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IMPROVING THE EFFICIENCY OF THE BINOMIAL OPTION PRICING METHOD

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Abstract

The binomial option pricing model is now widely used to value options, particularly where no analytic (closed form) solution exists. The aim of this paper is to present a modified binomial option pricing algorithm applied to the case of the American put option. These modifications, which are based on a set of propositions concerning the binomial method, considerably improve its computational efficiency. They have a particular application to the valuing of long lived options.

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Introduction

The aim of this paper is to present a modified binomial option pricing algorithm applied to the case of the American put option. These modifications considerably improve the computational efficiency of the method. They are based on a number of propositions concerning the binomial method. These propositions allow us to partition the set of 'nodes' in the binomial process into subsets, for each of which it is possible to determine whether the immediate exercise value of the option exceeds its holding value, or *vice versa*, without having to explicitly calculate and compare these values. To this extent these propositions contradict a long held belief in regard to methods of pricing the American put option: viz., the assertion [1] that the exercise boundary condition must be checked at every instant in the pricing model (see Merton 1974; Geske and Shastri 1985, pp. 48 and 64). In respect of the binomial model we show that the probability of early exercise in respect of certain periods of the model, even when the option is in the money, can be demonstrated to be zero. Before doing so, however, we begin with a brief recapitulation of the binomial model and introduce the notation to be used throughout the paper.

The Binomial Option Pricing Model

We deal in this paper with American put options written on non-dividend paying stock. We make the usual assumptions - namely that the risk free interest rate, r, and the annualised standard deviation of the underlying stock price, σ, are both non-stochastic and constant over the life of the option. We denote time by the index t (t = 0 ... T, the maturity date of the option), the stock price at t by S(t) and the exercise price by X.

In the binomial option pricing model, the life of the option is divided into N discrete time periods, during each of which the price of the underlying asset is assumed to make a single move, either up or down. The magnitude of these movements is given by the multiplicative parameters u and d. The probabilities [2] of an upward or downward movement (p and 1-p respectively) in the stock's price are given by

$$p = (q-d)/(u-d) \text{ and}$$

$$1-p = (u-q)/(u-d)$$

where q is the one period risk free rate, given by

$$q = r^{T/N}$$

The binomial method approximates the continuous change in the option's value through time by valuing the option at a discrete set of nodes which together make a cone shaped grid. We identify each node in the cone by <j,n> where j indicates the number of upward stock moves required to generate the option's immediate non-negative exercise value at that node, given by

$$B_{j,n} = \max(0\ ;\ X - u^j d^{n-j} S) \text{ (for a put)} \qquad (1)$$

and n is the period of the model ($n = 0 \ldots N$).

When valuing European options or American call options on non-dividend paying stock, it is only necessary to calculate the N+1 terminal exercise values of the option (i.e. the set B_{jN}, $j=0 \ldots N$ in our notation). Since there is no (rational) probability of early exercise in these cases the intermediate values of the binomial process (for $0 < n < N$) need not be computed. Instead the binomial formula is used to 'jump backwards' from the terminal values to the initial option value (at node $<0,0>$). In Geske and Shastri's (1985) analysis of approximation methods for option valuations, it was this feature of the binomial method that was chiefly responsible for its outperforming its competitors (finite difference methods) in terms of computing demands and expense by a considerable margin in the valuing of a call option on non-dividend paying stock (see, for example, Geske and Shastri 1985 table 2, p.60, and figure 1, p.61).

However, the application of the binomial method to the valuing of an American put option on non-dividend paying stock will be much less efficient. This is because the possibility of early exercise requires that both the holding value and the exercise value of the option be computed for each node in the process. To see this we define $A_{j,n}$ to be the holding value of the option at the jnth node. This is given by the weighted sum of the discounted values of the option at the two points to which it can move in the following period:

$$\frac{p}{q}\,[\max \emptyset;\ X - u^{j+1} d^{n_{+1}-j} S;\ A_{j+1,n+1}] + \frac{(1-p)}{q}\,[\max \emptyset; X - u^{j-1} d^{n-j} S;\ A_{j,n+1}] \qquad (2)$$

