B$ into $M(A \otimes_\alpha B)$, which is injective since $A \otimes_\alpha B$ is essential in $M(A) \otimes_\alpha M(B)$. For elementary tensors $x \otimes y \in M(A) \otimes_\alpha M(B)$ and $a \otimes b \in A \otimes_\alpha B$ we have

$$(\Theta(x \otimes y))(a \otimes b) = \Theta(x \otimes y)\Theta(a \otimes b) = \Theta(xa \otimes yb) = xa \otimes yb,$$

(since Θ is the identity on $A \otimes_\alpha B$), and similarly $(a \otimes b)(\Theta(x \otimes y)) = ax \otimes by$.

For elementary tensors $z_1 \otimes z_2 \in ZM(A) \otimes ZM(B)$ and $a \otimes b \in A \otimes_\alpha B$ we have

$$\Theta(z_1 \otimes z_2)(a \otimes b) = z_1 a \otimes z_2 b = az_1 \otimes bz_2 = (a \otimes b)\Theta(z_1 \otimes z_2),$$

from which it follows that for any $z \in ZM(A) \otimes ZM(B)$ and $c \in A \otimes_\alpha B$, $\Theta(z)c = c\Theta(z)$. Hence by (5.4.1) we see that $\Theta(ZM(A) \otimes ZM(B)) \subseteq ZM(A \otimes_\alpha B)$. □

We remark that it was shown in [1, Theorem 3.8] that if A is σ-unital and non-unital, and B is infinite dimensional, then Θ is not surjective. In what follows, we will suppress mention of Θ and simply consider $M(A) \otimes_\alpha M(B) \subseteq M(A \otimes_\alpha B)$.

The following result gives an equivalent formulation of the Dauns-Hofmann isomorphism of equation (2.3.1).

Corollary 5.4.2. *[21, III Corollary 8.16] For any C^*-algebra A, there is a homeomorphism of $\mathrm{Prim}\,(ZM(A))$ onto $\beta\mathrm{Glimm}(A)$, and hence a $*$-isomorphism $\mu_A : C(\beta\mathrm{Glimm}(A)) \to ZM(A)$. Moreover, μ_A satisfies*

$$\mu_A(f)a - f(p)a \in G_p, \text{ for all } f \in C(\beta\mathrm{Glimm}(A)), p \in \mathrm{Glimm}(A), a \in A,$$

where $G_p = \bigcap\{P \in \mathrm{Prim}(A) : \rho_A(P) = p\}$ is the Glimm ideal of A corresponding to p.

Applying this identification to $A \otimes_\alpha B$ together with the homeomorphism ψ of Theorem 5.3.3 allows us to determine $ZM(A \otimes_\alpha B)$ in terms of $\mathrm{Glimm}(A)$ and $\mathrm{Glimm}(B)$:

Theorem 5.4.3. *Let A and B be C^*-algebras and denote by ψ the homeomorphism of Theorem 5.3.3. For each point $p \in \mathrm{Glimm}(A \otimes_\alpha B)$ let G_p denote the Glimm ideal of $A \otimes_\alpha B$ corresponding to p. Then there is a canonical $*$-isomorphism Θ_α : $C(\beta(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})) \to ZM(A \otimes_\alpha B)$ with the property that*

$$\Theta_\alpha(f)c - (f \circ \psi)(p)c \in G_p,$$

for all $f \in C(\beta(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})), p \in \mathrm{Glimm}(A \otimes_\alpha B)$ and $c \in A \otimes_\alpha B$.

Proof. Since ψ is a homeomorphism the induced map ψ^* is a $*$-isomorphism. Denote by μ_α the $*$-isomorphism of Corollary 5.4.2 applied to $A \otimes_\alpha B$, and by Θ_α the composition of the $*$-isomorphisms

$$C\left(\beta(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})\right) \xrightarrow{\psi^*} C\left(\beta\mathrm{Glimm}(A \otimes_\alpha B)\right) \xrightarrow{\mu_\alpha} ZM(A \otimes_\alpha B).$$

Then Θ_α clearly has the required properties since μ_α does. □

On the other hand, applying the identification of Corollary 5.4.2 to A and B separately gives $*$-isomorphisms

$$\begin{array}{ccc} C\left(\beta\mathrm{Glimm}(A) \times \beta\mathrm{Glimm}(B)\right) & \xrightarrow{\nu} & C\left(\beta\mathrm{Glimm}(A)\right) \otimes C\left(\beta\mathrm{Glimm}(B)\right) \\ & & \downarrow{\scriptstyle \mu_A \otimes \mu_B} \\ & & ZM(A) \otimes ZM(B), \end{array}$$

where ν is the canonical identification satisfying $\nu^{-1}(f \otimes g)(x, y) = f(x)g(y)$ for all elementary tensors $f \otimes g$ and $(x, y) \in \beta\text{Glimm}(A) \times \beta\text{Glimm}(B)$.

Let X and Y be completely regular spaces. Then since the product $\beta X \times \beta Y$ is compact, the universal property of the Stone-Čech compactification [29, Theorem 6.5 (I)] ensures that the inclusion $\iota : X \times Y \to \beta X \times \beta Y$ has a continuous extension to $\iota^\beta : \beta(X \times Y) \to \beta X \times \beta Y$. Moreover, since ι has dense range, compactness of $\beta(X \times Y)$ implies that ι^β is necessarily surjective.

Considering $\text{Glimm}(A) \subseteq \beta\text{Glimm}(A)$ and $\text{Glimm}(B) \subseteq \beta\text{Glimm}(B)$, Lemma 5.2.1 gives a continuous map

$$\begin{array}{ccc} \phi : (\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}) & \longrightarrow & (\text{Glimm}(A) \times \text{Glimm}(B), \tau_p) \\ & & \big\downarrow \\ & & \beta\text{Glimm}(A) \times \beta\text{Glimm}(B), \end{array}$$

and ϕ is a homeomorphism onto its range if and only if $\tau_p = \tau_{cr}$ on $\text{Glimm}(A) \times \text{Glimm}(B)$. Again by the universal property of the Stone-Čech compactification, ϕ extends to a continuous surjection

$$\phi^\beta : \beta\left(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}\right) \to \beta\text{Glimm}(A) \times \beta\text{Glimm}(B).$$

Dual to this map is an injective $*$-homomorphism

$$(\phi^\beta)^* : C\left(\beta\text{Glimm}(A) \times \beta\text{Glimm}(B)\right) \to C\left(\beta\left(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}\right)\right),$$

sending $f \mapsto f \circ \phi^\beta$.

The situation is summarised in the following diagram:

$$\begin{array}{ccc} C\left(\beta\text{Glimm}(A) \times \beta\text{Glimm}(B)\right) & \xhookrightarrow{(\phi^\beta)^*} & C\left(\beta(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})\right) \\ \Big\downarrow \nu & & \Big\downarrow \psi^* \\ C\left(\beta\text{Glimm}(A)\right) \otimes C\left(\beta\text{Glimm}(B)\right) & \hookrightarrow & C\left(\beta\text{Glimm}(A \otimes_\alpha B)\right) \\ \Big\downarrow \mu_A \otimes \mu_B & & \Big\downarrow \mu_\alpha \\ ZM(A) \otimes ZM(B) & \hookrightarrow & ZM(A \otimes_\alpha B) \end{array}$$

Corollary 5.4.4. *For any C*-algebras A and B, $ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B)$ if and only if the canonical map*

$$\phi^\beta : \beta\left(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}\right) \to \beta\text{Glimm}(A) \times \beta\text{Glimm}(B)$$

is injective. Moreover, when this occurs we have $\tau_p = \tau_{cr}$ *on* $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$.

Proof. We first show that the preceding diagram commutes; that is, for any $h \in C(\beta\mathrm{Glimm}(A) \times \beta\mathrm{Glimm}(B))$ the multipliers z_1 and z_2 of $A \otimes_\alpha B$ given by

$$z_1 = (\mu_\alpha \circ \psi^* \circ (\phi^\beta)^*)(h), \; z_2 = ((\mu_A \otimes \mu_B) \circ \nu)\,(h)$$

are equal. By linearity and continuity it suffices to check equality for functions of the form $h = \nu^{-1}(f \otimes g)$, where $f \in C(\beta\mathrm{Glimm}(A))$, $g \in C(\beta\mathrm{Glimm}(B))$.

Consider an elementary tensor $a \otimes b \in A \otimes_\alpha B$, and a pair $(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$. We will show that $(z_1 - z_2)(a \otimes b) + \Phi(P, Q) = 0$. Set $(p, q) = (\rho_A \times \rho_B)(P, Q)$, and note that by Theorem 5.3.3 $(\psi \circ \rho_\alpha \circ \Phi)(P, Q) = (p, q)$. In particular it follows that

$$(\psi^* \circ (\phi^\beta)^*)(h)(\rho_\alpha(\Phi(P, Q))) = (\phi^\beta)^*(h)(\psi \circ \rho_\alpha \circ \Phi(P, Q)) = h(p, q) = f(p)g(q).$$

Firstly by applying the $*$-isomorphism of Corollary 5.4.2 to the element $(\psi^* \circ (\phi^\beta)^*)(h)$ of $C(\beta(\mathrm{Glimm}(A \otimes_\alpha B))$ see that

$$\begin{aligned} z_1(a \otimes b) + \Phi(P, Q) &= (\mu_\alpha \circ \psi^* \circ (\phi^\beta)^*)(h)(a \otimes b) + \Phi(P, Q) \\ &= f(p)g(q)\,(a \otimes b) + \Phi(P, Q) \\ &= f(p)a \otimes g(q)b + \Phi(P, Q). \end{aligned}$$

On the other hand, applying Corollary 5.4.2 to $f \in C(\beta\mathrm{Glimm}(A))$ gives $\mu_A(f)a - f(p)a \in P$, so that

$$(\mu_A(f)a - f(p)a) \otimes g(q)b = \mu_A(f)a \otimes g(q)b - f(p)a \otimes g(q)b \in \ker(q_P \otimes q_Q) = \Phi(P, Q),$$

from which it follows that $\mu_A(f)a \otimes g(q)b + \Phi(P, Q) = f(p)a \otimes g(q)b + \Phi(P, Q)$. A similar argument applied to B gives $\mu_A(f)a \otimes g(q)b + \Phi(P, Q) = \mu_A(f)a \otimes \mu_B(g)b + \Phi(P, Q)$, and we conclude that

$$\begin{aligned} z_1(a \otimes b) + \Phi(P, Q) &= \mu_A(f)a \otimes \mu_B(g)b + \Phi(P, Q) \\ &= (\mu_A \otimes \mu_B)(f \otimes g)(a \otimes b) + \Phi(P, Q) \\ &= (\mu_A \otimes \mu_B \circ \nu)(h)(a \otimes b) + \Phi(P, Q) \\ &= z_2(a \otimes b) + \Phi(P, Q). \end{aligned}$$

In particular $(z_1 - z_2)(a \otimes b) \in \Phi(P, Q)$ for all $a \in A, b \in B$ and $(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$. Since $\bigcap\{\Phi(P, Q) : (P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)\} = \{0\}$, it follows that

$(z_1 - z_2)(a \otimes b) = 0$ for all $a \in A, b \in B$. Thus $(z_1 - z_2)(A \otimes_\alpha B) = \{0\}$, that is, $z_1 = z_2$.

Since the vertical arrows of the diagram all describe $*$-isomorphisms, the inclusion $ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$ will be surjective if and only if $(\phi^\beta)^*$ is. By [29, Theorem 10.3], $(\phi^\beta)^*$ is surjective if and only if ϕ^β is a homeomorphism.

But then ϕ^β, being a continuous surjection from a compact Hausdorff space to a Hausdorff space, is thus a homeomorphism if and only if it is injective.

The final assertion follows from the fact that ϕ^β is the identity on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$.

□

Let X and Y be completely regular spaces. The question of establishing conditions on X, Y and $X \times Y$ for which canonical surjection $\iota^\beta : \beta(X \times Y) \to \beta X \times \beta Y$ is injective (and hence a homeomorphism) has been studied by several authors. If either X or Y is finite, then this is trivially true. The most well-known characterisation in the infinite case is due to Glicksberg [30].

Definition 5.4.5. Let X be a completely regular space. We say that X is *pseudocompact* if every $f \in C(X)$ is bounded.

Glicksberg's Theorem [30, Theorem 1] states that, for infinite completely regular spaces X and Y the canonical map $\beta(X \times Y) \to \beta X \times \beta Y$ is a homeomorphism if and only if $X \times Y$ is pseudocompact.

Theorem 5.4.6. *For any C^*-algebras A and B, $ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B)$ if and only if one of the following conditions hold:*

(i) $\mathrm{Glimm}(A)$ *or* $\mathrm{Glimm}(B)$ *is finite, or*

(ii) $\tau_p = \tau_{cr}$ *on* $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ *and* $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ *is pseudocompact.*

Proof. If (i) holds, w.l.o.g. $\mathrm{Glimm}(B)$ is finite, hence discrete and compact. In particular ρ_B is an open map, so by Proposition 5.2.9(ii), $\tau_p = \tau_{cr}$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$. Then by [55, Proposition 8.2] the map ϕ^β is a homeomorphism and hence $ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B)$ by Corollary 5.4.4.

If $\mathrm{Glimm}(A)$ and $\mathrm{Glimm}(B)$ are infinite then by [30, Theorem 1], $\beta\left((\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_p)\right)$ is canonically homeomorphic to $\beta\mathrm{Glimm}(A) \times$

βGlimm(B) if and only if (Glimm$(A) \times$ Glimm$(B), \tau_p)$ is pseudocompact. Hence in the infinite case, Corollary 5.4.4 gives $ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B)$ if and only if (ii) holds. □

Clearly if $M(A) \otimes_\alpha M(B) = M(A \otimes_\alpha B)$ then $ZM(A) \otimes ZM(B) = Z(M(A) \otimes_\alpha M(B)) = ZM(A \otimes_\alpha B)$. We will show in Example 5.7.2 that the converse is not true; we construct C*-algebras A and B such that $ZM(A) \otimes ZM(B) = Z(M(A \otimes_\alpha B))$, but $M(A) \otimes_\alpha M(B) \subsetneq M(A \otimes_\alpha B)$.

Remark 5.4.7. It is easily seen that the continuous image of a pseudocompact space is pseudocompact. In particular if X and Y are completely regular spaces such that $X \times Y$ is pseudocompact, then since the projection maps π_X and π_Y are continuous we have necessarily that both X and Y are pseudocompact.

In the other direction, it is not always true that a product of pseudocompact spaces is pseudocompact; see [29, Example 9.15] for a counterexample. However, for a product of pseudocompact spaces X and Y, one of which is also locally compact, then $X \times Y$ is pseudocompact by [55, Proposition 8.21].

5.5 Glimm ideals

We now turn to the question of determining the Glimm ideals of $A \otimes_\alpha B$ in terms of those of A and B. More precisely Theorem 5.5.9 shows that, when the Glimm spaces are considered as sets of ideals of A,B and $A \otimes_\alpha B$, then the map Δ of Equation (5.3.2) satisfies $\Delta = \psi^{-1}$.

We define a new map $\Psi : \mathrm{Id}'(A \otimes_\alpha B) \to \mathrm{Id}'(A) \times \mathrm{Id}'(B)$, which is a left inverse of the map Φ of Equation (5.3.1). For $M \in \mathrm{Id}'(A \otimes_\alpha B)$ we define closed two-sided ideals M^A and M^B of A and B respectively via

$$M^A = \{a \in A : a \otimes B \subseteq M\}, M^B = \{b \in B : A \otimes b \subseteq M\}. \qquad (5.5.1)$$

The assignment $\Psi(M) = (M^A, M^B)$ gives a map $\Psi : \mathrm{Id}'(A \otimes_\alpha B) \to \mathrm{Id}'(A) \times \mathrm{Id}'(B)$.

We define the topological space Fac(A) analogously to Prim(A) (Definition 2.1.1).

Definition 5.5.1. For a C*-algebra A, we denote by Fac(A) the set of kernels of factorial representations of A. For $I \in \mathrm{Id}'(A)$ we let

$$\mathrm{hull}_f(I) = \{M \in \mathrm{Fac}(A) : M \supseteq I\},$$

and for $T \subseteq \mathrm{Id}'(A)$ we let

$$k(T) = \bigcap\{I : I \in T\},$$

the *kernel* of T.

The *hull-kernel topology* on $\mathrm{Fac}(A)$ is defined via the closure operation $\mathrm{cl}(T) = \mathrm{hull}_f(k(T))$ for all $T \subseteq \mathrm{Fac}(A)$.

Proposition 5.5.2. *Let A and B be C*-algebras and $A \otimes_\alpha B$ their minimal C*-tensor product. Then the map $\Psi : \mathrm{Id}'(A \otimes_\alpha B) \to \mathrm{Id}'(A) \times \mathrm{Id}'(B)$ satisfies the following properties:*

(i) $\Psi \circ \Phi$ is the identity on $\mathrm{Id}'(A) \times \mathrm{Id}'(B)$,

(ii) $\Psi(\mathrm{Fac}(A \otimes_\alpha B)) = \mathrm{Fac}(A) \times \mathrm{Fac}(B)$,

(iii) The restriction of Ψ to $\mathrm{Fac}(A \otimes_\alpha B)$ is continuous in the hull-kernel topologies,

(iv) For any $M \in \mathrm{Fac}(A \otimes_\alpha B)$, the inclusion $M \subseteq \Phi \circ \Psi(M)$ holds.

Proof. (i) and (iii) are shown in the proof of [42, Theorem 2.6]. To prove (ii), [31, Proposition 1] shows that $\Psi(\mathrm{Fac}(A \otimes_\alpha B) \subseteq \mathrm{Fac}(A) \times \mathrm{Fac}(B)$. Surjectivity then follows from Proposition 5.3.2(iii) and part (i).

As for (iv), it is shown in [17, Lemma 2.13(iv)] that for any prime ideal M of $A \otimes_\alpha B$ we have $M \subseteq \Phi \circ \Psi(M)$. But then [12, Proposition II.6.1.11] shows that every factorial ideal of a C*-algebra is prime, from which (iv) follows.

□

We remark that Proposition 5.5.2 (ii) shows that Ψ maps $\mathrm{Prim}(A \otimes_\alpha B)$ to $\mathrm{Fac}(A) \times \mathrm{Fac}(B)$. It is not known in general whether Ψ maps $\mathrm{Prim}(A \otimes_\alpha B)$ onto $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$. For this reason, we will need to use an alternative construction of the space of Glimm ideals of a C*-algebra, which was first considered by Kaniuth in [37].

It is shown in [37, Section 2] how for any C*-algebra A, $\mathrm{Glimm}(A)$ can be constructed as $\rho(\mathrm{Fac}(A))$. For $I, J \in \mathrm{Fac}(A)$ we write $I \approx_f J$ if $f(I) = f(J)$ for all $f \in C^b(\mathrm{Fac}(A))$, and denote by $[I]_f$ the equivalence class of I in $\mathrm{Fac}(A)$. We remark that when A is separable the spaces $\mathrm{Fac}(A)$ and $\mathrm{Prim}(A)$ coincide [12, Propositions II.6.1.11 and II.6.5.15].

Proposition 5.5.3. *Let A be a C*-algebra. Then the relation $\approx_f$ on $\mathrm{Fac}(A)$ has the following properties:*

(i) *For* $I \in \mathrm{Fac}(A)$ *and* $P \in \mathrm{hull}(I)$ *we have*

$$[I]_f \cap \mathrm{Prim}(A) = [P] \text{ and } k\left([I]_f\right) = k\left([P]\right),$$

(ii) $\mathrm{Fac}(A)/\approx_f$ *is homeomorphic to* $\mathrm{Prim}(A)/\approx$ *via the map* $[I]_f \mapsto [P]$, *where* $P \in \mathrm{hull}(I)$, *when both spaces are considered with the quotient topology,*

(iii) *Each Glimm ideal of* A *is of the form* $G_I = k\left([I]_f\right)$ *for some* $I \in \mathrm{Fac}(A)$.

(iv) *The equivalence classes of* $\approx_f$ *satisfy*

$$[I]_f = \mathrm{hull}_f(G_I).$$

Proof. Parts (i) and (ii) are shown in [37, Lemma 2.2]. (iii) is immediate from (i).

To prove (iv) take $I \in \mathrm{Fac}(A)$. It follows from the definition of $\approx_f$ that the equivalence class $[I]_f$ is a closed subset of $\mathrm{Fac}(A)$. By the definition of the hull-kernel topology and by part (iii) we then have

$$[I]_f = \mathrm{hull}_f\left(k([I]_f)\right) = \mathrm{hull}_f(G_I).$$

□

As a consequence of Proposition 5.5.3(ii), we shall consider the set of equivalence classes $\mathrm{Fac}(A)/\approx_f$ as $\mathrm{Glimm}(A)$, and denote by $\rho_A^f : \mathrm{Fac}(A) \to \mathrm{Glimm}(A)$ the corresponding quotient map. Moreover, we may unambiguously speak of the quotient topology τ_q on $\mathrm{Glimm}(A)$ as the strongest topology on this space for which either ρ_A or ρ_A^f is continuous.

For C*-algebras A and B and two pairs of ideals $(P,Q),(R,S) \in \mathrm{Fac}(A) \times \mathrm{Fac}(B)$, we will write $(P,Q) \approx_f (R,S)$ when $g(P,Q) = g(R,S)$ for all $g \in C^b(\mathrm{Fac}(A) \times \mathrm{Fac}(B))$. By Lemma 5.2.1, this is equivalent to saying $P \approx_f R$ and $Q \approx_f S$.

Lemmas 5.5.4, 5.5.5 and Proposition 5.5.6 below relate equivalence classes of the relation $\approx_f$ in $\mathrm{Fac}(A) \times \mathrm{Fac}(B)$ with those in $\mathrm{Fac}(A \otimes_\alpha B)$, via the maps Φ and Ψ.

Lemma 5.5.4. *Let* $(I,J) \in \mathrm{Fac}(A) \times \mathrm{Fac}(B)$, *and let* $G_I = k\left([I]_f\right)$ *and* $G_J = k\left([J]_f\right)$ *be the corresponding Glimm ideals of* A *and* B *respectively. Then* $G_{\Phi(I,J)} := k\left([\Phi(I,J)]_f\right)$, *the Glimm ideal of* $A \otimes_\alpha B$ *corresponding to* $[\Phi(I,J)]_f$, *satisfies* $G_{\Phi(I,J)} \subseteq \Phi(G_I, G_J)$.

