MULTIDIMENSIONAL SECOND ORDER GENERALISED STOCHASTIC PROCESSES ON LOCALLY COMPACT ABELIAN GROUPS

By

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This thesis has not been submitted as an exercise for a degree at this or any other University.

This thesis is entirely my own work, except where indicated in the text.

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Bernard Keville

To My Parents

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Summary

This thesis is concerned with the harmonic analysis of multidimensional generalised stochastic processes on locally compact Abelian groups. A multidimensional generalised stochastic process is a continuous linear operator from a test function space into a space of $\mathcal{H}$-valued random variables, where $\mathcal{H}$ is a separable Hilbert space. The remarkable properties and very simple structure of the Feichtinger algebra $S_0(G)$ make it very suitable as a test function space in this respect. Classical representation theorems for stationary and harmonisable processes on locally compact Abelian groups which have been extended to infinite dimensions can be proved in in much more compact way, avoiding much of the technical machinery associated with operator valued integration and the theory of operator valued bimeasures.

Chapter 1 considers the Feichtinger algebra and its applications in abstract harmonic analysis. In particular, Bochner's theorem is extended to the dual space $S_0(G)'$ and a representation theorem is proved which is an essential prerequisite for the proof of the spectral representation theorem for stationary generalised stochastic processes.

Chapter 2 uses the results of chapter 1 to treat the harmonic analysis of scalar valued generalised stochastic processes, extending some of the classical results for stochastic processes, whose proof then is much simpler than heretofore, avoiding the technicalities associated to the theory of integration.

Chapter 3 is concerned with the abstract harmonic analysis of infinite dimensional stochastic processes. Again, in this setting, the properties of $S_0(G)$ prove to be very useful in deriving concise and elementary proofs of some of the results obtained by Kakihara in his monograph.

Chapter 1

$S_0(G)$ and the Bochner-Schwartz-Godemont Theorem

1.1 Introduction and general concepts

The Feichtinger algebra $S_0(G)$ has many remarkable properties, which, allied to its simple structure and the fact that its dual space is a space of tempered distributions which contains many of the classical spaces of interest in abstract harmonic analysis, make it very suitable as a space of test functions on locally compact Abelian groups. The formation of tensor products facilitates the harmonic analysis of vector-valued structures; in particular, the projective tensor product of $S_0(G)$ with a suitable Hilbert space of random variables yields a test space of second order stochastic processes. Central to the analysis of stationary second order stochastic processes are various interrelated notions in harmonic analysis such as positive-definiteness, the Bochner theorem, translation-invariant Hilbert spaces, unitary representations etc. We will show how the algebra $S_0(G)$ can be used to prove and extend the results necessary for later analysis of second order stochastic processes. In particular, the

Bochner theorem is extended to a space of unbounded measures and a closely related representation theorem for diagonally-invariant translation-bounded quasimeasures on $G \times G$ is proved which is later employed for the harmonic analysis of stationary second order stochastic processes with distributional kernels. We consider initially some basic concepts and establish notational conventions. We begin with some background material drawn from [2, 16, 30, 31] on analysis on locally compact Abelian groups.

1.1.1 Measure and integration on locally compact Abelian groups

The algebra of continuous functions with support in a compact subset K of a locally compact Abelian group G, equipped with the uniform norm, is denoted $C_K(G)$. The union of all such spaces,

$$\mathcal{K}(G) = \bigcup_K C_K(G),$$

is a locally convex function algebra [30] when equipped with its natural inductive limit topology. $\mathcal{K}(G)$ is continuously embedded into its dual space, the space of *Radon measures*,

$$\mathcal{R}(G) = \bigcap_K C_K(G)',$$

equipped with its natural projective limit topology. Pointwise multiplication may be extended by transposition to make $\mathcal{R}(G)$ a locally convex topological $\mathcal{K}(G)$-module. Given a Radon measure, $\mu \in \mathcal{R}(G)$, and $\phi \in \mathcal{K}(G)$, the value $\langle \phi, \mu \rangle = \mu(\phi)$ is an *integral*, denoted

$$\int_G \phi(x) d\mu(x).$$

A Radon measure μ is *positive* if it acts non-negatively on the positive cone of non-negative functions in $\mathcal{K}(G)$. The group G acts through translation on $\mathcal{K}(G)$; given $x \in G$, the corresponding translation operator L_x is defined by

$$L_x\phi(y) = \phi(y - x).$$

This action of G extends through transposition to $R(G)$ and it is well known that there exists a positive translation-invariant Radon measure which is unique up to multiplication by a scalar. We denote this *Haar measure* by dx. $\mathcal{K}(G)$ is also a topological algebra with respect to convolution, which is defined by

$$\phi \star \psi(x) = \int_G \phi(x - y)\psi(y)dy$$

and $\mathcal{R}(G)$ is a topological $\mathcal{K}(G)$-module with respect to convolution, defined by transposition. The involutions on $\mathcal{K}(G)$ defined by $\tilde{\phi}(x) = \phi(-x)$ and $\phi^\star(x) = \overline{\phi}(-x)$ extend to $\mathcal{R}(G)$ by transposition. The uniform completion of $\mathcal{K}(G)$ is the algebra $C_0(G)$ of *continuous functions vanishing at infinity.* Since $\mathcal{K}(G)$ is continuously and densely embedded into $C_0(G)$, it is clear that any Radon measure $\mu \in \mathcal{R}(G)$ which is uniformly bounded on $\mathcal{K}(G)$ in the sense that

$$sup_K \|\mu\|_{C_K(G)'} < \infty$$

extends to an element of the dual space of $C_0(G)$. $C_0(G)' = M(G)$ is the space of *bounded Radon measures* and is a commutative $\star$-Banach algebra with respect to convolution.

1.1.2 The Fourier transform

The completion of the image of the continuous embedding,

$$\phi \mapsto \phi dx,$$

of $\mathcal{K}(G)$ into $M(G)$ is $L^1(G)$, the algebra of *absolutely continuous bounded measures* or *integrable functions.*

The set of $L^1(G)$-homomorphisms, equipped with the weak-$\star$ topology, is a locally compact Abelian group, $\widehat{G}$, the *dual group*, whose elements are the *characters* of G. Each $\phi \in L^1(G)$ defines a function, $\widehat{\phi}$, on the dual group according to

$$\widehat{\phi}(\widehat{x}) = \int_G \phi(x)\overline{\langle x, \widehat{x}\rangle} dx, \widehat{x} \in \widehat{G}.$$

The linear mapping

$$\mathcal{F}_G : \phi \mapsto \widehat{\phi}$$

thus defined is the *Fourier transform*; its isometric image is the *Fourier algebra*, $A(\widehat{G})$. The identity

$$\langle \check{\phi}, \widehat{\psi} \rangle = \langle \phi, \psi \rangle, \phi \in L^1(G), \psi \in L^\infty(G),$$

where

$$\check{\phi}(\widehat{x}) = \int_G \phi(x)\langle x, \widehat{x}\rangle dx, \widehat{x} \in \widehat{G},$$

defines an extension of the Fourier transform by duality to an isometry of $L^\infty(G)$ onto $A(\widehat{G})'$, the algebra of *pseudomeasures.*

Definition 1. $h \in L^\infty(G)$ is of *positive type* if it is positive on the algebra $L^1(G)$ in the sense that

$$\langle \phi \star \phi^\star, h \rangle \geq 0, \forall \phi \in L^1(G).$$

Functions of positive type are characterised by the classical *Bochner theorem.*

Theorem 1. *(The Bochner Theorem)*
A function h on the locally compact Abelian group G is of positive type if and only if there exists a positive, bounded measure $\mu_h \in M(\widehat{G})$ such that

$$h(x) = \int_{\widehat{G}} \langle x, \widehat{x} \rangle \; d\mu_h(\widehat{x}).$$

Given a positive definite function h and μ_h as above, we have that

$$\|h\|_{L^\infty(G)} = h(0) = \|\mu_h\|_{M(\widehat{G})},$$

and we note that the character $\widehat{x}$ is the function of positive type on G associated to the *Dirac measure* $\delta_{\widehat{x}}$ on $\widehat{G}$.

