

**Complexity of Holographic Flavours
and
causality in QFTs with
Gauss-Bonnet dual**

Doctorate in Philosophy (Mathematics)

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14, 02, 2019

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The following thesis is the result of my 3 years Doctorate Programme. It is composed of two separate projects: an already published one [1] (Chapter 2) and a still ongoing project (Chapter 3)

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Summary

This thesis is the compilation of two different projects undertaken during my PhD programme.

Chapter 2 covers the work on quantum complexity. Quantum complexity of a thermofield double state in a strongly coupled quantum field theory has been argued to be holographically related to the action evaluated on the Wheeler-DeWitt patch [21]. The growth rate of quantum complexity in systems dual to Einstein-Hilbert gravity saturates a bound which follows from the Heisenberg uncertainty principle. This work, published in [1], considers corrections to the growth rate in models with flavor degrees of freedom. These are realized by adding a small number of flavor branes to the system. Holographically, such corrections come from the DBI action of the flavor branes evaluated on the Wheeler-DeWitt patch. After relating corrections to the growth of quantum complexity to corrections to the mass of the system, it is observed that the bound on the growth rate is never violated.

Chapter 3 covers the still ongoing project of causality in RG flows of systems with a Gauss-Bonnet gravity holographic dual. In order for the dual field theory to have no causality problems the speed of gravitons near the boundary of AdS must be bounded above by the speed of light. This bound is checked along the RG flow for QFTs that have AdS Gauss-Bonnet spacetime duals. It is found that, for certain values of the Gauss-Bonnet parameter, the field theory becomes acausal when sufficiently far away from the UV.

Acknowledgements

I would like to start thanking my supervisor, Andrei Parnachev, and Manuela Kulaxizi for all these years of enlightening discussions and the great atmosphere around them. I will also surely miss the weekly football games (not like we ever won any, but still).

Secondly, thanks to my fellow PhD students, professors and literally everyone in the department for making it such an enjoyable place to work at. Special thanks to Emma, Helen and Karen, who manage to make all bureaucratic procedures incredibly smooth for all of us.

Next is to my friends, who came visit me many times^① and have always kept in touch, making it feel like I never left them behind when I moved to Ireland. Special shout-out goes to Gagan Vasisht, with whom I shared my first house and made me feel like I was still at home.

And lastly, to my family, for being so supportive during all these years.

Oh, and also to Ireland, for being so cloudy, rainy and windy that it feels like I never left Asturias.

^① Ironically, it was because some of them that I got to know several places in Dublin right next to my door which I never even knew existed.

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Chapter 1

Introduction

1.1 The holographic principle

The holographic principle (also known as gauge/gravity duality) is one of the most powerful tools in the modern theoretical physics arsenal. First introduced by Maldacena in his now famous^① paper [2], it relates gravity and gauge theories. This correspondence originates from a low energy limit of string theory and can be heuristically understood by doing some analysis of D-branes physics.

String theory doesn't only contain strings, but also extended objects called D-branes. A D_p -brane is a p -dimensional membrane-like object where the end points of open strings are forced to be attached. As will be elaborated on later, a D_p -brane spans a $(p + 1)$ -worldvolume, so it is useful to consider D3-branes in order to end up with a 4-dimensional theory. Consider then a stack of N of these D3-branes in type IIB string theory in 10 dimensions. In the low energy limit we can study this from the point of view of open strings living on the D3 branes, or from the point of view of closed strings, where a stack of D3 branes is a background solution in type IIB supergravity. In both cases there are two decoupled systems:

A - strings point of view: D-branes are the places where open string endpoints are forced to end. Now, if two open strings lie on a D-brane, their endpoints can meet, forming a closed string. This is no longer confined to the D-brane (since it has no endpoints) and is free to move in the bulk of spacetime. This system is then described by an action of

^① so famous it is currently the most cited paper in Physics history, with almost 14.000 citations by September 2018.

the type $S = S_{bulk} + S_{brane} + S_{int}$, where S_{bulk} describes the closed string that propagate through the bulk, S_{brane} the open strings stretching between the branes and S_{int} the interaction terms between these two. In the low energy limit S_{int} becomes negligible, whereas the now decoupled open and closed systems become.

- **A1:** open strings on the D3-branes. The key idea here is that the lowest states in the spectrum of strings stretching between two of these branes consist of $N^2 - 1$ interacting massless gauge fields. In the low energy limit then, when these modes are the only ones that can be excited, the stack of N D3-branes has a $SU(N)$ gauge theory living in its worldvolume (so $d = 4$ spacetime dimensions in this case). Moreover, the original string theory contained 32 supersymmetry generators, but since the D-branes are BPS objects we are left with just half the supersymmetry. All together, S_{brane} reduces to the SYM lagrangian and so the open strings on the stack of D3-branes in the low energy limit are equivalent to $SU(N)$ $\mathcal{N} = 4$ SYM in 4 dimensions.
- **A2:** closed strings in the bulk. S_{bulk} is the action of 10d supergravity and as such it reduces to purely free gravity when the low energy limit is taken.

B - supergravity point of view: consider now the supergravity approach, where we have a p-brane described by

$$ds^2 = H^{-1/2} dx_{//}^2 + H^{1/2} (dr^2 + r^2 d\Omega_5^2), H(r) = 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g_s N \alpha'^2, \quad (1.1)$$

where $x_{//}$ denotes the coordinates parallel to the brane. Note that there is a redshift effect taking place here: the energy E_r measured by an observer at some point r is seen by an observer at infinity as $E \propto \sqrt{-g_{00}} E_r$. When close to the brane $r \rightarrow 0$, so the energy measured at infinity $E \propto r E_p$ approaches zero. In the low energy limit of this configuration, from the point of view of an observer at infinity, we can have two different types of excitations

- **B1:** particles propagating close to the brane at $r \ll R$. In this region one can approximate $H \sim \frac{R^4}{r^4}$, and so the metric (1.1) becomes that of $AdS_5 \times S^5$.
- **B2:** far away from the brane at $r = 0$. In this regime gravity becomes free.

In the low energy limit ($\alpha' \rightarrow 0$, with $\frac{\alpha'}{r^2}$ kept fixed)^② these systems are also decoupled. This can be seen from the fact that an excitation near the brane cannot overcome the gravitational potential and escape to infinity, whereas the low energy absorption cross section behaves like $\sigma \sim \omega^3 \mathcal{R}^8$, which implies that excitations far away from the brane cannot interact with the near brane region.

We thus find the remarkable fact that systems A2 and B2 are the same! It is then reasonable to conclude that $A1 \sim B1$. Since the $r \rightarrow 0$ limit of the metric (1.1) is $AdS_5 \times S^5$ we find that $\mathcal{N} = 4$ SYM in 4d is equivalent to weakly coupled gravity in $AdS_5 \times S^5$.

Now, keep in mind that for the above to be accurate the supergravity limit of string theory must be valid. This is the case when the curvature l of spacetime is much bigger than the string length $l_s = \sqrt{\alpha'}$, i.e $l/l_s = (g_s N)^{1/4} \gg 1$. So we need $g_s N \gg 1$. Also, we need quantum corrections to string theory to be negligible, so we must require that the string coupling goes to zero, $g_s \rightarrow 0$. For these two conditions to be met we must have

$$N \rightarrow \infty, \quad \text{and} \quad g_s \rightarrow 0 \quad \text{while keeping} \quad \lambda = g_s N \quad \text{fixed and large.} \quad (1.2)$$

This duality receives stringy corrections of order $1/\sqrt{\lambda}$. However, since λ is kept large one usually does not keep these corrections. From the Super Yang-Mills point of view, this amounts to working at strong coupling. The coupling constant in the Yang-Mills theory is related to the string coupling through $g_{YM}^2 = g_s$. This can be understood from the fact that *two* open strings, whose interaction is controlled by g_{YM} , can merge to form *one* closed string, which is governed by g_s . The interesting point to remark here is that the effective coupling in the SYM theory is given precisely by λ . Since $\lambda \gg 1$ the SYM is strongly coupled, while the gravity theory is free ($g_s \rightarrow 0$). This finally gives us the famous relation

$$\mathcal{N} = 4 \text{ planar strongly coupled SYM in 4d is equivalent to weakly coupled gravity in } AdS_5 \times S^5.$$

The fact that the two elements of this correspondence live in two opposite regimes is one of the reasons why AdS/CFT is so useful (and, in fact, also the reason why it is called

② On the one hand, taking α' to zero decouples the open and closed string sectors we have been talking about. On the other hand, keeping the ratio $\frac{\alpha'}{r^2}$ fixed allows for arbitrarily excited states to exist in the near horizon region of the geometry.

a duality).

The AdS/CFT correspondence is encoded in a dictionary that relates observables in the CFT to observables in the gravity side. One of the most commonly discussed quantities are n-point functions, which can be computed using the holographic statement that

$$Z_{CFT} = Z_{grav}, \quad (1.3)$$

where Z above stands for the field theory and gravitational (AdS) partition functions respectively. In the large-N limit one can use the saddle-point approximation of the gravitational action to write this as

$$\left\langle e^{i \int \phi^{(0)} \mathcal{O}} \right\rangle = e^{i S_{class}(\phi|_{u=0}=\phi^{(0)})}. \quad (1.4)$$

This equation is usually referred to as the Witten/GKP formula. In the field theory side (LHS) \mathcal{O} is an operator and $\phi^{(0)}$ acts as a source, making the LHS of this equation the usual generating functional for the operator. In the RHS ϕ is a bulk field which asymptotes to $\phi^{(0)}$ at the AdS boundary (which sits at $u = 0$). The action in the RHS is just the classical gravitational action, and since it is on-shell it reduces to a boundary integral (thus becoming 4D). It is this boundary action what can be used as the generating functional of \mathcal{O} instead of the LHS of (1.4) to compute n-point functions in a much simpler way.

One of the major breakthroughs in building up this connection came up when Ryu and Takayanagi were able to relate entanglement entropy of a region in the CFT with the area of a given surface in AdS [3]. Why was this so important? Entanglement entropy is a field theoretical quantity with a simple definition: "consider a quantum system described by a density matrix ρ and split it into two subsystems A and B. The entanglement entropy S_A of the region A, with boundary ∂A , with the rest of the system is defined by

$$S_A = Tr \rho_A \log \rho_A, \quad (1.5)$$

where $\rho_A = Tr_B \rho$ is the reduced density matrix of subsystem A". The definition is quite simple^③ but, even with the help of very clever tools like the replica trick, computing

③ This splitting of the system into two spatial regions A and B is indeed simple when one is dealing with a theory whose physical degrees of freedom are localized. This is not the case when one deals with gauge theories though, and this splitting of the Hilbert space is much more complicated (check for example reference [4]).

S_A can be quite a challenging task even for simple subsystems. On the other hand, the holographic principle allows us to find S_A by finding the minimal area surface in AdS whose boundary is precisely ∂A . This method is much simpler (although not trivial by any means), which shows one of the great advantages holography has brought forward: we can now compute very complicated quantities in a CFT (gravity) by studying their simpler gravity (CFT) counterparts.

Entanglement entropy is a good example of how AdS/CFT has made possible to simplify some of the hardest computations one encounters in theoretical physics. Chapter 2 in this thesis shows an example that goes beyond that: holography made it possible to have (some) access to observables whose CFT definition is still not even fully understood. Quantum computational complexity is a quantity with a very clear definition for systems of qbits, but how to generalize it to the continuum QFT case is a task that has not been fully achieved yet. Some remarkable progress has been recently achieved though [5]-[14] (to be elaborated on in this chapter). Through a new proposed entry in the holographic dictionary [21] we now have a tool that allows us to study this concept. The fact that progress in our understanding of computational complexity is much faster now than it would have ever been without the gauge/gravity duality showcases why this correspondence has been at the core of the most important discoveries in the recent history of Physics.

Holography is not only the means by which hard computations can be simplified. More importantly, it is also a tool that has completely changed the way problems are tackled, providing us with a novel and fundamental understanding of a great variety of systems.

1.2 Classical and quantum computational complexity

Computational complexity is a very important concept in computer science. In classical computation theory, the complexity of a given problem/task is defined as the time it takes for the fastest possible algorithm to compute it. Problems are then classified according to how their complexity scales when the size of the input is increased. In this context, size refers to bit size i.e. if the input is some number X , then its size is defined as the number of bits $Z = \log_2 X$ required to store its value.

Although the concept of complexity is, naively, quite simple there are still character-

istics that are not completely understood. The most famous one is probably the **P** vs **NP** problem (which is actually one of the Millennium Problems). On the one hand, tasks whose complexity scales with some polynomial of Z are called of type **P** (from polynomial). On the other hand, tasks whose solutions can be checked to be correct in polynomial time are called **NP**. It is clear that $\mathbf{P} \subseteq \mathbf{NP}$, but whether **P** is a subset or actually equal to **NP** is still unclear. This is the so-called **P** vs **NP** problem.

Note that, if **P** is not equal to **NP**, then there exists a class of problems for which finding a solution takes a non-polynomial time (which, in practice, makes them unsolvable) but which solutions are verifiable in polynomial time. The factorization of a big number into its prime factors seems to be one of the problems belonging to this class: no classical algorithm has been found yet that solves this in polynomial time.

It is for this reason that factorization plays a key role in the most widely used encryption algorithm used up to date: the RSA encryption algorithm. The underlying principle of RSA is that Alice can pick two large prime numbers p and q and construct their product $n = pq$ and the totient function $\phi(n) = (p - 1)(q - 1)$ of this product^④. One then needs to find a number e fulfilling both that: $a) 1 < e < \phi(n)$ and $b) e$ is coprime with both n and $\phi(n)$. Once e is chosen, the last step is to compute d , the multiplicative inverse of $e \bmod \phi(n)$. This means to solve

$$ed = 1 \pmod{\phi(n)}$$

for d . After this last step is completed, Alice chooses the pair $P = (e, n)$ as the public key (anyone can know it), whereas the pair $S = (d, n)$ is kept secret. If Bob wants to send a message M to Alice he computes $M' \equiv M^e \pmod{n}$ and sends that instead^⑤. Alice can recover the original message by simply computing $(M')^d \pmod{n}$, which equals precisely $M \pmod{n}$. The key to this algorithm is that, in order to break it, an eavesdropper Eve would need to find the number d , for which she first needs to factor the number n of the public key. Since the factorization problem is not solvable in polynomial time this

④ The Euler totient function $\phi(n)$ counts the number of integers smaller than n that are coprime with n .

⑤ With this method there only up to n different messages that can be codified, so it would look like this is not a very realistic method to implement in real life scenarios. On a daily basis, almost all secured connections are encrypted with a symmetric key, RSA being used to codify the symmetric key sharing process that needs to be done during the first interaction between Alice and Bob.

cryptographic system is then *expected* to be unbreakable in practice. Note that this is just an expectation though. Since the **P** vs **NP** conjecture has not been solved there is no guarantee that no algorithm faster than any one we have found so far cannot exist.