Proof. The fact that $\Phi(I,J) \in \mathrm{Fac}(A \otimes_\alpha B)$ follows from Proposition 5.3.2 (iii). Taking $(P,Q) \in \mathrm{hull}(I) \times \mathrm{hull}(J)$ we have $\Phi(P,Q) \in \mathrm{hull}\,(\Phi(I,J))$ by Proposition 5.3.2 (i). In this case Proposition 5.5.3(i) gives

$$k([I]_f) = k([P]), k([J]_f) = k([Q]) \text{ and } k\left([\Phi(I,J)]_f\right) = k\left([\Phi(P,Q)]\right),$$

and so we may replace (I,J) with (P,Q).

Note that if $(R,S) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ such that $(R,S) \approx (P,Q)$, and $f \in C^b(\mathrm{Prim}(A \otimes_\alpha B)$ then $f \circ \Phi \in C^b(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$, hence $f(\Phi(R,S)) = f(\Phi(P,Q))$, so that $\Phi(R,S) \approx \Phi(P,Q)$. It then follows from Proposition 5.3.2(v) that

$$\begin{aligned}
\Phi\left(G_I, G_J\right) = \Phi\left(k([P]), k([Q])\right) &= \bigcap\{\Phi(R,S) : (R,S) \in \mathrm{hull}(k[P]) \times \mathrm{hull}(k[Q])\} \\
&= \bigcap\{\Phi(R,S) : (R,S) \approx (P,Q)\} \\
&\supseteq \bigcap\{M \in \mathrm{Prim}(A \otimes_\alpha B) : M \approx \Phi(P,Q)\} \\
&= k\left([\Phi(P,Q)]\right) = G_{\Phi(I,J)}
\end{aligned}$$

□

Lemma 5.5.5. *For any $M \in \mathrm{Fac}(A \otimes_\alpha B)$, $\Phi \circ \Psi(M) \in \mathrm{Fac}(A \otimes_\alpha B)$ and $M \approx_f \Phi \circ \Psi(M)$.*

Proof. The fact that $\Phi \circ \Psi(M) \in \mathrm{Fac}(A \otimes_\alpha B)$ follows from Propositions 5.3.2(iii) and 5.5.2(ii). By Proposition 5.5.2(iv), we have $M \subseteq \Phi \circ \Psi(M)$. Hence $\Phi \circ \Psi(M) \in \mathrm{hull}_f(M) = \overline{\{M\}}$, so that $M \approx_f \Phi \circ \Psi(M)$.

□

Note that the proof of Proposition 5.5.6 below requires that we base the definition of Glimm ideals on the complete regularisation of the space of factorial ideals (since Ψ maps factorial ideals to factorial ideals).

Proposition 5.5.6. *Let $(I,J) \in \mathrm{Fac}(A) \times \mathrm{Fac}(B)$, $M \in \mathrm{Fac}(A \otimes_\alpha B)$ and let (M^A, M^B) denote $\Psi(M)$. Then $M \approx_f \Phi(I,J)$ if and only if $(M^A, M^B) \approx_f (I,J)$. Hence with G_I, G_J and $G_{\Phi(I,J)}$ as defined in Lemma 5.5.4, we have*

$$M \in \mathrm{hull}_f\left(G_{\Phi(I,J)}\right) \text{ if and only if } (M^A, M^B) \in \mathrm{hull}_f(G_I) \times \mathrm{hull}_f(G_J)$$

Proof. Suppose $M \approx_f \Phi(I,J)$, and take $g \in C^b(\mathrm{Fac}(A) \times \mathrm{Fac}(B))$. Using Proposition 5.5.2 (ii) and (iii), we have $g \circ \Psi \in C^b(\mathrm{Fac}(A \otimes_\alpha B))$. Hence

$$g(M^A, M^B) = (g \circ \Psi)(M) = (g \circ \Psi)(\Phi(I,J)) = g(I,J),$$

since $\Psi \circ \Phi$ if the identity on $\mathrm{Fac}(A) \times \mathrm{Fac}(B)$ by Proposition 5.5.2 (i). It follows that $(M^A, M^B) \approx_f (I,J)$.

Since $G_{\Phi(I,J)} = k([\Phi(I,J)]_f)$, Proposition 5.5.3(iv) shows that $[\Phi(I,J)]_f = \mathrm{hull}_f(G_{\Phi(I,J)})$. Similarly $[I]_f = \mathrm{hull}_f(G_I)$ and $[J]_f = \mathrm{hull}_f(G_J)$.

To prove the converse, suppose that $(M^A, M^B) \approx_f (I,J)$. Then by Lemma 5.2.1, $M^A \approx_f I$ and $M^B \approx_f J$, so that $M^A \supseteq G_I$ and $M^B \supseteq G_J$. Together with Lemma 5.5.4, this gives the inclusion

$$\Phi \circ \Psi(M) = \Phi(M^A, M^B) \supseteq \Phi(G_I, G_J) \supseteq G_{\Phi(I,J)},$$

and since by Proposition 5.3.2(iii) $\Phi \circ \Psi(M) \in \mathrm{Fac}(A \otimes_\alpha B)$, it follows from Proposition 5.5.3(iv) that $\Phi \circ \Psi(M) \approx_f \Phi(I,J)$. Then by Lemma 5.5.5 we have $M \approx_f \Phi(I,J)$.

The final assertion of the statement follows from Proposition 5.5.3(iv). □

In what follows we make use of the map $\Delta : \mathrm{Id}'(A) \times \mathrm{Id}'(B) \to \mathrm{Id}'(A \otimes_\alpha B)$ defined via (5.3.2). For $(I,J) \in \mathrm{prime}(A) \times \mathrm{prime}(B)$, $(\Psi \circ \Delta)(I,J) = (I,J)$ [17, Lemma 2.13(i)]. We will extend this to general $(I,J) \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$ in Lemma 5.5.7 below. On the other hand, if $K \in \mathrm{Id}'(A \otimes_\alpha B)$ then

$$(\Delta \circ \Psi)(K) = K^A \otimes_\alpha B + A \otimes_\alpha K^B \subseteq K \tag{5.5.2}$$

by the definition of K^A and K^B in (5.5.1).

Lemma 5.5.7. *Let I and J be proper ideals of A and B respectively. Then*

(i) $\mathrm{hull}_f \Delta(I,J) = \Psi^{-1}(\mathrm{hull}_f(I) \times \mathrm{hull}_f(J))$.

(ii) $\Psi \circ \Delta(I,J) = (I,J)$

Proof. To show (i), take $F \in \mathrm{hull}_f \Delta(I,J)$, then $\Psi(F) = (F^A, F^B) \in \mathrm{Fac}(A) \times \mathrm{Fac}(B)$ and $F^A \supseteq I, F^B \supseteq J$. Hence $\Psi(F) \in \mathrm{hull}_f(I) \times \mathrm{hull}_f(J)$.

On the other hand, suppose $F \in \mathrm{Fac}(A \otimes_\alpha B)$ and $\Psi(F) \in \mathrm{hull}_f(I) \times \mathrm{hull}_f(J)$. Then using (5.5.2), $\Delta(I,J) \subseteq \Delta(F^A, F^B) \subseteq F$, as required.

To prove (ii), let $K = \Delta(I, J)$ and $(K^A, K^B) = \Psi(K)$. Then if $a \in I$, $a \otimes B \subseteq I \otimes_\alpha B \subseteq K$, so $a \in K^A$ and hence $I \subseteq K^A$. On the other hand, suppose $a \in K^A$ and $b \in B \backslash J$, so that $a \otimes b \in K$. Choose a bounded linear functional λ on B vanishing on J such that $\lambda(b) = 1$. Let $L_\lambda : A \otimes_\alpha B \to A$ be the corresponding left slice map defined via $L_\lambda(a' \otimes b') = \lambda(b')a'$ on elementary tensors and extended to $A \otimes_\alpha B$ by linearity and continuity [54, Theorem 1]. Then $L_\lambda(A \odot J) = \{0\}$ and $L_\lambda(I \odot B) \subseteq I$, so that $L_\lambda(K) \subseteq I$. In particular $L_\lambda(a \otimes b) = a \in I$, hence $K^A \subseteq I$ and so $K^A = I$. A similar argument shows that $K^B = J$, which completes the proof. □

Corollary 5.5.8. *Let $(I, J) \in \mathrm{Fac}(A) \times \mathrm{Fac}(B)$. Then with G_I, G_J and $G_{\Phi(I,J)}$ defined as in Lemma 5.5.4, we have*

$$G_{\Phi(I,J)} = G_I \otimes_\alpha B + A \otimes_\alpha G_J$$

Proof. Take $M \in \mathrm{Fac}(A \otimes_\alpha B)$. By Proposition 5.5.6, $M \supseteq G_{\Phi(I,J)}$ if and only if $M \in \Psi^{-1}(\mathrm{hull}_f(G_I) \times \mathrm{hull}_f(G_J))$. Hence by Lemma 5.5.7(i), $M \supseteq G_{\Phi(I,J)}$ if and only if $M \supseteq \Delta(G_I, G_J)$, so $G_{\Phi(I,J)} = \Delta(G_I, G_J)$. □

As a consequence of Corollary 5.5.8, we are now in a position to prove a similar result to [37, Theorem 2.3]:

Theorem 5.5.9. *Let A and B be C*-algebras and denote by $\psi : \mathrm{Glimm}(A \otimes_\alpha B) \to (\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})$ the homeomorphism of Theorem 5.3.3. Then identifying the Glimm spaces with the corresponding sets of ideals we have*

(i) $\Delta = \psi^{-1}$ *on* $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$*, hence* Δ *is a homeomorphism of* $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})$ *onto* $\mathrm{Glimm}(A \otimes_\alpha B)$*,*

(ii) ψ *is given by the restriction of* Ψ *to* $\mathrm{Glimm}(A \otimes_\alpha B)$*.*

Proof. Following the notation of Theorem 5.3.3, Proposition 5.5.3(i) and Corollary 5.5.8 show that the diagram

$$\begin{array}{ccc} \mathrm{Prim}(A) \times \mathrm{Prim}(B) & \xhookrightarrow{\ \Phi\ } & \mathrm{Prim}(A \otimes_\alpha B) \\ {\scriptstyle \rho_A \times \rho_B}\big\downarrow & & \big\downarrow{\scriptstyle \rho_\alpha} \\ \mathrm{Glimm}(A) \times \mathrm{Glimm}(B) & \xrightarrow{\ \Delta\ } & \mathrm{Glimm}(A \otimes_\alpha B) \end{array}$$

commutes, i.e., that $\Delta \circ (\rho_A \times \rho_B) = \rho_\alpha \circ \Phi$. Therefore if we can show that $\psi^{-1} \circ (\rho_A \times \rho_B) = \rho_\alpha \circ \Phi$ also, then it will follow necessarily that $\Delta = \psi^{-1}$ (since $\rho_A \times \rho_B$ is surjective).

From Theorem 5.3.3, the extension $\overline{(\rho_A \times \rho_B)}$ of $\rho_A \times \rho_B$ to $\mathrm{Prim}(A \otimes_\alpha B)$ satisfies $\overline{(\rho_A \times \rho_B)} \circ \Phi = \rho_A \times \rho_B$ and $\psi \circ \rho_\alpha = \overline{(\rho_A \times \rho_B)}$, so we have

$$\begin{aligned} \psi^{-1} \circ (\rho_A \times \rho_B) &= \psi^{-1} \circ \overline{(\rho_A \times \rho_B)} \circ \Phi \\ &= \psi^{-1} \circ (\psi \circ \rho_\alpha) \circ \Phi \\ &= \rho_\alpha \circ \Phi, \end{aligned}$$

which proves (i).

Assertion (ii) is then immediate from (i) and Lemma 5.5.7(ii). □

Suppose that A and B are C*-algebras such that $A \otimes_\alpha B$ satisfies Tomiyama's property (F). Under this assumption, Kaniuth's result [37, Theorem 2.3] shows that the map $\Delta : (\mathrm{Glimm}(A), \tau_q) \times (\mathrm{Glimm}(B), \tau_q) \to (\mathrm{Glimm}(A \otimes_\alpha B), \tau_q)$ is an open bijection, where τ_q denotes the quotient topology on the Glimm space as discussed in Remark 5.2.11.

In order to extend this to arbitrary minimal tensor products, we first need some Lemmas:

Lemma 5.5.10. *Suppose that the complete regularisation maps ρ_A and ρ_B are open with respect to either τ_q or τ_{cr} on* $\mathrm{Glimm}(A)$ *and* $\mathrm{Glimm}(B)$ *respectively. Then*

(i) $\tau_q = \tau_{cr}$ on each of $\mathrm{Glimm}(A)$ *and* $\mathrm{Glimm}(B)$,

(ii) $\tilde{\tau}_q = \tau_{cr} = \tau_p$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, *where $\tilde{\tau}_q$ is the topology induced by the product map $\rho_A \times \rho_B$,*

(iii) $\tau_q = \tau_{cr}$ on $\mathrm{Glimm}(A \otimes_\alpha B)$.

Proof. (i) is shown in the discussion in [7, p. 351].

(ii) As a consequence of (i), we may consider τ_p as the product of the quotient topologies. Since $\rho_A \times \rho_B$ is necessarily τ_p continuous, τ_p is always weaker than $\tilde{\tau}_q$.

Consider a $\tilde{\tau}_q$ open subset $\mathcal{U}$ of $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ and let $(x, y) \in \mathcal{U}$. Choose $(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ with $(\rho_A \times \rho_B)(P, Q) = (x, y)$. Then since $(\rho_A \times \rho_B)^{-1}(\mathcal{U})$

is an open subset of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$, we can find open neighbourhoods $\mathcal{W}$ of P and $\mathcal{S}$ of Q such that $\mathcal{W} \times \mathcal{S} \subseteq (\rho_A \times \rho_B)^{-1}(\mathcal{U})$. Then we have

$$(\rho_A \times \rho_B)(P, Q) = (x, y) \in \rho_A(\mathcal{W}) \times \rho_B(\mathcal{S}) \subseteq \mathcal{U},$$

and $\rho_A(\mathcal{W}) \times \rho_B(\mathcal{S})$ is τ_p-open since ρ_A and ρ_B are both τ_q-open. In particular (x, y) is a τ_p-interior point of $\mathcal{U}$, and hence $\mathcal{U}$ is τ_p-open.

The fact that $\tau_p = \tau_{cr}$ follows from condition (ii) of Proposition 5.2.9.

As for (iii), it is always true that τ_q is stronger than τ_{cr}, thus we need to prove that any τ_q open subset $\mathcal{U}$ of $\mathrm{Glimm}(A \otimes_\alpha B)$ is τ_{cr}-open. By part (ii) and Corollary 5.3.4, ψ is a homeomorphism of $(\mathrm{Glimm}(A \otimes_\alpha B), \tau_{cr})$ onto $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tilde{\tau}_q)$. So given a τ_q-open subset $\mathcal{U}$ of $\mathrm{Glimm}(A \otimes_\alpha B)$, it will suffice to prove that $\psi(\mathcal{U})$ is $\tilde{\tau}_q$-open, that is, that $(\rho_A \times \rho_B)^{-1}(\psi(\mathcal{U}))$ is open.

Let $\mathcal{W} = (\rho_A \times \rho_B)^{-1}(\psi(\mathcal{U}))$. We will show that $\mathcal{W} = \Phi^{-1}\left(\rho_\alpha^{-1}(\mathcal{U})\right)$. Since $\mathcal{U}$ is τ_q-open and Φ is continuous on $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ by Proposition 5.3.2(ii), this will imply that $\mathcal{W}$ is open. For any $(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$, Theorem 5.3.3 gives

$$\psi \circ \rho_\alpha \circ \Phi(P, Q) = (\rho_A \times \rho_B)(P, Q),$$

so that $(P, Q) \in \mathcal{W}$ if and only if $\rho_\alpha \circ \Phi(P, Q) \in \psi^{-1}(\psi(\mathcal{U})) = \mathcal{U}$. It follows that $\mathcal{W} = \Phi^{-1}\left(\rho_\alpha^{-1}(\mathcal{U})\right)$, so that $\psi(\mathcal{U})$ is $\tilde{\tau}_q$-open and hence $\mathcal{U}$ is τ_{cr}-open.

□

Lemma 5.5.11. *Let ρ_A^f, ρ_B^f and ρ_α^f denote the complete regularisation maps of* $\mathrm{Fac}(A)$, $\mathrm{Fac}(B)$ *and* $\mathrm{Fac}(A \otimes_\alpha B)$ *respectively, and let* $\psi : \mathrm{Glimm}(A \otimes_\alpha B) \to \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ *be the map of Theorem 5.3.3. Then it holds that*

$$\psi \circ \rho_\alpha^f \circ \Phi = \rho_A^f \times \rho_B^f$$

on $\mathrm{Fac}(A) \times \mathrm{Fac}(B)$.

Proof. Let $(I, J) \in \mathrm{Fac}(A) \times \mathrm{Fac}(B)$ and $(p, q) = (\rho_A^f \times \rho_B^f)(I, J)$. Take $(P, Q) \in \mathrm{hull}(I) \times \mathrm{hull}(J)$, so that $(\rho_A \times \rho_B)(P, Q) = (p, q)$ by Proposition 5.5.3(i). Then by Proposition 5.3.2(i), $\Phi(P, Q) \in \mathrm{hull}\,(\Phi(I, J))$, so that $(\rho_\alpha \circ \Phi)(P, Q) = (\rho_\alpha^f \circ \Phi)(I, J)$ by Proposition 5.5.3(i) applied to $A \otimes_\alpha B$.

Finally, Theorem 5.3.3 gives

$$(\psi \circ \rho_\alpha \circ \Phi)(P, Q) = (\rho_A \times \rho_B)(P, Q),$$

so that

$$(\psi \circ \rho_\alpha^f \circ \Phi)(I, J) = (\rho_A^f \times \rho_B^f)(I, J),$$

as required. □

We are now in a position to extend [37, Theorem 2.3], which required the assumption that $A \otimes_\alpha B$ satisfies property (F).

Theorem 5.5.12. *The map ψ of Theorem 5.3.3 defines a continuous bijection of* $(\mathrm{Glimm}(A \otimes_\alpha B), \tau_q)$ *onto the product space* $(\mathrm{Glimm}(A), \tau_q) \times (\mathrm{Glimm}(B), \tau_q)$, *where τ_q denotes the quotient topology induced by the complete regularisation map. It follows that its inverse Δ is an open bijection. Moreover, Δ is a homeomorphism whenever the complete regularisation maps ρ_A and ρ_B are open with respect to the quotient topologies on* $\mathrm{Glimm}(A)$ *and* $\mathrm{Glimm}(B)$.

Proof. Let $\mathcal{U} \times \mathcal{V}$ be a basic open subset of the product space $(\mathrm{Glimm}(A), \tau_q) \times (\mathrm{Glimm}(B), \tau_q)$. Then by Proposition 5.5.3(ii), the preimages $\mathcal{W} := (\rho_A^f)^{-1}(\mathcal{U})$ and $\mathcal{S} := (\rho_B^f)^{-1}(\mathcal{V})$ are open subsets of $\mathrm{Fac}(A)$ and $\mathrm{Fac}(B)$ respectively. We claim that $\psi^{-1}(\mathcal{U} \times \mathcal{V})$ is a τ_q-open subset of $\mathrm{Glimm}(A \otimes_\alpha B)$, that is, that $(\rho_\alpha^f)^{-1}\left(\psi^{-1}(\mathcal{U} \times \mathcal{V})\right)$ is an open subset of $\mathrm{Fac}(A \otimes_\alpha B)$. Since the map Ψ is continuous by Proposition 5.5.2(iii), it will suffice to show that

$$(\rho_\alpha^f)^{-1}\left(\psi^{-1}(\mathcal{U} \times \mathcal{V})\right) = \Psi^{-1}(\mathcal{W} \times \mathcal{S}).$$

Let $M \in \Psi^{-1}(\mathcal{W} \times \mathcal{S})$. By Lemma 5.5.5 $M \approx_f \Phi \circ \Psi(M)$, so that $\rho_\alpha^f(M) = \rho_\alpha^f(\Phi \circ \Psi(M))$. Then Lemma 5.5.11 gives

$$\psi \circ \rho_\alpha^f(M) = (\psi \circ \rho_\alpha^f \circ \Phi)(\Psi(M)) = (\rho_A^f \times \rho_B^f)(\Psi(M)) \in \mathcal{U} \times \mathcal{V}.$$

Hence $\rho_\alpha^f(M) \in \psi^{-1}(\mathcal{U} \times \mathcal{V})$, and we have $\Psi^{-1}(\mathcal{W} \times \mathcal{S}) \subseteq (\rho_\alpha^f)^{-1}\left(\psi^{-1}(\mathcal{U} \times \mathcal{V})\right)$.

To show the reverse inclusion, let $M \in (\rho_\alpha^f)^{-1}\left(\psi^{-1}(\mathcal{U} \times \mathcal{V})\right)$, and $(p, q) = (\psi \circ \rho_\alpha^f)(M) \in \mathcal{U} \times \mathcal{V}$. Choose $(I, J) \in \mathcal{W} \times \mathcal{S}$ with $(\rho_A^f \times \rho_B^f)(I, J) = (p, q)$. Then invoking Lemma 5.5.11 again we have

$$(\psi \circ \rho_\alpha^f)(\Phi(I, J)) = (\rho_A^f \times \rho_B^f)(I, J) = (p, q).$$

Since ψ is injective and $(\psi \circ \rho_\alpha^f)(M) = (p, q)$, it follows that $\rho_\alpha^f(M) = \rho_\alpha^f(\Phi(I, J))$ and hence $M \approx_f \Phi(I, J)$. By Proposition 5.5.6, this implies that $\Psi(M) \approx_f (I, J)$, so that

$(\rho_A^f \times \rho_B^f)(\Psi(M)) = (p, q)$. In particular $\Psi(M) \in (\rho_A^f \times \rho_B^f)^{-1}(\mathcal{U} \times \mathcal{V}) = \mathcal{W} \times \mathcal{S}$, so that $M \in \Psi^{-1}(\mathcal{W} \times \mathcal{S})$ and hence $(\rho_\alpha^f)^{-1}\left(\psi^{-1}(\mathcal{U} \times \mathcal{V})\right) \subseteq \Psi^{-1}(\mathcal{W} \times \mathcal{S})$, as required.