Each $x \in G$ determines a character $\widehat{\widehat{x}}_x$ of $\widehat{G}$ according to

$$\langle \widehat{x}, \widehat{\widehat{x}}_x \rangle = \langle x, \widehat{x} \rangle.$$

In fact, all characters of the dual group are determined in this way, which is the content of the following theorem.

Theorem 2. *(Pontryagin duality theorem)*

$$x \mapsto \widehat{\widehat{x}}_x$$

is a homeomorphic isomorphism of the groups G and $\widehat{\widehat{G}}$.

Theorem 3. *(The Plancherel theorem) There exists a normalisation of the dual Haar measure $d\widehat{x}$ such that the Fourier transform is unitary from the dense subspace $L^1 \cap L^2(G)$ of $L^2(G)$ into $L^2(\widehat{G})$ and hence extends to a unitary isomorphism of $L^2(G)$ onto $L^2(\widehat{G})$.*

Hence, for each $\phi \in L^2(G)$, we have

$$\int_G |\phi(x)|^2 dx = \int_{\widehat{G}} |\widehat{\phi}(\widehat{x})|^2 d\widehat{x}.$$

Theorem 4. *(The inversion theorem) Given that the respective Haar measures on G and its dual have been normalised in order to ensure the validity of the Plancharel theorem, every $\phi \in L^1 \cap A(G)$ can be represented in the form*

$$\phi(x) = \int_{\widehat{G}} \widehat{\phi}(\widehat{x}) \langle x, \widehat{x} \rangle d\widehat{x}, x \in G.$$

1.2 Banach Gelfand triples

A Gelfand triple is a generalisation of the notion of a Hilbert space which is well-known in the literature, albeit under different names such as 'rigged Hilbert space', 'equipped Hilbert space' [18], 'resolution space'. Gelfand triples arising from dense Banach subspaces of a Hilbert space are considered in [14] and arise, at least implicitely, from integrable group representations in the theory of atomic decompositions of coorbit spaces [13].

1.2.1 Basic notions

Let $(\mathcal{H}, \langle \mid \rangle)$ be a separable Hilbert space with a dense Banach subspace, B. We denote the pairing of B with its dual B' by $\langle\ ,\ \rangle$ - ie

$$\langle\ ,\ \rangle : B \times B' \longrightarrow \mathbb{C}; (\phi, \mu) \mapsto \langle \phi, \mu \rangle.$$

If we denote the inclusion of B into $\mathcal{H}$ by i_B and the antilinear identification of $\mathcal{H}$ with $\mathcal{H}'$ by $\overline{I}_{\mathcal{H}}$, then the operator

$$\overline{I}_B = i_B^{\star} \overline{I}_{\mathcal{H}} i_B : B \longrightarrow B'; \phi \mapsto \overline{\phi}$$

is an antilinear embedding of B into its dual (see §0 of [32] and chapter 2 of [33] for a similar notion). We may write this fact as

$$\langle i_B \phi | i_B \psi \rangle = \langle \phi, \overline{I}_B \psi \rangle,$$

or simply,

$$\langle \phi | \psi \rangle = \langle \phi, \overline{\psi} \rangle.$$

The space

$$\overline{B} = \overline{I}_B B = \{\overline{\phi} : \phi \in B\},$$

equipped with the transported norm, is a dense Banach subspace of $\mathcal{H}'$ called the *antispace* of B and it is clear that

$$\overline{\overline{B}} = B.$$

The operator $\overline{I}_B$ extends by duality to an antilinear isomorphism of B' and $\overline{B'} = \overline{B}'$, the antidual of B - ie for $\phi \in B, \mu \in B'$,

$$\langle\overline{\phi}, \overline{\mu}\rangle = \overline{\langle\phi, \mu\rangle}.$$

Since i_B is an injection, $\overline{B}$ is w* dense in B' and the inner product may be extended to a sesquilinear form on $B \times B'$ or $(B' \times B)$ - ie

$$\langle\phi|\mu\rangle = \langle\phi, \overline{\mu}\rangle, \langle\mu|\phi\rangle = \overline{\langle\phi, \overline{\mu}\rangle}, \text{ where } \phi \in B, \mu \in B'. \tag{1.2.1}$$

Definition 2. A triple

$$((B, \mathcal{H}, B'), \langle\,|\,\rangle),$$

consisting of a Banach space B, continuously and densely embedded into a Hilbert Space $\mathcal{H}$, which, in turn, is weak* densely embedded into the dual space B', equipped with the inner product (1.2.1), is called a *Banach Gelfand triple.*

Definition 3. Given two Gelfand triples $(B_1, \mathcal{H}_1, B_1')$ and $(B_2, \mathcal{H}_2, B_2')$, an isomorphism V of B_1 onto B_2 extends to an isomorphism of the respective dual spaces. V is a *Gelfand triple isomorphism* if it extends to an isomorphism of $\mathcal{H}_1$ onto $\mathcal{H}_2$, in which case it is a *unitary Gelfand Triple isomorphism* if

$$\langle V\phi|V\psi\rangle_2 = \langle\phi|\psi\rangle_1, \forall\phi, \psi \in B_1.$$

1.2.2 Kernels and operators

The algebraic tensor product $B \otimes \overline{B}$ is a dense subspace of the Hilbert space tensor product $\mathcal{H} \otimes \overline{\mathcal{H}}$. The *projective tensor product* $B\widehat{\otimes}\overline{B}$ is the completion of $B \otimes \overline{B}$ with respect to the least cross-norm

$$\|\sigma\|_{\widehat{\otimes}} = inf\{\sum_{i\,\in I} \|\phi_i\|_B\|\psi_i\|_B : \sigma = \sum_{i\,\in I} \phi_i \otimes \overline{\psi}_i\},$$

and is a Banach $\star$-algebra when equipped with the multiplication

$$\phi_1 \otimes \overline{\psi}_1 . \phi_2 \otimes \overline{\psi}_2 = \langle \phi_2, \psi_1 \rangle \phi_1 \otimes \overline{\psi}_2$$

and involution

$$(\phi \otimes \overline{\psi})^\star = \psi \otimes \overline{\phi}. \tag{1.2.2}$$

Since $B \widehat{\otimes} \overline{B}$ is continuously embedded into $\mathcal{H} \otimes \overline{\mathcal{H}}$, it generates the Gelfand triple

$$(B \widehat{\otimes} \overline{B}, \mathcal{H} \otimes \overline{\mathcal{H}}, (B \widehat{\otimes} \overline{B})').$$

If, for $\phi, \psi, \varphi \in B$, we let

$$\phi \star \psi^\star \varphi = \langle \varphi | \psi \rangle \phi,$$

then the mapping $\phi \otimes \overline{\psi} \mapsto \phi \star \psi^\star$ extends to an $\star$-isometry $\sigma \mapsto T_\sigma$ of $B \widehat{\otimes} \overline{B}$ onto an operator algebra which we denote by $B \widehat{\otimes} B^\star$. $B \widehat{\otimes} B^\star$ is a two-sided ideal in $\mathcal{L}(B)$, the algebra of bounded linear operators on B. We denote the inverse isometry by

$$\kappa : T \mapsto \kappa(T).$$

Where B is a function space on a locally compact Abelian group G, T is an integral operator, and the function $\kappa(T)$ on $G \times G$ is called the *kernel* of T.