Thus the value of the American put option at <jn> is

$$V_{jn} = \max (A_{jn};\ B_{jn}) \qquad (3)$$

Typically the use of the binomial method to value the American put would require the following set of calculations. First, calculate the terminal values of the option at n=N via the equation:

$$V_{jn} = B_{jn} = \max(\emptyset;\ X - u^j d^{n-j} S) \quad \text{for all } j\ (=\emptyset \ldots n)$$

Then use the values so obtained to estimate the values of the option at n=N-1, discounting accordingly:

$$A_{jn} = V_{j,n+1}\,(1-p)/q + V_{j+1,n+1}\,p/q \quad \text{for all } j\ (=\emptyset \ldots n)$$

These are the values of holding the option to the next period. However, the immediate exercise value of the option is given by:

$$B_{jn} = \max(\emptyset;\ X - u^j d^{n-j} S) \quad \text{for all } j\ (=\emptyset \ldots n)$$

Thus the value of the option at period N-1 is

$$V_{jn} = \max (A_{jn};\ B_{jn}) \quad \text{for all } j\ (=\emptyset \ldots n)$$

Repeat this process until the final calculation for n=Ø which is the value of the option.

The virtues of this approach are that it is easy to program and is closely tied to the economic interpretation of the process. Algorithms using the binomial method are fast and cheap to run. In Geske and Shastri's (1985) comparative analysis, the binomial was found to be the cheapest among the set of methods they examined for the valuing of a single call or put option (all methods were equally accurate). As the number of options to be valued increased, however, Geske and Shastri (1985) found that finite difference methods became more cost effective. In the case of the call option, the cost of valuation per option was least under the binomial method up to 300 options: in the case of the put the binomial's relative cheapness lasted only up to the valuing of nine options. As the authors note, this difference is due to the infeasibility of 'time jumping' in the put case.

Propositions

All the following propositions refer to the binomial process.

Proposition 1: let S^* be any value of S, s.t. $X - uS^* > 0$. Then

$$X-S^* > (X-dS^*)(1-p)/q + (X-uS^*)p/q \qquad (4)$$

Proof:

$(X-dS^*)(1-p)/q + (X-uS^*)p/q = -((1-p)d + pu)S^*/q + X/q$

since $((1-p)d + pu) = q$

$$= -S^* + X/q$$

Therefore

$$X-S^* > -S^* + X/q$$

which is true iff $q>1$ (i.e. interest rates positive).

This proposition tells us that if both of the nodes to which the process can move in the next period have a value equal to their immediate exercise value, then the value of the option at its current node will equal its immediate exercise value.

Proposition 2: let $X-S^* = B_{jn}$ and

$$A_{j+1,n+1} > B_{j+1,n+1} = X-uS^* \quad (5a)$$
$$A_{j,n+1} > B_{j,n+1} = X-dS^* \quad (5b)$$

then

$$X-S^* < V_{j+1,n+1}\, p/q + V_{j,n+1}\,(1-p)/q \quad (5c)$$

Proof: see the appendix.

This proposition says that if both of the nodes to which the process can move in the next period have a value equal to their holding value, then the value of the option at its current node will equal its holding value.

Two corollaries follow from these propositions.

(a) (from propositions 1 and 2):

$$B_{j,n+1} > A_{j,n+1}$$

is a necessary condition for

$$B_{jn} > A_{jn} \geqslant 0$$

(b) (from proposition 1):

Let $S^* = u^j d^{n-j} S$. If $X - u^{N-n} S^* > 0$ then $B_{jn} > A_{jn}$

Proposition 3: Let $S^*=u^j d^{n-j} S$:

$$\text{if } X-d^{N-n}S^* \leq 0 \text{ then } V_{jn} = 0$$

Proof: $X-d^{N-n}S^* = X-u^j d^{N-j} S \leq 0$

so, $X-u^k d^{m-k} S \leq 0$ for all $k \geq j$

In other words, all nodes to which the process can move from <j,n> have holding and exercise values less than or equal to zero, hence V_{jn} is zero.

This proposition says that if, given the value of the share price at any node, there is no probability of the share price falling below the exercise price within the binomial process, then the option will have zero value at that node.