A simple example of RSA works is useful to illustrate how it works. Let's pick the prime numbers $p = 2$ and $q = 7$. This yields $n = 14$, $\phi(n) = 6$ and $1, 3, 5, 9, 11, 13$ as the 6 integers that are coprime with n . To find e we need a number from that list which is smaller than and coprime with $\phi(n) = 6$. In this simple case, the only choice is to pick $e = 5$. The last step then is to find a d satisfying

$$5d = 1 \pmod{6}. \quad (1.6)$$

A solution to the equation is $d = 11$. This means that we end up a public key pair $P = (5, 14)$ and a private key pair $S = (11, 14)$. If Bob now wants to send Alice the message "B", he can convert that letter in the integer $M = 2$ by using the A1Z28 coding and then computing

$$M' = M^e \pmod{n} \rightarrow M' = 2^5 \pmod{14} = 4 \pmod{14},$$

so Bob would be sending the letter D (the one corresponding to 4 in the A1Z28 coding) to Alice. She would then decrypt the message by computing

$$M'' = M'^d \pmod{n} \rightarrow M'' = 4^{11} \pmod{14} = 2 \pmod{14},$$

getting back the original message B .

But, besides this elusive classical algorithm, there is another way of theoretically solving the factorization problem in polynomial time: quantum computation. This prompts the question of how is complexity defined in a quantum system. If the discussion is restricted to qubit systems (which would be the ones appearing in a quantum computer) then the definition is quite simple

- **states:** given a state $|\phi\rangle$ and a reference state $|0\rangle$, the complexity of $|\phi\rangle$ is defined as the minimum number of 2-gates that one needs to apply to obtain $|\phi\rangle$ from $|0\rangle$
- **operators:** given an operator \mathcal{O} , its complexity is defined as the minimum number of gates one needs to apply in order to implement the action of \mathcal{O} .

Note that these definitions are not unique since there is some arbitrariness in the choices one can make. First of all, one can choose the reference *vacuum* state in many different ways (although there may be some physically motivated candidates which are more suitable for this role). Second, the set of gates that we can choose is also arbitrary. It is well known that any unitary operator acting on a set of qubits can be approximated with arbitrary accuracy using only the set of gates $\{H,S,T, \text{CNOT}\}$, where

- H is the **Hadamard** gate: single qubit gate defined by $|0\rangle \rightarrow |0\rangle + |1\rangle$ and $|1\rangle \rightarrow |0\rangle - |1\rangle$
- S is the **Phase** gate: single qubit gate defined by $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow i|1\rangle$
- T is $\pi/8$ gate: single qubit gate defined by $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow e^{i\pi/4}|1\rangle$
- CNOT is the **Controlled not** gate: 2-gate that performs a NOT operation (i.e. takes $|0\rangle$ to $|1\rangle$ and viceversa) on the second qubit if the first one is in the state $|1\rangle$, and does nothing if the first qubit is in the state $|0\rangle$.

This is the natural set of gates to consider since every single unitary can be built up using them^⑥, but that doesn't mean that any unitary can be efficiently implemented. The reason why these gates form a universal set is because of two results: a) any arbitrary unitary operation on qubits can be built using single qubit and CNOT gates and, b) any single qubit operation can be built using the Hadarmard and $\pi/8$ gates. Thus, these 3 gates combined allow us to build any unitary operator on the qubits.

Another set of universal gates could be chosen that reduces the size of the circuit needed to implement a particular operator U , thus modifying its complexity. Even so, to know the quantum complexity of a given operator is crucial for practical purposes, since it can make a quantum algorithm relying on it feasible or impracticable.

Another thing worth mentioning is the fact that while in classical computer each operation performed on the classical bits moves the system from a state to an orthogonal one (since the states "0" and "1" of the bit are orthogonal and a state of N classical bits is just a product state) this is no longer the case for a quantum computer. Given N qubits on a particular state, applying a gate to a pair of them will not generally result (and most likely won't) in a new state that is orthogonal to the initial one. This then implies that

^⑥ The phase gate is actually not necessary since it can be built using two T gates. The reason it is included in the list is simply because of the predominant role it plays in error correction constructions.

it is not too clear how efficiently the computational bounds discussed in Chapter 2 will apply to a quantum computer.

If one moves outside the scope of qubits into the case of a continuum QFT, how to define the complexity of a state/operator is not yet very clear. There are however some promising results. The first attempt to systematically define this concept in field theories was introduced by Nielsen and collaborators in [5]-[7]. The idea presented therein has been further developed by Myers and collaborators in [8],[14]. The approach here is to first introduce an ultraviolet regulator ^⑦. One then defines a reference state $|\psi_0\rangle$ and considers a target state $|\psi_F\rangle$ which is obtained by the application of some unitary operator U to the reference state: $|\psi_F\rangle = U |\psi_0\rangle$. The idea it is then to find a time dependent Hamiltonian that determines U through

$$U = \overleftarrow{P} e^{\int_0^1 dt H(t)}, \quad H(t) = \sum_n Y^n(t) M_n, \quad (1.7)$$

where M_n is the set of gates (operators) we choose as basis to write down the Hamiltonian and \overleftarrow{P} is a time ordering operator that ensures that the operators that apply at earlier times sit in the rightmost side of the product. The functions Y_n are called the control functions and form a $(4^n - 1)$ -dimensional vector space, where we can see

$$U(t) = \overleftarrow{P} e^{\int_0^t ds H(s)}, \quad H(t) = \sum_n Y^n(t) M_n, \quad (1.8)$$

as the tangent vector to a given trajectory in the space of unitaries. The objective is then to introduce a cost functional^⑧

$$D[U] = \int_0^1 dt F(U, \dot{U}), \quad (1.9)$$

and then find out which circuit path $U(t)$ minimises it. The length of this minimal path would then be equated to the complexity of the state $|\psi_F\rangle$ obtained using U . The geometrical approach to computational complexity is still in its very early stages, but it has already shown some promising agreement [8] with the holographic proposals of Susskind et al. [21].

^⑦ It is believed no consistent definition of quantum computational complexity can be obtained without this short-scale cut-offs [55].

^⑧ subject to some physical requirements, like positivity and homogeneity.

1.3 Renormalization group in field theories

The renormalization group is the framework used in field theories to study how observables change when the energy/distance scale is modified. This idea was first proposed by Gell-Mann et al. in [107] and later further developed by Wilson in [108], [109].

The principle underlying the RG is closely related to the averaging of degrees of freedom we do in thermodynamics. Assume, for example, that we have a well defined and understood theory in the ultraviolet. This means that we know, among other things, the coupling constants and interactions at very high energies μ_{UV} (or very small distances). However, if we are not concerned with such small scales but rather want to understand some macroscopic behaviour we probably are not too interested in the UV coupling constants but in their lower energy versions. These are obtained by coarse graining our system and averaging the degrees of freedom that are not accessible at the new scale μ_{IR} we are interested in. In more concrete terms, if one has an effective theory defined up to some energy scale Λ_1 , we can obtain the lower energy version, valid up to energy scale $\Lambda_2 < \Lambda_1$, by integrating out the degrees of freedom in the region $[\Lambda_1, \Lambda_2]$.

The RG information of a field theory is encoded in its beta function

$$\beta(g) \equiv \frac{\partial g}{\partial \log(\mu)}, \quad (1.10)$$

which expresses how the coupling constant g changes when the energy scale μ is modified. Note that, in spite of its name, the coupling constant is no longer constant but it runs with the energy scale. Fixed points of the RG flow are the configurations with a vanishing beta function and are characterised by the underlying field theory being scale invariant.

Conformal field theories are theories invariant under the conformal symmetry group, which involves the usual Poincare group plus dilatations and special conformal transformations. It can be the case though, that a classically conformal theory doesn't have the expected quantum behaviour. Since a CFT is invariant under re-scaling of the metric one expects that the trace of the energy-momentum tensor $T_{\mu\nu}$ vanishes. However, this is not the case if the quantum theory is put in a curved background. For example in $D = 4$, which is the relevant case in this thesis, one finds that this quantity is proportional to the Weyl tensor and the Euler density

$$\langle T \rangle \propto c C^2 + a E_4, \quad (1.11)$$

where E_4 is the Euler density in 4 dimensions and C the Weyl tensor, which is generally defined as

$$C_{\mu\nu\rho\sigma} = \mathcal{R}_{\mu\nu\rho\sigma} - \frac{2}{d-2} \left(g_{a[c} \mathcal{R}_{d]b} - g_{b[c} \mathcal{R}_{d]a} \right) + \frac{2}{(d-1)(d-2)} \mathcal{R} g_{a[c} g_{d]b}. \quad (1.12)$$

The quantities a and c are the conformal anomaly coefficients, which are important and related to the RG flow. The c -theorem [111],[112] states that, in 2D, the quantity c monotonically decreases along the RG flow (so that $c_{IR} \leq c_{UV}$). In 4D, a similar theorem for the a coefficient (called the a-theorem) also exists [110]. These two quantities are also relevant to determining the causality properties of a given theory, as studied in [118], where they showed that the value a/c has to be within a specific window for the energy correlators to behave in the proper way.

Chapter 2

Complexity of holographic flavours

Quantum complexity \mathcal{C} is a quantity defined for a quantum system, where unitary operations, called gates, are applied to pairs of qubits^①. For a state $|\psi\rangle$ complexity is defined as the minimum number of such gates that have to be applied to a simple reference state to produce $|\psi\rangle$. It has been argued in [16], due to the Heisenberg uncertainty principle, and in [17], on general quantum dynamics grounds, that quantum complexity obeys a bound on its growth rate:

$$\frac{d\mathcal{C}}{dt} \leq \frac{2\mathcal{M}}{\pi}, \quad (2.1)$$

where \mathcal{M} is the mass of the system (See references [18]-[21] for some violations of this bound^②). The argument in [17] goes as follows: any quantum mechanical state can be written as $|\psi_0\rangle = \sum_n a_n |E_n\rangle$, where $|E_n\rangle$ are energy eigenstates. If this state is let to evolve during some time t it will become $|\psi_t\rangle = \sum_n a_n e^{-iE_n t/\hbar} |E_n\rangle$. One can then compute how much time t will at least take for $|\psi_t\rangle$ to become orthogonal to the starting state by looking at their overlap $S(t) = \langle\psi_0|\psi_t\rangle$. Using the inequality $\cos x \geq 1 - \frac{2}{\pi}(x + \sin x)$ (valid for $x > 0$) one finds out that the overlap

$$S(t) = \sum_n |a_n|^2 e^{-iE_n t/\hbar}, \quad (2.2)$$

satisfies

$$\begin{aligned} \operatorname{Re}(S) &= \sum_n |a_n|^2 \cos\left(\frac{E_n t}{\hbar}\right) \geq \sum_n |a_n|^2 \left[1 - \frac{2}{\pi} \left(\frac{E_n t}{\hbar} + \sin\left(\frac{E_n t}{\hbar}\right)\right)\right] \\ &= 1 - \frac{2E}{\pi\hbar} t + \frac{2}{\pi} \operatorname{Im}(S). \end{aligned} \quad (2.3)$$

① We will consider the case of two-gates, but one can easily generalize the discussion to the k-gates.

② This violations happen before the late time limit is approached. In the references it is shown that the rate at which complexity changes becomes constant in the late time limit (this is the rate shown in the equation above), but this is approached from above, thus violating the bound.

Since we want both real and imaginary parts of the overlap S to be zero this then yields $0 \geq 1 - \frac{2E}{\pi\hbar}$, which results in the time to orthogonality fulfilling

$$t \geq \frac{\hbar}{4E}. \quad (2.4)$$

In other words, since each operation in a computer requires the bits to change from one state to another, its computational speed is limited by the total energy E in the system (expressed as the mass \mathcal{M} of the system, in equation (2.1)).

Recently a holographic recipe has been proposed [22, 21] to compute complexity for thermofield double states in strongly coupled quantum field theories. (For related work, including a few lecture notes, see [23]-[75].) The proposal of [22, 21], which we refer to as Complexity-Action (CA) proposal, makes use of the holographic representation of the thermofield double state in a strongly coupled quantum field theory in terms of the eternal asymptotically AdS black hole [76]. On this spacetime one can define the Wheeler-DeWitt patch, shown in Fig. 2.1. The patch is anchored at boundary times t_L and t_R , and the proposal of [22, 21] equates the complexity of the thermofield dual state $|\psi(t_L, t_r)\rangle$ with the action evaluated on the Wheeler-DeWitt patch S_{WdW} :

$$\mathcal{C}(\psi(t_L, t_R)) = \frac{S_{WdW}}{\pi\hbar}, \quad (2.5)$$

It was also shown in [22, 21] that for the Einstein-Hilbert action, AdS black holes saturate the bound (2.1).

In this paper we add massless matter in the fundamental representation to $\mathcal{N} = 4$ super Yang-Mills and compute the corresponding corrections to $d\mathcal{C}/dt$. We achieve this by adding a small number of flavor branes to the stack of the D3 branes. At strong 't Hooft coupling, we need to study flavor branes propagating in asymptotically $AdS_5 \times S^5$ background. The action of D-branes is just the DBI action, and thus the CA correspondence identifies the correction to quantum complexity with the DBI action evaluated on the Wheeler-DeWitt patch^③

$$\delta\mathcal{C} = \frac{S_{DBI,WdW}}{\pi\hbar}, \quad (2.6)$$

Note that the variational problem for the DBI action is well defined and there is no need to introduce boundary terms in (2.6). We will see that $\delta\mathcal{C}$ can be written as a

③ We consider the action proposal here and not the volume one (check [25]) because there's no clear way on how to generalise the latter to introduce flavour fields.

function of temperature times the contribution of the flavor degrees of freedom to the total mass of the system, δM . One may wonder whether the growth rate of the total quantum complexity still obeys the inequality (2.1),

$$\frac{d\mathcal{C}_{tot}}{dt} \equiv \frac{d\mathcal{C}}{dt} + \frac{d(\delta\mathcal{C})}{dt} \stackrel{?}{\leq} \frac{2\mathcal{M}_{tot}}{\pi} = 2 \left(\frac{\mathcal{M}}{\pi} + \frac{\delta\mathcal{M}}{\pi} \right). \quad (2.7)$$

We will show that the corrections have the form

$$\frac{d(\delta\mathcal{C})}{dt} = -K(x) \frac{\delta\mathcal{M}}{\pi}, \quad x = \pi LT, \quad (2.8)$$

with $K(x)$ a monotonically increasing function. It is important to note that this correction is negative because of the overall minus sign that appears in front of the Lorentzian DBI action. Hence, the flavor corrections reduce the rate at which complexity grows and the bound (2.1) is no longer saturated. In our computations we neglected the back reaction from the flavor branes (which corresponds to the small number of flavors), focussed only on trivial embeddings and considered the late-time limit. Note that the flavor corrections are parametrically small and thus the complexification rate cannot become negative.

The rest of the chapter is organized as follows. In Section 2.1 we review the proposal of [22, 21]; Section 2.2 covers some generalities of the $D3/Dq$ systems. In section 2.3 we compute corrections to the complexity growth and to the mass of the system. We conclude in Section 2.6.

2.1 The complexity-action proposal

A concrete way for computing complexity in QFTs is not yet known. However, for some strongly coupled QFTs, such as $\mathcal{N} = 4$ super Yang-Mills, an equivalent gravitational description is available. One may then hope that a geometric prescription for evaluating complexity will be easier to define. In this article, we will use the proposal of [22, 21].

The authors of [22, 21] provide a prescription for evaluating the complexity of the thermofield double state in the dual gauge theory. For a conformal field theory (CFT) with a holographic dual, the finite temperature state is described by the AdS-Schwarzschild spacetime. (We are considering temperatures above the Hawking-Page transition [79]) An important role in the proposal is played by the Wheeler-DeWitt patch, denoted as WdW patch from now on (see Figure 2.1). The proposal states that the complexity \mathcal{C} of the thermofield double-state is given by (2.5) where S_{WdW} is the Einstein-Hilbert action,

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} (\mathcal{R} - 2\Lambda) + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{h} \mathcal{K}, \quad (2.9)$$

evaluated over the WdW patch. As usual, the Einstein-Hilbert action is supplemented by the York-Gibbons-Hawking term (YGH), for the variational problem to be well defined.