The operator algebra $\mathcal{H} \widehat{\otimes} \mathcal{H}^\star$ is the algebra of *trace class operators* on $\mathcal{H}$ and denoted $\tau(\mathcal{H})$. It is equipped with the inner product

$$\langle S | T \rangle = tr S T^\star,$$

where the *trace functional*, $tr \in \tau(\mathcal{H})'$, is defined by

$$tr\ \phi \otimes \psi^\star = \langle \phi | \psi \rangle.$$

The Hilbert space completion of $\tau(\mathcal{H})$ is the *Hilbert-Schmidt* space, $\mathcal{HS}(\mathcal{H})$. Given that $\mathcal{H}'$ may be identified with $\overline{\mathcal{H}}$, it is a simple consequence of the theory of projective tensor products that $\sigma \mapsto T_\sigma$ extends to a unitary isomorphism of the Gelfand triple

$$(\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}, \mathcal{H} \otimes \overline{\mathcal{H}}, (\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}})')$$

onto the Gelfand triple of operator spaces.

$$(\tau(\mathcal{H}), \mathcal{HS}(\mathcal{H}), \mathcal{L}(\mathcal{H})).$$

More generally, since $B\widehat{\otimes}B^\star$ dense in $\tau(\mathcal{H})$, $\sigma \mapsto T_\sigma$ extends to a unitary isomorphism of the Gelfand triple

$$(B\widehat{\otimes}\overline{B}, \mathcal{H} \otimes \overline{\mathcal{H}}, (B\widehat{\otimes}\overline{B})')$$

onto the Gelfand triple of operator spaces

$$(B\widehat{\otimes}B^\star, \mathcal{HS}(\mathcal{H}), \mathcal{L}(B, \overline{B}')),$$

where the identification between an operator $T \in \mathcal{L}(B, \overline{B}')$ and $\kappa(T) \in (B\widehat{\otimes}\overline{B})'$ is given by

$$\langle \phi \otimes \overline{\psi}, \kappa(T)\rangle = \langle \overline{\psi}, T\phi\rangle.$$

Definition 4. With the notation above, we call $\kappa(T)$ the *kernel* of the operator T.

1.2.3 Banach Gelfand triples and reproducing kernel Hilbert spaces

Schwartz showed in [32] and [33] that there is a bijective correspondence between the set of positive definite kernels - the Schwartz reproducing kernels - associated to a separable locally convex topological vector space, E, and the set of Hilbert subspaces of E. Given a Banach Gelfand triple, $(B, \mathcal{H}, B')$, very elementary methods are used below to show that there exists a bijective correspondence between the Hilbert subspaces of B' and the set of positive definite elements of $(B\widehat{\otimes}\overline{B})'$, which we refer to as Schwartz reproducing kernels.

Definition 5. The involution (1.2.2) extends by duality to an involution of $(B\widehat{\otimes}\overline{B})'$. $\kappa(T) \in (B\widehat{\otimes}\overline{B})'$ is *Hermitian* if it is invariant under involution, ie $\kappa(T)^\star = \kappa(T)$, and *positive-definite* if

$$\langle \phi \otimes \overline{\phi}, \kappa(T) \rangle \geq 0, \ \forall \phi \in B.$$

The operator T corresponding to an Hermitian or positive-definite $\kappa(T)$ is called an Hermitian or positive-definite operator, respectively.

The set, $(B\widehat{\otimes}\overline{B})'^{+}$, of positive-definite elements of $(B\widehat{\otimes}\overline{B})'$, is a w*-closed convex cone, on which an ordering, $\geq$, is defined by

$$\kappa(T) \geq \kappa(S) \iff \kappa(T) - \kappa(S) \geq 0.$$

If we denote the space

$$\overline{I}_B\mathcal{L}(B, \overline{B'}) = \{\overline{I}_B T : T \in \mathcal{L}(B, \overline{B'})\}$$

of antilinear operators from B into its dual by $\overline{\mathcal{L}}(B, B')$, then the antilinear identification between $\kappa(T) \in (B\widehat{\otimes}\overline{B})'$ and an antilinear operator $\overline{T} \in \overline{\mathcal{L}}(B, B')$ is given by

$$\langle \psi, \overline{T}\phi \rangle = \overline{\langle \phi \otimes \overline{\psi}, \kappa(T) \rangle}.$$

Definition 6. Let $(B, \mathcal{H}, B')$ be a Banach Gelfand triple. A Hilbert space continuously embedded into B' is called a *Hilbert subspace of* B'; the set of all such subspaces is denoted $Hilb(B')$.

Definition 7. A Hilbert subspace $(\mathcal{J}, \langle \mid \rangle)$ of B' is a *reproducing kernel Hilbert space* if there exists an antilinear operator $\overline{T} \in \overline{\mathcal{L}}(B, B')$ such that

1. $\overline{T}\phi \in \mathcal{J}, \forall \phi \in B$ and

2. $\langle \phi, \mu \rangle = \langle \mu | \overline{T}\phi \rangle, \forall \mu \in \mathcal{J}, \phi \in B.$

In the following two propositions, we show that every Hilbert subspace of B' is a reproducing kernel Hilbert space. This result is new for Gelfand triples; similar results in the context of reproducing kernel Hilbert subspaces of $\mathcal{D}'(\mathbb{R}^n)$ can be found in [25] and in a more general context in [33].

Initially we show that every $\kappa(T) \in (B\widehat{\otimes}\overline{B})'^{+}$ generates a Hilbert subspace, $\mathcal{H}_T$ of B'.

Proposition 1. *Given $\kappa(T) \in (B\widehat{\otimes}\overline{B})'^{+}$, let $\mathcal{H}_T$ be the completion of the subspace*

$$\overline{T}B = \{\overline{T}\phi : \phi \in B\}$$

of B' with respect to the seminorm

$$\|\overline{T}\phi\| = \sqrt{\langle \phi \otimes \overline{\phi}, \kappa(T) \rangle}.$$

Then $(\mathcal{H}_T, \langle\ |\ \rangle_T)$, where

$$\langle \overline{T}\phi | \overline{T}\psi \rangle_T = \overline{\langle \phi \otimes \overline{\psi}, \kappa(T) \rangle},$$

is a Hilbert subspace of B'.

Proof. $\langle\ |\ \rangle_T$ is an inner product since from the Schwartz inequality,

$$|\langle \psi, \overline{T}\phi \rangle| \leq \|\overline{T}\phi\| \|\overline{T}\psi\|,$$

it is clear that

$$\|\overline{T}\phi\| = 0 \Rightarrow \langle \psi, \overline{T}\phi \rangle = 0, \forall \psi \in B,$$

and hence that $\overline{T}\phi$ is the zero element in B'. The linear operator $\tilde{T} \in \mathcal{L}(B, \overline{\mathcal{H}_T})$ defined by $\tilde{T}\phi = T\phi = \overline{\overline{T}\phi}$ has dense range and hence $\tilde{T}^{\star}$ is a continuous linear embedding of $\overline{\mathcal{H}}_T' = \mathcal{H}_T$ into B'. □

Conversely, the following proposition shows every element of $Hilb(B')$ is generated by an element of $(B\widehat{\otimes}\overline{B})'^{+}$.

Proposition 2. *Given a Banach Gelfand triple, $(B, \mathcal{H}, B')$ and $\mathcal{J} \in Hilb(B')$, there exists a unique $\kappa(T) \in (B\widehat{\otimes}\overline{B})'^{+}$ such that $\mathcal{J} = \mathcal{H}_T$.*

Proof. If we denote the embedding of $\mathcal{J}$ into B' by $i_{\mathcal{J}}$, then $i^{\star}_{\mathcal{J}} \in \mathcal{L}(B, \mathcal{J}')$ and has dense range. If we denote the antilinear identification of $\mathcal{J}'$ and $\mathcal{J}$ by $\overline{I}_{\mathcal{J}'}$, then

$$\langle i_{\mathcal{J}}\overline{I}_{\mathcal{J}'}i^{\star}_{\mathcal{J}}\phi | i_{\mathcal{J}}\overline{I}_{\mathcal{J}'}i^{\star}_{\mathcal{J}}\psi\rangle_{\mathcal{J}} = \overline{\langle \psi, i_{\mathcal{J}}\overline{I}_{\mathcal{J}'}i^{\star}_{\mathcal{J}'}\phi\rangle}, \text{ for } \phi, \psi \in B,$$

and it is clear that

$$\mathcal{J} = \mathcal{H}_T, \text{ where } \overline{T} = i_{\mathcal{J}'}\overline{I}_{\mathcal{J}}i^{\star}_{\mathcal{J}'}.$$

□

From propositions 1 and 2 above, there is a bijective correspondence,

$$\kappa(T) \leftrightarrow \mathcal{H}_T,$$

between the $Hilb(B')$ and $(B\widehat{\otimes}\overline{B})'^{+}$.