If in addition the complete regularisation maps ρ_A and ρ_B are open, then by Lemma 5.5.10 we have $\tau_q = \tau_{cr}$ on each of Glimm(A), Glimm(B) and Glimm$(A \otimes_\alpha B)$. Applying Corollary 5.3.4 and Theorem 5.5.9 it follows that Δ is a homeomorphism of $(\text{Glimm}(A), \tau_q) \times (\text{Glimm}(B), \tau_q)$ onto Glimm$(A \otimes_\alpha B)$. □

5.6 Sectional representation

In this section, we study $A \otimes_\alpha B$ as the section algebra of the canonical upper-semicontinuous C*-bundle over Glimm$(A \otimes_\alpha B)$ defined by the Dauns-Hofmann Theorem (Theorem 2.3.6). In particular, we relate this bundle to the corresponding bundles over Glimm(A) and Glimm(B) associated with A and B respectively.

As a consequence of Theorems 2.3.6 and 5.5.9, we can see that for any C*-algebras A and B the canonical bundle associated with $A \otimes_\alpha B$ has base space homeomorphic to $(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})$ (or its Stone-Čech compactification), and fibre algebras $*$-isomorphic to the quotient C*-algebras

$$\frac{A \otimes_\alpha B}{\Delta(G_p, G_q)}$$

for $(p, q) \in \text{Glimm}(A) \times \text{Glimm}(B)$, and zero otherwise.

Lee's Theorem [44, Theorem 4] implies that this bundle is a continuous C*-bundle if and only if the complete regularisation map ρ_A is open (see also Proposition 2.3.3(iii) and [7, Theorem 2.1]). Thus in the case of the minimal tensor product of C*-algebras, it is natural to ask whether ρ_α being open is equivalent to ρ_A and ρ_B being open. It follows from [37, Lemma 2.2 and Theorem 2.3] that this is indeed the case when $A \otimes_\alpha B$ satisfies property (F). We now consider this question under more general hypotheses.

It is well known that $A \otimes_\alpha B$ satisfies property (F) if and only if $\Phi(I, J) = \Delta(I, J)$ for all $(I, J) \in \text{Id}'(A) \times \text{Id}'(B)$; see [54, Theorem 5 (2)] for example. The assumption that $\Phi(G, H) = \Delta(G, H)$ for all Glimm ideals of A and B is weaker in general. For example, if H is an infinite dimensional Hilbert space then $B(H) \otimes_\alpha B(H)$ does not satisfy property (F) [57, Corollary 7]. However, Glimm$(B(H))$ is a one point space consisting of the zero ideal, and clearly $\Phi(\{0\}, \{0\}) = \Delta(\{0\}, \{0\})$. It appears to be unknown whether there exist C*-algebras A and B and Glimm ideals $(G, H) \in \text{Glimm}(A) \times \text{Glimm}(B)$ such that $\Delta(G, H) \subsetneq \Phi(G, H)$.

The condition that $\Delta = \Phi$ on Glimm(A) is equivalent to requiring that the fibre algebras of the canonical bundle associated with $A \otimes_\alpha B$ are given by the minimal tensor products of the corresponding fibres of the bundles of A and B, that is

$$\{(A/G_p) \otimes_\alpha (B/G_q) : (p,q) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)\}, \tag{5.6.1}$$

(with topology inherited from $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})$ by Theorem 5.5.9). Indeed one may always consider an element $c \in A \otimes_\alpha B$ as a cross-section of the fibred space given by (5.6.1) via $(p,q) \mapsto c + \Phi(G_p, G_q)$ for $(p,q) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ [6, p. 136-137]. In the case that the bundles of A and B are continuous, it follows from [6, Corollary 3.1] that this representation of $A \otimes_\alpha B$ defines a continuous C*-bundle over $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_p)$ in the obvious way if and only if $\Delta = \Phi$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$.

Theorem 5.6.2, together with Lee's Theorem quoted above, asserts that under the assumption $\Delta = \Phi$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, the Dauns-Hofmann representation of $A \otimes_\alpha B$ defines a continuous C*-bundle over $\mathrm{Glimm}(A \otimes_\alpha B)$ if and only if A and B define continuous C*-bundles over Glimm(A) and Glimm(B) respectively.

Lemma 5.6.1. *Let A and B be C*-algebras such that $\Delta(G,H) = \Phi(G,H)$ for all $(G,H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, and let $\mathcal{U} \subseteq \mathrm{Prim}(A \otimes_\alpha B)$ be open. Then*

$$\rho_\alpha(\mathcal{U}) = \rho_\alpha\left(\mathcal{U} \cap \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))\right).$$

Proof. Let $m \in \rho_\alpha(\mathcal{U})$ and choose $M \in \mathcal{U}$ such that $\rho_\alpha(M) = m$. Let $G_m = k([M])$ be the corresponding Glimm ideal of $A \otimes_\alpha B$. Then there exist $(p,q) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ and corresponding Glimm ideals G_p and G_q of A and B respectively with $G_m = \Phi(G_p, G_q) = \Delta(G_p, G_q)$ by Theorem 5.5.9.

Now $M \in \mathrm{hull}(G_m) \cap \mathcal{U}$, which is a nonempty relatively open subset of $\mathrm{hull}(G_m)$. By Proposition 5.3.2(iv), $\Phi(\mathrm{hull}(G_p) \times \mathrm{hull}(G_q))$ is dense in $\mathrm{hull}(G_m) = \rho_\alpha^{-1}(\{m\})$. Hence there exists $(P,Q) \in \mathrm{hull}(G_p) \times \mathrm{hull}(G_q)$ such that $\Phi(P,Q) \in \mathrm{hull}(G_m) \cap \mathcal{U}$. In particular $\Phi(P,Q) \in \mathcal{U} \cap \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$, and $\rho_\alpha \circ \Phi(P,Q) = m$. It follows that $\rho_\alpha(\mathcal{U}) \subseteq \rho_\alpha(\mathcal{U} \cap \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B)))$, and the reverse inclusion is trivial.

□

Theorem 5.6.2. *Let A and B be C*-algebras such that $\Delta(G,H) = \Phi(G,H)$ for all $(G,H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$. Then the following are equivalent:*

(i) ρ_α *is an open map with respect to* τ_{cr} *on* $\mathrm{Glimm}(A \otimes_\alpha B)$,

(ii) ρ_α *is an open map with respect to* τ_q *on* $\mathrm{Glimm}(A \otimes_\alpha B)$,

(iii) ρ_A *and* ρ_B *are open maps with respect to* τ_{cr} *on* $\mathrm{Glimm}(A)$ *and* $\mathrm{Glimm}(B)$ *respectively,*

(iv) ρ_A *and* ρ_B *are open maps with respect to* τ_q *on* $\mathrm{Glimm}(A)$ *and* $\mathrm{Glimm}(B)$ *respectively.*

Proof. Note that by Lemma 5.5.10(i), (i) is equivalent to (ii), and (iii) is equivalent to (iv). We will show that (i) implies (iv) and that (iii) implies (i).

Suppose that (i) holds. We first claim that $\rho_A \times \rho_B$ is open as a map into $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})$. Take an open subset $\mathcal{U} \subseteq \mathrm{Prim}(A) \times \mathrm{Prim}(B)$. Then since the restriction of Φ to $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ is a homeomorphism onto its image by Proposition 5.3.2(ii), there is an open subset $\mathcal{U}' \subseteq \mathrm{Prim}(A \otimes_\alpha B)$ with $\mathcal{U}' \cap \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B)) = \Phi(\mathcal{U})$. By Lemma 5.6.1 $\rho_\alpha(\mathcal{U}') = \rho_\alpha(\Phi(\mathcal{U}))$. Then by Theorem 5.3.3

$$(\rho_A \times \rho_B)(\mathcal{U}) = (\psi \circ \rho_\alpha \circ \Phi)(\mathcal{U}) = (\psi \circ \rho_\alpha)(\mathcal{U}'),$$

which is τ_{cr}-open since ρ_α is an open map and ψ is a homeomorphism.

As in the proof of [7, p.351], τ_{cr} must agree with the quotient topology $\tilde{\tau}_q$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ induced by the map $\rho_A \times \rho_B$. In particular $\rho_A \times \rho_B$ is a $\tilde{\tau}_q$-open map. To see that ρ_A is open, let $\mathcal{W} \subseteq \mathrm{Prim}(A)$ be open. Then

$$(\rho_A \times \rho_B)(\mathcal{W} \times \mathrm{Prim}(B)) = \rho_A(\mathcal{W}) \times \mathrm{Glimm}(B)$$

is $\tilde{\tau}_q$-open, hence $(\rho_A \times \rho_B)^{-1}(\rho_A(\mathcal{W}) \times \mathrm{Glimm}(B))$ is open. By Lemma 5.2.1 we have

$$(\rho_A \times \rho_B)^{-1}(\rho_A(\mathcal{W}) \times \mathrm{Glimm}(B)) = \rho_A^{-1}(\rho_A(\mathcal{W})) \times \mathrm{Prim}(B),$$

so that in particular, $\rho_A^{-1}(\rho_A(\mathcal{W}))$ is open. It follows that ρ_A is a τ_q-open map. A similar argument shows that ρ_B is τ_q-open, hence (i) implies (iv).

Assume that (iii) holds and take an open subset $\mathcal{U} \subseteq \mathrm{Prim}(A \otimes_\alpha B)$. Then $\mathcal{U} \cap \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$ is a relatively open subset of $\Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$. Again by Proposition 5.3.2(ii), there is an open subset $\mathcal{V} \subseteq \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ such that $\Phi(\mathcal{V}) = \mathcal{U} \cap \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$. By Theorem 5.3.3 we then have

$$(\psi \circ \rho_\alpha)(\mathcal{U} \cap \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))) = (\psi \circ \rho_\alpha \circ \Phi)(\mathcal{V}) = (\rho_A \times \rho_B)(\mathcal{V}),$$

which is τ_{cr}-open since $\rho_A \times \rho_B$ is a τ_{cr}-open map by Lemma 5.5.10(ii). Together with Lemma 5.6.1, this shows that

$$\rho_\alpha(\mathcal{U}) = \rho_\alpha\left(\mathcal{U} \cap \Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))\right) = \psi^{-1}\left((\rho_A \times \rho_B)(\mathcal{V})\right),$$

which is open since ψ is a homeomorphism. Hence ρ_α is an open map.

□

Following a suggestion of R.J. Archbold, below we give a similar result to [6, Proposition 4.1]. Under the assumption that A and B each have at least one Glimm quotient containing a nonzero projection, we show in Proposition 5.6.4 that the implication (i)⇒(iii) of Theorem 5.6.2 does not require $\Delta = \Phi$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$. We establish as a corollary that under the same assumptions on A and B, if $A \otimes_\alpha B$ is quasi-standard then A and B must be quasi-standard.

Lemma 5.6.3. *Let X and Y be topological spaces. Then for any $y_0 \in Y$, the map sending $\rho_X(x) \mapsto \rho_{X\times Y}(x, y_0)$ is a homeomorphic embedding of ρX into $\rho(X \times Y)$, with respect to the corresponding τ_{cr}-topologies on each space.*

Proof. By Lemma 5.2.1 we may identify $\rho(X \times Y)$ with $(\rho X \times \rho Y, \tau_{cr})$ under the canonical mapping $\rho_{X\times Y}(x, y) \mapsto (\rho_X(x), \rho_Y(y))$. Clearly the map sending $\rho_X(x) \mapsto (\rho_X(x), \rho_Y(y_0))$ is a homeomorphic embedding of ρX into $\rho X \times \rho Y$ with the product topology τ_p. Thus we must show that the restrictions of the τ_p and τ_{cr} topologies to the subspace $\rho X \times \{\rho_Y(y_0)\}$ are equal. Since $\tau_p \leq \tau_{cr}$ it will suffice to show that for any τ_{cr}-open subset $\mathcal{O}$ of $\rho X \times \rho Y$ and $x_0 \in X$ such that $(\rho_X(x_0), \rho_Y(y_0)) \in \mathcal{O}$, there is a cozero set neighbourhood $\mathcal{U}$ of $\rho_X(x_0)$ in ρX such that $\mathcal{U} \times \{\rho_Y(y_0)\} \subseteq \mathcal{O}$.

Since $\mathcal{O}$ is τ_{cr}-open there is $g \in C^b(X \times Y)$ such that $\mathrm{coz}(g^\rho)$ is a neighbourhood of $(\rho_X(x_0), \rho_Y(y_0))$ contained in $\mathcal{O}$. Define $f \in C^b(X)$ via $f(x) = g(x, y_0)$, then $f^\rho \in C^b(\rho X)$ and $f^\rho(\rho_X(x)) = g^\rho(\rho_X(x), \rho_Y(y_0))$ for all $x \in X$. In particular , $f^\rho(\rho_X(x)) = 0$ if and only if $g^\rho(\rho_X(x), \rho_Y(y_0)) = 0$, so that

$$\mathrm{coz}(f^\rho) \times \{\rho_Y(y_0)\} = \mathrm{coz}(g^\rho) \cap (\rho X \times \{\rho_Y(y_0)\}),$$

as required.

□

Proposition 5.6.4. *Suppose that A and B are C^*-algebras such that the complete regularisation map $\rho_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to \mathrm{Glimm}(A \otimes_\alpha B)$ is open. If there exists a* *point* $q_0 \in \mathrm{Glimm}(B)$ *(resp.* $p_0 \in \mathrm{Glimm}(A)$*) such that the quotient* C^**-algebra* B/G_{q_0} *(resp.* A/G_{p_0}*) contains a nonzero projection, then* ρ_A *(resp.* ρ_B*) is open.*

Proof. Let $e \in B$ such that $e + G_{q_0}$ is a nonzero projection in B/G_{q_0}. Then the map $\Theta_p : A \to (A \otimes_\alpha B)/\Delta(G_p, G_{q_0})$ defined by

$$\Theta_p(a) = a \otimes e + \Delta(G_p, G_{q_0})$$

is a $*$-homomorphism for each $p \in \mathrm{Glimm}(A)$. We first claim that $\ker\Theta_p = G_p$.

Indeed, it is clear that if $a \in G_p$ then $a \otimes e \in \Delta(G_p, G_{q_0})$, so that $G_p \subseteq \ker\Theta_p$. Now choose a state λ of B vanishing on G_{q_0} such that $\lambda(e) = 1$, and consider the associated left slice map $L_\lambda : A \otimes_\alpha B \to A$ defined on elementary tensors via $L_\lambda(a \otimes b) = \lambda(b)a$, and extended to $A \otimes_\alpha B$ by linearity and continuity. Then since $L_\lambda(A \odot G_{q_0}) = \{0\}$ and $L_\lambda(G_p \odot B) \subseteq G_p$, we have $L_\lambda(\Delta(G_p, G_{q_0})) \subseteq G_p$. In particular, if $a \in \ker\Theta_p$ then $a \otimes e \in \Delta(G_p, G_{q_0})$, so that

$$L_\lambda(a \otimes e) = \lambda(e)a = a \in G_p,$$

hence $\ker\Theta_p = G_p$. It follows that for any $a \in A$ and $p \in \mathrm{Glimm}(A)$, $\|a + G_p\| = \|\Theta_p(a)\|$.

By [7, Theorem 2.1 (i) $\Rightarrow$ (ii)] the function on $\mathrm{Glimm}(A \otimes_\alpha B)$ sending $x \mapsto \|a \otimes e + G_x\|$ is continuous. Since by Theorem 5.5.9, $\Delta : (\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr}) \to \mathrm{Glimm}(A \otimes_\alpha B)$ is a homeomorphism, the map sending $(p, q) \mapsto \|a \otimes e + \Delta(G_p, G_q)\|$ is τ_{cr}-continuous on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ for every $a \in A$.

Finally, by Lemma 5.6.3 the map $p \mapsto (p, q_0)$ is a homeomorphic embedding of $\mathrm{Glimm}(A)$ into $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})$. It follows that for each $a \in A$ the function $p \mapsto \|a + G_p\|$ agrees with the composition of continuous maps given by

$$p \mapsto (p, q_0) \mapsto \Delta(G_p, G_{q_0}) \mapsto \|a \otimes e + \Delta(G_p, G_{q_0})\|$$

$$\mathrm{Glimm}(A) \hookrightarrow (\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr}) \longrightarrow \mathrm{Glimm}(A \otimes_\alpha B) \longrightarrow \mathbb{R},$$

hence is continuous. By [7, Theorem 2.1 (ii) $\Rightarrow$ (i)], this implies that ρ_A is open. □

Definition 5.6.5. An ideal I of a C*-algebra A is said to be *primal* if given $n \geq 2$ and ideals $J_1, \ldots, J_n$ of A such that $J_1 J_2 \ldots J_n = \{0\}$, then there is an index $1 \leq i \leq n$ with $J_i \subseteq I$. A C*-algebra A is called *quasi-standard* if

(i) the Dauns-Hofmann representation of A is a continuous C*-bundle over Glimm(A), and

(ii) For each $p \in$ Glimm(A), the Glimm ideal G_p is a primal ideal of A.

For separable A, quasi-standardness of A is equivalent to the condition that A is a continuous $C_0(X)$-algebra such that there exists a dense subset D of X with J_x primitive for all $x \in D$. There are several other equivalent definitions of quasi-standard C*-algebras, see [7, Theorems 3.3 and 3.4] for example.

Corollary 5.6.6. *Suppose that A and B are C*-algebras such that $A \otimes_\alpha B$ is quasi-standard. If there exists a point $q_0 \in$ Glimm(B) (resp. $p_0 \in$ Glimm(A)) such that the quotient C*-algebra B/G_{q_0} (resp. A/G_{p_0}) contains a nonzero projection, then A (resp. B) is quasi-standard.*

Proof. Since $A \otimes_\alpha B$ is quasi-standard, the Glimm ideals of A and B are primal by [6, Lemma 4.1]. The fact that ρ_A and ρ_B are open under the respective hypotheses then follows from Proposition 5.6.4. □

Corollary 5.6.7. *Let A and B be C*-algebras such that $Z(A)$ and $Z(B)$ are nonzero.*

(i) If ρ_α is open then ρ_A and ρ_B are open,

(ii) If $A \otimes_\alpha B$ is quasi-standard then A and B are quasi-standard.

Proof. Since $Z(B) \neq \{0\}$, there is $q_0 \in$ Glimm(B) such that $Z(B) \not\subseteq G_{q_0}$ (otherwise $Z(B) \subseteq \bigcap\{G_q : q \in \text{Glimm}(B)\} = \{0\}$). It then follows from [8, Proposition 2.2(ii)] that B/G_{q_0} is in fact unital, and in particular contains a nonzero projection. Similarly, there is $p_0 \in$ Glimm(A) for which A/G_{p_0} is unital. Assertions (i) and (ii) then follow from Proposition 5.6.4 and Corollary 5.6.6 respectively. □

We remark that the condition $Z(B) \neq \{0\}$ is not necessary to ensure that B has a Glimm quotient containing a nonzero projection. This can be seen by taking $B = K(H)$ for a separable infinite dimensional Hilbert space H. Then Glimm(B) consists of the zero ideal, so that B is a Glimm quotient of itself. We have $Z(B) = \{0\}$, while B contains all of the finite rank projections on H.

5.7 Examples

Our first example shows that the topologies τ_p and τ_{cr} can indeed differ for the complete regularisation of a product of primitive ideal spaces when condition (ii) of Theorem 5.2.6 fails. We show that the primitive ideal space of the (separable) C*-algebra A of [21, III, Example 9.2] admits a point P_0 for which no cozero set neighbourhood of P_0 in $\mathrm{Prim}(A)$ has w-compact closure. Further we exhibit a cozero set neighbourhood $\mathcal{U}$ of (P_0, P_0) in $\mathrm{Prim}(A) \times \mathrm{Prim}(A)$ which does not contain a product $\mathcal{V} \times \mathcal{W}$ of cozero sets $\mathcal{V}, \mathcal{W} \subseteq \mathrm{Prim}(A)$. Thus $\rho_A \times \rho_V(\mathcal{U})$ is a τ_{cr}-open subset of $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ which is not τ_p-open. In particular we deduce that $\mathrm{Glimm}(A \otimes_\alpha A)$ is not homeomorphic to $(\mathrm{Glimm}(A) \times \mathrm{Glimm}(A), \tau_p)$.

Example 5.7.1. *Let A be the C*-algebra constructed in Example 2.3.8. We claim that none of the points $0^{(n)}$ have a cozero set neighbourhood in* $\mathrm{Prim}(A)$ *with w-compact closure. Since* $\mathrm{Glimm}(A)$ *is not locally compact, it follows from Lemma 5.2.8 that there is a point in* $\mathrm{Prim}(A)$ *for which no cozero set neighbourhood has w-compact closure. Since every point in $[-1,0) \cup \bigcup_{n=1}^{\infty} (0^{(n)}, 1^{(n)}]$ has such a neighbourhood, it must be the case that there is some $m \in \mathbb{N}$ such that $0^{(m)}$ does not.*

For $m, n \in \mathbb{N}$ the bijection $\theta_{m,n} : \mathrm{Prim}(A) \to \mathrm{Prim}(A)$ exchanging $t^{(m)}$ and $t^{(n)}$ for $0 \le t \le 1$, and fixing every other point, is clearly a homeomorphism. Thus if $\mathrm{coz}(f)$ were a neighbourhood of $0^{(n)}$ with w-compact closure, $\mathrm{coz}(f \circ \theta_{m,n})$ would be a neighbourhood of $0^{(m)}$ with w-compact closure. Hence $0^{(n)}$ does not have a cozero set neighbourhood with w-compact closure for any n.

We now show that the product topology τ_p on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(A)$ is strictly weaker than τ_{cr}. Note first that

$$\mathrm{Prim}(A) \times \mathrm{Prim}(A) = [-1,0) \times [-1,0) \cup \left([-1,0) \times \bigcup_{n=1}^{\infty} I_n \right) \cup \left(\bigcup_{m=1}^{\infty} I_m \times [-1,0) \right) \cup \left(\bigcup_{m,n=1}^{\infty} I_m \times I_n \right).$$

The neighbourhood basis of the following types of points will be of interest:

- $(0^{(m)}, 0^{(n)})$*: sets of the form*

$$\left((-\delta, 0) \cup [0^{(m)}, \delta^{(m)}) \right) \times \left((-\varepsilon, 0) \cup [0^{(n)}, \varepsilon^{(n)}) \right)$$

where $\delta, \varepsilon > 0$.