Proposition 4: if $S_{jn} < X$, $B_{jn} = 0$ and $V_{jn}=A_{jn}$

This is self evident. The proposition says that if, at any node, the option is out of the money, its value will equal its (possibly zero) holding value.

Partitioning the Binomial Cone

Define a to equal the maximum number of upward movements the share price can make and still finish (at N) in the money. Proposition 3 tells us that at all nodes, <k,n>, where k > a, the option will have zero value. This allows us to define a triangular area in the upper right hand corner of the binomial cone comprising zero valued nodes. In the example given in figure 1, the nodes in this area are indicated by zeros.

[figure 1 about here]

Proposition 1 and corollary (b) tell us that for all nodes <j,m>, m $\geqslant$ N-a, $B_{jm} > A_{jm}$. This allows us to define a triangular area in the bottom right hand corner of the cone in which all nodes will take their immediate exercise value. In figure 1 this comprises the nodes indicated with crosses and the first (i.e. lowest) diagonal values which lie on the boundary of this area. More significantly, proposition 1 also tells us that we need calculate only the values for nodes lying on this boundary ($B_{a,N}$, $B_{a-1,N-1}$, $B_{a-2,N-2}$ and so on) and not those within the boundary of this area.

The effect of defining these areas is that calculation of option values need only take place within the rectangular central area of the cone defined by the boundaries of these two areas and including the boundary of the second area. Within this central area we calculate the holding and exercise values of nodes by taking each diagonal (rather than each period) at a time. So the first diagonal is the boundary diagonal of area (2) for which we calculate holding values only. For the next diagonal (<a+1,N-1>, <a,N-2>, <a-1,N-3> and so on) we calculate both the holding and exercise values of each node. [3] However, in the next diagonal we need calculate only the holding value of the first node (<a+2,N-2>) (by proposition (4)) and the holding and exercise values of the remaining nodes. In the next diagonal we calculate holding values only for the first two nodes, and holding and exercise values for the remainder. As we progress to later diagonals, the number of nodes for which both sets of values must be calculated declines because, although the size of the diagonals remains constant (at N+1-a), the number of nodes which are out of the money increases by one each time. The value of the option at these nodes is a function of values which have already been computed.

This process can be carried on until we estimate the value of the option as the last node in the first (or final, taking the order of our computations) diagonal. In itself this reduces the required number of computations considerably when compared with the unmodified binomial method. However, proposition 2 and corollary (a) allow us to reduce further the number of computations required, insofar as they permit jumping over nodes in a manner comparable to that used in valuing calls. If, in any diagonal, the <0,n> node takes its holding value rather than its exercise value, then the values of that diagonal can be used to compute the value of the option directly. This is because, if $B_{0n} \leq A_{0n}$, then for all $m \leq n$, $B_{jm} \leq A_{jm}$. In practice, then, we compute the value of the nodes in successive diagonals, as described, halting the process at the first diagonal for which <0,n> takes its holding value. We then treat the values in that diagonal in the same way as we treat the terminal values in valuing a call option: that is, we multiply them by their probability of occurence and take their sum to yield the option's value.

Binomial Coefficients for Diagonal Values

If such jumping backwards from a set of diagonal values is possible, then to compute the value of the option we need to weight these values by the appropriate 'binomial coefficients' (i.e. the number of 'paths' linking the nodes of the diagonal to the <0,0> node) and the values of (1-p)/q and p/q. In this section we describe the computation of these coefficients.

The binomial coefficients linking <j,n> to <0,0> are given by

$$n!/(j!\ (n-j)!)$$

In the case of the nodes in a diagonal, this number will be reduced by the value of the binomial coefficient for the lower adjacent node, <j-1,n-1>, given by

$$\frac{(n-1)!}{(j-1)!(n-j)!}$$

This means that the adjusted binomial coefficient for each node in a diagonal will be given by

$$\frac{n!}{j!(n-j)!} - \frac{(n-1)!}{(j-1)!\ (n-j)!}$$

which is

$$\frac{n!}{j!\ (n-j)!}(1 - j/n) \qquad (6)$$

For each <j,n> in the diagonal the appropriate values of (1-p)/q and p/q are given by

$$(p/q)^{j}((1-p)/q)^{n-j} \qquad (7)$$

Accordingly, the option value is given by weighting the values of the first diagonal in the backward stepping procedure for which $V_{0n} = A_{0n}$ by (6) and (7) and summing.[4]