Definition 8. The positive-definite kernel $\kappa(T) \in (B\widehat{\otimes}\overline{B})'^{+}$ is called the *Schwartz reproducing kernel* of the Hilbert subspace $\mathcal{H}_T$ of B'.

1.2.4 The convex cone $Hilb(B')$

We describe briefly the remarkable structure of $Hilb(B')$.

1. *There exists a law of multiplication by non-negative scalars.* Given $\mathcal{J} \in Hilb(B')$ and a positive real number c, $c\mathcal{J}$ is the set consisting of the elements of the original Hilbert space $\mathcal{J}$ equipped with the scalar product $\langle\ ,\ \rangle_{c\mathcal{J}}$, where

$$\langle h, k\rangle_{c\mathcal{J}} = 1/c\langle h, k\rangle_{\mathcal{J}}.$$

   Where $c = 0$, we let $c\mathcal{J} = \{0\}$.

2. *There exists a law of addition on $Hilb(B')$.* Given Hilbert subspaces $\mathcal{J}, \mathcal{K} \in Hilb(B')$ the Hilbert space $\mathcal{J} + \mathcal{K}$ is the completion of the space

$$\{\mu = j + k : j \in \mathcal{J}, k \in \mathcal{K}\},$$

with respect to the norm

$$\|\mu\|_{\mathcal{J}+\mathcal{K}} = inf\{\sqrt{\|j\|^2_{\mathcal{J}} + \|k\|^2_{\mathcal{K}}} : j \in \mathcal{J}, k \in \mathcal{K}\}.$$

Equivalently,

$$\mathcal{J} + \mathcal{K} = (\mathcal{J} \oplus \mathcal{K})/ker\Phi,$$

where

$$\Phi : \mathcal{J} \oplus \mathcal{K} \longrightarrow B'; (h, k) \mapsto h + k.$$

3. *An order relation is defined on* $Hilb(B')$.

$$\mathcal{J} \leq \mathcal{K} \iff B_{\mathcal{J}} \subset B_{\mathcal{K}},$$

where $B_{\mathcal{J}}, B_{\mathcal{K}}$ are the closed unit balls in $\mathcal{J}, \mathcal{K} \in Hilb(B')$ respectively.

We are now in a position to state the fundamental result concerning the relationship between $(B\widehat{\otimes}\overline{B})'^{+}$ and $Hilb(B')$, the proof of which may be found in [32, 33].

Proposition 3. *The mapping* $\kappa(T) \mapsto \mathcal{H}_T$ *is an isomorphism of the convex cone* $(B\widehat{\otimes}\overline{B})'^{+}$ *onto* $Hilb(B')$. *More explicitly:*

1. $c.\kappa(T)$ *is the Schwartz reproducing kernel of* $c\mathcal{H}_T$*;*
2. $\kappa(S) + \kappa(T)$ *is the Schwartz reproducing kernel of* $\mathcal{H}_S + \mathcal{H}_T$*;*
3. $\mathcal{H}_S \leq \mathcal{H}_T \iff \kappa(S) \leq \kappa(T)$.

1.2.5 Direct integrals in $Hilb(B')$ over locally compact Abelian groups

For each $x \in G$, let T_x be an element of $(B\widehat{\otimes}\overline{B})'^{+}$ and let $x \mapsto \kappa(T_x)$ be weakly measurable in the sense that the function $x \mapsto \langle \phi \otimes \overline{\psi}, \kappa(T_x)\rangle$ is Borel measurable for all $\psi, \phi \in B$. Each $\phi \in B$ defines a vector field

$$\tilde{\phi} : G \longrightarrow \prod_{x \in G} \mathcal{H}_{T_x}; x \mapsto \overline{T}_x \phi.$$

Hence if $\{\phi_i\}_{i \in I}$ is a countably dense subset of B, the following is true:

1. $x \mapsto \langle \tilde{\phi_i}(x), \tilde{\phi_j}(x) \rangle_{H_{T_x}}$ is measurable for all $i, j \in I$;

2. the linear span of $\{\tilde{\phi_i}(x)\}_{i \in I}$ is dense in $\mathcal{H}_{T_x}$ for each $x \in G$.

Consequently, the vector fields $\{\tilde{\phi_i}\}_{i \in I}$ make $\{\mathcal{H}_{T_x}\}_{x \in G}$ into a *measurable field* in $Hilb(B')$ over G [16].

Definition 9. Given a positive Radon measure μ on G, $\{\mathcal{H}_{T_x}\}_{x \in G} \subset Hilb(B')$ is μ*-summable* if

$$\int_G \|\overline{T}_x \phi\|^2_{\mathcal{H}_{T_x}} d\mu(x) < \infty, \forall \phi \in B.$$

Given a positive Radon measure μ and a μ-summable field $\{\mathcal{H}_{T_x}\}_{x \in G} \subset Hilb(B')$, the Hilbert space completion of the set of vector fields $\{\tilde{\phi} : \phi \in B\}$, equipped with the inner product

$$\langle \tilde{\phi} | \tilde{\psi} \rangle = \int_G \langle \tilde{\phi}(x) | \tilde{\psi}(x) \rangle_{\mathcal{H}_{T_x}} d\mu(x),$$

is an element of $Hilb(B')$ called the *direct integral* of the spaces $\{\mathcal{H}_{T_x}\}_{x \in G}$ with respect to the measure μ and is denoted by

$$\int_G^{\oplus} \mathcal{H}_{T_x} d\mu(x). \tag{1.2.3}$$

The Schwartz reproducing kernel of the direct integral 1.2.3 is

$$\int_G \kappa(T_x) d\mu(x) \in (B \widehat{\otimes} \overline{B})'^{+}$$

defined by

$$\langle \phi \otimes \overline{\psi}, \int_G \kappa(T_x) d\mu(x) \rangle = \int_G \langle \phi \otimes \overline{\psi}, \kappa(T_x) \rangle d\mu(x).$$

1.3 The Schwartz-Bruhat space and tempered distributions on G

Fourier analysis has been extended to larger classes of objects using distributional methods associated to various spaces of test functions. One such space is the Schwartz-Bruhat space [4] of rapidly decreasing functions, $\mathcal{S}(G)$, which we describe purely for the sake of comparison and completeness.

Let G be an elementary group, ie

$$G = \mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^r \times F,$$

where p, q and r are non-negative integers and F is a finite Abelian group. A function ϕ is an element of $\mathcal{S}(G)$ if and only if the function and all its derivatives with respect to the real and toral subgroups remain bounded when multiplied with a polynomial whose coefficients consist of infinitely differentiable functions on the torus. The topology on $\mathcal{S}(G)$ is derived from the corresponding seminorms in the usual manner. In the case of an arbitrary locally compact Abelian group, G, a Bruhat pair consists of an open, compactly generated subgroup H of G and a compact subgroup L of H such that H/L is an elementary group.

Definition 10. $\mathcal{S}(G)$ is the inductive limit of the spaces $\mathcal{S}(H/L)$ over all Bruhat pairs (H, L).