This proposal allows one to directly compute $d\mathcal{C}/dt$ and check whether or not the bound (2.1) is respected. Differentiating the holographic complexity is straightforward. Suppose t_L evolves for an infinitesimal amount δt . Such an evolution changes the WdW patch as shown in Figure 2.1. To compute the change in the action, one needs to evaluate it on the four regions denoted in Figure 2.1. However, as already noted in [22, 21], the action evaluated on region 2 is cancelled by that on region 3, while region 4 shrinks to zero in the limit $t_L \gg \beta$. We will be interested in precisely this limit (large time behavior of the complexity growth). So only region 1, the region behind the future singularity, contributes to the rate of change of the holographic complexity. The result presented in [22, 21] is the remarkably simple answer

$$\frac{d\mathcal{C}}{dt} = \frac{2\mathcal{M}}{\pi}, \quad (2.10)$$

which exactly saturates the bound (2.1).

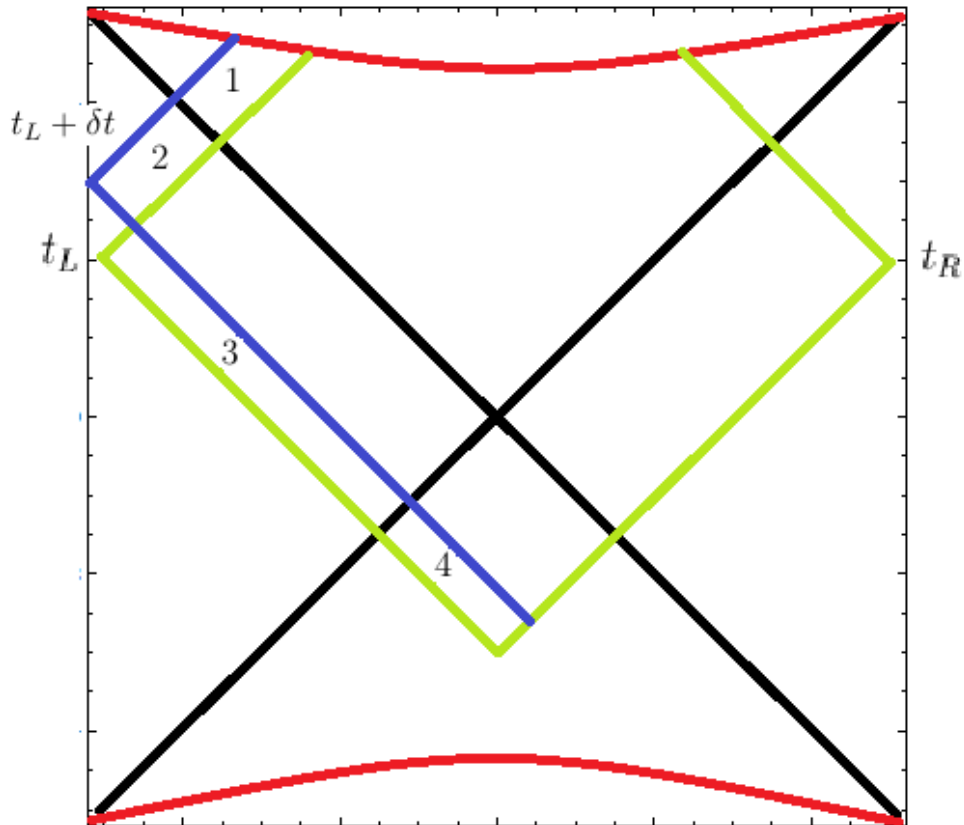


Figure 2.1: Penrose diagram of an AdS-Schwarzschild black hole. The red lines represent the future and past singularities, while the black lines crossing the diagram are the event horizons. The area enclosed by the green lines and the future singularity is the WdW patch at times t_L and t_R . If t_L is let to evolve infinitesimally the result is the patch shown in blue. This evolution then makes the patch lose regions 3 and 4 while gaining regions 1 and 2.

2.2 Generalities of D3/Dq systems

In this article, we are interested in studying the holographic complexity for a strongly coupled gauge theory with fundamental matter fields (fields transforming under the fundamental representation of the gauge group). To this end, we consider D3/Dq systems [77]. These systems are made out of a stack of N_c D3-branes and a number N_f of Dq-branes (the flavour branes). Strings stretching between the N_c D3-branes give rise to $\mathcal{N} = 4$ SYM, while strings stretching between the D3-branes and the flavour Dq-branes introduce fields that transform in the fundamental representation of the gauge group. To simplify the discussion, we will focus on the probe limit, where the number of flavor branes is much smaller than that of the color branes: $N_c \gg N_f$. In this limit, the Dq-branes can be treated as probes, propagating in the spacetime created by the stack of the D3-branes, *i.e.*, $AdS_5 \times S^5$, without backreaction.

The Dq-branes span a $(q+1)$ -dimensional worldvolume and thus wrap a $(q+1)$ -subspace

of $AdS_5 \times S^5$. There are in principle many ways of embedding an $AdS_n \times S^m$ ($n, m \leq 5$) into the background $AdS_5 \times S^5$, *i.e.*, several ways of choosing an S^m inside the S^5 or an AdS_n inside the AdS_5 . The embedding is usually specified by a set of scalar functions determining how the subspaces are chosen inside the 5-sphere and AdS_5 . For example, for the case of the $D3/D7$ configuration one can consider any of the following embeddings: $AdS_5 \times S^3$, $AdS_4 \times S^4$ or $AdS_3 \times S^5$.

However, not all possible embeddings preserve supersymmetry. In general, the endpoints of an open string stretching between a Dp and a Dq -brane will satisfy different boundary conditions depending on the specific arrangement of these two branes. Specifically, the endpoints could satisfy NN boundary conditions (both endpoints are Neumann), DD (both Dirichlet), ND or DN. The brane embedding preserves supersymmetry only if the difference between the number of ND and DN boundary conditions $\nu = ND - DN$ is a multiple of 4. Two such examples are the $AdS_5 \times S^3$ that arises from the D3/D7 configuration and the $AdS_4 \times S^2$ from the D3/D5 (these are the two cases that will be explicitly studied here). For a longer list of supersymmetric configurations check for example Table 1 in [78]. Note that the $AdS_n \times S^m$ that arise from these brane embeddings are always such that $|m - n| = 2$.

We will be interested in evaluating the complexity of the thermofield double. In the dual gravitational language, this can be achieved by considering Dq branes propagating in the AdS-Schwarzschild spacetime, which describes $\mathcal{N} = 4$ Super Yang Mills at finite temperature. Its metric is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 + L^2 d\Omega_5^2; \quad (2.11)$$

$$f(r) = 1 + \frac{r^2}{L^2} - \frac{M}{r^2},$$

where $M = \frac{8G}{3\pi} \mathcal{M}$. Apart from the dependence on L , the radius of curvature of both the AdS_5 and the S^5 spaces, the AdS-Schwarzschild metric also depends on an additional parameter M which is proportional to the mass of the black hole. The Penrose diagram of the AdS-Schwarzschild spacetime is depicted in Fig.1.

To evaluate the contribution to the complexity of the state from the flavor degrees of freedom in the large N_c and large 't Hooft coupling λ limit, we simply need to evaluate the action for the propagation of the probe Dq branes in the AdS-Schwarzschild background on the WdW patch. The action which governs the propagation of the Dq branes is the

DBI action:^④

$$S_{DBI} = -N_f T_{Dq} \int \sqrt{-g_{Dq}}, \quad (2.12)$$

where the tension of the Dq -brane is given by

$$T_{Dq} = \frac{1}{(2\pi l_s)^q g_s l_s}. \quad (2.13)$$

The string length l_s and the string coupling constant g_s are related to the 't Hooft coupling λ and the colour degrees of freedom N_c through

$$\lambda = g_{YM}^2 N_c = 2\pi g_s N_c, \quad L^4 = 4\pi g_s N_c l_s^4, \quad (2.14)$$

where L denotes the AdS radius of curvature as above. In (2.12) g_{Dq} denotes the determinant of the induced metric of the Dq branes, which depends on the details of the embedding.

The embeddings we consider in this article, are the trivial embeddings, and correspond to adding massless flavor matter in the $N = 4$ SYM Lagrangian. As explained above, the asymptotic form of the induced metric will be $AdS_m \times S^n$. Evaluating the DBI action on asymptotically AdS geometries leads to divergences which can be treated with holographic renormalization [80]. Holographic renormalization for the case of D3/Dq systems was studied in [81]. For technical reasons it is convenient to express the AdS-Schwarzschild metric in Fefferman-Graham coordinates

$$ds^2 = L^2 \left\{ \frac{dz^2}{z^2} + \frac{L^2}{4z^2} \left[1 - \frac{z^4}{L^4} \left(1 + 4\frac{M}{L^2} \right) \right]^2 \frac{d\tau^2}{F(z, M)} + \frac{F(z, M)}{4z^2} d\Omega_3^2 + d\Omega_5^2 \right\}, \quad (2.15)$$

where

$$F(z, M) = L^2 - 2z^2 + \frac{z^4}{L^2} \left(1 + 4\frac{M}{L^2} \right). \quad (2.16)$$

The boundary of AdS is now at $z = 0$, while the horizon is mapped to

$$z_H \equiv z(r = r_H) = \frac{L^2}{\sqrt{L^2 + 2r_H^2}}. \quad (2.17)$$

^④ The Euclidean DBI action has a positive sign. Also, we will denote the Euclidean action as I instead of S to avoid confusion with entropy. Note that the variation of this action is proportional to just $\delta g_{\mu\nu}$, so no boundary terms are needed here to make the variational problem well defined.

The radial coordinates (z, r) are related to one another as follows:

$$z(r) = \frac{L^2}{\left[L^2 + 2r^2 + 2\sqrt{r^4 + L^2r^2 - L^2M}\right]^{1/2}}, \quad r^2 = L^2 \frac{F(z, M)}{4z^2}. \quad (2.18)$$

The trivial embeddings considered in this paper are described by induced metrics with asymptotics of the form $AdS_m \times S^n$, where $m + n = q + 1$ and

$$ds_{Dq}^2 = L^2 \left\{ \frac{dz^2}{z^2} + \frac{L^2}{4z^2} \left[1 - \frac{z^4}{L^4} \left(1 + 4\frac{M}{L^2} \right) \right]^2 \frac{d\tau^2}{F(z, M)} + \frac{F(z, M)}{4z^2} d\Omega_{n-2}^2 + d\Omega_{q-n+1}^2 \right\}. \quad (2.19)$$

As explained above, we will use Holographic Renormalization in order to deal with the divergent contributions in $\int \sqrt{g_{Dq}}$. The procedure consists of the following steps: firstly, we introduce a cutoff surface at $z = \epsilon$ and define covariant counterterms on the $z = \epsilon$ surface such that the divergences are cancelled. Then, we take the limit $\epsilon \rightarrow 0$ to remove the cutoff. The appropriate counterterms were worked out in [81] and are of two classes; the ones needed to regulate the volume part of the integral and the ones required to regulate the contributions from the embedding functions. For trivial embeddings only the former type of counterterms appear since the embedding functions are zero. As a result, for the induced metrics quoted in (2.19) the following counterterms are required:

$$I^{ren} = I_{DBI} + I_{count}; \quad I_{count} = N_f T_{Dq} \int \sqrt{\gamma} (L_1 + L_2) = N_f T_{Dq} \int \sqrt{\gamma} (-a + b\mathcal{R}_\gamma)$$

$$a = \begin{cases} L/4 & \text{for } AdS_5 \\ L/3 & \text{for } AdS_4 \\ L/2 & \text{for } AdS_3 \\ L & \text{for } AdS_2 \end{cases} \quad b = \begin{cases} L^3/48 & \text{for } AdS_5 \\ L^3/12 & \text{for } AdS_4 \\ 0 & \text{for } AdS_3 \\ 0 & \text{for } AdS_2 \end{cases} \quad (2.20)$$

where \mathcal{R}_γ is the Ricci scalar associated with the induced metric γ on the constant z surface.

2.3 Complexity and Energy of D3/D7 systems

The thermodynamic properties of a system are derived from its Euclidean action, which in this case is the DBI action, I_{D7} . The correction to the free energy of the black hole is given by $\delta F = T I_{D7}$ and the energy is obtained from the thermodynamic relation

$$\delta \mathcal{M} = \delta F + T \delta S, \quad \delta S = \frac{\partial \delta F}{\partial T}. \quad (2.21)$$

In terms of inverse temperature $\beta = 1/T$, the above relation can be expressed as

$$\delta \mathcal{M} = \delta F - \beta \frac{\partial \delta F}{\partial \beta}. \quad (2.22)$$

To compute $\delta \mathcal{M}$ we thus need to evaluate the Euclidean DBI action on the $D7$ -brane configuration:

$$\begin{aligned} I_{D7} &= N_f T_{D7} \frac{L^9}{16} \int_0^\beta d\tau \int d\Omega_3 \int d\Omega_3 \int_0^{z_H} \frac{F(z)}{z^5} \left[1 - \frac{z^4}{L^2} \left(1 + 4 \frac{M}{L^2} \right) \right] \\ &= N_f T_{D7} \frac{L^9}{16} V_{\Omega_3}^2 \beta \left[\frac{-L^2}{4z^4} + \frac{1}{z^2} + \frac{(L^2 + 4M)z^2}{L^6} - \frac{(L^2 + 4M)^2 z^4}{4L^{10}} \right]_0^{z_H}. \end{aligned} \quad (2.23)$$

As anticipated above, the action diverges when $z \rightarrow 0$. Introducing a cutoff at $z = \epsilon$ and evaluating the relevant counterterms from (2.20) yields

$$I_{count} = N_f T_{D7} V_{\Omega_3}^2 \beta \left[-\frac{L^{11}}{64\epsilon^4} + \frac{L^9}{16\epsilon^2} + \mathcal{O}(\epsilon^2) \right], \quad (2.24)$$

which exactly cancels the divergences of I_{D7} without introducing any finite contribution.

The final result is

$$I_{D7}^{ren} = N_f T_{D7} \frac{L^9}{16} V_{\Omega_3}^2 \beta \left[\frac{-L^2}{4z_H^4} + \frac{1}{z_H^2} + \frac{(L^2 + 4M)z_H^2}{L^6} - \frac{(L^2 + 4M)^2 z_H^4}{4L^{10}} \right]. \quad (2.25)$$

To compute the thermodynamic quantities we're interested in, we need to write I_{D7}^{ren} as a function of β . To do so we use (2.17) to relate z_H with r_H , where r_H is the position of the horizon of the AdS-Schwartzchild black hole in the original coordinates (2.11) and is related to the temperature as [82],

$$r_H(\beta) = \frac{L^2 \pi + \sqrt{L^4 \pi^2 - 2L^2 \beta^2}}{2\beta} = L \frac{x + \sqrt{x^2 - 2}}{2}. \quad (2.26)$$

Note that there is a minimum temperature allowed, namely $T = \frac{\sqrt{2}}{\pi L}$. This is the temperature below which black holes cannot exist.