- $(x, y) \in \{0^{(m)}\} \times (0^{(n)}, 1^{(n)}]$: *neighbourhoods of the form*

$$\Big((-\delta, 0) \cup [0^{(n)}, \delta)\Big) \times \Big(((y-\varepsilon)^{(n)}, (y+\varepsilon)^{(n)})\Big)$$

where $0 < \delta < 1, 0 < \varepsilon < |y|$.

- $(x, y) \in (0^{(m)}, 1^{(m)}] \times \{0^{(n)}\}$: *neighbourhoods of the form*

$$\Big((x-\delta)^{(m)}, (x+\delta)^{(m)}\Big) \times \Big((-\varepsilon, 0) \cup [0^{(n)}, \varepsilon^{(n)})\Big)$$

For each $m, n \in \mathbb{N}$ *define* $f_{m,n} : I_m \times I_n \to [0, 1]$ *via*

$$f_{m,n}(x, y) = \max(1 - mnx, 1 - mny, 0).$$

Then $f_{m,n}(x, y) > 0$ *when* $x < \frac{1}{mn}$ *or* $y < \frac{1}{mn}$, *i.e.* $\operatorname{coz}(f_{m,n}) = [0^{(m)}, (\frac{1}{mn})^{(m)}) \times [0^{(n)}, (\frac{1}{mn})^{(n)})$. *Now define* $f : \operatorname{Prim}(A) \times \operatorname{Prim}(A) \to [0, 1]$ *via*

$$f(x, y) = \begin{cases} f_{m,n}(x, y) & \text{if } (x, y) \in I_m \times I_n \\ 1 & \text{otherwise} \end{cases}$$

Then f *is continuous since* $f_{m,n}(0^{(m)}, 0^{(n)}) = f_{m,n}(0^{(m)}, y^{(n)}) = f_{m,n}(x^{(m)}, 0^{(n)}) = 1$, *and by the neighbourhood bases of these points constructed above. Moreover, the cozero set of* f *is*

$$[-1, 0) \times [-1, 0) \cup \left([-1, 0) \times \bigcup_{n=1}^{\infty} I_n\right) \cup \left(\bigcup_{m=1}^{\infty} I_m \times [-1, 0)\right)$$
$$\cup \left(\bigcup_{m,n=1}^{\infty} [0^{(m)}, (\tfrac{1}{mn})^{(m)}) \times [0^{(n)}, (\tfrac{1}{mn})^{(n)})\right)$$

We show that $\operatorname{coz}(f)$ *is not a union of cozero set rectangles. To this end, let* $i, j \in \mathbb{N}$ *and let* U *and* V *be cozero set neighbourhoods of* $0^{(i)}$ *and* $0^{(j)}$ *respectively. Then* $U = \rho^{-1}(W)$, *where* W *is a cozero set neighbourhood of* 0 *in* $\operatorname{Glimm}(A)$, *hence* U *is also neighbourhood of* $0^{(k)}$ *for all* $k \in \mathbb{N}$. *In particular, for every* $k \in \mathbb{N} \cup \{0\}$ *there is* $\varepsilon_k > 0$ *such that*

$$U' := (-\varepsilon_0, 0) \cup \bigcup_{n=1}^{\infty} [0^{(n)}, \varepsilon_n^{(n)}) \subseteq U.$$

Thus we get open sets $U' \subseteq U$ and $V' \subseteq V$ defined via

$$U' = (-\delta_0, 0) \cup \bigcup_{m=1}^{\infty} [0^{(m)}, \delta_m^{(m)}) \text{ and } V' = (-\varepsilon_0, 0) \cup \bigcup_{n=1}^{\infty} [0^{(n)}, \varepsilon_n^{(n)}),$$

where $\delta_j, \epsilon_j > 0$ for all $j \geq 0$. If $U \times V \subseteq \mathrm{coz}(f)$ then it would follow that $U' \times V' \subseteq \mathrm{coz}(f)$. In particular this would imply that

$$[0^{(m)}, \delta_m^{(m)}) \times [0^{(n)}, \varepsilon_n^{(n)}) \subseteq [0^{(m)}, (\tfrac{1}{mn})^{(m)}) \times [0^{(n)}, (\tfrac{1}{mn})^{(n)})$$

for all $m, n \geq 1$. In other words, $\delta_m \leq \frac{1}{mn}$ for all $n \geq 1$ and $\varepsilon_n \leq \frac{1}{mn}$ for all $m \geq 1$. But then $\delta_m = \varepsilon_n = 0$ for all $m, n \geq 1$.

Together with Theorem 5.2.2, this shows that $\tau_{cr} \neq \tau_p$ on Glimm(A) × Glimm(A).

In what follows we denote by ω_0 the first infinite ordinal and by ω_1 the first uncountable ordinal. For $i = 0, 1$ we let $[0, \omega_i)$ be the space of all ordinals $\gamma < \omega_i$ and $[0, \omega_i] = [0, \omega_i + 1)$. These spaces will be considered with the order topology, with basic open sets given by

$$(\alpha, \beta) := \{\gamma \in [0, \omega_i] : \alpha < \gamma < \beta\},$$

where $\alpha, \beta \in [0, \omega_i]$ for $i = 0, 1$.

It follows from [29, 5.11(c) and 5.12(c)] that the space $[0, \omega_1)$ is a non-compact pseudocompact space. On the other hand $[0, \omega_0)$ is homeomorphic to $\mathbb{N}$, which being infinite and discrete cannot be pseudocompact.

Our second example is a nontrivial application of Theorem 5.4.6. First we describe the C*-algebra A of [41, Appendix], which has the property that Glimm(A) is pseudocompact but non-compact. We then construct a (non-unital) σ-unital C*-algebra B with Glimm(B) compact, such that $M(A) \otimes_\alpha M(B) \neq M(A \otimes_\alpha B)$, while $ZM(A) \otimes ZM(B) = ZM(A \otimes B)$.

Example 5.7.2. *Let $Y = [0, \omega_1] \times [0, \omega_0] \backslash \{\{\omega_1\} \times \{\omega_0\}\}$ and $S = \{\omega_1\} \times [0, \omega_0)$ and $T = [0, \omega_1) \times \{\omega_0\}$. Define a new space $X = Y \cup \{y\}$, where $y \notin Y$ with topology such that Y is embedded homeomorphically into X, and $\{y\}$ is an open set whose closure is $S \cup \{y\}$.*

Let $C = C_0(Y)$, $D = C_0(S)$ and let $\pi_1 : C \to D$ be the restriction map. Let H be an infinite dimensional separable Hilbert space and $\{p_n\}$ a sequence of infinite dimensional mutually orthogonal projections on H. Define an injective $*$-homomorphism $\lambda : D \to$

$B(H)$ via $\lambda(f) = \sum_{n=1}^{\infty} f(n)p_n$ (identifying S with $\mathbb{N}$ in the obvious way), and note that $\lambda(D) \cap K(H) = \{0\}$.

Set $E = \lambda(D) + K(H)$ and let $\pi_2 : E \to D$ be the quotient map. Let $A = \{c \oplus e \in C \oplus E : \pi_1(c) = \pi_2(e)\}$. Then $\mathrm{Prim}(A)$ is homeomorphic to X.

The complete regularisation map ρ_A maps $Y \backslash S$ to itself, and $S \cup \{y\}$ to a single point which we will denote by z. Thus $\mathrm{Glimm}(A) = (Y \backslash S) \cup \{z\}$, where a neighbourhood basis of z is given by the collection of sets of the form $(\bigcup_{n=1}^{\omega_0} (\alpha_n, \omega_1) \times \{n\}) \cup \{z\}$, for some ordinals $0 < \alpha_n < \omega_1$ for all $1 \leq n \leq \omega_0$.

Note that $Y \backslash S = [0, \omega_1) \times [0, \omega_0]$, being the product of a pseudocompact space and a compact space, is necessarily pseudocompact. It follows that $\mathrm{Glimm}(A)$ is pseudocompact.

Consider the C-algebra B of sequences $(T_n) \in B(H)$ such that $T_n \to T_\infty \in K(H)$, with pointwise operations and supremum norm. Then $\mathrm{Prim}(B)$ consists of the ideals*

$$P_{n_0} = \{(T_n) : T_{n_0} = 0\}, K_{n_0} = \{(T_n) : T_{n_0} \in K(H)\}$$

for $n_0 \in \mathbb{N}$, and $P_\infty = \{(T_n) : T_\infty = 0\}$. The $\approx$-equivalence classes in $\mathrm{Prim}(B)$ then consist of pairs $\{P_{n_0}, K_{n_0}\}$ for $n_0 \in \mathbb{N}$, and $\{P_\infty\}$. As in the proof of [7, Proposition 3.6], the complete regularisation map $\rho_B : \mathrm{Prim}(B) \to \mathrm{Glimm}(B)$ is open and $\mathrm{Glimm}(B)$ is homeomorphic to $\mathbb{N} \cup \{\infty\}$, with $G_q = P_q$ for all $1 \leq q \leq \infty$.

We claim that B is a σ-unital C-algebra. Fix an orthonormal basis $\{e_m : m \in \mathbb{N}\}$ of H. For each $n \in \mathbb{N}$ let 1_n denote the projection onto the n-dimensional subspace of H spanned by $e_1, \ldots, e_n$. Then $\{1_n : n \in \mathbb{N}\}$ is an increasing approximate identity for $K(H)$.*

Now define sequences $E^{(n)} = (E_m^{(n)})$ in B via

$$E_m^{(n)} = \begin{cases} 1 & \text{if} \quad m \leq n \\ 1_n & \text{if} \quad m > n \end{cases}$$

for each $n \in \mathbb{N}$. We will show that the sequence $\{E^{(n)}\}_n$ is an approximate identity for B. Take $T = (T_m) \in B$ and let $\varepsilon > 0$ be given. Note that for each n, $\|T - E^{(n)}T\| = \sup_{m \geq 1} \|T_m - E_m^{(n)} T_m\| = \sup_{m > n} \|T_m - 1_n T_m\|$, by definition of $E^{(n)}$. Then

- *there exists $m_0 \geq 1$ such that $\|T_m - T_\infty\| < \frac{\varepsilon}{4}$ whenever $m \geq m_0$, and*
- *there exists $n_1 \geq 1$ such that $\|T_\infty - 1_n T_\infty\| < \frac{\varepsilon}{4}$ whenever $n \geq n_1$ (since $T_\infty \in K(H)$).*

Set $n_0 = \max(m_0, n_1)$*. Then if* $n \geq n_0$ *and* $m > n$ *we have*

$$\begin{aligned}\|T_m - 1_n T_m\| &\leq \|T_m - T_\infty\| + \|T_\infty - 1_n T_m\| \\ &\leq \|T_m - T_\infty\| + \|T_\infty - 1_n T_\infty\| + \|1_n T_\infty - 1_n T_m\| \\ &\leq \|T_m - T_\infty\| + \|T_\infty - 1_n T_\infty\| + \|T_\infty - T_m\| < \frac{3\varepsilon}{4}\end{aligned}$$

In particular, for all $n \geq n_0$ *we have* $\|T - E^{(n)}T\| = \sup_{m>n} \|T_m - 1_n T_m\| \leq \frac{3\varepsilon}{4} < \varepsilon$*, so that* $\{E^{(n)}\}_n$ *is a countable approximate identity for* B*.*

Now take the tensor product $A \otimes_\alpha B$*. Since* B *is* σ*-unital, the inclusion* $M(A) \otimes_\alpha M(B) \subseteq M(A \otimes_\alpha B)$ *is strict by [1, Theorem 3.8]. Since* ρ_B *is open, Proposition 5.2.9 (ii) shows that* $\tau_{cr} = \tau_p$ *on* $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$*. Hence by Theorem 5.3.3* $\mathrm{Glimm}(A \otimes_\alpha B)$ *is homeomorphic to* $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ *with the product topology. Moreover, as a product of a pseudocompact space and a compact space,* $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ *is pseudocompact [55, Proposition 8.21]. It then follows from Theorem 5.4.6 that* $ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B)$*.*

Chapter 6

Exact C*-algebras and $C_0(X)$-structure

This chapter is concerned with the study of tensor products of a $C_0(X)$-algebra (A, X, μ_A) and a $C_0(Y)$-algebra (B, Y, μ_B). We describe how their minimal tensor product $A \otimes_\alpha B$ gives rise to a $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$, and we investigate the structure of the associated upper-semicontinuous C*-bundles.

Our approach differs from the usual notion of 'fibrewise tensor products' of C*-bundles introduced by Kirchberg and Wassermann [40]. Indeed, we show in Section 6.4 that continuity of the bundles we describe here is a strictly weaker property than continuity of the corresponding fibrewise tensor product bundle.

Given a fixed continuous $C_0(X)$-algebra (A, X, μ), we seek conditions on A for which $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is continuous for all continuous $C_0(Y)$-algebras (B, Y, μ_B). We show that this occurs precisely when A is an exact C*-algebra (as was the case for the tensor product bundles considered in [40]).

Finally, we apply these results to the study of quasi-standard C*-algebras and those C*-algebras A with Prim(A) Hausdorff. We show that neither of these classes are stable under taking minimal tensor products; in each case, the subclass of exact C*-algebras is the largest class that is stable under this operation. As a consequence, we obtain some new characterisations of exact C*-algebras.

Sections 6.1 to 6.6 have appeared in the article [46].

6.1 Introduction

Tensor products of continuous bundles of C*-algebras are known to exhibit pathological behaviour. The earliest examples of this were given by Kirchberg and Wassermann in [40], who showed that continuity of a C*-bundle was in general not preserved by tensoring fibrewise with a fixed C*-algebra B. Moreover, it was shown that such an operation preserves continuity for all C*-bundles precisely when B is exact. Archbold later obtained a localisation of this result in [6], where continuity at a point was characterised in terms of a weaker exactness-type condition. Similar questions have been studied extensively by Blanchard and coauthors in [14], [13], [16], [15] and [18].

Here we are concerned with the stability of certain well-behaved classes of C*-bundles under the operation of forming tensor products. In particular, we study this question for continuous $C_0(X)$-algebras, the *quasi-standard* C*-algebras (Definition 5.6.5) introduced by Archbold and Somerset [7], and C*-algebras with Hausdorff primitive ideal space (maximal full algebras of operator fields, using Fell's terminology [27]).

In Chapter 5, we constructed the Glimm ideal space of the minimal tensor product of two C*-algebras in terms of those of the factors. As a consequence, it is possible to construct the Dauns-Hofmann bundle of $A \otimes_\alpha B$ in terms of that of A and B, although the fibre algebras of this bundle remain difficult to describe without additional assumptions on A and B. In particular, it is not immediate from our results in Section 5.6 whether or not this bundle agrees with the fibrewise tensor product of Kirchberg and Wassermann. Moreover, it remains difficult to show in general whether or not certain classes of C*-algebras with well-behaved Dauns-Hofmann representations, for example the quasi-standard C*-algebras, are stable under minimal tensor products.

In Section 6.3 we study a natural construction which equips the minimal tensor product $A\otimes_\alpha B$ of a $C_0(X)$-algebra A and a $C_0(Y)$-algebra B the structure of a $C_0(X\times Y)$-algebra. It has been observed previously that this bundle representation of $A\otimes_\alpha B$ may differ from the fibrewise tensor product [40], [15], and that exactness of A or B (Definition 6.3.1) plays a decisive role in these considerations.

While the results of Kirchberg, Wassermann and Archbold give necessary and sufficient conditions for the continuity of the fibrewise minimal tensor product, less is known regarding the tensor product bundle that we study in this work. Indeed, we show in

Section 6.4 that there are quasi-standard C*-algebras A and B such that $A \otimes_\alpha B$ is quasi-standard, while the fibrewise tensor product of A and B gives rise to a discontinuous C*-bundle.

Related work of Blanchard [15] (concerning the amalgamated $C_0(X)$-tensor product of two $C_0(X)$-algebras A and B) indicated that continuity may fail for the tensor product bundle that we define in Section 6.3 also. However, the argument used in [15, §3] relies on specific properties of the C*-algebras involved. We show in Section 6.6 that for an inexact continuous $C_0(X)$-algebra A, one can always construct a continuous $C_0(Y)$-algebra B such that $A \otimes_\alpha B$ is discontinuous as a $C_0(X \times Y)$-algebra. As a consequence it is shown in Theorem 6.5.6 that stability of continuity is in fact equivalent to exactness of A. Thus our tensor product construction identifies exactness in precisely the same way as the fibrewise tensor product of Kirchberg and Wassermann [40, Theorem 4.5].

In Section 6.6 we investigate the question of stability of the property of quasi-standardness under the operation of taking tensor products (in particular with respect to the minimal C*-norm). One consequence of this is the observation that, in general, the C*-bundle associated with the Dauns-Hofmann representation of such a tensor product is not given by the fibrewise tensor product of the corresponding bundles of the factors.

Until now, it appears that there were no known examples of a pair of quasi-standard C*-algebras whose minimal tensor product fails to be quasi-standard. It was shown by Kaniuth in [37] that if $A \otimes_\alpha B$ satisfies Tomiyama's property (F), then $A \otimes_\alpha B$ is quasi-standard if and only if A and B are quasi-standard. In particular this is the case whenever either A or B is exact. The assumption of property (F) was weakened in [45] to an assumption involving exact sequences related to Glimm ideals.

We show in Theorem 6.6.6 that if A is a quasi-standard C*-algebra which is not exact, then one can always construct a quasi-standard C*-algebra B for which $A \otimes_\alpha B$ is not quasi-standard. In particular it follows that a quasi-standard C*-algebra A is exact if and only if $A \otimes_\alpha B$ is quasi-standard for all quasi-standard B. This is consistent with the characterisation of exactness obtained by Kirchberg and Wassermann in [40], though perhaps surprising in light of the results of Section 6.4. Similarly, in the unital case we show in Theorem 6.6.10 that stability of the property of quasi-standardness under taking maximal tensor products (Definition 6.6.8) is equivalent to nuclearity.

6.2 Preliminaries on $C_0(X)$-algebras and C*-bundles

For a C*-algebra A we will denote by Min-Primal(A) its set of minimal (w.r.t. inclusion) primal ideals. The canonical topology τ on Min-Primal(A) is the weakest topology such that the norm functions $I \mapsto \|a + I\|$ on Min-Primal(A) are continuous for all $a \in A$. If A is a C*-algebra for which every $G \in \mathrm{Glimm}(A)$ is a primal ideal of A, then necessarily we have $\mathrm{Glimm}(A) = \text{Min-Primal}(A)$ as sets. From [7, Theorem 3.3], a C*-algebra A is quasi-standard if and only if $(\mathrm{Glimm}(A), \tau_{cr}) = (\text{Min-Primal}(A), \tau)$ (i.e., as sets of ideals and topologically).

The Glimm space of a C*-algebra appears as an intermediate step in any representation of a C*-algebra as a $C_0(X)$-algebra, due to the functorial property of the complete regularisation of a topological space (see the remarks following Definition 2.1.5). Indeed, if X is a completely regular space and $\phi : \mathrm{Prim}(A) \to X$ a continuous map, then ϕ induces a continuous map $\psi : \mathrm{Glimm}(A) \to X$ with $\phi = \psi \circ \rho_A$, i.e.,

$$\begin{array}{ccc} \mathrm{Prim}(A) & & \\ \downarrow \rho_A & \searrow \phi & \\ \mathrm{Glimm}(A) & \xrightarrow{\psi} & X \end{array}$$

commutes. Conversely, starting with a continuous map $\psi : \mathrm{Glimm}(A) \to X$, we may set $\phi = \psi \circ \rho_A$, so that $\phi : \mathrm{Prim}(A) \to X$ is continuous.

If in addition X is locally compact, then A is a $C_0(X)$ algebra if and only if there is a continuous map $\psi_A : \mathrm{Glimm}(A) \to X$. This fact is useful when working with tensor products of C*-algebras, since by Theorem 5.5.9 we may always construct $\mathrm{Glimm}(A \otimes_\alpha B)$ in terms of $\mathrm{Glimm}(A)$ and $\mathrm{Glimm}(B)$. The same is not true in general for the spaces $\mathrm{Prim}(-)$ and $\mathrm{Fac}(-)$.

In the remainder of this section we give some technical results on the structure of $C_0(X)$-algebras and Glimm spaces of C*-algebras which we will make reference to in subsequent sections.

Lemma 6.2.1. *Let A be a $C_0(X)$-algebra with base map $\phi_A : \mathrm{Prim}(A) \to X$, and denote by $\psi_A : \mathrm{Glimm}(A) \to X$ the induced continuous map with the property that $\psi_A \circ \rho_A = \phi_A$. Then for each $x \in X$, the ideal J_x of (2.3.2) is given by*

$$J_x = \bigcap \{G \in \mathrm{Glimm}(A) : \psi_A(G) = x\}.$$

Proof. Let $F = \psi_A^{-1}(\{x\}) \subseteq \mathrm{Glimm}(A)$, so that for $P \in \mathrm{Prim}(A)$, we have $\phi_A(P) = x$ if and only if $\rho_A(P) \in F$. Thus

$$\begin{aligned} J_x &= \bigcap\{P \in \mathrm{Prim}(A) : \rho_A(P) \in F\} \\ &= \bigcap_{p \in F} \left\{\bigcap\{P \in \mathrm{Prim}(A) : \rho_A(P) = p\}\right\} \\ &= \bigcap\{G \in \mathrm{Glimm}(A) : \psi_A(G) = x\}. \end{aligned}$$

□

As was the case in [37] and Chapter 5, we will make use of the space $\mathrm{Fac}(A)$ of factorial ideals of A, in order to define a $C_0(X)$-algebra structure on A. Lemma 6.2.2 and Proposition 6.2.3 below generalise parts of [37, lemmas 2.1 and 2.2] from Glimm ideals to the ideals J_x defined in the $C_0(X)$-algebra case.