The Schwartz-Bruhat space can also be defined without using structure theory. Osborne [28] defined a space of functions which decay rapidly off compact subsets of a locally compact Abelian group G as follows: A function $\phi \in L^\infty(G)$ is an element of $\mathcal{A}(G)$ if there exists a compact subset $C(\phi)$ of G with the property that for each positive integer n there is a constant M_n such that for each integer $k \geq 1$,

$$\|\phi|_{G-C(\phi)^k}\|_\infty \leq M_n k^{-n}.$$

The following theorem was proved in [28]:

Theorem 5. *The Schwartz-Bruhat space is the space*

$$\mathcal{S}(G) = \{\phi \in \mathcal{A}(G) | \widehat{\phi} \in \mathcal{A}(\widehat{G})\}.$$

The Schwartz-Bruhat space has a large number of very desirable properties from the point of view of harmonic analysis. In particular, it is a Fourier invariant, locally convex function space with a double module structure, continuously and densely embedded into many of the classical function spaces. The Fourier transform may be extended by duality to the space of tempered distributions, $\mathcal{S}'(G)$, and of particular interest in the context of the harmonic analysis of stochastic processes is the extension of Bochner's theorem to $\mathcal{S}'(G)$ [36]. The Schwartz-Bruhat space is, however, unwieldy and difficult to use. A much simpler and more accessible option is available, as we shall see.

1.4 The Feichtinger algebra $S_0(G)$ and applications

1.4.1 Quasimeasures

Given a compact subset K of a locally compact Abelian group G, we denote by

$$A_K(G) = A(G) \bigcap C_K(G)$$

the set consisting of elements of the Fourier algebra supported in K. The *compacted Fourier algebra*,

$$A_c(G) = \bigcup_K A_K(G),$$

equipped with its natural inductive limit topology, is a locally convex function algebra and a convolution ideal of $\mathcal{K}(G)$ containing positive functions with arbitrarily small support. Its dual space

$$Q(G) = \bigcap_K A_K(G)',$$

is the space of *quasimeasures.* The space of Radon measures, $\mathcal{R}(G)$, is continuously embedded into $Q(G)$, when both spaces are equipped with their natural projective limit topology. A quasimeasure is *positive* if it acts non-negatively on the cone of non-negative elements of the compacted Fourier algebra. Every positive quasimeasure is, in fact, a Radon measure; this is easily proved.

Proposition 4. *A positive quasimeasure is a (positive) Radon measure.*

Proof. Let a compact set $K \subset G$ be given. Pick a compact neighbourhood U of the identity. Since $A_c(G)$ is a convolution ideal in $\mathcal{K}(G)$ and contains positive functions with arbitrarily small support, we can arbitrarily uniformly approximate any positive function $\phi \in C_K(G)$ by a positive function $u \star \phi \in A_c(G)$ with support in $K + U$. An application of Theorem 2, Chap III, section 2 of [3] completes the proof. □

Definition 11. A quasimeasure μ *vanishes on an open set* $O \subset G$ if

$$\langle \phi, \mu \rangle = 0, \forall \phi \in A_c(G) \text{ with } supp\ \phi \subset O.$$

The support of a quasimeasure μ, denoted by *supp* μ, is the complement of the largest open set on which μ vanishes.

We note that every pseudomeasure is a quasimeasure and every compactly supported quasimeasure is a pseudomeasure. $Q(G)$ is a locally convex toplogical module over the compacted Fourier algebra with respect to pointwise multiplication, defined by transposition. In the case of convolution, the module action extends to a vaguely or $\sigma(\mathcal{R}(G), \mathcal{K}(G))$-continuous linear operator from $\mathcal{K}(G)$ into $R(G)$, which commutes with translations. Conversely, every operator of this type is a convolution with a quasimeasure. These facts were established in [17] where quasimeasures were introduced as the topological dual of a space of functions $D(G)$, which is shown in [6] to be isomorphic to the compacted Fourier algebra.

1.4.2 Bounded uniform partitions of unity in A(G)

Definition 12. Let U be a compact neighbourhood of the identity. A countable collection $\{x_i\}_{i\in I} \subset G$ is

1. *U-dense*, if
$$G = \bigcup_{i\in I}(x_i + U);$$
2. *relatively separated*, if, for any compact set $K \subset G$,
$$sup_{y\in G}|\{i \in I : (y + K) \cap (x_i + U) \neq \emptyset\}| = c_K < \infty.$$

A *bounded uniform partition of unity in* $A(G)$ *subordinate to* U - U-BUPU, for short - is a collection $\{\psi_i\}_{i\in I} \subset A(G)$ such that,

1.
$$\sum_{i\in I} \psi_i(x) = 1, \forall x \in G;$$
2.
$$supp\ \psi_i \subset (x_i + U), \forall i \in I;$$
3.
$$sup_{i\in I}\|\psi_i\|_{A(G)} < \infty,$$

where $\{x_i\}_{i\in I} \subset G$ is any U-dense, relatively separated set.

A bounded uniform partition of unity in $A(G)$ may be found for any compact neighbourhood of the identity [11].

1.4.3 Wiener amalgam spaces of quasimeasures

We present a simplified version of the theory of Wiener amalgam spaces, specialised to quasimeasures. The original presentation can be found in [9] and a detailed exposition in [7]. Given a compact neighbourhood of the identity, U, and an associated U-BUPU, $\{\psi_i\}_{i\in I}$, every quasimeasure, $\mu \in Q(G)$, has a representation,

$$\mu = \sum_{i\in I} \psi_i \mu, \tag{1.4.1}$$

as a locally-finite sum of compactly supported pseudomeasures. Conversely, any μ with the representation (1.4.1) is a quasimeasure [17]. The Wiener amalgam space, $W(A', l^p)(G)$, with *local component* A' and global component l^p consists of the set of quasimeasures, μ, such that $\{\|\psi_i\mu\|_{A(G)'}\}_{i\in I}$ is p-summable. Wiener amalgam spaces with a local component other than A' may also be defined.

Definition 13. Given a Banach $A(G)$-module B, which is continuously embedded into $Q(G)$, the space of *quasimeasures locally in* B is the space

$$B_{loc} = \{\mu \in Q(G) : \psi\mu \in B, \forall\psi \in A_c(G)\}.$$

Definition 14. Let $\{\psi_i\}_{i\in I}$ be a bounded uniform partition of unity in $A(G)$ subordinate to a compact neighbourhood of the identity. *The Wiener amalgam space of quasimeasures locally in B with global l^p behaviour*, for $1 \leq p \leq \infty$, is the space

$$W(B, l^p)(G) = \{\mu \in B_{loc} : \{\psi_i\mu\}_{i\in I} \in l^p\}.$$

When equipped with the amalgam norm

$$\|\mu\| = \|\{\psi_i\mu\}_{i\in I}\|_{l^p(B)},$$

$W(B, l^p)(G)$ is a Banach $A(G)$-module, continuously embedded into $Q(G)$. The definition is independent of the partition of unity used - different partitions of unity generate equivalent norms and, given any $h \in A_c(G)$, an equivalent "continuous" norm is defined by

$$\|\mu\|_{W(B,l^p)} = (\int_G \|L_x h.\mu\|_B^p \, dx)^{1/p}$$

for $1 \leq p < \infty$, with the obvious adjustment for $p = \infty$.

The following result, which is theorem 2 of [21], is very useful.

Theorem 6. *Let $1 \leq p < \infty$ and $1/p + 1/p' = 1$. Then the dual of the Wiener amalgam space $W(B, l^p)$ is the space*

$$W(B, l^p)(G)' = W(B', l^{p'})(G).$$

1.4.4 Segal algebras

Definition 15. A Banach space B, which continuously embedded into $L^1_{loc}(G)$, is *homogeneous* if it is

1. translation isometric - ie

$$\|L_x\phi\|_B = \|\phi\|_B, \forall\phi \in B, \forall x \in G;$$

2. strongly translation continuous - ie

$$lim_{x\to 0}\|L_x\phi - \phi\|_B = 0, \forall\phi \in B.$$

A *Segal algebra* is a dense, homogeneous subalgebra of $L^1(G)$.