Solving $f(r_H) = 0$, one finds that

$$r_H^2 = L^2 \frac{-1 + \sqrt{1 + 4M/L^2}}{2}, \quad (2.27)$$

which, together with (2.17), leads to

$$z_H = \frac{L^2}{(L^2 + 2r_H^2)^{1/2}} \quad \rightarrow \quad z_H = \frac{L}{\left(1 + 4\frac{M}{L^2}\right)^{1/4}}. \quad (2.28)$$

Substituting into our result for $I_{D7}^{ren}(z_H, M)$ results in

$$I_{D7}^{ren} = \frac{N_f T_{D7} L^7 V_{\Omega_3}^2 \beta}{32} \left[4 \left(1 + 2\frac{r_H^2}{L^2}\right) - \left(1 + 2\frac{r_H^2}{L^2}\right)^2 \right]. \quad (2.29)$$

It is easy to express $I_{D7}^{ren}(\beta)$ in terms of the inverse temperature β by using (2.26). Applying (2.22) then leads to the following expression for the energy of the $D7$ system

$$\begin{aligned} \delta \mathcal{M}_{D7} &= \frac{N_f T_{D7} L^7 V_{\Omega_3}^2}{32} H_{D7}(\beta). \\ H_{D7}(\beta) &\equiv 6 \left[\frac{L^4 \pi^4}{\beta^4} - \frac{L^2 \pi^2}{\beta^2} + \frac{L^2 \pi^3 \sqrt{L^4 \pi^2 - 2L^2 \beta^2}}{\beta^4} \right]. \end{aligned} \quad (2.30)$$

In the planar limit, $L/\beta \rightarrow \infty$, this agrees with eq. (4.28) in [83] (see also [84] for a similar computation for massive embeddings).

Complexity

Here we discuss the complexity computation. The Penrose diagram of the D3/D7 system is still the one shown in Figure 2.1, so our integral will split into the same 4 regions. The difference is that now our action is^⑤

$$\delta \mathcal{C} = S_{DBI} = -N_f T_{D7} \int_{WdW} \sqrt{-g}. \quad (2.31)$$

Note that no surface terms are needed since the variation δS_{DBI} contains no terms depending on $\delta(\partial_\sigma g_{\mu\nu}) \Big|_{\partial \mathcal{M}}$. With our action, the integrals from parts 2 and 3 again cancel each other out, and the region 4 doesn't contribute either because it shrinks to zero size^⑥. So we are only left with region 1, which is bounded by the surfaces $r = 0$ and $r = r_H$. Working with the metric as in (2.11), the integrand is

$$\sqrt{-g} = r^3 L^3. \quad (2.32)$$

^⑤ Recall that the Lorentzian action has negative sign.

^⑥ In the Einstein gravity case studied in [21] a topological argument is needed to rule this part out because the integrand there is \mathcal{R} ; since our integral is just a volume for us this argument is trivial.

The time derivative of the action is then simply

$$\begin{aligned} \frac{dS_{DBI}}{dt} &= -N_f T_{D7} \frac{d}{dt} \int \sqrt{-g} = -N_f T_{D7} L^3 \int dr r^3 \int d\Omega_3 \int d\Omega_3 \\ &= -N_f T_{D7} L^3 V_{\Omega_3}^2 \frac{r_H^4}{4} = -N_f T_{D7} L^7 V_{\Omega_3}^2 \frac{r_H^4}{4L^4}. \end{aligned} \quad (2.33)$$

We would like to express our result for the complexity as a function of the temperature and the energy of the system. To introduce the energy into the last equation we use (2.30) to write the overall factor in (2.33) as

$$N_f T_{D7} L^7 V_{\Omega_3}^2 = \frac{32 \delta M_{D7}}{H_{D7}(\beta)}. \quad (2.34)$$

So, using (2.34) and (2.26) yields

$$\begin{aligned} \frac{d(\delta\mathcal{C})}{dt} &= \frac{dS_{DBI}}{dt} = -\frac{\delta\mathcal{M}}{\pi} K_{D7}(x), \\ K_{D7}(x) &\equiv \frac{8r_H(\beta)^4}{H_{D7}(\beta)} = \frac{1}{12} \frac{x^2 \left[1 + \sqrt{1 - \frac{2}{x^2}}\right]^4}{x^2 \left[1 + \sqrt{1 - \frac{2}{x^2}}\right] - 1}, \quad x = \pi L T. \end{aligned} \quad (2.35)$$

Note that there is a minimum value x can take, being $x_{min} = \sqrt{2}$. The function $K(x)$ is plotted on Figure 2.2. The function is monotonically increasing, positive and ranging between the value $1/6$ at the minimum and asymptotically approaching $2/3$.

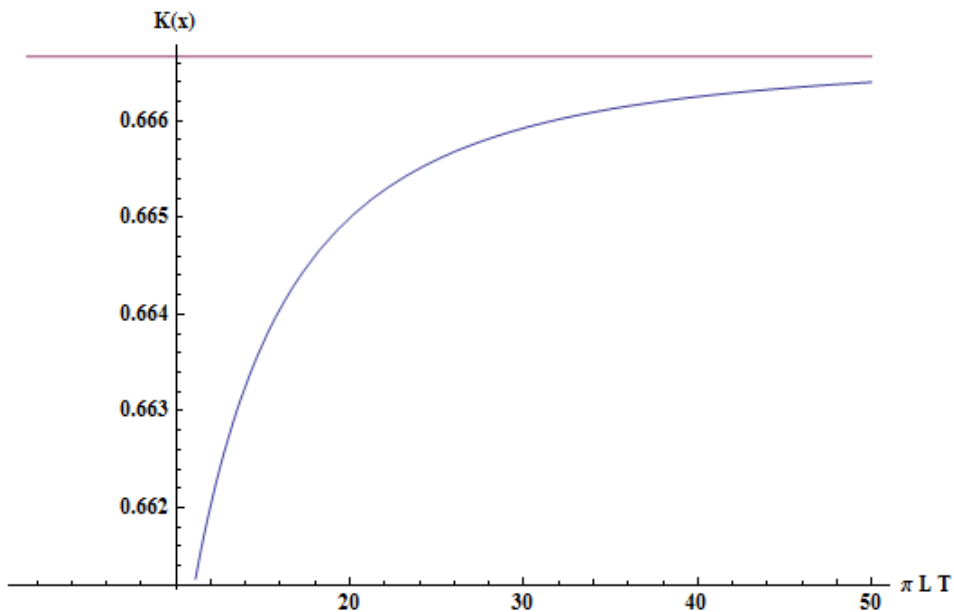


Figure 2.2: Plot of the function $K_{D7}(x)$ starting from the minimum value $x_{min} = \sqrt{2}$. The horizontal orange line is the value to which it asymptotes, namely $2/3$.

Due to the minus sign present in (2.35) the correction lowers the speed at which the system complexifies, so the bound is respected but not saturated.

2.4 Complexity and Energy of D3/D5 systems

To compute the correction to the energy of the D3/D5 due to the flavor D5 branes in the probe limit, we will follow exactly the same steps as in section 4.1. The Euclidean action is in this case given by

$$\begin{aligned} I_{D5} &= T_{D5} N_f \int \sqrt{g} = T_{D5} N_f \beta V_{\Omega_2}^2 \frac{L^7}{8} \int_0^{z_H} dz \frac{\left[1 - \frac{z^4}{L^4} \left(1 + 4\frac{M}{L^2}\right)\right]}{z^4} \sqrt{F(z)} \\ &= -\frac{T_{D5} N_f \beta V_{\Omega_2}^2 L^7}{8} \left[\frac{F(z)^{3/2}}{3L^2 z^3} \right]_{\epsilon}^{z_H}, \end{aligned} \quad (2.36)$$

with divergent terms of the form

$$I_{D5}^{div} = -T_{D5} N_f \beta V_{\Omega_2}^2 \left[\frac{L^8}{24\epsilon^3} - \frac{L^6}{8\epsilon^2} + \mathcal{O}(\epsilon) \right]. \quad (2.37)$$

The relevant counterterms are

$$\begin{aligned} I_{count} &= N_f T_{D5} \int \sqrt{\gamma} (L_1 + L_2) \quad \longrightarrow \quad I^{ren} = I + I_{count} \\ L_1 &= \frac{-L}{3}, \quad L_2 = \frac{L^3}{12} \mathcal{R}_{\gamma}. \end{aligned} \quad (2.38)$$

Just as in the D3/D7 case, the holographic renormalization procedure removes the divergent parts without adding any finite terms. The final result is:

$$I_{D5}^{ren} = -\frac{T_{D5} N_f \beta V_{\Omega_2}^2 L^5}{8} \left[\frac{F(z_H)^{3/2}}{3z_H^3} \right]. \quad (2.39)$$

Using (2.28) it's immediate to see that

$$F(z_H) = L^2 \frac{4r_H^2/L^2}{1 + 2r_H^2/L^2}, \quad z_H^3 = \frac{L^3}{(1 + 2r_H^2/L^2)^{3/2}}, \quad (2.40)$$

which allows us to write the renormalized action as

$$I_{D5}^{ren} = -\frac{T_{D5} N_f \beta V_{\Omega_2}^2 L^5}{3} \frac{r_H^3}{L^3}. \quad (2.41)$$

The correction to the free energy of the D3/D5 system is

$$\delta F_{D5} = -\frac{T_{D5} N_f \beta V_{\Omega_2}^2 L^5}{3} \frac{r_H^3}{L^3}. \quad (2.42)$$

With the help of (2.26) we obtain the free energy as a function of the inverse temperature, $F_{D5}(\beta)$ and use the standard thermodynamic relations (2.22) to obtain

$$\begin{aligned}\delta\mathcal{M} &= \frac{T_{D5}N_fV_{\Omega_2}^2L^5}{3}H_{D5}(x), \\ H_{D5}(x) &= \frac{2x^4 + 2x^3\sqrt{x^2-2} - 2x^2 - 1}{2\sqrt{x^2-2}}, \quad x \equiv \pi LT.\end{aligned}\tag{2.43}$$

Complexity

Let's now see how the complexity is related to the energy in the D3/D5 system. The arguments made in section 4.1.2 regarding the contribution of the different parts of the WdW patch are still valid, and clearly the first equality in (2.33) is still true (changing $T_{D7} \leftrightarrow T_{D5}$), the only difference being the explicit form of $\sqrt{-g}$. The induced metric is in this case asymptotically $AdS_4 \times S^2$:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_2^2 + L^2d\Omega_2^2,\tag{2.44}$$

with the determinant

$$\sqrt{-g} = r^2L^2.\tag{2.45}$$

Following exactly the same steps which lead us to (2.33) and dividing by V_x to obtain a density, leads to

$$\frac{dS_{DBI}}{dt} = -\frac{N_fT_{D5}V_{\Omega_2}^2L^5}{3}\frac{r_H^3}{L^3}.\tag{2.46}$$

Similar to the D3/D7 case, the factor multiplying r_H^3/L^3 in the equation above can be expressed in terms of the energy of the system

$$\frac{T_{D5}N_fV_{\Omega_2}^2L^5}{3} = \frac{\delta\mathcal{M}}{H_{D5}(x)}.\tag{2.47}$$

This together with (2.26) transforms the equation for dS_{DBI}/dt into

$$\begin{aligned}\frac{d\delta\mathcal{C}}{dt} &= -\frac{\delta\mathcal{M}}{\pi}K_{D5}(x), \\ K_{D5}(x) &= \left(\frac{x + \sqrt{x^2-2}}{2}\right)^3 H_{D5}^{-1}(x), \quad \text{with } x = \pi LT.\end{aligned}\tag{2.48}$$

Note again, that there is a minimum value allowed for x , namely $x = \sqrt{2}$. The function is positive, monotonically increasing and ranges between 0 at the minimum and the asymptotic value $1/2$.

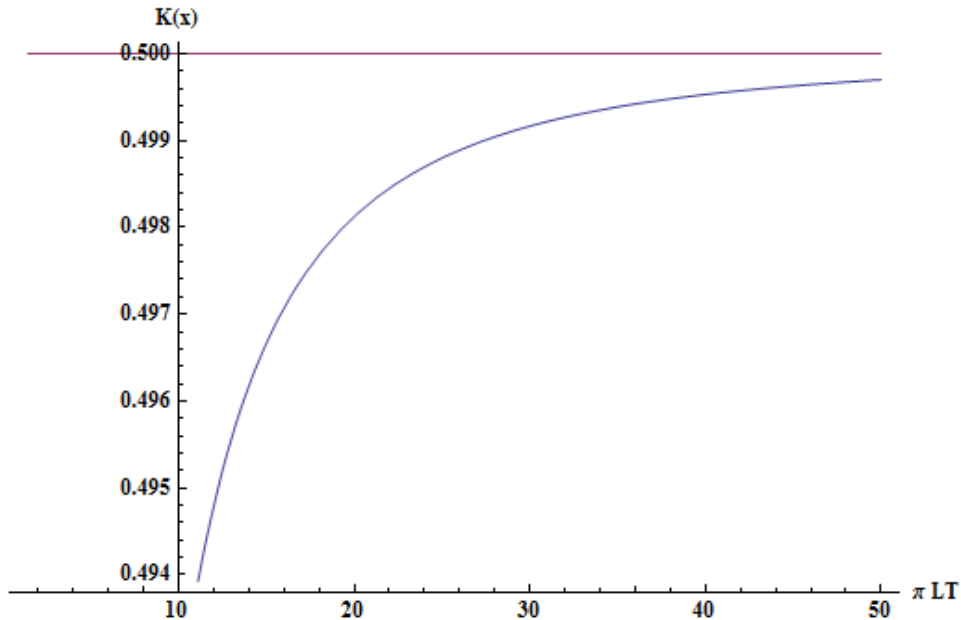


Figure 2.3: The function K_{D5} vs temperature, starting from the minimum value $x_{min} = \sqrt{2}$. The horizontal orange line is the value to which it asymptotes, namely $1/2$.

2.5 The general case: Complexity and Energy of D3/Dq systems

Having gained some insight from the detailed study of the $D3/D5$ and the $D3/D7$ systems, we move on to consider the generic $D3/Dq$ system. As we will see, the qualitative features of the complexity of the thermofield double state in the presence of flavour matter fields, remain the same for both stable and unstable (non-supersymmetric) configurations.

As discussed above, the different embeddings of the Dq -branes are submanifolds of the $AdS_5 \times S^5$ generated by the background $D3$ -branes, with the asymptotic form of $AdS_n \times S^m$ where $m + n = q + 1$. Regarding the energy computation, all the divergent parts in the Euclidean action come from the AdS_n part of the manifold. The induced metric on the Dq branes is given in (2.19) and its determinant is equal to:

$$\sqrt{g} = \frac{L^{q+2}}{2^{n-1}} z^{-n} \left[1 - \frac{z^4}{L^4} \left(1 + 4 \frac{M}{L^2} \right) \right] F(z)^{\frac{n-3}{2}}. \quad (2.49)$$

It is straightforward to evaluate the Euclidean DBI action I_{Dq} to obtain

$$I_{Dq} = N_f T_{Dq} \int \sqrt{g} = N_f T_{Dq} \left[- \frac{L^q}{2^{n-1}(n-1)} \frac{F(z)^{\frac{n-1}{2}}}{z^{n-1}} \right]_0^{z_H} \beta V_{\Omega_{n-2}} V_{\Omega_{q-n+1}} \quad (2.50)$$

To proceed it will be convenient to separately analyze the cases where the AdS_n part

of the embedding is of even or odd dimensionality.