Lemma 6.2.2. *Let A be a C*-algebra and X a locally compact Hausdorff space. Then any continuous map $\phi_A : \mathrm{Prim}(A) \to X$ has a unique extension to a continuous map $\phi_A^f : \mathrm{Fac}(A) \to X$. For $I \in \mathrm{Fac}(A)$ and $P \in \mathrm{hull}(I)$ we have $\phi_A^f(I) = \phi_A(P)$.*

Proof. By [42, Lemma 3.1] there is a unique continuous map $\tilde{\phi}_A : \mathrm{prime}(A) \to X$ extending ϕ_A. Set $\phi_A^f = \tilde{\phi}_A\big|_{\mathrm{Fac}(A)}$, then ϕ_A^f is continuous since $\tilde{\phi}_A$ is. Uniqueness follows from the fact that $\mathrm{Prim}(A)$ is dense in $\mathrm{Fac}(A)$.

Take $I \in \mathrm{Fac}(A)$ and $P \in \mathrm{hull}(I)$, so that $P \in \mathrm{cl}\{I\}$. Then since ϕ_A^f is continuous and extends ϕ_A, we necessarily have that $\phi_A^f(I) = \phi_A^f(P) = \phi_A(P)$. □

Proposition 6.2.3. *Let A be a C*-algebra and X a locally compact Hausdorff space.*

(i) A is a $C_0(X)$-algebra if and only if there exists a continuous map $\phi_A^f : \mathrm{Fac}(A) \to X$,

(ii) For $x \in \mathrm{Im}\phi_A$, $J_x = \bigcap\{I \in \mathrm{Fac}(A) : \phi_A^f(I) = x\}$,

(iii) For $I \in \mathrm{Fac}(A)$ and $x \in X$, we have $I \supseteq J_x$ if and only if $\phi_A^f(I) = x$.

Proof. (i) If A is a $C_0(X)$-algebra with base map $\phi_A : \mathrm{Prim}(A) \to X$ then ϕ_A has a unique continuous extension to a map $\phi_A^f : \mathrm{Fac}(A) \to X$ by Lemma 6.2.2. Conversely, if such a map exists then setting $\phi_A = \phi_A^f\big|_{\mathrm{Prim}(A)}$ defines a base map.

(ii) For $x \in \mathrm{Im}\phi_A$, we have

$$J_x = \bigcap\{P \in \mathrm{Prim}(A) : \phi_A(P) = x\} \supseteq \bigcap\{I \in \mathrm{Fac}(A) : \phi_A^f(I) = x\}$$

since ϕ_A^f extends ϕ_A. Take $I \in \mathrm{Fac}(A)$ such that $\phi_A^f(I) = x$. Then for all $P \in \mathrm{hull}(I)$, $\phi_A(P) = x$, hence $P \supseteq J_x$ for all such P. Hence $J_x \subseteq k\,(\mathrm{hull}(I)) = I$ for all $I \in \mathrm{Fac}(A)$ with $\phi_A^f(I) = x$, so that

$$J_x \subseteq \bigcap\{I \in \mathrm{Fac}(A) : \phi_A^f(I) = x\}.$$

as required.

(iii) It follows from (ii) that if $\phi_A^f(I) = x$ then $I \supseteq J_x$. Now suppose that $I \supseteq J_x$ and take $P \in \mathrm{hull}(I)$. Then $P \supseteq J_x$ and so $\phi_A(P) = x$ by [8, p. 74]. It then follows from Lemma 6.2.2 that

$$\phi_A^f(I) = \phi_A(P) = x.$$

□

6.3 Tensor Products of C*-bundles

Definition 6.3.1. A sequence of $*$-homomorphisms between C*-algebras of the form

$$0 \longrightarrow J \xrightarrow{\iota} B \xrightarrow{q} (B/J) \longrightarrow 0, \tag{6.3.1}$$

where J is an ideal of B, ι the inclusion of J into B, and q the quotient $*$-homomorphism, is called a *short exact sequence of C*-algebras*. We say that a C*-algebra A is *exact* if the sequence

$$0 \longrightarrow A \otimes_\alpha J \xrightarrow{\mathrm{id}\otimes\iota} A \otimes_\alpha B \xrightarrow{\mathrm{id}\otimes q} A \otimes_\alpha (B/J) \longrightarrow 0 \tag{6.3.2}$$

is exact for every short exact sequence of the form (6.3.1).

A C*-algebra A is exact if and only if $A \otimes_\alpha B$ has property (F) for all C*-algebras B. Clearly if A is such that $A \otimes_\alpha B$ satisfies (F) for all B, then for any short exact sequence of the form (6.3.1), exactness of (6.3.2) follows from the fact that $\Delta(\{0\}, J) = \Phi(\{0\}, J)$. The converse is shown in [17, Proposition 2.17]. We will make use of this equivalence repeatedly in the sequel.

The following theorem lists some of the known properties of tensor products of C*-bundles, and their relation to the maps Φ and Δ.

Theorem 6.3.2. *Let* $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ *and* $\mathscr{B} = (Y, B, \sigma_y : B \to B_y)$ *be C*-bundles over locally compact spaces* X *and* Y *respectively. Then*

(i) the fibrewise tensor product $\mathscr{A} \otimes_\alpha \mathscr{B}$ *of* $\mathscr{A}$ *and* $\mathscr{B}$ *defined via*

$$\mathscr{A} \otimes_\alpha \mathscr{B} = (X \times Y, A \otimes_\alpha B, \pi_x \otimes \sigma_y : A \otimes_\alpha B \to A_x \otimes_\alpha B_y)$$

is a C-bundle over* $X \times Y$ *[6, p.136,137].*

If in addition $\mathscr{A}$ *and* $\mathscr{B}$ *are continuous, then*

(ii) $\mathscr{A} \otimes_\alpha \mathscr{B}$ *is lower-semicontinuous over* $X \times Y$ *[40, Proposition 4.9], and*

(iii) for $(x_0, y_0) \in X \times Y$*, the function*

$$(x, y) \mapsto \|(\pi_x \otimes \sigma_y)(c)\|$$

is continuous for all $c \in A \otimes_\alpha B$ *at* (x_0, y_0) *if and only if*

$$\ker(\pi_{x_0} \otimes \sigma_{y_0}) = \ker(\pi_{x_0}) \otimes_\alpha B + A \otimes_\alpha \ker(\sigma_{y_0}),$$

that is, if and only if $\Phi(\ker(\pi_{x_0}), \ker(\sigma_{y_0})) = \Delta(\ker(\pi_{x_0}), \ker(\sigma_{y_0}))$ *[6, Theorem 3.3].*

We now introduce an alternative approach to defining a C*-bundle structure on the tensor product of two (upper-semicontinuous) C*-bundles, based on the ideal structure of $A \otimes_\alpha B$ rather than the fibrewise tensor product. This construction was considered previously in [15], in the case where the base spaces are compact, however, we will need additional information on the interplay between the base and structure maps involved.

Suppose that (A, X, ϕ_A) is a $C_0(X)$-algebra and (B, Y, ϕ_B) a $C_0(Y)$-algebra, where $\phi_A : \mathrm{Prim}(A) \to X$ and $\phi_B : \mathrm{Prim}(B) \to Y$ are their base maps. Then we get a continuous map $\phi_A \times \phi_B : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to X \times Y$. By a theorem of Lazar [42, Theorem 3.2], we get a unique continuous map $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to X \times Y$ such that

$$\begin{array}{ccc} \mathrm{Prim}(A) \times \mathrm{Prim}(B) & \overset{\Phi}{\hookrightarrow} & \mathrm{Prim}(A \otimes_\alpha B) \\ & \underset{\phi_A \times \phi_B}{\searrow} & \downarrow \phi_\alpha \\ & & X \times Y \end{array}$$

commutes, that is, $\phi_\alpha \circ \Phi = \phi_A \times \phi_B$. Thus, taking ϕ_α as the base map, $A \otimes_\alpha B$ becomes a $C_0(X \times Y)$-algebra, $(A \otimes_\alpha B, X \times Y, \phi_\alpha)$.

The structure maps $\mu_A : C_0(X) \to ZM(A), \mu_B : C_0(Y) \to ZM(B)$ and $\mu_\alpha : C_0(X \times Y) \to ZM(A \otimes_\alpha B)$ are then uniquely determined by ϕ_A, ϕ_B and ϕ_α. We will show in Proposition 6.3.3 that in fact μ_α may be identified with the map $\mu_A \otimes \mu_B : C_0(X) \otimes C_0(Y) \to ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$.

For $x \in X, y \in Y$ we shall fix the following notation for the ideals defined in equation (2.3.2):

$$\begin{aligned} I_x &= \mu_A(\{f \in C_0(X) : f(x) = 0\})\,A \\ &= \bigcap\{P \in \mathrm{Prim}(A) : \phi_A(P) = x\} \\ J_y &= \mu_B(\{g \in C_0(Y) : g(y) = 0\})\,B \\ &= \bigcap\{Q \in \mathrm{Prim}(A) : \phi_B(Q) = y\} \\ K_{x,y} &= \mu_\alpha(\{h \in C_0(X \times Y) : h(x,y) = 0\})\,(A \otimes_\alpha B) \\ &= \bigcap\{M \in \mathrm{Prim}(A \otimes_\alpha B) : \phi_\alpha(M) = (x,y)\}. \end{aligned}$$

By [1], there is a canonical injective $*$-homomorphism $\iota : M(A) \otimes_\alpha M(B) \to M(A \otimes_\alpha B)$, and by Lemma 5.4.1 this map satisfies

$$\iota(x \otimes y)(a \otimes b) = (xa) \otimes (yb) \text{ and } (a \otimes b)(\iota(x \otimes y)) = (ax) \otimes (by),$$

for all elementary tensors $x \otimes y \in M(A) \otimes_\alpha M(B), a \otimes b \in A \otimes_\alpha B$, so that the image $\iota(ZM(A) \otimes ZM(B))$ is contained in $ZM(A \otimes_\alpha B)$. We will suppress mention of ι and simply consider $ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$.

Proposition 6.3.3. *Suppose (A, X, ϕ_A) is a $C_0(X)$-algebra and (B, Y, ϕ_B) a $C_0(Y)$ algebra. Then with the above notation:*

(i) $A \otimes_\alpha B$ is a $C_0(X \times Y)$-algebra with base map $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to X \times Y$ satisfying $\phi_\alpha \circ \Phi = \phi_A \times \phi_B$,

(ii) The structure map $\mu_\alpha : C_0(X \times Y) \to ZM(A \otimes_\alpha B)$ corresponding to ϕ_α may be identified with $(\mu_A \otimes \mu_B) \circ \eta$, where $\eta : C_0(X \times Y) \to C_0(X) \otimes C_0(Y)$ is the canonical $*$-isomorphism and we identify $ZM(A) \otimes ZM(B)$ with its canonical image in $ZM(A \otimes_\alpha B)$.

(iii) Denoting by ϕ_A^f, ϕ_B^f and ϕ_α^f the extensions of ϕ_A, ϕ_B and ϕ_α to $\mathrm{Fac}(A), \mathrm{Fac}(B)$ and $\mathrm{Fac}(A \otimes_\alpha B)$ respectively, we have $\phi_\alpha^f \circ \Phi = \phi_A^f \times \phi_B^f$ on $\mathrm{Fac}(A) \times \mathrm{Fac}(B)$,

(iv) For any $M \in \mathrm{Fac}(A \otimes_\alpha B)$ and $(x,y) \in X \times Y$, $\phi_\alpha^f(M) = (x,y)$ if and only if $(\phi_A^f \times \phi_B^f)(\Psi(M)) = (x,y)$,

(v) For all $(x,y) \in X \times Y$ we have $K_{x,y} = \Delta(I_x, J_y)$.

Proof. (i) is shown in the remarks preceding the proposition.

(ii) For $f \in C_0(X \times Y)$, we have $\mu_\alpha(f) = \theta_\alpha(f \circ \phi_\alpha)$, where $\theta_\alpha : C^b(\mathrm{Prim}(A \otimes_\alpha B)) \to ZM(A \otimes_\alpha B)$ is the Dauns-Hofmann isomorphism of equation (2.3.1). For $f \otimes g \in C_0(X) \otimes C_0(Y)$ and $(x,y) \in X \times Y$, the $*$-isomorphism η satisfies $\eta^{-1}(f \otimes g)(x,y) = f(x)g(y)$. Thus by linearity and continuity, it suffices to show that for all $f \otimes g \in C_0(X) \otimes C_0(Y)$ and $a \otimes b \in A \otimes_\alpha B$ we have

$$\mu_\alpha(\eta^{-1}(f \otimes g))(a \otimes b) = (\mu_A \otimes \mu_B)(f \otimes g)(a \otimes b) = \mu_A(f)a \otimes \mu_B(g)b.$$

Take $(P,Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$, then the Dauns-Hofmann $*$-isomorphism θ_α of (2.3.1) gives

$$\begin{aligned}
(\mu_\alpha \circ \eta^{-1})(f \otimes g)(a \otimes b) + \Phi(P,Q) &= \theta_\alpha\left(\eta^{-1}(f \otimes g) \circ \phi_\alpha\right)(a \otimes b) + \Phi(P,Q) \\
&= \eta^{-1}(f \otimes g) \circ \phi_\alpha(\Phi(P,Q))\,(a \otimes b + \Phi(P,Q)) \\
&= (f \circ \phi_A)(P)(g \circ \phi_B)(Q)(a \otimes b + \Phi(P,Q)),
\end{aligned}$$

and since $A \otimes_\alpha B/\Phi(P,Q) \equiv (A/P) \otimes_\alpha (B/Q)$, the last line becomes

$$((f \circ \phi_A)(P)(a+P)) \otimes ((g \circ \phi_B)(Q)(b+Q))\,.$$

On the other hand, applying the isomorphisms θ_A and θ_B of (2.3.1) associated with A and B respectively we get

$$\begin{aligned}
(\mu_A \otimes \mu_B)(f \otimes g)(a \otimes b) + \Phi(P,Q) &= (\mu_A(f)a + P) \otimes (\mu_B(g)b + Q) \\
&= (\theta_A(f \circ \phi_A)a + P) \otimes (\theta_B(g \circ \phi_B)b + Q) \\
&= ((f \circ \phi_A)(P)(a+P)) \otimes ((g \circ \phi_B)(Q)(b+Q))\,.
\end{aligned}$$

Thus for all $(P,Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ we have

$$\left((\mu_\alpha \circ \eta^{-1})(f \otimes g) - (\mu_A \otimes \mu_B)(f \otimes g)\right)(a \otimes b) \in \Phi(P,Q),$$

and since $\bigcap\{\Phi(P,Q) : (P,Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)\} = \{0\}$, it follows that

$$(\mu_\alpha \circ \eta^{-1})(f \otimes g) = (\mu_A \otimes \mu_B)(f \otimes g).$$

(iii) Let $(I,J) \in \mathrm{Fac}(A) \times \mathrm{Fac}(B)$ and $(P,Q) \in \mathrm{hull}(I) \times \mathrm{hull}(J)$. Then by (i) we have $(\phi_A \times \phi_B)(P,Q) = (\phi_\alpha \circ \Phi)(P,Q)$. By [42, Corollary 2.3] we have $\Phi(P,Q) \in \mathrm{hull}(\Phi(I,J))$, so that Lemma 6.2.2 gives

$$(\phi_A^f \times \phi_B^f)(I,J) = (\phi_A \times \phi_B)(P,Q) = (\phi_\alpha \circ \Phi)(P,Q) = (\phi_\alpha^f \circ \Phi)(I,J).$$

(iv) By Propositions 5.3.2 and 5.5.2, $(\Phi \circ \Psi)(M) \in \mathrm{Fac}(A \otimes_\alpha B)$ with $M \supseteq (\Phi \circ \Psi)(M)$, so that $(\Phi \circ \Psi)(M) \in \mathrm{cl}\{M\}$ and hence

$$\phi_\alpha^f(M) = \phi_\alpha^f\left((\Phi \circ \Psi)(M)\right).$$

By (iii), the latter is precisely $(\phi_A^f \times \phi_B^f)(\Psi(M))$.

(v) By (iv) and Proposition 6.2.3(iii), we have $M \in \mathrm{hull}_f(K_{x,y})$ if and only if $\Psi(M) \in \mathrm{hull}_f(I_x) \times \mathrm{hull}_f(J_y)$. But then since $\Psi^{-1}(\mathrm{hull}_f(I_x) \times \mathrm{hull}_f(J_y)) = \mathrm{hull}_f(\Delta(I_x, J_y))$
by Lemma 5.5.7(i), it follows that $K_{x,y} = \Delta(I_x, J_y)$. □

Definition 6.3.4. For a $C_0(X)$-algebra (X, A, μ_A) and a $C_0(Y)$-algebra (B, Y, μ_B) we will denote by $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ the $C_0(X \times Y)$-algebra defined by Proposition 6.3.3, and we will consider this construction as the natural (minimal) tensor product in the category of $C_0(-)$-algebras,

$$(A, X, \mu_A) \otimes_\alpha (B, Y, \mu_B) \equiv (A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B).$$

The tensor product construction of definition 6.3.4 does not agree in general with the fibrewise tensor product bundle studied by Kirchberg and Wassermann in [40]. This fact may be deduced from [40, Lemma 2.3 and Proposition 4.3], and is demonstrated explicitly in [15, Proposition 3.1].

We now introduce some properties which characterise when these two notions of the tensor product of a pair of C*-bundles coincide. For C*-algebras A and B we define the properties $(\mathrm{F_{Gl}})$ and $(\mathrm{F_{MP}})$ on $A \otimes_\alpha B$ as follows:

$$\Phi(G,H) = \Delta(G,H) \text{ for all } (G,H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B) \qquad (\mathrm{F_{Gl}})$$

$$\Phi(I,J) = \Delta(I,J) \text{ for all } (I,J) \in \text{Min-Primal}(A) \times \text{Min-Primal}(B). \qquad (\mathrm{F_{MP}})$$

If in addition (A, X, μ_A) is a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra, we will say that the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ satisfies *property* $(\mathrm{F}_{X,Y})$ if the equation

$$\Phi(I_x, J_y) = \Delta(I_x, J_y) \text{ for all } (x,y) \in X \times Y \qquad (\mathrm{F}_{X,Y})$$

holds. For convenience, we will refer to $(\mathrm{F}_{X,Y})$ as a property of $A \otimes_\alpha B$, rather than $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$, when the context is clear.

We remark that if $A \otimes_\alpha B$ satisfies Tomiyama's property (F) then clearly $A \otimes_\alpha B$ satisfies properties $(\mathrm{F}_{X,Y})$, $(\mathrm{F}_{\mathrm{Gl}})$ and $(\mathrm{F}_{\mathrm{MP}})$, cf [37, Theorem 1.1 and Theorem 2.3]. The converse is not true in general; indeed, let A and B be C*-algebras such that $A \otimes_\alpha B$ does not satisfy property (F), and let $X = \{x\}$ and $Y = \{y\}$ be one-point spaces. Regarding A as a $C_0(X)$-algebra and B as a $C_0(Y)$-algebra in the obvious (trivial) way, we have $I_x = \{0\}$ and $J_y = \{0\}$. Then it is evident that $\Delta(I_x, J_y) = \Phi(I_x, J_y) = \{0\}$, so that $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{X,Y})$, hence this property does not imply (F).

To see that $(\mathrm{F}_{\mathrm{MP}})$ and $(\mathrm{F}_{\mathrm{Gl}})$ do not imply (F), let $A = B = B(H)$, where H is a separable infinite dimensional Hilbert space. Then $\mathrm{Glimm}(B(H)) = \text{Min-Primal}(B(H)) = \{0\}$, so that as before, $B(H) \otimes_\alpha B(H)$ satisfies $(\mathrm{F}_{\mathrm{MP}})$ and $(\mathrm{F}_{\mathrm{Gl}})$, but does not satisfy (F) by [57]. Other examples are discussed in [6, p. 140-141].

The following Theorem relates the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$, its corresponding upper-semicontinuous C*-bundle, and the fibrewise tensor product of the bundles associated with A and B.

Theorem 6.3.5. *Let (A, X, μ_A) be a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra, and let $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ and $\mathscr{B} = (Y, B, \sigma_y : B \to B_y)$ be the associated upper-semicontinuous C*-bundles over X and Y respectively. Then*

(i) the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ defines an upper-semicontinuous C-bundle*

$$\left(X \times Y, A \otimes_\alpha B, \gamma_{(x,y)} : A \otimes_\alpha B \to (A \otimes_\alpha B)_{(x,y)}\right),$$

where $(A \otimes_\alpha B)_{(x,y)} = A \otimes_\alpha B / \Delta(I_x, J_y)$ for all $(x,y) \in X \times Y$,

(ii) the bundle $\left(X \times Y, A \otimes_\alpha B, \gamma_{(x,y)} : A \otimes_\alpha B \to (A \otimes_\alpha B)_{(x,y)}\right)$ agrees with the fibrewise tensor product bundle $\mathscr{A} \otimes_\alpha \mathscr{B}$ if and only if $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{X,Y})$,

(iii) If $\mathscr{A}$ and $\mathscr{B}$ are continuous C-bundles and $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{X,Y})$, then $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is a continuous $C_0(X \times Y)$-algebra.*

Proof. (i) is immediate from Proposition 6.3.3(v) and the equivalence of $C_0(X)$-algebras (resp. $C_0(Y)-, C_0(X \times Y)-$) and upper-semicontinuous C*-bundles over X (resp. $Y, X \times Y$).

By definition of the maps Φ and Δ, for all $(x,y) \in X \times Y$ we have $\Phi(I_x, J_y) = \Delta(I_x, J_y)$ if and only if $(A \otimes_\alpha B)_{(x,y)} \equiv A_x \otimes_\alpha B_y$, from which (ii) follows.

If $\Phi(I_x, J_y) = \Delta(I_x, J_y)$ for all $(x,y) \in X \times Y$, then (iii) follows from (ii) and Theorem 6.3.2(iii). □

Remark 6.3.6. (i) Given continuous $C_0(X)$-algebras (A, X, μ_A) and (B, Y, μ_B), it is natural to ask whether or not the converse of Theorem 6.3.5(iii) holds; that is, is property $(\mathrm{F}_{X,Y})$ a necessary condition for the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ to be continuous. The analogous result for the fibrewise tensor product is true by Theorem 6.3.2(iii). We will show in Section 6.4 that this is not the case; we can construct such pairs (A, X, μ_A) and (B, Y, μ_B) such that $A \otimes_\alpha B$ does not satisfy property $(\mathrm{F}_{X,Y})$ but $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is a continuous $C_0(X \times Y)$-algebra. One interesting consequence of this fact is that continuity of the associated fibrewise tensor product is a strictly stronger property than continuity of $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$.