Reiter's ideal theorem (Theorem 6.2.9 of [30]) states that there is a bijective correspondence between the closed ideals of a Segal algebra $S(G)$ and those of $L^1(G)$. More precisely, any closed ideal in $L^1(G)$ is the closure of an closed ideal I_S of $S(G)$; conversely, given any closed ideal I_S of $S(G)$, there a unique closed ideal I of $L^1(G)$ such that

$$I_S = I \cap S(G).$$

Since the group characters are the annihilators or *co-ideals* in $L^\infty(G)$ of the maximal closed ideals in $L^1(G)$, it is a consequence of the ideal theorem that the maximal ideal space of a Segal algebra is the dual group $\widehat{G}$.

Definition 16. A Segal algebra $S(G)$ is *strongly character-invariant* if it is modulation isometric - ie

$$\|M_{\widehat{x}}\phi\|_B = \|\phi\|_B, \forall\phi \in B, \forall\widehat{x} \in \widehat{G},$$

in which case it is, in addition, strongly modulation continuous.

1.4.5 The Feichtinger algebra

Definition 17. The *Feichtinger algebra* is the Wiener amalgam space $W(A, l^1)(G)$.

The symbol $S_0(G)$ is used for the Feichtinger algebra to signify that it is the minimal strongly character invariant Segal algebra [12]. Its dual space

$$S_0'(G) = W(A', l^\infty)(G)$$

is the space of *translation-bounded quasimeasures.* We note from [12] that $S_0(G)$ contains the Schwartz-Bruhat space as a dense subspace and from [10] that $S_0'(G)$ is a subspace of the space of tempered distributions.

1.4.6 The Wiener algebra

$S_0(G)$ is continuously and densely embedded into another Segal algebra of interest, the Wiener algebra [8],

$$W(G) = W(C_0, l^1)(G),$$

whose dual,

$$T(G) = W(M, l^\infty)(G),$$

is a Banach subspace of $S_0'(G)$, the space of *translation-bounded measures.*

1.4.7 The extended Fourier transform and the Plancherel theorem

One of the most useful properties of S_0 is its Fourier invariance - Feichtinger proved in [12] that the Fourier transform is an isomorphism of $S_0(G)$ onto $S_0(\widehat{G})$. The Fourier transform may hence be extended by duality to an isomorphism of the dual spaces. The following extension of the Plancherel theorem is then a simple consequence of the density of $S_0(G)$ in $L^2(G)$ and the classical Plancherel theorem.

Theorem 7. *(The Extended Plancherel Theorem) The Fourier transform is a unitary Gelfand triple isomorphism of*

$$(S_0, L^2, S_0')(G) \text{ onto } (S_0, L^2, S_0')(\widehat{G}).$$

1.4.8 Positivity, distributions of positive type and the Bochner theorem

We may use the Fourier transform on the space of distributions $S_0'(G)$ to extend the classical Bochner theorem to distributions of positive type. The version of the Bochner theorem presented here for S_0' is completely new, although the theorem has already been extended to spaces of distributions. The prototype result in this respect is the Bochner-Schwartz theorem - theorem 3, p. 157 of [18] and theoreme XVIII of [34]. Other results of interest include a Bochner type result for positive-definite measures on locally compact Abelian groups - theorem 4.5 in [2] and a similar result for transformable measures - theorem 4.1 of [1]. Theorem 1, p.231 of [18] and theorems 21.4, p.493 and 2.7, p.1012 of [15] are representation theorems in a similar vein for positive functionals on commutative Banach *-algebras. We begin by characterising the positive elements of $S_0(G)'$.

Proposition 5. *Every positive element μ of $S_0'(G)$ is a positive, translation-bounded measure and*

$$\|\mu\|_{T(G)} = \|\mu\|_{S_0'(G)}.$$

Proof. Let μ be a positive element of $S_0'(G)$ and $\{\psi_i\}_{i\in I}$ be a BUPU in $A(G)$. For each $i \in I$, $\psi_i\mu$ is a positive, bounded measure and hence we have

$$\|\psi_i\mu\|_{M(G)} = \|\widehat{\psi_i} \star \widehat{\mu}\|_{L^\infty(\widehat{G})}.$$

Hence

$$\begin{aligned}
\|\mu\|_{T(G)} &= sup_{i\in I}\|\psi_i\mu\|_{M(G)} \\
&= sup_{i\in I}\|\widehat{\psi_i} \star \widehat{\mu}\|_{L^\infty(\widehat{G})} \\
&= sup_{i\in I}\|\psi_i\mu\|_{A'(G)} \\
&= \|\mu\|_{S_0'(G)}.
\end{aligned}$$

□

Since $S_0(G)$ is Fourier invariant, it is clear that in addition to being a Segal algebra, it is a Banach function algebra.

Definition 18. A distribution $\mu \in S_0'(G)$ is *multiplicatively positive* if it is positive on the function algebra $S_0(G)$ - ie

$$\langle |\phi|^2, \mu \rangle \geq 0, \forall \phi \in S_0(G).$$

It is clear that every positive element of $S_0'(G)$ is multiplicatively positive, but not obvious that the converse is true. An example of a linear space where multiplicative positivity does not imply positivity can be found in [18].

Definition 19. A distribution $\mu \in S_0'(G)$ is of *positive type* if it is positive on the Segal algebra $S_0(G)$ - ie if

$$\langle \phi \star \phi^\star, \mu \rangle \geq 0, \forall \phi \in S_0(G).$$

We note that the Fourier Transform of a multiplicatively positive distribution is a distribution of positive type. The following theorem demonstrates that positivity and multiplicative positivity are equivalent.

Theorem 8. *(Bochner's Theorem) $\mu \in S_0'(G)$ is of positive type if and only if its Fourier transform is a positive, translation-bounded measure.*

Proof. The Fourier transform of a positive, translation-bounded measure is an element of $S_0'(G)$ of positive type.

Conversely, let $\mu \in S_0'(G)$ be of positive type. Given an approximate identity $\{u_\alpha\} \subset S_0(G)$, μ is the w*-limit

$$\mu = lim_\alpha u_\alpha \star u_\alpha^\star \star \mu$$

of functions of positive type. By the classical Bochner theorem and the w*-w* continuity of the extended Fourier transform, $\widehat{\mu}$ is the w*-limit in $S_0'(\widehat{G})$ of a net of positive, bounded measures and is hence positive. An application of proposition 5 completes the proof. □

1.4.9 Sets of spectral synthesis for $S_0(G)$

Given a closed subgroup H of a locally compact Abelian group G, we consider the relationship between the space of distributions $S_0'(H)$ and the set of elements of $S_0'(G)$ supported in H. In particular, we show that $S_0'(H)$ is continuously embedded into $S_0'(G)$ and that every distribution with support in H can be represented by an element of $S_0'(H)$. This is, in fact, equivalent to saying a closed subgroup is a *set of spectral synthesis* for the Banach function algebra $S_0(G)$. The result is new in the context of $S_0(G)$, although Hoermann has proved in [20] that, given locally compact Abelian groups G_1 and G_2, the subgroup $\{0_1\} \times G_2$ of $G_1 \times G_2$ is a set of spectral synthesis for $S_0(G_1 \times G_2)$. Background information on sets of spectral synthesis can be found in [30], from which the following definition is taken.

Definition 20. The *cospectrum* of an ideal I in the function algebra $S_0(G)$ is the set

$$Cosp\ I = \{x \in G : \phi(x) = 0, \forall \phi \in I\}.$$

Given a closed set $E \subset G$, it is clear that every ideal I_S in the function algebra $S_0(G)$ with E as cospectrum is contained in the closed ideal

$$I_E = \{\phi \in S_0(G) : \phi(x) = 0, \forall x \in E\},$$

and Proposition 2.1.16 of [30] shows that any such ideal I_S contains the ideal

$$J_E = \{\phi \in S_0(G) : E \cap supp\ \phi = \emptyset\}.$$

Definition 21. A closed set $E \subset G$ is a *set of spectral synthesis* for the function algebra $S_0(G)$ if

$$\overline{J_E} = I_E,$$

or, equivalently, if there exists precisely one closed ideal of the function algebra $S_0(G)$ with E as cospectrum.