When n is an even integer. As one can see from (2.50), for n even, the Euclidean action is given in terms of the metric function $F(z)$ elevated to a half-integer power. The behaviour of I_{Dq} for small z can be split into two types of contributions

$$I_{Dq}^{even} \Big|_{z \rightarrow 0} \rightarrow f(z) + g\left(\frac{1}{z}\right), \quad (2.51)$$

where $f(z)$ and $g(\frac{1}{z})$ represent polynomial functions in z and $\frac{1}{z}$ respectively, with vanishing zeroth order terms. $f(z)$ then vanishes when evaluated at $z \rightarrow 0$, while $g(\frac{1}{z})$ is divergent but its divergences are exactly cancelled by the relevant counterterms and no constant piece is introduced. The result is then given by contributions from just the horizon as

$$\begin{aligned} I_{Dq} &= -N_f T_{Dq} \frac{L^q}{2^{n-1}(n-1)} \frac{F(z_H)^{\frac{n-1}{2}}}{z_H^{n-1}} \beta V_{\Omega_{n-2}} V_{\Omega_{q-n+1}} \\ &= -N_f T_{Dq} \frac{L^q}{n-1} \left(\frac{r_H}{L}\right)^{n-1} \beta V_{\Omega_{n-2}} V_{\Omega_{q-n+1}}, \end{aligned} \quad (2.52)$$

where we have used (2.40) in the last equality.

We can now write the free energy $F_{Dq} = T I_{Dq}$ and use (2.22) to obtain the energy of the system as,

$$\begin{aligned} \delta \mathcal{M} &= N_f T_{Dq} \frac{L^q}{(n-1)} H_{Dq}(x) V_{\Omega_{n-2}} V_{\Omega_{q-n+1}}, \\ H_{Dq}(x) &= \left(\frac{r_H(x)}{L}\right)^{n-2} \frac{2 + (n-2)x\sqrt{x^2-2} + (n-2)x^2}{2\sqrt{x^2-2}}, \quad \frac{r_H(x)}{L} = \frac{x + \sqrt{x^2-2}}{2}. \end{aligned} \quad (2.53)$$

When n is an odd integer. In this case $F(z)$ is elevated to an integer power, and the result is a polynomial in even powers of z , *i.e.*,

$$F(z)^{\frac{n-1}{2}} = A_0 + A_2 z^2 + \dots + A_{2(n-1)} z^{2(n-1)}, \quad (2.54)$$

This implies that the quantity $F^{\frac{n-1}{2}}/z^{n-1}$ in (2.50) contains a constant term, independent from z . Once more the divergent terms at the boundary $z = 0$ are precisely cancelled by the relevant counterterms and the Euclidean action is given by

$$I_{Dq} = -N_f T_{Dq} \frac{L^q}{2^{n-1}(n-1)} \left[\frac{F(z_H)^{\frac{n-1}{2}}}{z_H^{n-1}} - c_0 \right] \beta V_{\Omega_{n-2}} V_{\Omega_{q-n+1}}. \quad (2.55)$$

Clearly the constant term, indicated by c_0 , is cancelled by the same z -independent term in $\frac{F(z_H)^{\frac{n-1}{2}}}{z_H^{\frac{n-1}{2}}}$.

In practice, there exist only two non-trivial embeddings in this class: those which asymptote to AdS_3 and those which asymptote to AdS_5 . The latter case was addressed in the context of the $D3/D7$ system, we only need to consider the AdS_3 case. From (2.19) and (2.49) we can see that we are now working with

$$ds_{Dq}^2 = L^2 \left\{ \frac{dz^2}{z^2} + \frac{L^2}{4z^2} \left[1 - \frac{z^4}{L^4} \left(1 + 4\frac{M}{L^2} \right) \right]^2 \frac{d\tau^2}{F(z, M)} + \frac{F(z, M)}{4z^2} d\theta^2 + d\Omega_m^2 \right\}, \quad (2.56)$$

$$\sqrt{g_{Dq}} = \frac{L^4}{4} z^{-3} \left[1 - \frac{z^4}{L^4} \left(1 + 4\frac{M}{L^2} \right) \right] dz d\tau d\theta (L^m d\Omega_m).$$

It is straightforward to apply the general result above to the case $n = 3$ to obtain:

$$I_{Dq} = -\frac{N_f T_{Dq} L^q}{4} \beta V_{\Omega_1} V_{\Omega_{q-2}} \left(1 + \frac{2r_H^2}{L^2} \right), \quad (2.57)$$

where we used the relation between $(z_H, F(z_H))$ and r_H from (2.17). Evaluating (2.22) then yields

$$\delta\mathcal{M} = \frac{N_f T_{Dq} L^2 V_{\Omega_1}}{2} H_{Dq}(x), \quad (2.58)$$

$$H_{Dq}(x) = \frac{x^3 + x^2 \sqrt{x^2 - 2}}{\sqrt{x^2 - 2}}.$$

Complexity of the $D3/Dq$ system.

When n is an even integer. We follow exactly the same steps as in the previous sections to evaluate the time derivative of the DBI action $S_{DBI} = -N_f T_{Dq} \int \sqrt{-g}$, which is given by

$$\frac{dS_{DBI}}{dt} = -\frac{N_f T_{Dq} L^q}{n-1} \left(\frac{r_H}{L} \right)^{n-1} V_{\Omega_{n-2}} V_{\Omega_{q-n+1}}. \quad (2.59)$$

As usual, we can solve (2.53) for $N_f T_{Dq} L^q$ to write this derivative as

$$\frac{d\delta\mathcal{C}}{dt} = -\frac{\delta\mathcal{M}}{\pi H_{Dq}(x)} \left(\frac{r_H}{L} \right)^{n-1} = -\frac{\sqrt{x^2 - 2} (x + \sqrt{x^2 - 2})}{2 - (n-2)x\sqrt{x^2 - 2} + (n-2)x^2} \frac{\delta\mathcal{M}}{\pi} \equiv -K_{Dq}(x) \frac{\delta\mathcal{M}}{\pi} \quad (2.60)$$

When n is an odd integer. For odd n we only need to consider $n = 3$ and focus on embeddings which asymptote to AdS_3 as in 4.3.1. Similarly to the previous sections we obtain

$$\frac{d\delta\mathcal{C}}{dt} = \frac{dS_{DBI}}{\pi dt} = -\frac{N_f T_{Dq} L^q}{2\pi} \left(\frac{r_H}{L}\right)^2 V_{\Omega_1} V_{\Omega_{q-2}}, \quad (2.61)$$

which coincides with equation (2.59) for $n = 3$. As usual, we can solve the energy equation to express the numerator as a function of δM . This produces the final result

$$\frac{d\delta\mathcal{C}}{dt} = -\frac{r_H^2}{L^2} \frac{\delta\mathcal{M}}{\pi H_{Dq}(x)} = -\frac{\sqrt{x^2-2}(x+\sqrt{x^2-2})}{4x^2} \frac{\delta\mathcal{M}}{\pi} \quad (2.62)$$

Clearly, the correction to the complexity due to the probe, flavor branes is negative and monotonically decreasing for all the D3/Dq systems.

2.6 Conclusions

Introducing fundamental matter leads to a correction term to the left-hand side of (2.10), which is negative. It is interesting that the growth of quantum complexity in systems with fundamental matter seems to be slower than that with just adjoint matter. It would be interesting to compare this with a direct computation in field theory. Note that the presence of extra matter in the bulk was shown to reduce the rate of complexity growth in [21].

It would be interesting to compute the flavor corrections to the complexification rate using the complexity-volume proposal [24]. It is not immediately clear to us how to generalize this proposal to include flavor corrections.

It would also be interesting to study the behavior of the quantum complexity growth in non-conformal field theories. In gravity, one could investigate asymptotically AdS domain wall solutions or general Dp/Dq systems.

Chapter 3

Causality in RG flows of QFTs with a Gauss-Bonnet holographic dual

This chapter covers a still ongoing project [86] where causality constraints in RG flows of general QFTs are studied through the gauge/gravity duality. The approach here will be to track the value of a given causally-bounded quantity along the flow from the UV. This way we will find if a given theory, which is naively consistent in the UV, can become unphysical at some point. So, what is the scenario we need to study RG flows and which bound will we analyse?

In a holographic setup renormalization flows are controlled by the holographic (radial) coordinate, which is the analogue of the energy scale of the associated quantum field theory (See references [87]-[106] for the early works in holographic RG flows). Since AdS is the holographic analogue of conformal field theories one cannot consider this spacetime to study RG flows (the field theory is not conformal outside the fixed points). This is the reason why domain wall spacetimes need to be introduced, since they are constructed in such a way that their boundary is that of AdS (and so we have a CFT in the UV) while the bulk is not AdS (allowing for the description of a general QFT).

Regarding the causality-constrained quantity, we will look at the speed of gravitons near the boundary. Take any two points in the AdS boundary (and so also in its associated CFT). There two types of trajectories a particle could travel to go from one to the other: either never leaving the boundary or travelling through the bulk. If the velocity of the bulk gravitons near the boundary is too fast then this would create a causality violation from the point of view of a boundary observer (since a graviton would have somehow travelled between the two points faster than it would have through the boundary). For this problem to be absent one needs the gravitons to travel slower than the speed of light

through the bulk. This will be the quantity we will look at along the flow, checking if the gravitons speed exceed this limit at some point.

3.1 The holographic RG flow

Asymptotically AdS_{d+1} spacetimes can be used to study the renormalization group of the d -dimensional field theory that lives on the boundary. This flow can be understood as triggered by either a deformation of the original field theory (given by the introduction of a relevant operator) or by setting a different vacuum (by introducing some nonzero vev).

For our purposes it is enough to work with the effective field theory of supergravity, that is, to deal with scalars coupled to gravity. The base idea here is that the radial holographic coordinate in the bulk works as the rough equivalent of the energy scale in the boundary field theory. This can be understood from the fact that the two-point function of primary operators with conformal dimension greater than one in the field theory is computed by

$$\langle \mathcal{O}(x_i) \mathcal{O}(y_i) \rangle_{boundary} \sim e^{-mD(x,y)}, \quad (3.1)$$

where $D(x, y)$ is the length of the bulk geodesic that connects the two boundary points x and y . As one starts separating the two operators apart (and so going to smaller energies due to the $E \sim 1/\delta x$ relation) the geodesic starts probing further and further into the bulk. It is in this way that the holographic coordinate can be understood as the energy scale of the field theory, the boundary of AdS representing the UV. Keeping this in mind, the RG flow is described as a domain-wall metric that interpolates between several different AdS vacua representing the fixed points of the flow. Several field quantities, like the beta function or the Weyl anomalies, can be computed through holography as elaborated on in the next section.

3.1.1 Scalar fields in AdS and Hamilton-Jacobi formalism

Let's first recap the behaviour of scalar fields in AdS and introduce the Hamilton-Jacobi formalism. Consider the action of scalars coupled to gravity

$$S = \int \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right). \quad (3.2)$$

The equation of motion is just the Klein-Gordon equation. Performing a separation of variables $\phi(r)\phi(x)$, where r is the holographic coordinate, the r -equation of motion in AdS reads

$$\left[-r^{d+1}\partial_r(r^{-d+1}\partial_r) + m^2L^2 + k^2r^2\right]\phi(r)_k = 0, \quad (3.3)$$

where k is a constant that appears during the separation of variables process and the boundary is at $r = 0$. After some work the solutions can be found in terms of modified Bessel functions as

$$\phi_k(kr) = a_k(kr)^{\nu/2}K_\nu(kr) + b_k(kr)^{\nu/2}I_\nu(kr), \quad \nu = \sqrt{\frac{d}{2} + m^2L^2}. \quad (3.4)$$

Imposing regularity of the solution in the bulk forces us to set b_k to zero since $I_\nu(r) \sim e^{kr}$. Looking at the boundary now, the remaining Bessel function asymptotic behaviour yields

$$\begin{aligned} \phi(r) &\approx \phi_0(k)r^{\Delta_-} + \phi_1(k)r^{\Delta_+}, \\ \Delta_\pm &= \frac{d}{2} \pm \nu = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}, \end{aligned} \quad (3.5)$$

where ϕ_0 and ϕ_1 are constants whose specific form is not relevant for this discussion. Solutions that asymptote to ϕ_0 correspond to deformations of the field theory, whereas those that asymptote to ϕ_1 correspond to theories with a different vacuum.

Let's now discuss why this set-up allows you to analyse RG flows. The idea is to take the scalars coupled to gravity action, use the Hamilton-Jacobi formalism, determine its classical version and show that one can derive beta function equations from it (and other RG flow related quantities, like the Callan-Symanzik equation). Consider scalar fields $\phi^a(x, r)$ coupled to gravity in a $(d + 1)$ -dimensional manifold,

$$S = \int_{\mathcal{M}} d^{d+1}x \sqrt{g} \left(\frac{1}{2} L_{ab}(\phi) g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b - \mathcal{R} + V(\phi) \right) - 2 \int_{\partial\mathcal{M}} d^d x \sqrt{g} K, \quad (3.6)$$

where K is the extrinsic curvature of the boundary and is needed to make the variational problem well defined (See the Appendix A for more details on variational problems in spacetimes with boundaries). For the purpose of this chapter it is helpful to work using the ADM formalism. That is, spacetime is foliated into a family of spacelike hypersurfaces and the metric takes the form

$$ds^2 = N(x, \tau)d\tau^2 + g_{ij}(x, \tau) \left(dx^i + \lambda^i(x, \tau)d\tau \right) \left(dx^j + \lambda^j(x, \tau)d\tau \right). \quad (3.7)$$

Objects with Greek indices e.g. $g_{\mu\nu}$ live in the $(d+1)$ -dimensional bulk spacetime, while objects with Latin indices e.g. g_{ij} live in the d -dimensional hypersurfaces. In this formalism, the gravity degrees of freedom are encoded in the metric g_{ij} of the hypersurfaces and their conjugate momenta π_{ij} , whereas the parameters N and λ^i function as Lagrange multipliers. The action now is considered to be a functional of all these: $S = S[g_{ij}, \pi^{ij}; \phi^a, \pi_a; N, \lambda^i]$. Variations of the action with respect to the Lagrange multipliers must yield zero, so

$$\begin{aligned} 0 &= \frac{1}{\sqrt{g}} \frac{\delta S}{\delta N} \equiv \mathcal{H} \\ 0 &= \frac{1}{\sqrt{g}} \frac{\delta S}{\delta \lambda^i} \equiv \mathcal{P}^i. \end{aligned} \tag{3.8}$$

These two conditions $\mathcal{H} = \mathcal{P}^i = 0$ are known as the Hamiltonian and momentum constraints

$$\begin{aligned} \mathcal{H} &= \frac{1}{d-1} (\pi_i^i)^2 - \pi_{ij}^2 - \frac{1}{2} L^{ab} \pi_a \pi_b + V(\phi) - \mathcal{R} + \frac{1}{2} L_{ab} g^{ij} \partial_i \phi^a \partial_j \phi^b, \\ \mathcal{P}^i &= 2 \nabla_j \pi^{ij} - \pi_a \nabla^i \phi^a, \end{aligned} \tag{3.9}$$

where ∇ here is the covariant derivative associated with the full $(d+1)$ -dimensional bulk metric. The interesting point is that the classical action is independent of g_{ij} and is completely specified by these two constraints.