(ii) A special case of Proposition 6.3.3 arises as follows; let A be a $C_0(X)$-algebra and B any C*-algebra. Then we may regard B as a $C(Y)$-algebra where $Y = \{y\}$ is a one-point space, so that $X \times Y = X$ and $A \otimes_\alpha B$ is also a $C_0(X)$-algebra. The base map $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to X$ is the extension of $\phi_A \circ p_1 : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to X$ to $\mathrm{Prim}(A \otimes_\alpha B)$, where p_1 is the projection onto the first factor. The corresponding structure map is given by $\mu_A \otimes 1 : C_0(X) \to ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$, where $\mu_A \otimes 1(f) = \mu_A(f) \otimes 1$ for all $f \in C_0(X)$.

Thus by Corollary 6.3.5(i) we get an upper-semicontinuous C*-bundle $(X, A \otimes_\alpha B, \gamma_x : A \otimes_\alpha B \to (A \otimes_\alpha B)_x)$, where $(A \otimes_\alpha B)_x = (A \otimes_\alpha B)/(I_x \otimes_\alpha B)$ for all $x \in X$. The analogous construction in the fibrewise tensor product case is as follows: for a C*-bundle $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ and a C*-algebra B, we define the C*-bundle $\mathscr{A} \otimes_\alpha B = (X, A \otimes_\alpha B, \pi_x \otimes \mathrm{id} : A \otimes_\alpha B \to A_x \otimes_\alpha B)$. The two bundles agree precisely when $\Phi(I_x, \{0\}) = \Delta(I_x, \{0\})$ for all $x \in X$, by Corollary 6.3.5(ii). We will make use of this special case as an intermediate step in the construction of the tensor product of two C*-bundles in subsequent sections.

6.4 Comparison with the fibrewise tensor product

In this section we show that the assumption of property $(\mathrm{F}_{X,Y})$ in Theorem 6.3.5(iii) is not necessary in general. More precisely, we show that for any inexact C*-algebra B, there is a continuous $C_0(X)$-algebra (A, X, μ_A) such that

(i) the fibrewise tensor product $\mathscr{A} \otimes_\alpha B$, where $\mathscr{A}$ is the continuous C*-bundle associated with (A, X, μ_A), is discontinuous, while

(ii) the $C_0(X)$-algebra $(A \otimes_\alpha B, X, \mu_A \otimes 1)$ is continuous.

This shows that the analogue of Archbold's result [6, Theorem 3.3] for the bundles constructed in Section 6.3 is untrue. In particular, we deduce that for continuous $C_0(X)$-algebras (A, X, μ_A) and (B, Y, μ_B), the assumption that the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is equal to the fibrewise tensor product (i.e. $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{X,Y})$), is not a necessary condition for continuity.

Lemma 6.4.1. *Let (A, X, μ_A) be a $C_0(X)$-algebra, and denote by $\phi_A^f : \mathrm{Fac}(A) \to X$ the base map. For any closed subset $F \subseteq X$, setting*

$$I_F = \bigcap \{I_x : x \in F\},$$

we have

(i) For $M \in \mathrm{Fac}(A)$ $M \supseteq I_F$ if and only if $\phi_A^f(M) \in F$.

(ii) For any C-algebra B we have*

$$I_F \otimes_\alpha B = \bigcap_{x \in F} (I_x \otimes_\alpha B).$$

Proof. (i) Let $m = \phi_A^f(M)$ and suppose first that $m \notin F$. Choose $a \in A$ with $\|a + M\| = 1$ and $f \in C_0(X)$ with $f(m) = 1$ and $f(F) = \{0\}$. Then

$$\mu_A(f)a + M = f(\phi_A^f(M))(a + M) = f(m)(a + M) = a + M,$$

so that $\mu_A(f)a \notin M$. On the other hand, for all $x \in F$,

$$\mu_A(f)a + I_x = f(x)(a + I_x) = 0,$$

so that $\mu_A(f)a \in I_F$. In particular, $M \not\supseteq I_F$.

Now suppose that $m \in F$. Then by Proposition 6.2.3(iii), $M \supseteq I_m$ and so $M \supseteq \bigcap_{x\in F} I_x = I_F$.

(ii) We will regard $A \otimes_\alpha B$ as a $C_0(X)$-algebra as in Remark 6.3.6(ii); the base map $\phi_\alpha : \text{Prim}(A \otimes_\alpha B) \to X$ being the unique extension to $\text{Prim}(A \otimes_\alpha B)$ of $\phi_A \circ p_1 : \text{Prim}(A) \times \text{Prim}(B) \to X$, with p_1 the projection onto the first factor.

Let $M \in \text{Fac}(A \otimes_\alpha B)$ and let $(M^A, M^B) = \Psi(M)$. We first show that $M \supseteq I_F \otimes_\alpha B$ if and only if $M^A \supseteq I_F$. By Lemma 5.5.7(ii) we have $\Psi(I_F \otimes_\alpha B) = (I_F, \{0\})$, and since Ψ is order-preserving, it is clear that if $M \supseteq I_F \otimes_\alpha B$ then $M^A \supseteq I_F$. On the other hand, since Δ is also order-preserving, if $M^A \supseteq I_F$ then using (5.5.2) we see that

$$M \supseteq \Delta(M^A, M^B) \supseteq \Delta(I_F, \{0\}) = I_F \otimes_\alpha B.$$

By (i) $M^A \supseteq I_F$ if and only if $\phi_A^f(M^A) \in F$. But then by Proposition 6.3.3(iv), we have

$$\phi_\alpha^f(M) = (\phi_A^f \times \phi_B^f)(M^A, M^B) = (\phi_A^f)(M^A)$$

and the conclusion follows.

□

Lemma 6.4.2. *Let X be an extremally disconnected compact Hausdorff space. Then any $C(X)$-algebra (A, X, μ_A) is continuous.*

Proof. For each $x \in X$ we have

$$\|a + I_x\| = \inf\{\|(1 - \mu_A(f) + f(x))a\| \, f \in C(X)\}$$

by [14, Lemme 1.10]. Moreover, it is easily seen that for a given $f \in C(X)$ and $a \in A$, the norm function $x \mapsto \|(1 - \mu_A(f) + f(x))a\|$ is continuous on X. Since X is extremally disconnected and compact, $C(X)$ is monotone complete, and so the above infimum belongs to $C(X)$. □

Proposition 6.4.3. *Let B be a C*-algebra, M a von Neumann algebra and $(M, \text{Glimm}(M), \theta_M)$ the $C(\text{Glimm}(M))$-algebra associated with the Dauns-Hofmann representation of M. Then $(M \otimes_\alpha B, \text{Glimm}(M), \theta_M \otimes 1)$ is a continuous $C(\text{Glimm}(M))$-algebra.*

Proof. Since $Z(M)$ is a von Neumann algebra, $\mathrm{Glimm}(M) = \mathrm{Prim}(Z(M))$ is an extremally disconnected compact Hausdorff space. Continuity of $(M \otimes_\alpha B, \mathrm{Glimm}(M), \theta_M \otimes 1)$ then follows from Lemma 6.4.2. □

Theorem 6.4.4. *Let B be an inexact C^*-algebra. Then there is a von Neumann algebra M, whose Dauns-Hofmann representation $(M, \mathrm{Glimm}(M), \theta_M)$ and associated continuous C^*-bundle $(\mathrm{Glimm}(M), M, \pi_p : M \to M_p)$ satisfy*

(i) $(M \otimes_\alpha B, \mathrm{Glimm}(M), \theta_M \otimes 1)$ is a continuous $C(\mathrm{Glimm}(M))$-algebra, and

(ii) $(M \otimes_\alpha B, \mathrm{Glimm}(M), \pi_p \otimes \mathrm{id}_B : M \otimes_\alpha B \to M_p \otimes_\alpha B)$ is a discontinuous C^*-bundle.

If in addition B is a prime C^*-algebra (e.g. if B primitive), then $M \otimes_\alpha B$ is quasi-standard.

Proof. Let $M = \prod_{n=1}^\infty M_n(\mathbb{C})$. Then M is a von Neumann algebra, hence it is quasi-standard by [5, Section 5]. Moreover, $Z(M)$ consists of the sequences $(\lambda_n 1_n)_{n=1}^\infty \in M$, where $\lambda_n \in \mathbb{C}$ and 1_n is the $n \times n$ identity matrix. It follows that $\mathrm{Glimm}(M) = \mathrm{Prim}(Z(M))$ is canonically homeomorphic to $\beta\mathbb{N}$.

Since B is inexact, the sequence

$$0 \to I_0 \otimes_\alpha B \to M \otimes_\alpha B \to (M/I_0) \otimes_\alpha B \to 0 \qquad (\dagger)$$

is inexact by [39]. We claim that there is some $q \in \beta\mathbb{N}$ for which $G_q \otimes_\alpha B \subsetneq \ker(\pi_q \otimes \mathrm{id}_B)$. Suppose not. By [42, Lemma 2.2]

$$\ker(\pi_0 \otimes \mathrm{id}_B) = \bigcap_{q \in \beta\mathbb{N}\backslash\mathbb{N}} \ker(\pi_q \otimes \mathrm{id}_B),$$

and by Lemma 6.4.1

$$I_0 \otimes_\alpha B = \bigcap_{q \in \beta\mathbb{N}\backslash\mathbb{N}} I_q \otimes_\alpha B.$$

Thus if $I_q \otimes_\alpha B = \ker(\pi_q \otimes \mathrm{id}_B)$ for all $q \in \beta\mathbb{N}\backslash\mathbb{N}$, the above intersections would agree, which would imply that (†) was exact, which is not the case.

Thus $(M \otimes_\alpha B, \beta\mathbb{N}, \pi_q \otimes \mathrm{id}_B : M \otimes_\alpha B \to M_q \otimes_\alpha B)$ is discontinuous at some point $p \in \beta\mathbb{N}\backslash\mathbb{N}$ by [40, Proposition 2.7].

On the other hand, by Proposition 6.4.3, the $C(\mathrm{Glimm}(M))$-algebra $(M \otimes_\alpha B, \mathrm{Glimm}(M), \theta_M \otimes 1)$ is continuous.

Under the additional assumption that the zero ideal of B is prime, then necessarily we have $\mathrm{Glimm}(B) = \{0\}$ by [7, Lemma 2.2], and so $\mathrm{Glimm}(M \otimes_\alpha B)$ is homeomorphic to $\beta\mathbb{N}$ in the obvious way. In particular, $(M \otimes_\alpha B, \mathrm{Glimm}(M), \theta_M \otimes 1)$ corresponds to the Dauns-Hofmann representation of $M \otimes_\alpha B$, and the fibre algebras are prime throughout a dense subset of $\mathrm{Glimm}(M \otimes_\alpha B)$, namely the points of $\mathbb{N}$. By [7, Theorem 3.4 (iii)⇒(i)], $M \otimes_\alpha B$ is quasi-standard.

□

6.5 Continuity and exactness of the $C_0(X \times Y)$-algebra $A \otimes_\alpha B$

In this section we investigate the relationship between exactness of a continuous $C_0(X)$-algebra (A, X, μ_A) and its minimal tensor product with an arbitrary continuous $C_0(Y)$-algebra (B, Y, μ_B). The corresponding result regarding continuity of fibrewise tensor products of C*-bundles was obtained by Kirchberg and Wassermann in [40]:

Theorem 6.5.1. (E. Kirchberg, S. Wassermann [40, Theorem 4.5]) *The following conditions on a C*-algebra B are equivalent:*

(i) B is exact,

(ii) For every locally compact Hausdorff space X and continuous C-bundle* $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ *over X, the fibrewise tensor product* $\mathscr{A} \otimes_\alpha B$ *is continuous,*

(iii) For every separable, unital continuous C-bundle* $\mathscr{A} = (\hat{\mathbb{N}}, A, \pi_n : A \to A_n)$ *over* $\hat{\mathbb{N}}$*, the fibrewise tensor product* $\mathscr{A} \otimes_\alpha B$ *is continuous.*

While Theorem 6.4.4 shows that continuity of a $C_0(X \times Y)$-algebra of the form $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is a strictly weaker property than continuity of the corresponding fibrewise tensor product, a discontinuous example was already exhibited by Blanchard in [15]. The construction of this counterexample depends heavily on the specific properties of the algebras involved. Our main result of this section, Theorem 6.5.6, shows that this pathology is in some sense universal; more precisely, we show that (A, X, μ_A) is exact if and only if the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is continuous for each continuous $C_0(Y)$-algebra (B, Y, μ_B).

The following two lemmas are known, we include a proof for completness.

Lemma 6.5.2. *Let A and B be C*-algebras and $(I, J) \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$. Then the quotient C*-algebra $A \otimes_\alpha B/\Delta(I, J)$ is naturally isomorphic to $(A/I) \otimes_\gamma (B/J)$, where $\|\cdot\|_\gamma$ is a C*-norm on $(A/I) \odot (B/J)$. Moreover, $\|\cdot\|_\gamma = \|\cdot\|_\alpha$ if and only if $\Phi(I, J) = \Delta(I, J)$.*

Proof. Let $\pi_I : A \to A/I$ and $\pi_J : B \to B/J$ be the quotient maps. We remark that if $(\pi_I \odot \pi_J) : A \odot B \to (A/I) \odot (B/J)$ denotes the canonical algebraic $*$-homomorphism, then the closure of its kernel in $A \otimes_\alpha B$ is $\overline{\ker(\pi_I \odot \pi_J)} = \Delta(I, J)$.

Take $z = \sum_{i=1}^n x_i \otimes y_i \in (A/I) \odot (B/J)$ and choose $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ such that $\pi_I(a_i) = x_i$ and $\pi_J(b_i) = y_i$ for $1 \leq i \leq n$ and set $c = \sum_{i=1}^n a_i \otimes b_i$, so that $(\pi_I \odot \pi_J)(c) = z$. Define $\gamma : (A/I) \odot (B/J) \to [0, \infty)$ via $\gamma(z) = \|c + \Delta(I, J)\|$. Then γ is well-defined since if $c' \in A \odot B$ also satisfies $(\pi_I \odot \pi_J)(c') = z$, then $c - c' \in \ker(\pi_I \odot \pi_J) \subseteq \Delta(I, J)$, hence

$$\gamma(c') = \gamma(c' - c + c) = \|c' - c + c + \Delta(I, J)\| = \|c + \Delta(I, J)\| = \gamma(c).$$

Clearly γ is a seminorm, and since $z^*z = (\pi_I \odot \pi_J)(c^*c)$, we have

$$\gamma(z^*z) = \|c^*c + \Delta(I, J)\| = \|(c + \Delta(I, J))^*(c + \Delta(I, J))\| = \|c + \Delta(I, J)\|^2 = \gamma(z)^2,$$

(by the C*-condition on the quotient norm), so γ is a C*-seminorm. Finally, if $\gamma(z) = 0$ then $c \in \Delta(I, J) \cap A \odot B = \ker(\pi_I \odot \pi_J)$, so that $z = 0$ in $(A/I) \odot (B/J)$. It follows that γ is a well-defined C*-norm on $(A/I) \odot (B/J)$.

It follows that $\pi_I \odot \pi_J : A \odot_\alpha B \to (A/I) \odot_\gamma (B/J)$ is a bounded, surjective $*$-homomorphism of normed $*$-algebras, and hence extends to a $*$-homomorphism $\pi_I \otimes_\gamma \pi_J : A \otimes_\alpha B \to (A/I) \otimes_\gamma (B/J)$. Since the range of $\pi_I \otimes_\gamma \pi_J$ is closed and contains the dense set $(A/I) \odot (B/J)$, it is surjective. We claim that $\ker(\pi_I \otimes_\gamma \pi_J) = \Delta(I, J)$.

Since $\ker(\pi_I \otimes_\gamma \pi_J)$ is closed and contains $I \odot B + A \odot J$, it must also contain $\Delta(I, J)$. To show the reverse inclusion, let $d \in \ker(\pi_I \otimes_\gamma \pi_J)$ and $\varepsilon > 0$. Then there is $c \in A \odot B$ with $\|c - d\| < \frac{\varepsilon}{2}$. By the definition of $\|\cdot\|_\gamma$,

$$\|c + \Delta(I, J)\| = \|(\pi_I \otimes_\gamma \pi_J)(c)\| = \|(\pi_I \otimes_\gamma \pi_J)(c - d)\| < \frac{\varepsilon}{2}$$

(since $d \in \ker(\pi_I \otimes_\gamma \pi_J)$). It follows that

$$\|d + \Delta(I, J)\| = \|d - c + c + \Delta(I, J)\| \leq \|d - c\| + \|c + \Delta(I, J)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε was arbitrary, $d \in \Delta(I, J)$. We have shown that $\ker(\pi_I \otimes_\gamma \pi_J) = \Delta(I, J)$, and hence we can conclude that $(A \otimes_\alpha B)/\Delta(I, J)$ is canonically $*$-isomorphic to $(A/I) \otimes_\gamma (B/J)$.

For the final assertion, note that $\|\cdot\|_\gamma = \|\cdot\|_\alpha$ if and only if $\pi_I \otimes_\gamma \pi_J$ is the canonical $*$-homomorphism $\pi_I \otimes \pi_J : A \otimes_\alpha B \to (A/I) \otimes_\alpha (B/J)$, whose kernel is by definition $\Phi(I, J)$. Hence the two norms are equal if and only if $\ker(\pi_I \otimes_\gamma \pi_J)$ $(=\Delta(I, J))$ is equal to $\Phi(I, J)$. □

Lemma 6.5.3. *Let $\mathcal{A} = (X, A, \pi_x : A \to A_x)$ be a C*-bundle and $x_0 \in X$. Then for each $a \in A$*

$$\|a + I_{x_0}\| = \inf_W \sup_{x \in W} \|\pi_x(a)\|,$$

as W ranges over all open neighbourhoods of x_0 in X.

Proof. Fix an open neighbourhood U of x_0 in X. We first claim that

$$\sup_{x \in U} \|a + I_x\| = \sup_{x \in U} \|\pi_x(a)\|.$$

It is clear that $\ker(\pi_x) \supseteq I_x$, hence we have $\|\pi_x(a)\| \leq \|a + I_x\|$ for all $x \in U$ and so the supremum on the left is always greater than or equal to that on the right.

Let $x_1 \in U$ and choose $f \in C_0(X)$, $0 \leq f \leq 1$, with $f(x_1) = 1$ and $f(X \backslash U) \equiv 0$, then $\|a + I_{x_1}\| = \|f \cdot a + I_{x_1}\|$. Moreover,

$$\|f \cdot a\| = \sup_{x \in X} \|\pi_x(f \cdot a)\| = \sup_{x \in U} \|\pi_x(f \cdot a)\| = \sup_{x \in U} |f(x)| \cdot \|\pi_x(a)\| \leq \sup_{x \in U} \|\pi_x(a)\|,$$

whence

$$\|a + I_{x_1}\| = \|f \cdot a + I_{x_1}\| \leq \|f \cdot a\| \leq \sup_{x \in U} \|\pi_x(a)\|.$$

It follows that

$$\sup_{x \in U} \|a + I_x\| \leq \sup_{x \in U} \|\pi_x(a)\|,$$

and so

$$\sup_{x \in U} \|a + I_x\| = \sup_{x \in U} \|\pi_x(a)\|.$$

Suppose for a contradiction that

$$\alpha := \inf_W \sup_{x \in W} \|\pi_x(a)\| > \|a + I_{x_0}\|.$$

Then since $x \mapsto \|a + I_x\|$ is upper semicontinuous on X, we could find an open neighborhood U of x_0 such that

$$\|a + I_x\| < \left(\frac{\alpha + \|a + I_{x_0}\|}{2}\right) \text{ for all } x \in U.$$

But this would then imply that

$$\sup_{x \in U} \|a + I_x\| \leq \left(\frac{\alpha + \|a + I_{x_0}\|}{2}\right) < \alpha \leq \sup_{x \in U} \|\pi_x(a)\|,$$

contradicting the fact that these suprema must be equal for all open neighbourhoods U of x_0. □

Proposition 6.5.4. *Let B be an inexact C*-algebra. Then there is a separable unital $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$ such that the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous at ∞.*

Proof. Since B is inexact, by [40, Proposition 4.2] there is a separable, unital continuous C*-bundle $\mathcal{C} = (C, \hat{\mathbb{N}}, \sigma_n : C \to C_n)$ with the property that the minimal fibrewise tensor product $\mathcal{C} \otimes_\alpha B = (C \otimes_\alpha B, \hat{\mathbb{N}}, \sigma_n \otimes \mathrm{id} : C \otimes_\alpha B \to C_n \otimes_\alpha B)$ is discontinuous at ∞.

Since C is continuous, for each $n \in \hat{\mathbb{N}}$ we have $I_n = \ker \sigma_n$. By [6, Theorem 3.3], we have $I_\infty \otimes_\alpha B \subsetneq \ker(\sigma_\infty \otimes \mathrm{id})$, while $I_n \otimes_\alpha B = \ker(\sigma_n \otimes \mathrm{id})$ for all $n \in \mathbb{N}$. It follows in particular that $(C \otimes_\alpha B)/(I_\infty \otimes_\alpha B)$ is canonically $*$-isomorphic to $C_\infty \otimes_\gamma B$, where $\|\cdot\|_\gamma$ is a C*-norm on $C_\infty \odot B$ distinct from $\|\cdot\|_\alpha$ by Lemma 6.5.2.

There is thus $y = \sum_{i=1}^{\ell} c_\infty^{(i)} \otimes b^{(i)} \in C_\infty \odot B$ with the property that $\|y\|_\gamma > \|y\|_\alpha$. Choose $c^{(1)}, \ldots, c^{(\ell)} \in C$ such that $\sigma_\infty(c^{(i)}) = c_\infty^{(i)}$ for $1 \leq i \leq \ell$, and set $\overline{y} = \sum_{i=1}^{\ell} c^{(i)} \otimes b^{(i)} \in C \odot B$. Then

$$\|\overline{y} + I_\infty \otimes_\alpha B\| = \|y\|_\gamma > \|y\|_\alpha = \|(\sigma_\infty \otimes \mathrm{id})(y)\|,$$

by the definitions of $\|\cdot\|_\gamma$ and $\sigma_\infty \otimes \mathrm{id}$.