Given a closed set E, the operation of *restriction to* E is the quotient map

$$R_E : S_0(G) \longrightarrow S_0(G)/I_E; \phi \mapsto \phi + I_E.$$

The adjoint operator $R_E^\star$ is an isomorphism of the dual space $(S_0(G)/I_E)'$ onto the annihilator $I_E^\perp \subset S_0'(G)$. Given $\mu \in (S_0(G)/I_E)'$, we have for any $\phi \in S_0'(G)$ such that $supp\ \phi \cap E = \emptyset$,

$$\langle \phi, R_E^\star \mu \rangle = \langle R_E \phi, \mu \rangle = 0$$

and hence that $supp\ R_E^\star \mu \subset E$. Thus we have the following inclusion

$$I_E^\perp \subset \{\mu \in S_0'(G) : supp\ \mu \subset E\}.$$

The obvious question which arises here is whether inclusion can be replaced by equality; equivalently, whether every element of $S_0'(G)$ with support in the closed set $E \subset G$ is of the form $R_E^\star \mu$, for some $\mu \in (S_0(G)/I_E)'$.

Proposition 6. *Let E be a closed subset of the locally compact Abelian group G. If E is a set of spectral synthesis for the Banach function algebra $S_0(G)$, then*

$$R_E^\star (S_0(G)/I_E)' = \{\mu \in S_0'(G) : supp\ \mu \subset E\}.$$

Proof. From section 4.6 of [16], we note that it is a consequence of the Hahn-Banach theorem that

$$I \leftrightarrow I^\perp$$

defines a bijective correspondence between the closed ideals of $L^1(G)$ and the w*-closed, translation-invariant subspaces of $L^\infty(G)$. Hence, using Reiter's ideal theorem and the Fourier invariance of S_0, it is clear that

$$I_S \leftrightarrow I_S^\perp$$

defines a bijective correspondence between the closed ideals of the function algebra $S_0(G)$ and the w*-closed, modulation-invariant subspaces of $S_0'(G)$. Since

$$J_E \subset^\perp \{\mu \in S_0'(G) : supp\ \mu \subset E\}$$

and $\{\mu \in S_0'(G) : supp\ \mu \subset E\}$ is a w*-closed, modulation-invariant subspace of $S_0'(G)$, we have that

$$\overline{J_E} \subset^{\perp} \{\mu \in S_0'(G) : supp\ \mu \subset E\} \subset I_E.$$

Using the fact that E is a set of spectral synthesis for the function algebra $S_0(G)$, we now have that

$$^{\perp}\{\mu \in S_0'(G) : supp\ \mu \subset E\} = I_E$$

and hence

$$R_E^{\star}(S_0(G)/I_E)' = I_E^{\perp} = \{\mu \in S_0'(G) : supp\ \mu \subset E\}.$$

□

The next result is crucial for later work.

Proposition 7. *Closed subgroups are sets of spectral synthesis for $S_0(G)$.*

Proof. The statement follows from the ideal theorem and corollary 7.3.4 of [30]. □

1.4.10 Periodisation and restriction

The Feichtinger algebra exhibits invariance under the operation of restriction, R_H, and periodisation, P_H, over a closed subgroup H. We show that the corresponding adjoint operators $R_H^{\star}$ and $P_H^{\star}$ are isomorphisms of $S_0(H)'$ and $S_0(G/H)'$ onto the closed subspaces of $S_0(G)'$ consisting of elements which are supported in H and are H-periodic respectively. The proofs require results results of the previous section and are presented here for the first time, although the results have been stated in [10]. In addition, using the Bochner theorem for $S_0(G)'$, we present an original proof of the Poisson summation formula.

Proposition 8. *Given a closed subgroup H of the locally compact Abelian group G, the image of $S_0(G)$ under the restriction map R_H, equipped with the transported norm is $S_0(H)$, ie*

$$R_H(S_0(G)) = S_0(H).$$

Proof. See theorem 7 of [12]. □

Proposition 9. *Let H be a closed subgroup of a locally compact Abelian group G. Then*

$$\{\mu \in S_0'(G) : supp\ \mu \subset H\} = R_H^\star(S_0'(H));$$

equivalently, given any $\mu \in S_0'(G)$ with support in a closed subgroup H, there exists $\mu_H \in S_0'(H)$ such that

$$\langle \phi, \mu \rangle = \langle R_H \phi, \mu_H \rangle$$

Proof. The proposition follows from the propositions 6, 7 and 8. □

As a consequence of proposition 8, we have the *convolution tensor product representation* of $S_0(G)$.

Proposition 10. *(The convolution tensor product representation)*
Given $\varphi \in S_0(G)$ and $\epsilon > 0$, there exists $\{\phi_i\}_{i\in I}, \{\psi_i\}_{i\in I} \subset S_0(G)$ such that

$$\varphi = \sum_{i\in I} \phi_i \star \psi_i^\star$$

and

$$\sum_{i\in I} \|\phi_i\|_{S_0(G)} \|\psi_i\|_{S_0(G)} < \|\varphi\|_{S_0(G)} + \epsilon.$$

Proof. From proposition 8, the restriction of $S_0(G \times G)$ to the antidiagonal can be identified with $S_0(G)$ and the proof is completed by an appeal to the Fourier invariance of S_0. □.

Definition 22. The *periodisation operator* P_H over the closed subgroup H of a locally compact Abelian group G is defined by

$$P_H \phi(x + H) = \int_H \phi(x + h) dh.$$

Proposition 11. *The image of $S_0(G)$ under the* periodisation operator P_H, *equipped with the quotient norm, is isometrically isomorphic to the space $S_0(G/H)$. - ie*

$$P_H(S_0(G)) = S_0(G/H).$$

Proof. See theorem 7 of [12]. □

Definition 23. A distribution $\mu \in S_0'(G)$ is *H-periodic* if

$$L_h \mu = \mu, \forall h \in H.$$

Definition 24. Let H be a closed subgroup of a locally compact Abelian group G. The *annihilator subgroup* of H is the subgroup $H^{\perp}$ of $\widehat{G}$ defined by

$$H^{\perp} = \{h^{\perp} \in \widehat{G} : \langle h, h^{\perp} \rangle = 1, \forall h \in H\}.$$

The following result characterises the H-periodicity of a distribution in terms of the support of its Fourier transform.

Proposition 12. *Let H be a closed subgroup of a locally compact Abelian group, G.*

$$\mu \in S_0'(G) \text{ is } H\text{-periodic} \iff \text{supp } \widehat{\mu} \in H^{\perp}.$$

Proof. It is clear that a distribution with spectrum in $H^{\perp}$ is H-periodic.

Conversely, let $\mu \in S_0'(G)$ be H-periodic. Then, given a BUPU $\{\psi_i\}_{i\in I} \subset A(\widehat{G})$, μ has the representation

$$\mu = \sum_{i\in I} \widehat{\psi}_i \star \mu$$

as a sum of H-periodic, continuous $L^{\infty}(G)$ functions. An application of proposition 7.1.20 of [30] completes the proof. □

Proposition 13. *Let H be a closed subgroup of a locally compact Abelian group G. Then*

$$P_H^{\star}(S_0'(G/H)) = \{\mu \in S_0'(G) : \mu \text{ is } H-\text{periodic}\}.$$

Proof. The proposition follows directly from proposition 9, proposition 11 and proposition 13. □

Theorem 9. *(The Poisson Summation Formula) Let H be a closed subgroup of a locally compact Abelian group G. Then*

$$\mathcal{F}_G 1_H = 1_{H^{\perp}}.$$

Proof. 1_H is obviously H-invariant and, in addition, is of positive type, since

$$\begin{aligned} \langle \phi \star \phi^{\star}, 1_H \rangle &= \int_H R_H\phi \star (R_H\phi)^{\star}(h)dh \\ &= |\mathcal{F}_H(R_H\phi)|^2(0_{\widehat{H}}) \\ &\geq 0. \end{aligned}$$

1_H is invariant under the group of modulations $\{M_{h^{\perp}} : h^{\perp} \in H^{\perp}\}$ and hence $\mathcal{F}_G 1_H$ is a positive, $H^{\perp}$-invariant measure, supported in $H^{\perp}$ and is hence Haar measure on $H^{\perp}$. □

1.4.11 Wiener amalgams and kernels

Where B is a homogeneous function algebra on the locally compact Abelian group, G, Wiener amalgams of the form $W(B, l^1)(G)$ exhibit an invariance property under the formation of projective tensor products. The following result is new, although a special case - the invariance of S_0 under the formation of tensor products - has been proved in [12]..