Classical quantities will be denoted with a bar above them, so \bar{g}_{ij} for example represents the classical metric of the hypersurfaces. As it is usual in the Hamilton-Jacobi framework, the conjugate momenta are related to variations of the action,

$$\pi^{ij} = \frac{-1}{\sqrt{g}} \frac{\delta S}{\delta g_{ij}}, \quad \pi_a = \frac{-1}{\sqrt{g}} \frac{\delta S}{\delta \phi^a}. \tag{3.10}$$

This allows us to rewrite the Hamiltonian constraint (3.9) in terms of variations of the action as

$$\begin{aligned} \{S, S\} &= \mathcal{L}_d, \\ \{S, S\} &\equiv \frac{1}{g} \left[-\frac{1}{d-1} \left(g_{ij} \frac{\delta S}{\delta g_{ij}} \right)^2 + \left(\frac{\delta S}{\delta g_{ij}} \right)^2 + \frac{1}{2} L^{ab} \frac{\delta S}{\delta \phi^a} \frac{\delta S}{\delta \phi^b} \right], \end{aligned} \tag{3.11}$$

where \mathcal{L}_d is the Lagrangian in (3.6). This equation is known as the flow equation and it was introduced in [113]. This will be enough to specify the quantities we are interested in.

In order to solve the flow equation one needs to be careful though. Because AdS has infinite volume the usual holographic renormalization procedure has to be followed:

introduce a cut-off infinitesimally close to the boundary, define local counterterms on this cut-off surface, then push the cut-off to the boundary. The action can then be decomposed as

$$\frac{1}{2\kappa_{d+1}^2}S[g, \phi] = \frac{1}{2\kappa_{d+1}^2}S_{loc}[g, \phi] - \Gamma[g, \phi], \quad (3.12)$$

where S_{loc} represent the part containing the local counterterms and Γ the non-local part that is interpreted as the generating functional of the fields ϕ^a in the curved space given by g_{ij} . To solve the flow equation one then expands the local piece of the action by grouping terms with the same number of derivatives, i.e

$$S_{loc}[g, \phi] = \int d^d x \sqrt{g} \sum_{k=0,2,4,\dots} [\mathcal{L}_{loc}(x)]_k \quad (3.13)$$

where k represents the number of derivatives of each term. Plugging this expansion into the flow equation produces a series of equations relating the bulk action (3.6) with the classical one in (3.12) [114]. The lowest weight equation is given by

$$\mathcal{L}_d = [\{S_{loc}, S_{loc}\}]_0 + [\{S_{loc}, S_{loc}\}]_2. \quad (3.14)$$

If one parametrises the zero weight term as $[\mathcal{L}_{loc}]_0 = W(\phi)$ and the weight two one as $[\mathcal{L}_{loc}]_2 = -\Phi(\phi)\mathcal{R} + \frac{1}{2}M_{ab}g^{ij}\partial_i\phi^a\partial_j\phi^b$ then the equation above can be solved. Several relations and constraints among the quantities introduced arise from this, one of them being

$$V(\phi) = -\frac{d}{4(d-1)}W(\phi)^2 + \frac{1}{2}L^{ab}\partial_a W(\phi)\partial_b W(\phi). \quad (3.15)$$

This is precisely the relation between the potential V and superpotential W one uses when dealing with quantum field theories and RG flows. A generalization of this to Gauss-Bonnet gravity will be used later on in this chapter. Moreover, the weight d equation that arises from the flow equation

$$0 = 2[\{S_{loc}, \Gamma\}]_d - \frac{1}{2\kappa_{d+1}^2}[\{S_{loc}, S_{loc}\}]_d \quad (3.16)$$

yields the relation

$$\frac{2}{\sqrt{g}}g_{ij}\frac{\delta\Gamma}{\delta g_{ij}} - \beta^a(\phi)\frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta\phi^a} = -\frac{1}{2\kappa_{d+1}^2}\frac{2(d-1)}{W(\phi)}[\{S_{loc}, S_{loc}\}]_d. \quad (3.17)$$

The quantity $\beta^a(\phi)$ can actually be interpreted as the RG beta function of the boundary theory. Moreover, since the first term is nothing but the vacuum expectation value of the trace of the energy momentum tensor

$$\langle T^{ij}(x) \rangle_{g,\phi} = \frac{2}{\sqrt{g}} \frac{\delta \Gamma[g, \phi]}{\delta g_{ij}(x)}, \quad (3.18)$$

we can then see that choosing the beta functions to vanish (i.e choosing to sit in a fixed point) then the right-hand side of equation (3.17) computes the Weyl anomaly $\mathcal{W}_d = \langle T^{ij}(x) \rangle_{\beta(\phi)=0}$:

$$\mathcal{W}_d = -\frac{1}{2\kappa_{d+1}^2} \frac{2(d-1)}{W(\phi)} [\{S_{loc}, S_{loc}\}] \Big|_{\beta(\phi)=0}. \quad (3.19)$$

3.2 Gauss-Bonnet gravity and Domain Walls

3.2.1 Gauss-Bonnet gravity

Gauss-Bonnet is a higher derivative generalization of Einstein-Hilbert gravity [115]. The action of the latter is the integral of one of the simplest invariants we have at our disposal: the Ricci scalar. On a d -dimensional manifold one can, however, consider more general actions like

$$S_L \propto \int \sqrt{-g} \left(\mathcal{R} + \sum_{n=2}^{[d/2]} \lambda_n \mathcal{L}_n \right). \quad (3.20)$$

Here \mathcal{L}_n is the Euler density of a $2n$ -dimensional manifold, λ_n are the different coupling constants associated with each of these extra terms and $[d/2]$ is the integer part of $d/2$ (The sum stops at $[d/2]$ because, in d -dimensions, all Euler densities higher than that either vanish or are total derivatives). This extension is called Lovelock gravity. Gauss-Bonnet gravity is obtained by adding only \mathcal{L}_2 and no higher terms.^① Since \mathcal{L}_2 is the 4-dimensional Euler density χ_4 the action of Gauss-Bonnet is then

$$S = \frac{1}{\kappa} \int dx^5 \sqrt{-g} \left[\mathcal{R} + \frac{\lambda L^2}{2} \chi_4 \right]. \quad (3.21)$$

$$\chi_4 = \mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma} - 4\mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} + \mathcal{R}^2$$

A nice feature of GB gravity is that, although its action includes 4th order derivatives of the metric, only up to 2nd order derivatives survive in the equations of motion. This means

^① Note that GB gravity coincides with Lovelock gravity in dimensions lower than 6. Also, adding \mathcal{L}_2 to the action is only non-trivial in dimensions higher than four.

no extra degrees of freedom are introduced when considering this extension. Moreover, AdS spacetime is a solution to Gauss-Bonnet gravity, which allows it to be used in holography. The main reason Gauss-Bonnet is an interesting holographic setup is because Einstein-Hilbert gravity only allows us to study CFTs with equal conformal anomalies $a = c$. The introduction of the Gauss-Bonnet parameter λ permits the anomalies to be different.

3.3 Domain Wall spacetimes: Equations of motion and vacua

As explained at the beginning of this chapter, domain wall spacetimes are interesting set-ups to study because they allow use to understand RG flow in quantum field theories [87]-[106]. The metric of a DW is of the form

$$ds^2 = dr^2 + e^{A(r)} \left(-dt^2 + \eta_{ij} dx^i dx^j \right), \quad (3.22)$$

with the boundary condition that, at infinity, $A(r) \rightarrow r/L_{AdS}$. One then recovers AdS at the boundary, which is the holographic dual of a CFT (the UV CFT). The holographic coordinate r corresponds to the energy scale of the field theory. Moving into the bulk of the DW spacetime, towards smaller values of the radial coordinate, then takes us away from conformality. In other words, a radially ingoing trajectory is the holographic equivalent of the QFT flowing from the CFT in the UV towards the IR.

However, the above pure domain wall metric is not a solution to Gauss-Bonnet gravity (nor to Einstein-Hilbert, for that matter). To solve this one needs to add matter fields into the picture. This amounts to consider the following action,

$$S = \frac{1}{\kappa} \int dx^5 \sqrt{-g} \left[\mathcal{R} + \frac{\lambda L^2}{2} \chi_4 - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - V(\phi) \right]. \quad (3.23)$$

Here $\phi(r)$ is a dimensionless scalar field, λ the dimensionless Gauss-Bonnet coupling and χ_4 the 4-dimensional Euler density. The quantity L is related to the cosmological constant Λ through $2\Lambda = \frac{(d-1)(d-2)}{L^2}$. Note that the cosmological constant is not explicitly written in the action. This is because, when evaluated in the vacua of the system, the potential $V(\phi)$ provides a constant term playing this role.

Extremising this action produces second-order equations of motion. However, one can introduce an auxiliary function $W(\phi)$, known as the superpotential, to transform them into a system of first-order differential equations (see for example [116] or [93]). The

superpotential is related to $V(\phi)$ through^{②③}

$$V(\phi) = 2(W')^2 \left[1 - \frac{2\lambda L^2}{(d-2)^2} W^2 \right]^2 - \frac{(d-1)}{(d-2)} W^2 \left[1 - \frac{\lambda L^2}{(d-2)^2} W^2 \right], \quad (3.24)$$

and the equations of motion in term of $W(\phi)$ read

$$\begin{aligned} \frac{d\phi}{dr} &= 2W'(\phi) \left[1 - \frac{2\lambda L^2}{(d-2)^2} W(\phi)^2 \right], \\ \frac{dA(r)}{dr} &= -\frac{W(\phi)}{d-2}. \end{aligned} \quad (3.25)$$

The above relation (3.24) is a manifestation of a Hamiltonian constraint. Taking the Gauss-Bonnet action, one can introduce an arbitrary lapse-like function multiplying the time component of the Domain Wall metric. The variation of the action w.r.t to this lapse function then gives rise to the Hamiltonian constraint in terms of \dot{A} and the physical potential V . If one demands \dot{A} to be proportional to W (as it is in the Einstein-Hilbert case) and plugs that into the Hamiltonian constraint then the relation (3.24) arises.

The vacua of this system are defined as the configurations of constant ϕ which minimise the physical potential $V(\phi)$.

$$\text{vacua:} \quad \phi = \phi_k^* \quad \text{with} \quad V'(\phi_k^*) = 0. \quad (3.26)$$

Note that in these vacua the potential in (3.24) simplifies to^④

$$V(\phi_k^*) = -\frac{(d-1)}{(d-2)} W(\phi_k^*)^2 \left[1 - \frac{\lambda L^2}{(d-2)^2} W(\phi_k^*)^2 \right]. \quad (3.27)$$

This value of the potential will play the role of the cosmological constant. Note that, since we want this to equal $(d-1)(d-2)/L^2$ at the UV, it will impose some relations between the parameters of the superpotential.

To determine the length scale of the vacua one can look at the Ricci scalar associated with the DW metric,

$$\mathcal{R} = -2(d-1) \left[\frac{d}{2} A'(r) + A''(r) \right] = -2(d-1) \left[\frac{d}{2(d-2)^2} W^2 - \frac{1}{d-2} W' \dot{\phi} \right]. \quad (3.28)$$

② This superpotential is the same that appears in the Introduction 1 of this thesis.

③ The derivative of the superpotential here is w.r.t the scalar field, $W'(\phi) = \partial_\phi W(\phi)$. Derivatives w.r.t the radial coordinate will be denoted with a dot, e.x $\dot{\phi} = \partial_r \phi(r)$.

④ As explained below in (3.30), vacua correspond to either $W' = 0$ or $C_0 = 0$ (see (3.32) for its definition). Either way, the first term in (3.24) vanishes and we are left with the expression shown below.

The equations of motion have been used in the last equality. Note that the second term vanishes in the vacua and that in AdS the Ricci scalar has the form $\mathcal{R} = \frac{d(d-1)}{L_{AdS}^2}$. One then deduces that the length scale of each of the AdS vacua ϕ_k is given by

$$\frac{1}{L_{k,AdS}^2} = \frac{W(\phi_k)^2}{(d-2)^2} \equiv f_k(\phi^*). \quad (3.29)$$

To find these vacua one needs to minimise the potential V . Differentiating (3.24) yields

$$V'(\phi) = W'(\phi)\mathcal{C}_0\mathcal{F}, \quad (3.30)$$

where

$$\mathcal{C}_0 = 1 - \frac{2\lambda L^2}{9}W(\phi)^2, \quad \mathcal{F} = 2W''\mathcal{C}_0 - \frac{8\lambda L^2}{(d-2)^2}W'^2W - \frac{d-1}{d-2}W. \quad (3.31)$$

The quantity \mathcal{C}_0 plays a very important role since it is the coefficient multiplying the kinetic term of the gravitons when one expands the action [117]. It must be positive in order to have a ghost-free theory.

There are then three different types of vacua:

1. Solutions of $W'(\phi^*) = 0$, include the physical solutions and the unphysical ones (not ghost-free).
2. Solutions of $\mathcal{C}_0 = 0$, the so-called topological vacua.
3. Solutions of $\mathcal{F} = 0$, solutions that are not continuously connected to those that exist in Einstein gravity and will not be considered here.

In order for a vacuum solution ϕ^* to be physical it must fulfill:

1. $V(\phi^*) < 0$ or, equivalently, $f_k > 0$. This is the statement that the corresponding cosmological constant is negative and thus we are in AdS.
2. The function $\mathcal{C}_0(\phi_k)$ must be positive

$$\mathcal{C}_0(f_k) = 1 - 2\lambda \frac{W(\phi_k)^2}{(d-2)^2} > 0. \quad (3.32)$$

3. The scalar field fulfills the BF unitarity condition (Breitenlohner-Freedman)

$$-\frac{(d-1)^2}{4L_k} \leq m_\phi^2(\phi_k^*) = V''(\phi_k^*), \quad (3.33)$$

which ensures stability with respect to linear fluctuations of the scalar field (this provides a consistent quantization of the scalar on the AdS background)

The conditions $f_k > 0$ and $\mathcal{C}_0(f_k) > 0$ require all physical vacua to have

$$0 < f_k(\phi_k^*) < \frac{1}{2\lambda}. \quad (3.34)$$

In Gauss-Bonnet, the only solution to $C_0(f_k^{top}) = 0$ is given by

$$f_k^{top} = \frac{1}{2\lambda}. \quad (3.35)$$

This is above the possible physical values of f_k , so the topological vacua in Gauss-Bonnet are not physical.^⑤

3.4 The quartic superpotential: theoretical results

To study the causality of the QFT along the flow it is useful to pick a superpotential simple enough so that the majority of the computations can be done analytically. One such choice is given by the quartic superpotential

$$W(\phi) = -\frac{B}{L} [(\phi^2 - x_0)^2 + D]. \quad (3.36)$$

Keep in mind that one cannot freely chose all three parameters B, D, x_0 present here. Remember that, in each of the vacua, the potential (3.27) plays the role of the cosmological constant. Requiring that, in the UV, we get a cosmological constant equal to $(d-1)(d-2)/L^2$ imposes the relation

$$\frac{B^2 D^2}{(d-2)^2} = \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda}, \quad (3.37)$$

so the three parameters we can choose are x_0, λ and either B or D .

The physical vacua of this potential are the zeros of W' and sit at $\phi_{UV}^2 = x_0$ and $\phi_{IR}^2 = 0$.