Now let $D = C(\hat{\mathbb{N}}) \otimes_\alpha C_\infty$ be the trivial (hence continuous) bundle on $\hat{\mathbb{N}}$ with fibre C_∞, and denote by ε_n the evaluation maps, where $n \in \hat{\mathbb{N}}$. Let A be the pullback C*-algebra in the diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \sigma_\infty \\ D & \xrightarrow[\varepsilon_\infty]{} & C_\infty \end{array}$$

that is, the C*-subalgebra of $C \oplus D$ given by

$$A = \{c \oplus d \in C \oplus D : \sigma_\infty(c) = \varepsilon_\infty(d)\}.$$

Then we have a well-defined $*$-homomorphism $\pi_\infty : A \to C_\infty$ sending $c \oplus d$ to $\sigma_\infty(c) = \varepsilon_\infty(d)$ [50, 2.2].

For $n \in \mathbb{N}$ we may extend (keeping the same notation) σ_n and ε_n to A by setting

$$\sigma_n(c \oplus d) = \sigma_n(c), \varepsilon_n(c \oplus d) = \varepsilon_n(d).$$

A defines a continuous C*-bundle $(A, \hat{\mathbb{N}}, \pi_n : A \to A_n)$ as follows: for $n \in \mathbb{N}$ set $A_{2n-1} = C_n$ and $\pi_{2n-1} = \sigma_n$, $A_{2n} = C_\infty$ and $\pi_{2n} = \varepsilon_n$, and $\pi_\infty : A \to A_\infty = C_\infty$ as above. Continuity of A follows easily from that of C and D: for $c \oplus d \in A$ we have $\|\sigma_n(c)\|$ and $\|\varepsilon_n(d)\|$ both converge to $\|\pi_\infty(c \oplus d)\|$, it follows that $\|\pi_n(c \oplus d)\| \to \|\pi_\infty(c \oplus d)\|$.

Regarding A as a $C(\hat{\mathbb{N}})$-algebra, let $\mu_A : C(\hat{\mathbb{N}}) \to Z(A)$ be the structure map; where $f \in C(\hat{\mathbb{N}})$ acts by pointwise multiplication. Now $A \otimes B$ is a $C(\hat{\mathbb{N}})$-algebra with structure map $\mu_A \otimes 1 : C(\hat{\mathbb{N}}) \otimes \mathbb{C} \to ZM(A \otimes_\alpha B)$. For each $n \in \hat{\mathbb{N}}$ we let K_n be the ideal

$$(\mu \otimes 1)\left(\{f \in C(\hat{\mathbb{N}}) : f(n) = 0\} \otimes \mathbb{C}\right) A \otimes_\alpha B$$

of $A \otimes_\alpha B$, so that $n \mapsto \|x + K_n\|$ is upper-semicontinuous on $\hat{\mathbb{N}}$. Again it follows from [6, Theorem 3.3] that for $n \in \mathbb{N}$ we have $K_n = \ker(\pi_n \otimes \mathrm{id})$, and by Proposition 6.3.3(v) we have $K_\infty = I_\infty \otimes_\alpha B$.

On the other hand the (lower-semicontinuous) C*-bundle $\mathcal{A} \otimes_\alpha B = (\hat{\mathbb{N}}, A \otimes_\alpha B, \pi_n \otimes \mathrm{id} : A \otimes_\alpha B \to A_n \otimes_\alpha B)$ has fibres $A_n \otimes_\alpha B = C_{\frac{n+1}{2}} \otimes_\alpha B$ for n odd, $A_n \otimes_\alpha B = C_\infty \otimes_\alpha B$ for n even, and $A_\infty = C_\infty \otimes_\alpha B$.

For $1 \leq i \leq \ell$ let $d^{(i)} \in D$ be the constant section $\varepsilon_n(d^{(i)}) = c_\infty^{(i)}$, set $a^{(i)} =$

$c^{(i)} \oplus d^{(i)} \in A$, and let $x = \sum_{i=1}^{\ell} a^{(i)} \otimes b^{(i)}$. Then for $n \in \mathbb{N}$ we have

$$\begin{aligned}
(\pi_n \otimes \mathrm{id})(x) &= (\pi_n \otimes \mathrm{id}) \left(\sum_{i=1}^{\ell} a^{(i)} \otimes b^{(i)} \right) \\
&= \sum_{i=1}^{\ell} \pi_n(a^{(i)}) \otimes b^{(i)} \\
&= \begin{cases} \sum_{i=1}^{\ell} \sigma_{\frac{n+1}{2}}(c^{(i)}) \otimes b^{(i)} & \text{if } \quad n \text{ odd} \\ \\ \sum_{i=1}^{\ell} \varepsilon_{\frac{n}{2}}(d^{(i)}) \otimes b^{(i)} & \text{if } \quad n \text{ even} \end{cases} \\
&= \begin{cases} (\sigma_{\frac{n+1}{2}} \otimes \mathrm{id})(\overline{y}) & \text{if } \quad n \text{ odd} \\ \\ \sum_{i=1}^{\ell} c_{\infty}^{(i)} \otimes b^{(i)} & \text{if } \quad n \text{ even} \end{cases}
\end{aligned}$$

It then follows that

$$\|(\pi_n \otimes \mathrm{id})(x)\| = \begin{cases} \left\| (\sigma_{\frac{n+1}{2}} \otimes \mathrm{id})(\overline{y}) \right\| & \text{if } \quad n \text{ odd} \\ \\ \left\| \sum_{i=1}^{\ell} c_{\infty}^{(i)} \otimes b^{(i)} \right\|_{C_\infty \otimes_\alpha B} = \|y\|_{\alpha} & \text{if } \quad n \text{ even.} \end{cases}$$

Finally, note that by Lemma 6.5.3 we have

$$\begin{aligned}
\|x + K_\infty\| &= \max \left(\limsup_n \|(\pi_n \otimes \mathrm{id})(x)\|, \|(\pi_\infty \otimes \mathrm{id})(x)\| \right) \\
\|y\|_\gamma = \|\overline{y} + I_\infty \otimes_\alpha B\| &= \max \left(\limsup_n \|(\sigma_n \otimes \mathrm{id})(\overline{y})\|, \|(\sigma_\infty \otimes \mathrm{id})(\overline{y})\| \right).
\end{aligned}$$

Since we know that $\|\sigma_\infty \otimes \mathrm{id})(\overline{y})\| = \|y\|_\alpha < \|y\|_\gamma$, the second equality becomes

$$\|y\|_\gamma = \limsup_n \|(\sigma_n \otimes \mathrm{id})(\overline{y})\|.$$

In particular, we conclude that

$$\begin{aligned}
\|x + K_\infty\| &\geq \limsup_n \|(\pi_n \otimes \mathrm{id})(x)\| \\
&\geq \limsup \{ \|(\pi_n \otimes \mathrm{id})(x)\| : n \text{ odd} \} \\
&= \limsup \{ \left\| (\sigma_{\frac{n+1}{2}} \otimes \mathrm{id})(\overline{y}) \right\| : n \text{ odd } \} \\
&= \|y\|_\gamma.
\end{aligned}$$

Thus for all even n (since $K_n = \ker(\pi_n \otimes \mathrm{id})$ for $n \in \mathbb{N}$ by continuity at these points)

$$\|x + K_n\| = \|(\pi_n \otimes \mathrm{id})(x)\| = \|y\|_\alpha < \|y\|_\gamma \leq \|x + K_\infty\|,$$

and it follows that $n \mapsto \|x + K_n\|$ is discontinuous at ∞.

□

Corollary 6.5.5. *Let B be an inexact C*-algebra. Then there is a separable unital, continuous $C(\hat{\mathbb{N}})$-algebra $(\tilde{A}, \hat{\mathbb{N}}, \mu_{\tilde{A}})$, with $\mathrm{Prim}(\tilde{A})$ homeomorphic to $\hat{\mathbb{N}}$, such that the $C(\hat{\mathbb{N}})$-algebra $(\tilde{A} \otimes_\alpha B, \hat{\mathbb{N}}, \mu_{\tilde{A}} \otimes 1)$ is discontinuous at ∞. Moreover, $\mathrm{Prim}(\tilde{A})$ is canonically homeomorphic to $\hat{\mathbb{N}}$, and $\mu_{\tilde{A}}$ agrees with the Dauns-Hofmann $*$-isomorphism $\theta_{\tilde{A}} : C(\mathrm{Prim}(\tilde{A})) \to Z(\tilde{A})$.*

Proof. Let $(A, \hat{\mathbb{N}}, \mu_A)$ be the separable, unital continuous $C(\hat{\mathbb{N}})$-algebra of Proposition 6.5.4. Then by [16, Corollary 4.7] there is a unital continuous $C(\hat{\mathbb{N}})$-algebra $(\tilde{A}, \hat{\mathbb{N}}, \mu_{\tilde{A}})$ with simple fibres and a $C(\hat{\mathbb{N}})$-module $*$-monomorphism $\iota : A \to \tilde{A}$. By injectivity of the minimal tensor product we get a $*$-monomorphism $\iota \otimes \mathrm{id} : A \otimes_\alpha B \to \tilde{A} \otimes_\alpha B$.

Now, $\tilde{A} \otimes_\alpha B$ is a $C(\hat{\mathbb{N}})$-algebra with base map $\mu_{\tilde{A}} \otimes 1$. To show that $\iota \otimes \mathrm{id}$ is a $C(\hat{\mathbb{N}})$-module map, take $a \otimes b \in A \otimes_\alpha B$ and $f \in C(\hat{\mathbb{N}})$. Then we have

$$\begin{aligned}(\mu_{\tilde{A}} \otimes 1)(f)(\iota \otimes \mathrm{id})(a \otimes b) &= (\mu_{\tilde{A}}(f)\iota(a)) \otimes b \\ &= (\iota \otimes \mathrm{id})(\mu_A(f)a \otimes b) \in (\iota \otimes \mathrm{id})(A \otimes_\alpha B).\end{aligned}$$

In particular it follows from Theorem 2.3.12(iii) that $(\tilde{A} \otimes_\alpha B, \hat{\mathbb{N}}, \mu_{\tilde{A}} \otimes 1)$ is discontinuous, since it contains the discontinuous $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ as a $C(\hat{\mathbb{N}})$-submodule.

Denote by $\phi_{\tilde{A}} : \mathrm{Prim}(A) \to \hat{\mathbb{N}}$ the base map uniquely determined by $\mu_{\tilde{A}}$, and by

$$\tilde{I}_n = \mu_{\tilde{A}}\left(\{f \in C(\hat{\mathbb{N}}) : f(n) = 0\}\right)\tilde{A} = \bigcap\{P \in \mathrm{Prim}(\tilde{A}) : \phi_{\tilde{A}}(P) = n\}$$

for each $n \in \hat{\mathbb{N}}$. Then since each fibre $\tilde{A}/\tilde{I}_n$ is simple, it follows that $\tilde{I}_n$ is maximal (and in particular primitive) for all $n \in \hat{\mathbb{N}}$. Moreover, since every $P \in \mathrm{Prim}(\tilde{A})$ contains a unique $\tilde{I}_n$ for some $n \in \hat{\mathbb{N}}$, we see that $\tilde{I}_n \mapsto n$ is a bijection. The fact that this map is a homeomorphism then follows from Lee's theorem [44, Theorem 4]. □

Theorem 6.5.6. *The following conditions on a C*-algebra B are equivalent:*

(i) B is exact,

(ii) *for every separable, unital continuous $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$, the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is continuous,*

(iii) *for every separable, unital C*-algebra $\tilde{A}$ with $\mathrm{Prim}(\tilde{A})$ Hausdorff, the $C(\mathrm{Prim}(\tilde{A}))$-algebra $(\tilde{A} \otimes_\alpha B, \mathrm{Prim}(\tilde{A}), \theta_{\tilde{A}} \otimes 1)$ is continuous, where $\theta_{\tilde{A}} : C(\mathrm{Prim}(\tilde{A})) \to Z(\tilde{A})$ is the Dauns-Hofmann $*$-isomorphism,*

(iv) *for every locally compact Hausdorff space X and continuous $C_0(X)$-algebra (A, X, μ_A), the $C_0(X)$-algebra $(A \otimes_\alpha B, X, \mu_A \otimes 1)$ is continuous.*

Proof. (i)⇒(iv): Suppose that B is exact and let (A, X, μ_A) be a continuous $C_0(X)$-algebra. Then since $A \otimes_\alpha B$ has property (F), we have $\Delta(I_x, \{0\}) = \Phi(I_x, \{0\})$ for all $x \in X$. It then follows from Corollary 6.3.5(iii) that $(A \otimes_\alpha B, X, \mu_A \otimes 1)$ is continuous.

(iv)⇒(iii) and (iv)⇒(ii) are evident.

(ii)⇒(i): Suppose that B is inexact, then by Proposition 6.5.4 there is a separable, unital continuous $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$ such that $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous, so that (ii) fails.

(iii)⇒(ii): This follows similarly from Corollary 6.5.5.

□

Corollary 6.5.7. *The following conditions on a $C_0(Y)$-algebra (B, Y, μ_B) are equivalent:*

(i) *B is exact,*

(ii) *For every separable unital $C_0(X)$-algebra (A, X, μ_A), with $X = \hat{\mathbb{N}}$, the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ satisfies property $(\mathrm{F}_{X,Y})$,*

(iii) *For every $C_0(X)$-algebra (A, X, μ_A), the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ satisfies property $(\mathrm{F}_{X,Y})$,*

If in addition (B, Y, μ_B) is a continuous $C_0(Y)$-algebra, then (i) to (iii) are equivalent to:

(iv) *for every separable, unital continuous $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$, the $C(\hat{\mathbb{N}} \times Y)$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}} \times Y, \mu_A \otimes \mu_B)$ is continuous,*

(v) *for every continuous $C_0(X)$-algebra (A, X, μ_A), the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is continuous.*

Proof. The equivalence of (i),(ii) and (iii) is shown in the proof of [15, Proposition 3.1].

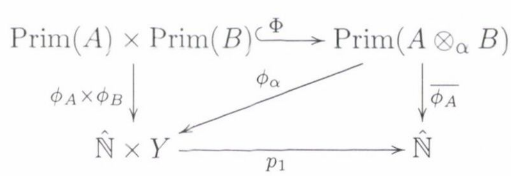
To see that (iv) implies (i), we argue by contradiction. Indeed, suppose that B is inexact, then by Proposition 6.5.4 there is a separable unital $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$ with the property that the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous. We will show that the $C(\hat{\mathbb{N}} \times Y)$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}} \times Y, \mu_A \otimes \mu_B)$ must also be discontinuous.

Let $\phi_A : \mathrm{Prim}(A) \to \hat{\mathbb{N}}$ and $\phi_B : \mathrm{Prim}(B) \to Y$ be the base maps corresponding to μ_A and μ_B respectively, which are open since both A and B are continuous. We will denote by $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to \hat{\mathbb{N}} \times Y$ and $\overline{\phi_A} : \mathrm{Prim}(A \otimes_\alpha B) \to \hat{\mathbb{N}}$ the base maps associated with $\mu_A \otimes \mu_B$ and $\mu_A \otimes 1$ respectively. Note that $\overline{\phi_A}$ is not an open mapping since the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is not continuous, and that

$$\overline{\phi_A} \circ \Phi = p_1 \circ (\phi_A \times \phi_B) = p_1 \circ \phi_\alpha \circ \Phi$$

on $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ by Proposition 6.3.3(i).

Consider now the diagram



where p_1 is the (open) projection onto the first factor. To see that the lower triangle of this diagram commutes, note that since $p_1 \circ \phi_\alpha$ agrees with $\overline{\phi_A}$ on $\Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$, both maps must agree on $\mathrm{Prim}(A \otimes_\alpha B)$ by the uniqueness part of [42, Theorem 3.2].

Then if ϕ_α were open, this would imply that $\overline{\phi_A} = p_1 \circ \phi_\alpha$ were open, which is impossible since $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous. In particular, it follows that $(A \otimes_\alpha B, \hat{\mathbb{N}} \times Y, \mu_A \otimes \mu_B)$ is discontinuous.

The fact that (v) implies (iv) is evident. To see that (i) implies (v), note that if (i) holds then $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ satisfies property $(\mathrm{F}_{X,Y})$ by the equivalence of (i) and (iii). Then by Theorem 6.3.5(iii), $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is a continuous $C_0(X \times Y)$-algebra.

□

6.6 Quasi-standard C*-algebras and Hausdorff primitive ideal spaces

This section is concerned with stability of the class of quasi-standard C*-algebras under tensor products. This question was first studied by Archbold in [6], where it was shown that if A and B are quasi-standard and $A \otimes_\alpha B$ satisfies property ($\mathrm{F_{Gl}}$), then $A \otimes_\alpha B$ is quasi-standard. We gave a partial converse to this result in Theorem 5.6.2. However, it is clear from Theorem 6.4.4 that property ($\mathrm{F_{Gl}}$) is not a necessary condition for $A \otimes_\alpha B$ to be quasi-standard.

We will show in Theorem 6.6.6 that a quasi-standard C*-algebra A is exact if and only if $A \otimes_\alpha B$ is quasi-standard for all quasi-standard B. As a related result, we show that a (not necessarily quasi-standard) C*-algebra A is exact if and only if $A \otimes_\alpha B$ satisfies property ($\mathrm{F_{Gl}}$) for all C*-algebras B, if and only $A \otimes_\alpha B$ satisfies property ($\mathrm{F_{MP}}$) for all C*-algebras B. The existence of C*-algebras A and B such that $A \otimes_\alpha B$ does not satisfy properties ($\mathrm{F_{Gl}}$) and ($\mathrm{F_{MP}}$) was previously unknown, thus our result answers a question posed by Archbold [6, p. 142] and Lazar [42, p. 250].

Proposition 6.6.1. *Let A and B be C*-algebras.*

(i) If $A \otimes_\alpha B$ satisfies ($\mathrm{F_{MP}}$), then $A \otimes_\alpha B$ satisfies ($\mathrm{F_{Gl}}$),

(ii) If (A, X, μ_A) is a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra, then $A \otimes_\alpha B$ satisfies property ($\mathrm{F_{Gl}}$) implies that $A \otimes_\alpha B$ satisfies property ($\mathrm{F}_{X,Y}$).

Proof. (i): We first show that for any C*-algebra A and $G_1 \in \mathrm{Glimm}(A)$,

$$G_1 = \bigcap \{P \in \text{Min-Primal}(A) : P \supseteq G_1\}.$$

Indeed, since each primitive ideal of A is primal, we necessarily have

$$G_1 = \bigcap \{P \in \mathrm{Primal}(A) : P \supseteq G_1\} \subseteq \bigcap \{P \in \text{Min-Primal}(A) : P \supseteq G_1\}.$$

Denote by H the ideal on the right, then if the above inclusion were strict, there would be some $Q \in \mathrm{Prim}(A)$ such that $Q \supseteq G_1$ but $Q \not\supseteq H$. Let R be a minimal primal ideal of A contained in Q, then by [7, Lemma 2.2] there is a unique Glimm ideal G_2 contained in R. Note $G_1 \neq G_2$ since otherwise R, and hence Q, would contain H. This in turn implies that $Q \not\supseteq G_1$, a contradiction.

Since $A \otimes_\alpha B$ satisfies (F_{MP}), we have $\Phi = \Delta$ on Min-Primal$(A) \times$ Min-Primal(B), and so [6, Theorem 4.1] shows that Δ is a homeomorphism of Min-Primal$(A) \times$ Min-Primal(B) onto Min-Primal$(A \otimes_\alpha B)$. For $(G, H) \in$ Glimm$(A) \times$ Glimm(B) we have $\Delta(G, H) \in$ Glimm$(A \otimes_\alpha B)$ by Theorem 5.5.9, which together with the above remarks gives

$$\begin{aligned}\Delta(G,H) &= \bigcap\{R \in \text{Min-Primal}(A \otimes_\alpha B) : R \supseteq \Delta(G,H)\}\\ &= \bigcap\{\Delta(I,J) : (I,J) \in \text{Min-Primal}(A) \times \text{Min-Primal}(B), \Delta(I,J) \supseteq \Delta(G,H)\}\end{aligned}$$

On the other hand by [42, Lemma 2.2] and the first part of the proof,

$$\begin{aligned}\Phi(G,H) &= \Phi\left(\bigcap\{I \in \text{Min-Primal}(A) : I \supseteq G\}, \bigcap\{J \in \text{Min-Primal}(B) : J \supseteq H\}\right)\\ &= \bigcap\{\Phi(I,J) : (I,J) \in \text{Min-Primal}(A) \times \text{Min-Primal}(B), I \supseteq G, J \supseteq H\}\\ &= \bigcap\{\Delta(I,J) : (I,J) \in \text{Min-Primal}(A) \times \text{Min-Primal}(B), I \supseteq G, J \supseteq H\}.\end{aligned}$$

Finally, since $\Psi \circ \Delta$ is the identity on $\text{Id}'(A) \times \text{Id}'(B)$ and since Ψ is order preserving, we see that

$$\Delta(I,J) \supseteq \Delta(G,H) \text{ if and only if } I \supseteq G \text{ and } J \supseteq H,$$

from which we conclude that $\Phi(G, H) = \Delta(G, H)$ for all $(G, H) \in$ Glimm$(A) \times$ Glimm(B). Hence $A \otimes_\alpha B$ satisfies property (F_{Gl}).

(ii): We will use the notation of Proposition 6.3.3. Note that for all $(x, y) \in X \times Y$, we have $K_{x,y} = \Delta(I_x, J_y)$ by Proposition 6.3.3(iv), so we will show that $K_{x,y} = \Phi(I_x, J_y)$. Let ψ_A, ψ_B and ψ_α denote the continuous maps on the Glimm spaces of A, B and $A \otimes_\alpha B$ induced by the base maps ϕ_A, ϕ_B and ϕ_α respectively. We first show that $\psi_\alpha \circ \Delta = \psi_A \times \psi_B$. Indeed, for all $(P, Q) \in$ Prim$(A) \times$ Prim(B) we have

$$(\phi_A \times \phi_B)(P,Q) = (\phi_\alpha \circ \Phi)(P,Q)$$

by Proposition 6.3.3(i). Hence by the definitions of ψ_A, ψ_B and ψ_α,

$$(\psi_A \times \psi_B) \circ (\rho_A \times \rho_B)(P,Q) = (\psi_\alpha \circ \rho_\alpha \circ \Phi)(P,Q).$$

Following the first paragraph of the proof of Theorem 5.5.9, we see that $\rho_\alpha \circ \Phi = \Delta \circ (\rho_A \times \rho_B)$, which shows that

$$(\psi_A \times \psi_B)(\rho_A(P), \rho_B(Q)) = (\psi_\alpha \circ \Delta)(\rho_A(P), \rho_B(Q))$$

for all $(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$.