An Example

Figure 1 shows a ten period binomial model valuing the put option with parameters

$$X=45,\ S=40,t=7/12,\sigma=.4,r=1.05$$

This is one of the values in the set of examples given by Cox and Rubinstein (1985, p.248) and for which the value after 150 periods is $7.39. The ten period binomial values the option at $7.48. The unbroken line in figure 1 separates nodes at which the option is in the money (below the line) from those where it is at or out of the money. The figures shown are the value of the option at each node, and the dashed line shows the early exercise boundary, so that nodes to the right of the line take as their value their immediate exercise value while those to the left take their holding value.

As can be seen, nodes in four diagonals of length 6 were valued. In the final (leftmost) diagonal node <0,2> took its holding value, allowing us to jump from this diagonal to the option's value. The calculations involved are set out in Table 1. Thus there was no need to value the nodes in the two remaining diagonals. In this example the total number of computations of holding and exercise values totalled 39. This compares with 121 which are required in the unmodified binomial (the general formula for this latter being that the number of computations = $(N+1)^2$). Clearly in a model with larger N the effect of such jumping would be much greater. The advantage of the modified binomial is only marginally reduced by the need to calculate the coefficients in the weighting of the final estimated diagonal since this is a computationally trivial task.

[tables 1 and 2 about here]

Table 2 shows the values of a sample of American put options on non-dividend paying stock arrived at by this method, together with some indication of the computational requirements, vis-a-vis the ordinary binomial. Here we have applied the method using 150 time periods. The computational savings are greatest for out of the money, low volatility, short maturity options, but in all cases the computational requirements are more than halved when compared with the unadjusted binomial method.

Conclusion

Because of the risk neutrality assumption the value of an option can be written as a function of its (rational) exercise values. The method presented here increases the efficiency of the binomial process by initially reducing the area in which to search for the early exercise opportunities. Furthermore, although we cannot know in advance the location of the exercise nodes which will determine the value of the option, we have shown that it is possible, using the binomial method, to determine at what point we have located them all.

It is this feature which allows us to jump backwards over redundant nodes of the process.

The backwards jumping feature of the method presented here is possible because of the convergence of the value of successive diagonals to the estimated put value. In other words, the discounted weighted value of the nodes in each diagonal form a sequence which converges uniformly to the estimated value of the put. The same is true of the sequence of values of the option at each period in the model (see Breen 1988 for the proof). If we define

$$P_a(S) = \sum_{j}^{N-a} \binom{N-a}{j} (p/q)^j \; ((1-p)/q)^{N-a-j} \; V_{j,N-a} \quad (8)$$

where V is defined as earlier, then for any binomial model with N periods, $P_a(S)$ defines a sequence of functions which converges to $P_N(S)$. Convergence occurs at $P_m(S)$, where m is the earliest period in the model for which $V_{\emptyset m}$ takes its immediate exercise value rather than its holding value. m will always be less than N unless the put should be exercised immediately. Thus the backward jumping process can be incorporated into the usual binomial model which values nodes period by period. [5] In addition, however, the convergence properties of (8) are the basis on which Breen (1988) has shown that it is possible to approximate the Geske and Johnson (1984) compound option model using the binomial. His method, termed the accelerated binomial, uses convergence acceleration techniques in conjunction with the binomial option pricing model. Thus, for large N, this will give a very accurate value of the put and reduces the computational requirements, when compared with the unmodified binomial, quite drastically. Typically, for example, the 150 period accelerated binomial would require the calculation of only 610 node values in a three point Richardson extrapolation.