These are the UV and IR of our RG flow, and their AdS radii are given by (3.29) as

$$\frac{L^2}{L_{UV}^2} = \frac{B^2 D^2}{(d-2)^2}, \quad \frac{L^2}{L_{IR}^2} = \frac{B^2 (D + x_0^2)^2}{(d-2)^2}. \quad (3.38)$$

^⑤ Topological vacua can be physical in higher derivative gravities involving cubic terms.

3.4.1 Solving the EOM

To work out the solutions $A(r)$ and $\phi(r)$ for our system we will need equations (3.25). We will take the ratio of these to find $\frac{d\phi^2}{dA}$ and solve for $A = A(\phi)$. Afterwards, the first equation can be solved for $r = r(\phi)$. Together they provide an implicit solution for $A = A(r)$.

The computation actually simplifies if one works with ϕ^2 , so the quotient we will look at is

$$\frac{d\phi^2}{dA} = \frac{d\phi^2/dr}{dA/dr} = 2\phi \frac{d\phi/dr}{dA/dr}. \quad (3.39)$$

Plugging in (3.25) this is

$$\frac{d\phi^2}{dA} = -12 \frac{\phi W'(\phi)}{W(\phi)} \left[1 - \frac{2\lambda L^2}{9} W^2(\phi) \right] \quad (3.40)$$

Note that for our superpotential W^2 and $\frac{\phi W'}{W}$ are functions of ϕ^2 (that being the reason why we chose this quotient). We can now integrate to get

$$\int dA = \frac{-1}{12} \int \frac{d\phi^2}{\frac{\phi W'}{W} \left[1 - \frac{2\lambda L^2}{9} W^2(\phi) \right]} \quad \longrightarrow \quad A = A(\phi). \quad (3.41)$$

The other equation we need is

$$\frac{d\phi^2}{dr} = 2\phi \frac{d\phi}{dr} = 4\phi W' \left[1 - \frac{2\lambda L^2}{9} W^2(\phi) \right], \quad (3.42)$$

which allows us to compute

$$\int dr = \int \frac{d\phi^2}{4\phi W' \left[1 - \frac{2\lambda L^2}{9} W^2(\phi) \right]} \quad \longrightarrow \quad r = r(\phi). \quad (3.43)$$

$\lambda > 0$ case Let's start by looking at the equation for $A = A(\phi)$,

$$\frac{d\phi^2}{dA} = -12 \frac{\phi W'(\phi)}{W(\phi)} \left[1 - \frac{2\lambda L^2}{9} W^2(\phi) \right]. \quad (3.44)$$

The difference with the Einstein-Hilbert case is that now we have a factor $(1 - 2\lambda L^2 W^2/9)$ multiplying the r.h.s of our ODE. This integral can be easily performed if one takes this polynomial and factorises it, writing it as

$$1 - \frac{2\lambda L^2}{9} W^2 = \text{const} \times \prod_{i=1}^4 (\phi^2 - \phi_i^2). \quad (3.45)$$

Note that this polynomial is precisely \mathcal{C}_0 , so its roots correspond to the topological vacua. Since they occur at $f_k^{\text{top}} = \frac{1}{2\lambda}$ their AdS scale is given by

$$\frac{L^2}{L_{top}^2} = \frac{1}{2|\lambda|}. \quad (3.46)$$

We have introduced an absolute value here preparing for the $\lambda < 0$ case. This way we avoid L/L_{top} becoming imaginary, which is traded by the appearance of the imaginary unit i in several parts of the equations.

Plugging in the explicit form of the superpotential one finds that the roots of C_0 sit at the ϕ_i values

$$\begin{aligned} \phi_i^2 &= x_0 + u_i, & u_1 &= -u_2 \equiv u, & u_3 &= -u_4 \equiv \tilde{u} \\ u &= \sqrt{D} \sqrt{-1 + \frac{L_{UV}}{L_{top}}}, & \tilde{u} &= i\sqrt{D} \sqrt{1 + \frac{L_{UV}}{L_{top}}}. \end{aligned} \quad (3.47)$$

The function C_0 can then be written as

$$1 - \frac{2\lambda B^2}{9} W^2 = -\frac{2\lambda B^2}{9} \prod_{i=1}^4 (\phi^2 - \phi_i^2). \quad (3.48)$$

With this factorization the EOM for $A(\phi)$ becomes

$$\int dA = \frac{3}{32\lambda B^2} \int d\phi^2 \frac{(\phi^2 - x_0)^2 + D}{\phi^2(\phi^2 - x_0)} \frac{1}{\prod_{i=1}^4 (\phi^2 - \phi_i^2)} \quad (3.49)$$

The integrand can then be separated into several fractions, each of which integrate to a logarithm. The result from this integral is

$$\begin{aligned} A(\phi^2) &= \frac{3}{32\lambda B^2} \left\{ \frac{D + x_0^2}{x_0 \prod_{i=1}^4 \phi_i^2} \log \phi^2 - \frac{D}{x_0 \prod_{i=1}^4 u_i} \log(\phi^2 - x_0) \right. \\ &\quad \left. - \sum_{k=1}^4 \frac{D + u_k^2}{u_k \phi_k^2 \prod_{i=1, i \neq k}^4 (u_k - u_i)} \log(\phi^2 - \phi_k^2) \right\} \end{aligned} \quad (3.50)$$

which can be compactly written as

$$A(\phi) = s_{IR} \log|\phi^2| + s_{UV} \log|\phi^2 - x_0| + \sum_{k=1}^4 s_k \log|\phi^2 - \phi_k^2|, \quad (3.51)$$

Note that this is an indefinite integral, so one has the freedom to add a constant term C . The constant must be chosen such that in the limit $\lambda \rightarrow 0$ the Einstein-Hilbert result is recovered. The tricky part here is that this constant turns out to not be finite, so it is not possible to deal with it in a numerical setup, as we plan to do. However, this constant term C drops out of the final result we want to check (to be shown later), so the computation can be carried out. This means that the $A(\phi)$ shown above is not complete (since it lacks this C term), but since A alone is not used anywhere we are safe to proceed keeping it as

presented here.

Plugging in the expressions for the topological vacua one can write the coefficients appearing in $A(\phi)$ as

$$\begin{aligned}
s_{IR} &= \frac{-9}{32\lambda B^3 x_0} \frac{L/L_{IR}}{(x_0^2 + D)^2 - D^2 \frac{L_{UV}^2}{L_{top}^2}}, \\
s_{UV} &= \frac{9}{32\lambda B^3 x_0} \frac{L/L_{UV}}{D^2 \left(1 - \frac{L_{UV}^2}{L_{top}^2}\right)}, \\
s_{1,2} &= \frac{1}{64B\phi_{1,2}^2} \frac{L_{UV}}{L_{top}} \frac{L/L_{top}}{-1 + L_{UV}/L_{top}}, \\
s_{3,4} &= \frac{-1}{64B\phi_{3,4}^2} \frac{L_{UV}}{L_{top}} \frac{L/L_{top}}{1 + L_{UV}/L_{top}},
\end{aligned} \tag{3.52}$$

It is important to note that, although the coefficients s_i in front of the topological vacua are complex, the final result for $A(\phi)$ is real-valued.

$\lambda < 0$ case Almost all the computations performed in the positive λ case carry-on for the negative case. The only difference is some sign changes and the appearance of an extra couple factors of i . The results for negative λ are summarised as

$$1 + \frac{2|\lambda|L^2}{9}W^2 = \frac{2\lambda L^2}{9} \prod_{i=1}^4 (\phi^2 - \phi_i^2). \tag{3.53}$$

$$\begin{aligned}
\phi_i^2 &= x_0 + u_i, & u_1 &= -u_2 \equiv u, & u_3 &= -u_4 \equiv \tilde{u} \\
u &= \sqrt{D} \sqrt{-1 + i \frac{L_{UV}}{L_{top}}}, & \tilde{u} &= i\sqrt{D} \sqrt{1 + i \frac{L_{UV}}{L_{top}}}.
\end{aligned} \tag{3.54}$$

$$A(\phi) = s_{IR} \log|\phi^2| + s_{UV} \log|\phi^2 - x_0| + \sum_{k=1}^4 s_k \log|\phi^2 - \phi_k^2|, \tag{3.55}$$

$$\begin{aligned}
s_{IR} &= \frac{9}{32|\lambda|B^3 x_0 \lambda} \frac{L/L_{IR}}{(x_0^2 + D)^2 + D^2 \frac{L_{UV}^2}{L_{top}^2}}, \\
s_{UV} &= \frac{-9}{32|\lambda|B^3 x_0} \frac{L/L_{UV}}{D^2 \left(1 + \frac{L_{UV}^2}{L_{top}^2}\right)}, \\
s_{1,2} &= \frac{-1}{64B\phi_{1,2}^2} \frac{L_{UV}}{L_{top}} \frac{L/L_{top}}{-1 + iL_{UV}/L_{top}}, \\
s_{3,4} &= \frac{1}{64B\phi_{3,4}^2} \frac{L_{UV}}{L_{top}} \frac{L/L_{top}}{1 + iL_{UV}/L_{top}},
\end{aligned} \tag{3.56}$$

The function $A(\phi)$ is still real-valued even if now all the topological vacua are complex.

3.4.2 The causality bounds

As shown in the coming paper [86], causality of the boundary theory imposes some bounds on the function $A(r)$. These are

$$\begin{aligned} 1 + \frac{2\lambda\partial_r A\partial_r f}{(1 - 2\lambda(\partial_r A)^2)f} &\geq 0 \\ 1 - \frac{2\lambda\partial_r A\partial_r f}{(1 - 2\lambda(\partial_r A)^2)f} &\geq 0 \\ 1 - \frac{\lambda\partial_r A\partial_r f}{(1 - 2\lambda(\partial_r A)^2)f} &\geq 0, \end{aligned} \tag{3.57}$$

where the function f is defined as

$$f(r) = \int_r^\infty \frac{e^{-4A(s)}}{1 - 2\lambda(\partial_s A(s))^2} ds. \tag{3.58}$$

Defining

$$\Omega \equiv \frac{2\lambda\partial_r A\partial_r f}{(1 - 2\lambda(\partial_r A)^2)f} \tag{3.59}$$

the constraints then read

$$-1 \leq \Omega \leq 1. \tag{3.60}$$

Note how, as announced earlier, the quantity Ω does not depend on the constant C we have been neglecting since (3.51). Since C appears only inside of A (and not in its derivatives) it enters the bound as the quotient $\partial_r f/f$. Now, from the definition of f above it is clear that C just contributes a constant factor, i.e instead of f we should have $e^C f(r)$. Since the r -derivative of f yields just the integrand in (3.58) we see that C enters $\partial_r f$ in the same fashion, i.e we should have $e^C \partial_r f$ instead of $\partial_r f$. The quotient that appears in Ω then is independent of the constant term C since both exponentials cancel each other out, and we are free to work out the bound without including C .

All these equations are written in terms of $A(r)$. However, what we computed is $A(\phi^2)$, so we're going to rewrite everything in terms of it. Let's first look at $f(r)$: The denominator can be written in terms of the superpotential using the EOM (3.25); the numerator is already known in terms of ϕ^2 , and the integration measure ds can be transformed into $d\phi^2$ using the chain rule $ds = \frac{ds}{d\phi^2} d\phi^2$. We then have

$$f(\phi^2) = \int_{\phi^2}^{x_0} \frac{e^{-4A(\phi^2)}}{1 - 2\lambda\frac{W(\phi^2)^2}{(d-2)^2}} \left(\frac{d\phi}{dr}\right)^{-1} d\phi^2. \tag{3.61}$$

Note that $d\phi/dr$ is also given in terms of W by (3.25). This integral can then be performed numerically to obtain the function f . On the other hand, the quantity $\partial_r f$ is given by minus the integrand in (3.58), so it is also completely known in terms of W and A . We then have all we need in order to compute the bound Ω .

3.5 The quartic superpotential: numerical results

Now that we have the general formula for Ω we can compute it for the quartic superpotential and let the theory flow. Our system is originally at the UV fixed point ($\phi^2 = x_0$) and starts flowing towards the IR ($\phi^2 = 0$). The position of the un-physical topological vacua ϕ_i relative to the UV and IR depends on the choice of values for the superpotential parameters we make. Having a look at equation (3.47) one can easily see that the topological vacuum ϕ_2 can sit in between the UV and IR for certain choices of these parameters (specifically, the choice that makes $0 < u < x_0$ in (3.47)). Since this vacuum is not physical, it is not good that the system runs into it. Note however that the key results that are presented here happen regardless of whether this topological vacuum is encountered during the flow or not. Moreover, even in the cases where it is present at some point during the flow, the violation shown here kicks in before ϕ_2 is reached^⑥.

For this superpotential, every single quantity we have presented so far is analytically computable except for $f(r)$ in (3.61). Note that every quantity inside this integral is either explicitly written in terms of $A(\phi)$ or $W(\phi)$ or it can be related to them through the EOM.

To carry on the numerical computations we need to fix the values of B , x_0 and λ . Changing the values of B and x_0 is not too relevant for the matter we're dealing with. Among other things it changes up until what point the flow can go (at some point, which depends on these parameters, the unphysical topological vacua enter into the flow, forcing it to stop), but in the region where the flow is valid it doesn't alter the general behaviour of $\Omega(\phi)$. Because of this we are going to set $B = 1/2$ and $x_0 = 1$ ^⑦ and study $\Omega(\phi)$ for different values of the Gauss-Bonnet parameter. Previous works in AdS Gauss-Bonnet ([118], [119]) showed that the theory does not violate causality if and only if $-\frac{7}{36} \leq \lambda \leq \frac{9}{100}$, so we will test values within this window. The bound for Ω is saturated in the UV when λ equals any of these two extreme values (specifically, $\Omega(\phi_{UV})$ becomes +1 when λ is set to

⑥ For the choice of parameters shown here the topological vacuum ϕ_2 does not appear in between the UV and IR fixed points.

⑦ The choice is not random. Values around these make the numerical computations more precise.

it's lowest allowed value and -1 when set to it's highest). It is then a good idea to start with one of these extreme values and check if our numerically computed Ω saturates the bound in the UV. Setting $\lambda = 9/100$ and running the numerics one obtains

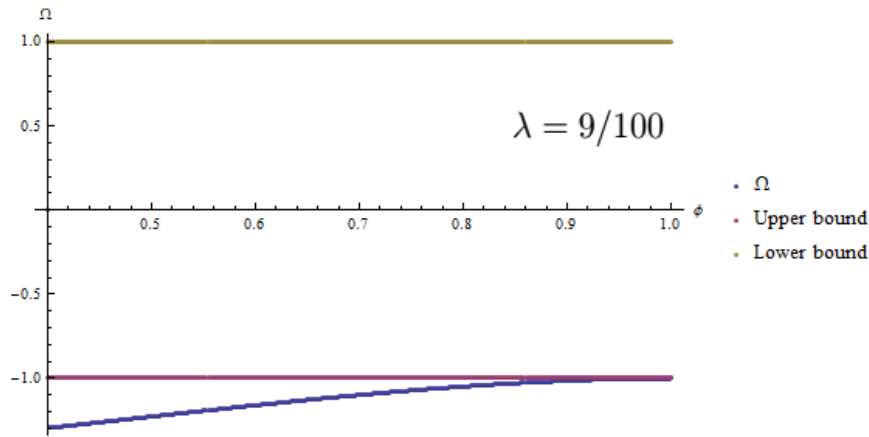


Figure 3.1: Values of $\Omega(\phi)$ for the extreme case $\lambda = 9/100$ with the parameter values set to $B = 1/2$ and $x_0 = 1$.