By Lemma 6.2.1,

$$K_{x,y} = \bigcap\{G \in \mathrm{Glimm}(A \otimes_\alpha B) : \psi_\alpha(G) = (x, y)\}.$$

Any such $G \in \mathrm{Glimm}(A \otimes_\alpha B)$ is the image $\Delta(G_p, G_q)$ of a pair of Glimm ideals of A and B by Theorem 5.5.9. Together with the fact that $\psi_\alpha \circ \Delta = \psi_A \times \psi_B$, this gives

$$K_{x,y} = \bigcap\{\Delta(G_p, G_q) : (p, q) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \psi_A(p) = x \text{ and } \psi_B(q) = y\}.$$

Using $\Phi = \Delta$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, [42, Lemma 2.2] shows that

$$\begin{aligned} K_{x,y} &= \bigcap\{\Phi(G_p, G_q) : \psi_A(p) = x, \psi_B(q) = y\} \\ &= \Phi\left(\cap\{G_p : \psi_A(p) = x\}, \cap\{G_q : \psi_B(q) = y\}\right) \\ &= \Phi(I_x, J_y), \end{aligned}$$

where the final equality follows from Lemma 6.2.1. Hence $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{X,Y})$. □

Proposition 6.6.2. *Let B be an inexact C^*-algebra. Then there is a separable C^*-algebra A with* $\mathrm{Prim}(A)$ *Hausdorff such that*

(i) there is a pair $(G, H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ *with* $\Delta(G, H) \subsetneq \Phi(G, H)$,

(ii) there is a pair $(I, J) \in \text{Min-Primal}(A) \times \text{Min-Primal}(B)$ *with* $\Delta(I, J) \subsetneq \Phi(I, J)$,

(iii) $\mathrm{Prim}(A \otimes_\alpha B)$ *is non-Hausdorff,*

(iv) The complete regularisation map $\rho_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to \mathrm{Glimm}(A \otimes_\alpha B)$ *is not open.*

Proof. To prove (i), let $(A, \hat{\mathbb{N}}, \mu_A)$ be the continuous $C(\hat{\mathbb{N}})$-algebra constructed in Corollary 6.5.5, so that the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous. We regard B as a (continuous) $C_0(Y)$-algebra over a one-point space $Y = \{y\}$ as in Remark 6.3.6(ii). Then since $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous, it follows from Theorem 6.3.5(iii) that $A \otimes_\alpha B$ does not satisfy property $(\mathrm{F}_{X,Y})$. Hence by Proposition 6.6.1(ii), it must follow that $\Phi \neq \Delta$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$. (ii) is immediate from (i) and Proposition 6.6.1(i).

(iii): Note that if $\mathrm{Prim}(B)$ is non-Hausdorff, then the same is true of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$. Since Φ maps $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ homeomorphically onto its image in $\mathrm{Prim}(A \otimes_\alpha B)$ [59, lemme 16], it then follows that the latter must also be non-Hausdorff.

Suppose now that $\mathrm{Prim}(B)$ is Hausdorff, then $\mathrm{Prim}(B) = \mathrm{Glimm}(B)$. Since $\mathrm{Prim}(A) = \mathrm{Glimm}(A)$ also, (i) implies that there are $(P,Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ such that $\Delta(P,Q) \subsetneq \Phi(P,Q)$. It follows that there is $R \in \mathrm{Prim}(A \otimes_\alpha B)$ such that $R \supseteq \Delta(P,Q)$ but $R \neq \Phi(P,Q)$. Since by Theorem 5.5.9, $\Delta(P,Q)$ is a Glimm ideal of $A \otimes_\alpha B$, we have $R \approx \Phi(P,Q)$, so that $\mathrm{Prim}(A \otimes_\alpha B)$ is non-Hausdorff.

(iv): If $\rho_B : \mathrm{Prim}(B) \to \mathrm{Glimm}(B)$ is not an open map, then since A is unital, ρ_α is not open by Corollary 5.6.7. Thus we will assume that ρ_B is open.

Since $A \otimes_\alpha B$ is a discontinuous $C(\hat{\mathbb{N}})$-algebra, the continuous mapping $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to \hat{\mathbb{N}}$ is not open. Moreover, ϕ_α is the unique extension of $\phi_A \circ p_1 : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to \hat{\mathbb{N}}$ to $\mathrm{Prim}(A \otimes_\alpha B)$, where p_1 is the projection onto the first factor. We will denote by $\psi_A : \mathrm{Glimm}(A) \to \hat{\mathbb{N}}$ the canonical homeomorphism, and by $\psi_\alpha : \mathrm{Glimm}(A \otimes_\alpha B) \to \hat{\mathbb{N}}$ the map induced by ϕ_α given by the universal property of the complete regularisation, so that $\phi_\alpha = \psi_\alpha \circ \rho_\alpha$.

Since $\mathrm{Prim}(A)$ is compact, the complete regularisation of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ is canonically identified with $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ with the product topology by Corollary 5.3.4. In particular, this implies that $\Psi : \mathrm{Glimm}(A \otimes_\alpha B) \to \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ is a homeomorphism by Theorem 5.5.9. Let $\tilde{p_1} : \mathrm{Glimm}(A) \times \mathrm{Glimm}(B) \to \mathrm{Glimm}(A)$ be the projection onto the first factor and consider now the diagram

$$\begin{array}{ccccc}
\mathrm{Prim}(A \otimes_\alpha B) & \xrightarrow{\rho_\alpha} & \mathrm{Glimm}(A \otimes_\alpha B) & \xrightarrow{\Psi} & \mathrm{Glimm}(A) \times \mathrm{Glimm}(B) \\
\downarrow{\scriptstyle \phi_\alpha} & & & & \downarrow{\scriptstyle \tilde{p_1}} \\
\hat{\mathbb{N}} & & \xleftarrow[\psi_A]{\qquad\qquad\qquad} & & \mathrm{Glimm}(A).
\end{array}$$

Then by Theorems 5.3.3 and 5.5.9(ii) we have $\Psi \circ \rho_\alpha \circ \Phi = \rho_A \times \rho_B$ on $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$, so that for all $(P,Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$,

$$\begin{aligned}
(\psi_A \circ \tilde{p_1} \circ \Psi \circ \rho_\alpha)(\Phi(P,Q)) &= (\psi_A \circ \tilde{p_1})(\rho_A(P), \rho_B(Q)) \\
&= (\psi_A \circ \rho_A)(P) \\
&= \phi_A(P) \\
&= (\phi_\alpha)(\Phi(P,Q)),
\end{aligned}$$

the final equality holding by Proposition 6.3.3(i). Since $\Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$ is dense in $\mathrm{Prim}(A \otimes_\alpha B)$, it follows by continuity that the above diagram commutes.

Note that $\Psi, \tilde{p}_1$ and ψ_A are all open mappings. Thus if ρ_α were open, it would follow that ϕ_α were, which would imply that $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ were a continuous $C(\hat{\mathbb{N}})$-algebra, a contradiction. □

Theorem 6.6.3. *The following conditions on a C*-algebra B are equivalent:*

(i) B is exact,

(ii) $A \otimes_\alpha B$ *satisfies property* $(\mathrm{F_{Gl}})$ *for all C*-algebras A,*

(iii) $A \otimes_\alpha B$ *satisfies property* $(\mathrm{F_{MP}})$ *for all C*-algebras A.*

Proof. (i)⇒(ii) and (iii): if B is exact then $A \otimes_\alpha B$ satisfies property (F) for all A, hence $A \otimes_\alpha B$ satisfies properties $(\mathrm{F_{Gl}})$ and $(\mathrm{F_{MP}})$ for all C*-algebras A.

To see that (ii) (resp. (iii)) implies (i), note that if B is inexact then there is by Proposition 6.6.2(i) (resp. (ii)) a C*-algebra A for which $A \otimes_\alpha B$ does not satisfy property $(\mathrm{F_{Gl}})$ (resp. $(\mathrm{F_{MP}})$). □

It was shown in [6] that if $A \otimes_\alpha B$ satisfies property $(\mathrm{F_{MP}})$, then Δ (equivalently, Φ) maps Min-Primal$(A) \times$ Min-Primal(B) homeomorphically onto Min-Primal$(A \otimes_\alpha B)$. The following corollary shows that if B is inexact and quasi-standard then this map may fail to be a homeomorphism.

Corollary 6.6.4. *Let B be an inexact, quasi-standard C*-algebra. Then there is a quasi-standard C*-algebra A for which the restriction of* Δ *to* Min-Primal$(A) \times$ Min-Primal(B) *is not a homeomorphism of this space onto* Min-Primal$(A \otimes_\alpha B)$.

Proof. Again, let A be the C*-algebra constructed in Corollary 6.5.5, so that $A \otimes_\alpha B$ is not quasi-standard by Proposition 6.6.2(iv). Since both A and B are quasi-standard, we have Glimm(A) = Min-Primal(A) and Glimm(B) = Min-Primal(B), both as sets and topologically [7, Theorem 3.3 (iii)]. By Corollary 5.3.4 and Theorem 5.5.9, Δ is a homeomorphism of Glimm$(A) \times$ Glimm(B) onto Glimm$(A \otimes_\alpha B)$. Thus if Δ were also a homeomorphism of Min-Primal$(A) \times$ Min-Primal(B) onto Min-Primal$(A \otimes_\alpha B)$, it would follow that Glimm$(A \otimes_\alpha B)$ = Min-Primal$(A \otimes_\alpha B)$, both as sets and topologically. This would imply that $A \otimes_\alpha B$ is quasi-standard by [7, Theorem 3.3 (iii)⇒(i)], which is a contradiction.

□

Example 6.6.5. *Let B be a primitive, inexact C^*-algebra, for example $B = B(H)$ [57] or $B = C^*(\mathbf{F}_2)$ [56] (the full group C^*-algebra of the free group on two generators), so that* $\text{Min-Primal}(B) = \{0\}$. *Then Corollary 6.6.4 gives a C^*-algebra A with* $\text{Prim}(A) = \text{Min-Primal}(A) = \{I_n : n \in \hat{\mathbb{N}}\}$ *for which Δ is not a homeomorphism of* $\text{Min-Primal}(A) \times \text{Min-Primal}(B)$ *onto* $\text{Min-Primal}(A \otimes_\alpha B)$.

We remark that by Theorem 6.3.2(iii), $\Delta(I_n, \{0\}) = \Phi(I_n, \{0\})$ for all $n \in \mathbb{N}$, so that $\Delta(I_n, \{0\}) \in \text{Prim}(A \otimes_\alpha B)$ for all such n. By [5, Proposition 4.5], $\Delta(I_n, \{0\}) \in \text{Min-Primal}(A \otimes_\alpha B)$ for all $n \in \mathbb{N}$. On the other hand, it is not clear whether or not the Glimm ideal $\Delta(I_\infty, \{0\})$ is a primal ideal of $A \otimes_\alpha B$.

Theorem 6.6.6. *Let B be a C^*-algebra. Then*

(i) If $\rho_B : \text{Prim}(B) \to \text{Glimm}(B)$ is an open map, then B is exact if and only if $\rho_\alpha : \text{Prim}(A \otimes_\alpha B) \to \text{Glimm}(A \otimes_\alpha B)$ is open for every C^*-algebra A with $\rho_A : \text{Prim}(A) \to \text{Glimm}(A)$ open.

(ii) If B is quasi-standard, then B is exact if and only if $A \otimes_\alpha B$ is quasi-standard for every quasi-standard C^*-algebra A.

(iii) If $\text{Prim}(B)$ is Hausdorff, then B is exact if and only if $\text{Prim}(A \otimes_\alpha B)$ is Hausdorff for every C^*-algebra A with $\text{Prim}(A)$ Hausdorff.

Proof. (i): If B is exact then $A \otimes_\alpha B$ satisfies property (F), hence property (F_{Gl}), for all C*-algebras A. Hence ρ_α is open for all C*-algebras A with ρ_A open by Theorem 5.6.2.

Conversely if B is inexact, then by Proposition 6.6.2(iv) there is a C*-algebra A with $\text{Prim}(A)$ Hausdorff, hence $\rho_A = \text{id}$ open, such that ρ_α is not open.

(ii): If B is exact, then it follows from [37, Corollary 2.5] that $A \otimes_\alpha B$ is quasi-standard for every quasi-standard A.

Conversely, if B is inexact then by Proposition 6.6.2(iv) there is a quasi-standard C*-algebra A for which $A \otimes_\alpha B$ is not quasi-standard.

(iii): We will make use of the fact that if A is a C*-algebra such that either $\text{Prim}(A)$ or $\text{Fac}(A)$ is Hausdorff, then $\text{Prim}(A) = \text{Fac}(A) = \text{Prime}(A)$, see e.g. [17, p. 474].

Suppose that B is exact and that A is a C*-algebra with $\text{Prim}(A)$ Hausdorff. Then since $A \otimes_\alpha B$ has property (F), [42, Proposition 5.1] shows that Δ is a homeomorphism of $\mathrm{Fac}(A) \times \mathrm{Fac}(B) = \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ onto $\mathrm{Fac}(A \otimes_\alpha B)$. Hence $\mathrm{Fac}(A \otimes_\alpha B)$ is Hausdorff, and thus the same is true of $\mathrm{Prim}(A \otimes_\alpha B)$.

Conversely, if B is inexact then by Proposition 6.6.2(iii) there is a separable C*-algebra A with $\mathrm{Prim}(A)$ Hausdorff for which $\mathrm{Prim}(A \otimes_\alpha B)$ is non-Hausdorff.

□

Example 6.6.7. *Let $M = \prod_{n\geq 1} M_n(\mathbb{C})$, so that M is quasi-standard as in Theorem 6.4.4. Moreover, M is inexact by [39, Theorem 1.1]. Then by Proposition 6.6.2(iv), there is a separable unital C*-algebra A with Hausdorff primitive ideal space* $\mathrm{Prim}(A)$ *homeomorphic to $\hat{\mathbb{N}}$ such that $A \otimes_\alpha M$ is not quasi-standard. In particular, the assumption that the zero ideal of the C*-algebra B of Theorem 6.4.4 is prime cannot be dropped.*

We will give an analogous result for maximal tensor products of (unital) C*-algebras in Theorem 6.6.10. We first recall the definition of $A \otimes_{\max} B$.

Definition 6.6.8. Let A and B be C* algebras, and for $c \in A \odot B$ let

$$\|c\|_{\max} = \sup\{\|c\|_\gamma : \|\cdot\|_\gamma \text{ a C*-norm on } A \odot B\}.$$

Then $\|\cdot\|_{\max}$ is a C*-norm on $A \odot B$, and we denote by $A \otimes_{\max} B$ the completion of $A \odot B$ with respect to this norm, the *maximal tensor product* of A and B.

The following proposition gives some (known) properties of the Dauns-Hofmann representation of the maximal tensor product of two unital C*-algebras.

Proposition 6.6.9. *Let A and B be unital C*-algebras, then*

(i) the map $(G, H) \mapsto G \otimes_{\max} B + A \otimes_{\max} H$ is a homeomorphism of $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ *onto* $\mathrm{Glimm}(A \otimes_{\max} B)$,

(ii) the Glimm quotients of $A \otimes_{\max} B$ are canonically $*$-isomorphic to $(A/G) \otimes_{\max} (B/H)$ for $(G, H) \in$ $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$,

(iii) The Dauns-Hofmann representation of $A \otimes_{\max} B$ defines canonically an upper-semicontinuous C-bundle*

$$(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), A \otimes_{\max} B, \pi_G \otimes_{\max} \sigma_H : A \otimes_{\max} B \to (A/G) \otimes_{\max} (B/H)),$$

where for $(G, H) \in$ $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ *, $\pi_G : A \to A/G$ and $\sigma_H : B \to B/H$ are the quotient maps.*

Proof. (i) is shown in [37, p. 304], and (ii) follows from (i) and [14, Proposition 3.15]. Since A and B are unital, $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ is compact, and so (iii) is then immediate from the Dauns-Hofmann Theorem (Theorem 2.3.6). □

Theorem 6.6.10. *Let B be a unital quasi-standard C*-algebra. Then B is nuclear if and only if $A \otimes_{\max} B$ is quasi-standard for every quasi-standard C*-algebra A.*

Proof. If B is nuclear, then B is exact, so that for any C*-algebra A we have that $A \otimes_{\max} B = A \otimes_{\alpha} B$ has property (F). Thus for all quasi-standard A, $A \otimes_{\max} B$ is quasi-standard by [37, Corollary 2.5].

Conversely, suppose that B is non-nuclear. For each $q \in \mathrm{Glimm}(B)$ let G_q be the corresponding Glimm ideal of B and denote by $(\mathrm{Glimm}(B), B, \sigma_q : B \to B_q)$ the corresponding continuous C*-bundle over $\mathrm{Glimm}(B)$, where $B_q = B/G_q$ for all $q \in \mathrm{Glimm}(B)$. By [14, Proposition 3.23], there is $p \in \mathrm{Glimm}(A)$ for which B_p is non-nuclear. As in the proof of [40, Theorem 3.2], one can construct a Hilbert space H, a unital C*-subalgebra $C \subseteq B(H)$ and $t = \sum_{i=1}^{\ell} r_i \otimes s_i \in C \odot B_p$ such that $\|t\|_{C \otimes_{\max} B_p} > \|t\|_{B(H) \otimes_{\max} B_p}$.

Denote by A the C*-algebra of sequences $(T_n) \subset B(H)$ such that T_n converges in norm to some element $T \in C$. Then, as in the proof of [7, Proposition 3.6], A is quasi-standard, $\mathrm{Glimm}(A)$ is homeomorphic to $\hat{\mathbb{N}}$, and the Glimm quotients A_n of A are given by

$$A_n = \begin{cases} B(H) & \text{if} \quad n \in \mathbb{N} \\ C & \text{if} \quad n = \infty \end{cases}$$

We denote by $(\hat{\mathbb{N}}, A, \pi_n : A \to A_n)$ the corresponding continuous C*-bundle over $\hat{\mathbb{N}}$. By Proposition 6.6.9 we may identify $\mathrm{Glimm}(A \otimes_{\max} B)$ with $\hat{\mathbb{N}} \times \mathrm{Glimm}(B)$, and the Glimm quotients of $A \otimes_{\max} B$ are isomorphic to $A_n \otimes_{\max} B_p$ for $(n,p) \in \hat{\mathbb{N}} \times \mathrm{Glimm}(B)$.

For $1 \leq i \leq \ell$ choose $\overline{r_i} \in A$ and $\overline{s_i} \in B$ such that $\pi_n(\overline{r_i}) = r_i$ for all $n \in \hat{\mathbb{N}}$, and $\sigma_p(\overline{s_i}) = s_i$. Then setting $\bar{t} = \sum_{i=1}^{\ell} \overline{r_i} \otimes \overline{s_i} \in A \odot B$, we have $\pi_\infty \otimes \sigma_G(\bar{t}) = t$. Then we have $\|(\pi_n \otimes_{\max} \sigma_p)(\bar{t})\| = \|t\|_{B(H) \otimes_{\max} B}$ for $n \in \mathbb{N}$, while $\|(\pi_\infty \otimes_{\max} \sigma_p)(\bar{t})\| = \|t\|_{C \otimes_{\max} B}$. Since $(n,p) \to (\infty, p)$ in $\hat{\mathbb{N}} \times \mathrm{Glimm}(B)$, it follows that $(n,q) \mapsto$ $\|(\pi_n \otimes_{\max} \sigma_q)(t)\|$ is discontinuous at (∞, p). In particular, $A \otimes_{\max} B$ is not quasi-standard. □

Notation and definitions

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$C_0(X)$-structure in C*-algebras, Multiplier Algebras and Tensor Products

David McConnell

Abstract

Our general theme concerns topological decompositions of C*-algebras and the interactions of these decompositions with multiplier algebras, tensor products and module structures. A primary focus is placed on modules known as $C_0(X)$-algebras.

Bundle structures, specifically C*-bundles, for a C*-algebra A (not usually unital) over a suitable base space X, are closely related to $C_0(X)$-algebras, and a natural consideration is to relate such structures on A to bundle representations of the multiplier algebra $M(A)$ of A over the Stone-Čech compactification βX of X. We discuss how the strict topology on $M(A)$ can be used in this context, and in the case of a $C_0(X)$-algebra A, relate it to the induced $C(\beta X)$-algebra structure on $M(A)$. Further preliminary results concern the ideal structure of $M(A)$ when A is a $C_0(X)$-algebra.

Sheaves of C*-algebras over X provide another approach which is partially equivalent to bundles when the topological space X is locally compact and Hausdorff. As a corollary, we show that a C*-bundle over a locally compact Hausdorff space X defines naturally a C*-bundle over βX in such a way that their algebras of continuous sections are naturally isomorphic. These results are applied to the particular case of the sheaf of multipliers of a $C_0(X)$-algebra A, which is shown to be canonically isomorphic to the sheaf of local sections arising from the $C(\beta X)$-structure on $M(A)$.

For our main results we consider the minimal tensor product $A \otimes_\alpha B$ of two C*-algebras A and B. Extending earlier results of Kaniuth [37], we obtain a complete description of the topological space of Glimm ideals of $A \otimes_\alpha B$ in terms of those of the factors (results published in [45]). As a consequence, we construct the Dauns-Hofmann bundle representation [21] of $A \otimes_\alpha B$ in terms of the corresponding representations of A and B, and describe the structure of the centre of the multiplier algebra of $A \otimes_\alpha B$ in this setting.

Given a $C_0(X)$-algebra A and a $C_0(Y)$-algebra B, we demonstrate how $A \otimes_\alpha B$ carries naturally the structure of a $C_0(X \times Y)$-algebra. We study the associated C*-bundle decomposition of $A \otimes_\alpha B$ over this space, and in particular we compare its structure to the fibrewise tensor product studied elsewhere. As a consequence, we obtain several new characterisations of the property of exactness in terms of the stability of certain classes of $C_0(X)$-algebras under the operation of forming tensor products.

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