The present method, however, is useful in two particular sets of circumstance - first in the case of options written on dividend paying stock; and, second, in situations where American options have very long lives - say over a year. The quadratic approximation of Macmillan (1987) cannot price options on dividend paying assets (except where the dividend yield is conceptualized as a constant stream of payments) while the two methods which utilise convergence acceleration techniques can do so only at the loss of some efficiency. The enhanced binomial presented here can handle constant proportionate dividends without difficulty. Where options are long lived the Geske-Johnson (1984) model and the Macmillan (1986) quadratic approximation return inaccurate results (see Barone-Adesi and Whaley 1987) as does the accelerated binomial (Breen 1988). This suggests that finite difference or binomial methods must be used in such cases. Hence the modified binomial presented here is particularly useful in such cases, since it is not an approximation to the binomial and does not break down with large values of T. Evidence of this is given in panel B of Table 2 where, for options with three years' maturity, the value returned by the present method is virtually identical to that arrived at using finite difference techniques. We might add also that the modified binomial, as well as handling dividends in the same manner as the ordinary binomial, can also value options on commodites, foreign exchange, futures, and so forth, simply by replacing q in the calculation of p by the one period cost of carry [6].

FOOTNOTES

1. Based on Merton's (1973) proof that such options have a positive probability of early exercise at every instant during their life.

2. By which we mean the probability within the binomial model. This corresponds to a real world probability only if investors are risk neutral.

3. Strictly speaking the routine outlined in the text applies only if $X-u^{a}d^{N-1-a}S > 0$ - in other words, if the set of terminal values of S does not include the largest value of S for which the option is still in the money. If this is so, then the largest such value will appear in the penultimate period of the model. If, on the other, this value is included among the terminal set, then, after calculating the value of the nodes in the boundary diagonal, we need only calculate the holding value of the first node in the next diagonal, rather than its holding and exercise values. In the next diagonal we need calculate only the holding values of the first two nodes - and so on.

4. Note that $(1-p)^{n-j}/q$ will be constant since for all nodes in the diagonal n-j is constant.

5. Diagonals are used in the method presented here only because they minimize the required number of computations given the existence of the two triangular areas of the binomial cone in which calculation of node values is redundant.

6. See Barone-Adesi and Whaley 1987 for a discussion of the value of the cost-of-carry parameter for different kinds of option.

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Appendix: Proof of Proposition Two

We wish to prove that if

$$V_{j+1,\ n+1} = A_{j+1,n+1} > X - uS^* \qquad \text{(a1)}$$

and

$$V_{j,n+1} = A_{j,n+1} > X - dS^* \qquad \text{(a2)}$$

then

$$X - S^* < A_{j+1,n+1}\ p/q + A_{j,n+1}\ (1-p)/q \qquad \text{(a3)}$$

For convenience we write $A_{j+1,n+1}$ as A and $A_{j,n+1}$ as A′.

(a1) implies that

$$A > (X - u^2S^*)\ p/q + (X-S^*)\ (1-p)/q \qquad \text{(a4)}$$

and likewise (a2) implies

$$A' > (X-S^*)\ p/q + (X - d^2S^*)\ (1-p)/q \qquad \text{(a5)}$$

Rearranging and summing yields

$$(A - (X-u^2S^*)\ p/q)\ \frac{q}{(1-p)} + \left(A' - (X-d^2S^*)\ \frac{(1-p)}{q}\right)\frac{q}{p} > 2(X-S^*) \qquad \text{(a6)}$$

To prove (a3) we show that

$$2\left(A\frac{p}{q} + A'\ \frac{(1-p)}{q}\right) > \left(A - (X-u^2S^*)\ \frac{p}{q}\right)\frac{q}{(1-p)} + \left(A' - (X-d^2S^*)\ \frac{(1-p)}{q}\right)\frac{q}{p} \qquad \text{(a7)}$$

Multiplying through by $\frac{(1-p)}{q}\ \left(\frac{p}{q}\right)$ and rearranging yields

$$2\frac{(1-p)}{q}\ \frac{p}{q}\left(A\frac{p}{q} + A'\ \frac{(1-p)}{q}\right) - A\frac{p}{q} - A'\ \frac{(1-p)}{q} > -(X-u^2S^*)\left(\frac{p}{q}\right)^2 - (X-d^2S^*)\left(\frac{1-p}{q}\right)^2$$

$$A\frac{p}{q} + A'\ \frac{1-p}{q} > -\ \frac{(X - u^2S^*)\ (p/q)^2 - (X - d^2S^*)\ ((1-p)/q)^2}{2\left(\frac{(1-p)}{q}\ \frac{p}{q}\right) - 1} \qquad \text{(a8)}$$