As we can see the bound is indeed saturated at the boundary $\phi_{UV} = 1$. When the theory is let to flow towards the IR though the bound becomes violated everywhere. Before drawing any conclusion let's compute $\Omega(\phi)$ for non-extremal values of λ . Three of such cases are shown in the following picture

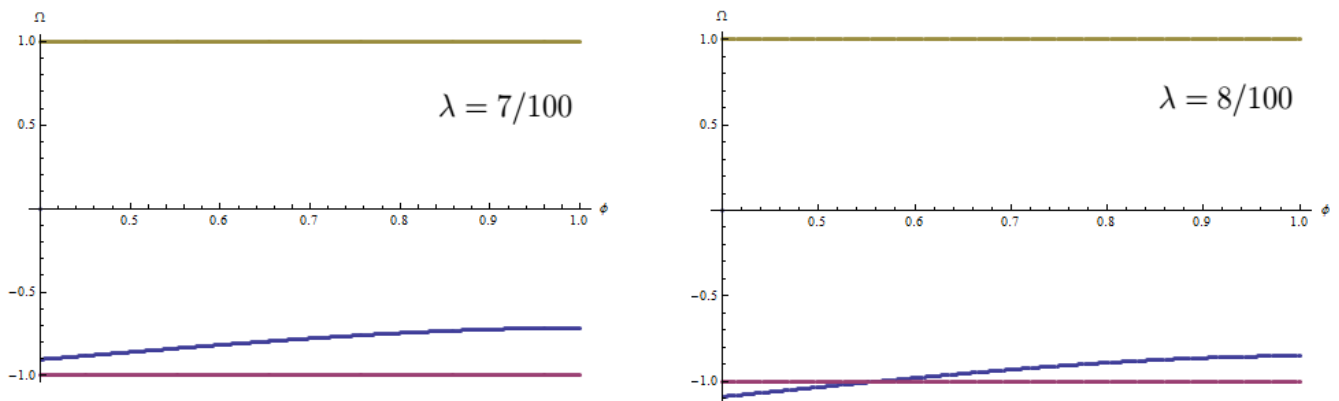


Figure 3.2: Values of $\Omega(\phi)$ for two non-extreme values of λ .

We can see that the violation is alleviated when the value of λ is not extremal. If the Gauss-Bonnet parameter is still big enough causality is still violated when a certain point in the RG flow is past, but if we keep lowering λ we end up having a consistent flow all the way down from the UV to the IR.

For negative values of λ (see graphs in the next page) the situation is basically the same. In the extreme case $\lambda = -7/36$ the bound is saturated at the UV and violated everywhere else, with bigger values of λ alleviating this behaviour as in the positive case.

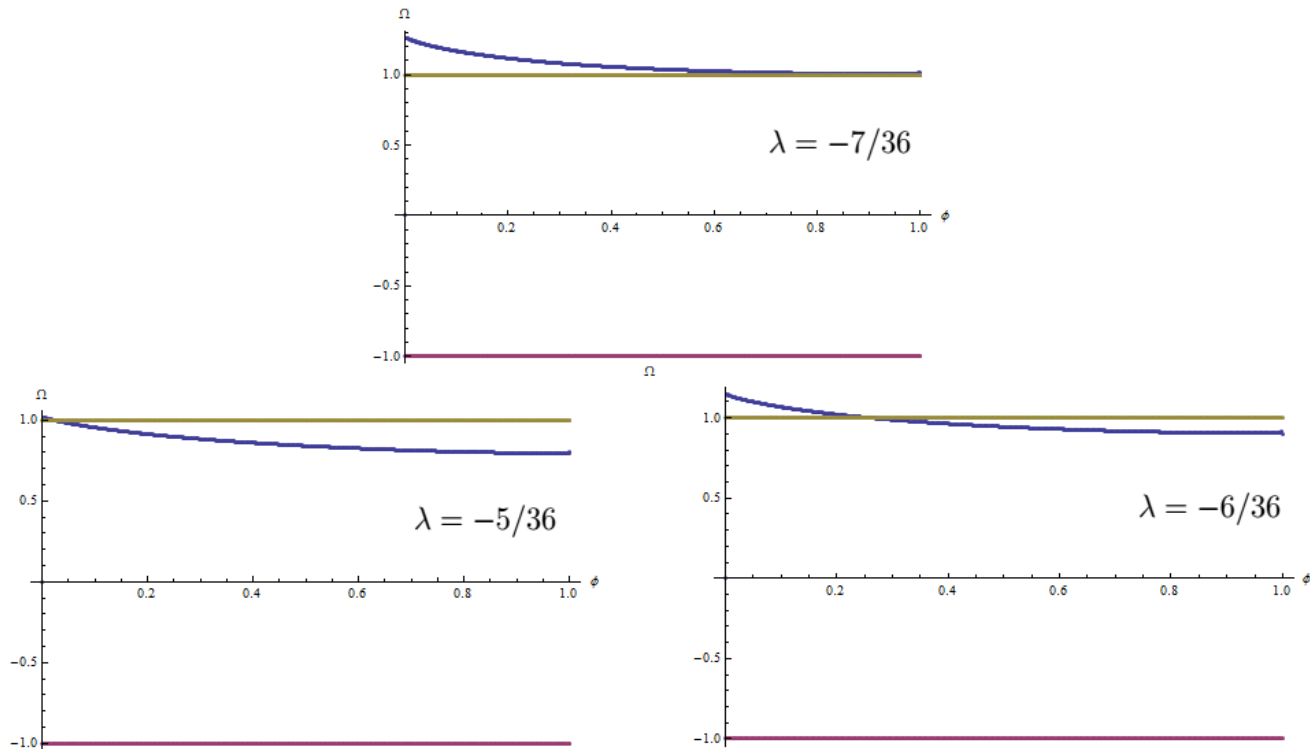


Figure 3.3: Values of $\Omega(\phi)$ for the extreme negative value and two non-extreme ones of λ .

The window of λ where the flow is fully consistent between the UV and IR is not universal. These values depend on the parameters of the superpotential.

3.6 Conclusions

The above analysis shows that, for some values of the Gauss-Bonnet coupling, a UV-consistent CFT can become inconsistent when it flows towards the IR. This is not the usual behaviour one expects, since we tend to assume that the effective field theories in the IR are well behaved when they are derived from a consistent UV theory. However, causality is usually understood at the level of 2-point functions, but it is possible that some higher point function behaves in an undesired way after some deformation is turned on in the CFT (thus starting the flow). This result shows that it may be necessary to pay closer attention to other quantities, beyond 2-point functions, in order to properly understand the behaviour of a theory along its RG flow.

It would be interesting to generalise this analysis to further superpotentials in order to check if this is just a feature of this quartic superpotential or it is shared by other flows.

Appendix A

Appendix: The Complexity-Action proposal and Gauss-Bonnet gravity

This appendix is devoted to bringing up the technical problems that arise when one tries to generalise the CA proposal to Gauss-Bonnet gravity. When studying the action functional of a gravity theory it is a well know fact that one needs to include some surface terms in order for the variational problem to be well defined [120]-[123]. The situation, however, is a bit trickier when the action is integrated over a finite region with a piecewise boundary containing lightlike surfaces. This setup was recently analysed, for the case of Einstein-Hilbert gravity, by Lehner et al. in [124]^①. As far as I know, how to deal with this non-smooth lightlike boundaries in the Gauss-Bonnet case has not been tackled yet. In the section below I will show the difficulties one finds when trying to generalise the results that appear in the papers cited here.

A.0.1 Variational problems and boundary terms

Consider a variational problem in gravity, where we are given an action functional of the form

$$S = \int_{\mathcal{M}} L(g)dx, \tag{A.1}$$

where \mathcal{M} is the manifold we are integrating over, x are coordinates on it and $L(g)$ the Lagrangian of the corresponding gravity theory. The aim here is to find if the variational problem is well defined. That is, if one can impose $\delta_g S = 0$ and get the equations of motion imposing *only* that $\delta g^{\mu\nu} = 0$ in the boundary $\partial\mathcal{M}$. It is clear that the action (A.1)

^① this was actually addressed to prove that the less rigorous approach taken in [21] yields the correct results.

by itself is not enough, the reason being that the variations of the Ricci scalar and the Christoffel connection are

$$\begin{aligned}\delta\mathcal{R} &= \nabla_\rho \left(g^{\mu\nu} \delta\Gamma_{\nu\mu}^\rho - g^{\mu\nu} \delta\Gamma_{\nu\mu}^\rho \right) \\ \delta\Gamma_{\delta\beta}^\alpha &= \frac{1}{2} g^{\alpha\lambda} \left(\nabla_\delta \delta g_{\lambda\beta} + \nabla_\beta \delta g_{\delta\lambda} - \nabla_\lambda \delta g_{\delta\beta} \right).\end{aligned}\tag{A.2}$$

Thus, even then variation of the simplest gravitational action $L = \mathcal{R} - 2\Lambda$ will involve terms of the form $\partial_\alpha(\delta g_{\mu\nu})$ and $\partial_\alpha\partial_\beta(\delta g_{\mu\nu})$. This immediately shows that the bulk action is not enough, since to make this to vanish one would also need to impose that the derivatives of the metric vanish on the boundary. One can, however, remedy this problem. Using Stokes theorem these kind of terms can be transformed into boundary terms. For example, terms with one derivative hitting the metric variation become

$$\begin{aligned}& \int_{\mathcal{M}} d^D x A^\alpha{}_{\mu\nu} \nabla_\alpha \delta g^{\mu\nu} \\ &= \int_{\mathcal{M}} d^D x \left[\nabla_\alpha \left(A^\alpha{}_{\mu\nu} \delta g^{\mu\nu} \right) - \nabla_\alpha A^\alpha{}_{\mu\nu} \delta g^{\mu\nu} \right] \\ &= \int_{\partial\mathcal{M}} d\Sigma_\alpha A^\alpha{}_{\mu\nu} \delta g^{\mu\nu} - \int_{\mathcal{M}} d^D x \nabla_\alpha A^\alpha{}_{\mu\nu} \delta g^{\mu\nu},\end{aligned}\tag{A.3}$$

where $d\Sigma_\mu = d\Sigma n_\mu$, n_μ and outgoing normal vector to the boundary hypersurface $\partial\mathcal{M}$ and A represents whatever terms arise when taking the variation of the action. The terms with two derivatives can be subject to the same type of process, but since now we can apply Stokes twice we will be able to push terms to the boundary of the boundary $\partial\partial\mathcal{M}$. For example, a term of the form $\nabla_\alpha \nabla_\beta \delta g^{\alpha\beta}$ yields a contribution of the form

$$\begin{aligned}& \int_{\mathcal{M}} d^D x A \nabla_\mu \nabla_\nu \delta g^{\mu\nu} = \int_{\mathcal{M}} d^D x \nabla_\nu \nabla_\mu A \delta g^{\mu\nu} \\ & - \int_{\partial\mathcal{M}} d\Sigma [n_\nu \nabla_\mu A + \nabla_\nu (n_\mu A)] \delta g^{\mu\nu} + \int_{\partial\partial\mathcal{M}} dS_\nu n_\mu A \delta g^{\mu\nu},\end{aligned}\tag{A.4}$$

where $dS_\mu = dS s_\mu$ is the outgoing surface element of the boundary of the boundary. Note that these extra terms are only relevant when the region of integration has a non-smooth boundary since, otherwise, $\partial\partial\mathcal{M}$ would be empty. When the boundary is piecewise, this boundary of the boundary regions are where the different boundary segments join, and thus terms integrated over $\partial\partial\mathcal{M}$ will be called joint terms from now on.

Taking all into account one ends up having a variation which, schematically, look as follows

$$\delta S = \int_{\mathcal{M}} d^D x E_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d\Sigma n_\mu \mathcal{A}_\nu \delta g^{\mu\nu} + \int_{\partial\partial\mathcal{M}} dS s_\mu \mathcal{B}_\nu \delta g^{\mu\nu}.\tag{A.5}$$

The variational problem is then well defined if one is able to find quantities A and B such that their variations precisely match $n_\mu \mathcal{A}_\nu \delta g^{\mu\nu}$ and $s_\mu \mathcal{B}_\nu \delta \mu^\nu$. If this is the case then we just need to complement the original bulk action with minus the boundary and joint integrals of these A and B . Since these new pieces will precisely cancel the unwanted boundary terms that stem from the bulk action we will end up with a properly defined variational problem.

This process works for any kind of gravitational theory one could consider, including Gauss-Bonnet gravity. The challenging and non-trivial part of the process is to find a quantity which variation has the desired form. This task has been fully completed in Einstein-Hilbert gravity, where one encounters the well-know boundary term given by the York-Gibbons-Hawking integral (the extrinsic curvature) and the not so well known joint terms shown in [122]-[124]. The case including light-like boundaries in EH gravity was also covered in the last reference.

It also important to mention that dealing with lightlike surfaces (like the ones present in the WdW patch) introduces several ambiguities in the computation. Specifically, the choice of parametrization of the null generators of the hypersurface and the function $\Phi(x^i)$ that defines it can be arbitrarily changed. It was shown in [124] that these ambiguities can be dealt with by imposing reasonable conditions (choosing the null generators to be affinely parametrised, imposing the gravitational action to be additive and choosing a normalization condition for the normal vector of the null surfaces near the boundary of AdS), after which one is guaranteed to find a univocal result. Armed with these formal results, Lehner et al. showed that the less rigorous approach taken by Brown et al. in [21] still yields the same result.

The case of Gauss-Bonnet gravity with non-smooth boundaries, however, has not been studied yet. Following the steps above yields a complicated variation with no obvious candidate A to choose such that δA mimics it. The joint terms, for example, yield

$$\begin{aligned} & \delta \int_{\mathcal{M}} d^D x \sqrt{-g} L_{GB} = \dots \\ & + \int_{\partial\partial\mathcal{M}} dS n_\sigma \left[2 \left(n_\gamma g^{\beta\sigma} g^{\gamma\alpha} - n_\lambda g^{\alpha\beta} g^{\lambda\sigma} \right) \mathcal{R} - 8n_\gamma \mathcal{R}^{\beta\sigma} g^{\gamma\alpha} \right. \\ & \quad \left. + 4n_\rho \mathcal{R}^{\alpha\beta} g^{\rho\sigma} - 4n_\gamma \mathcal{R}^{\sigma\gamma} g^{\alpha\beta} + 4n_\gamma \mathcal{R}^{\beta\sigma\gamma\alpha} \right] \delta g_{\alpha\beta}, \end{aligned} \quad (\text{A.6})$$

where the dots represent bulk and surface terms that have been omitted. Because we

don't have a well-defined variational problem to work with, nor can guarantee that the less formal approach of Brown et al. will still yield the correct answer when Gauss-Bonnet is considered, we ended up testing the *EH+fundamental matter* generalization of the complexity-action proposal instead of the Gauss-Bonnet one.

As a side remark, note that the DBI action considered in Chapter 2 is free from these problems even if we are still integrating over a region with a non-smooth boundary. This is because the action is simply $S_{DBI} \sim \int \sqrt{-g}$. Since the variation $\delta\sqrt{-g}$ only gives rise to terms proportional to $\delta g^{\mu\nu}$ no boundary term is needed to make this variational problem well defined.

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