By (a4) and (a5), (a8) holds if

$$\left((X-u^2S^*)\frac{p}{q} + (X-S^*)\frac{(1-p)}{q}\right)\frac{p}{q} + \left((X-d^2S^*)\frac{(1-p)}{q} + (X-S^*)\frac{p}{q}\right)\frac{(1-p)}{q}$$

$$\geq \frac{-1}{2\left(\frac{(1-p)}{q}\right)\frac{p}{q} - 1}\left((X-u^2S^*)\left(\frac{p}{q}\right)^2 + (X-d^2S^*)\left(\frac{1-p}{q}\right)^2\right) \qquad \text{(a9)}$$

Rearranging:

$$2(X-S^*)\,\frac{(1-p)}{q}\frac{p}{q} \geq \left(\frac{-1}{2\frac{(1-p)}{q}\frac{p}{q} - 1} - 1\right)\left((X-u^2S^*)\left(\frac{p}{q}\right)^2 + (X-d^2S^*)\left(\frac{1-p}{q}\right)^2\right) \qquad \text{(a10)}$$

$$\frac{2\frac{(1-p)}{q}\frac{p}{q} - 1}{-2\frac{(1-p)}{q}\frac{p}{q}}\left(2(X-S^*)\frac{(1-p)}{q}\frac{p}{q}\right) \geq (X-u^2S^*)\left(\frac{p}{q}\right)^2 + (X-d^2S^*)\left(\frac{1-p}{q}\right)^2$$

$$= X-S^* \geq (X-u^2S^*)\left(\frac{p}{q}\right)^2 + (X-d^2S^*)\left(\frac{1-p}{q}\right)^2 + 2(X-S^*)\left(\frac{1-p}{q}\right)^2\left(\frac{p}{q}\right)^2 \qquad \text{(a11)}$$

We deduce from Proposition 1 that (a11) holds, so proving (a9) which proves (a7) and thus (a3).

Q.E.D.

TABLES

Table 1 *Calculation of option value from final estimated diagonal*

a Diagonal Values	b 'Binomial Coefficients'	c $(p/q)^j$	d $((1-p)/q)^{n-j}$	Total (=a*b*c*d)
12.27	1	1	.258	3.17
9.44	2	.489	.258	2.38
6.70	3	.239	.258	1.24
4.19	4	.117	.258	0.51
2.10	5	.057	.258	0.16
0.66	6	.028	.258	0.03

Option Value = sum of totals = 7.48

p/q = .489

(1-p)/q = .508

Table 2 *Values and Computational Requirements of American Put Options Using Enhanced Binomial (N=150)*

A. S = 40, r = 1.05

	X	*P*	*Nodes Valued*	*Nodes Valued as percentage of number value in unmodified binomial*
s = .2	30	.00	2340	10.13
T = 0833	35	0.01	4483	19.66
	40	0.85	7029	30.83
	45	5.00	9223	40.45
	50	10.00	9069	39.77
s = .3	30	0.10	5340	23.42
T = .333	35	0.70	6308	27.67
	40	2.48	7239	31.75
	45	5.71	8109	35.56
	50	10.05	8894	39.01
s = .4	30	0.82	6112	26.81
T = .5833	35	2.16	6709	29.42
	40	4.35	7239	31.75
	45	7.39	7764	34.05
	50	11.12	8207	35.99

B. X = 100, s = .2, r = 1.0833, R = 3.00

S	*P (Finite Difference)*	*P (Binomial)*	*Nodes Valued*	*Nodes Valued as percentage of nodes value in unmodified binomial*
80	20.00	20.00	8899	39.03
90	11.69	11.68	8609	37.76
100	6.93	6.92	8160	35.79
110	4.15	4.15	7860	34.47
120	2.51	2.51	7431	32.59

Finite Difference Values from Barone-Adesi and Whaley 1987, Table V p. 317.

FIGURES

Figure 1: Node Values for example of Binomial Process given in text