Proposition 14. *Let B be a homogeneous function algebra on the locally compact Abelian group, G. Then*

$$W(B, l^1)(G)\widehat{\otimes}W(B, l^1)(G) = W(B\widehat{\otimes}B, l^1)(G \times G).$$

Proof. Given a BUPU $\{\psi_i\}_{i\in I} \subset A(G)$, $\{\psi_i \otimes \psi_j\}_{i,j\in I}$ is a BUPU in $A(G\times G)$. Hence, given $\mu \in W(B, l^1)(G)\widehat{\otimes}W(B, l^1)(G)$ with the representation

$$\mu = \sum_n f_n \otimes g_n,$$

the estimate

$$\sum_{i,j} \|\psi_i \otimes \psi_j \mu\|_{\widehat{\otimes}} \leq \sum_n \sum_i \|\psi_i f_n\|_B \sum_j \|\psi_j g_n\|_B$$

implies the inclusion

$$W(B, l^1)(G)\widehat{\otimes}W(B, l^1)(G) \subset W(B\widehat{\otimes}B, l^1)(G \times G).$$

An application of corollary 4 of [11] implies the reverse inclusion and completes the proof. □

The $S_0(G)$ kernel theorem, which is stated without proof in [10], is proved below very easily. The proof should be compared to that of the kernel theorem for the Schwartz-Bruhat space [4].

Theorem 10. *The mapping $\sigma \mapsto T_\sigma$ defined by*

$$\langle f \otimes \overline{g}, \sigma\rangle = \langle \overline{g}, T_\sigma f\rangle$$

is an isometric isomorphism of the space of kernels $S_0'(G \times G)$ onto the space of bounded linear operators $\mathcal{L}(S_0(G), S_0'(G))$.

Proof. As a simple consequence of proposition 14, we have the representation

$$S_0(G \times G) = S_0(G)\widehat{\otimes}S_0(G)$$

and, since $S_0(G)$ is conjugation invariant, the proof is completed by appealing to lemma III.B.26 of [37]. □

If we denote the isometric image of $S_0(G \times G)$ under $\sigma \mapsto T_\sigma$ by $\mathcal{B}$, then since $S_0(G)$ is dense in $L^2(G)$, $\mathcal{B}$ is a proper, dense subalgebra of the space of trace class operators $\tau(L^2(G))$ and we can extend the kernel theorem as follows:

Proposition 15. *The mapping $\sigma \mapsto T_\sigma$ is a unitary Gelfand triple isomorphism of*

$$(S_0(G \times G), L^2(G \times G), S_0'(G \times G)) \text{ onto } (\mathcal{B}, \mathcal{HS}(L^2(G)), \mathcal{B}').$$

1.4.12 Bimeasures

Definition 25. A *bimeasure* on a locally compact Abelian group G is continuous linear mapping

$$B : \mathcal{K}(G) \longrightarrow \mathcal{R}(G),$$

where $\mathcal{R}(G)$ is equipped with its natural projective limit topology.

The dual of the *Varopoulos algebra* $V_0(G \times G) = C_0(G)\widehat{\otimes}C_0(G)$ is the *bimeasure algebra* of bounded bimeasures [19],

$$BM(G \times G) = \mathcal{L}(C_0(G), M(G)).$$

$$W_0(G \times G) = W(C_0, l^1)(G)\widehat{\otimes}W(C_0, l^1)(G)$$

is a proper subalgebra of the $V_0(G \times G)$. An application of proposition 14 yields

$$W_0'(G \times G) = W(BM, l^\infty)(G \times G),$$

the space of *translation-bounded bimeasures.*

1.4.13 Fourier multipliers

The following definition is from [10].

Definition 26. $T \in \mathcal{B}'$ is a *Fourier multiplier* if it commutes with translations - ie

$$L_x^\star T L_x = T, \forall x \in G.$$

We denote by

$$M(S_0(G), S_0'(G))$$

the subspace of $\mathcal{B}'$ consisting of Fourier multipliers. The following result characterises the space $M(S_0(G), S_0'(G))$ was stated without proof in [10].

Proposition 16. *$T \in \mathcal{B}'$ is a Fourier multiplier if and only if there exists $\mu_T \in S_0'(G)$ such that*

$$T\phi = \phi \star \mu_T.$$

Furthermore,

$$T \mapsto \mu_T$$

is an isomorphism of $M(S_0(G), S_0'(G))$ onto $S_0'(G)$ - ie

$$\|T\|_{\mathcal{B}'} = \|\mu_T\|_{S_0'(G)}.$$

Proof. Any operator generated through convolution with an element of $S_0'(G)$ is a Fourier multiplier in $\mathcal{B}'$.

Conversely, from the kernel representation,

$$\langle \overline{\psi}, T\phi \rangle = \langle \phi \otimes \overline{\psi}, \kappa(T) \rangle,$$

and proposition 13 it is clear that

$$\begin{aligned} \text{T is a Fourier multiplier} &\iff \kappa(T) \text{ is } \Delta_G-\text{invariant} \\ &\iff supp\ \mathcal{F}_{G\times G}\kappa(T) \subset \nabla_{\widehat{G}}, \end{aligned}$$

where Δ_G and ∇_G denote the diagonal and antidiagonal subgroups of $G \times G$ respectively and we use the fact that $\Delta_{\widehat{G}}^\perp = \nabla_{\widehat{G}}$.

Since $\nabla_{\widehat{G}}$ is naturally isomorphic to $\widehat{G}$, we have from proposition 7 and the Fourier invariance of $S_0(G)$ that there exists $\mu_T \in S_0'(G)$ such that

$$\begin{aligned} \langle \overline{\psi}, T\phi \rangle &= \langle \widehat{\overline{\psi}}, \widehat{\mu}_T \widehat{\phi} \rangle \\ &= \langle \overline{\psi}, \mu_T \star \phi \rangle, \end{aligned}$$

which yields the representation

$$T\phi = \mu_T \star \phi$$

as required. If we denote the identification of $S_0'((G \times G)/\Delta_G)$ and $S_0'(G)$ by i_{Δ_G}, then, by proposition 13, the operator $P^{\star}_{\Delta_G} i_{\Delta_G}$ is an isomorphism of $S_0'(G)$ onto the subspace of diagonally invariant kernels in $S_0'(G \times G)$. □

1.4.14 Translation-invariant Hilbert Subspaces of $S_0'(G)$ and the Bochner-Schwartz-Godement Theorem

The main result in the following section - the Bochner-Schwartz-Godement theorem - is required for a representation theorem for stochastic processes which will be proved in the next chapter. We prove here that positive-definite diagonally invariant kernels in $S_0(G \times G)'$ can be represented by positive translation-bounded measures on the dual group; equivalently that every unitary representation of G in a Hilbert subspace of $S_0'(G)$ is generated by a positive translation-bounded measure on $\widehat{G}$. The results here are new, but similar in spirit to those in chapter 5 of [1]; section 6.3 in [16]; theorem 6, p.169 of [18] and, particularly in the case of translation-bounded bimeasures, the final theorem is essentially theorem 2.5 of [26].

Since $(S_0, L^2, S_0')(G)$ is a Banach Gelfand triple, every Hilbert subspace of $S_0'(G)$ is a reproducing kernel Hilbert space with Schwartz reproducing kernel

$$\kappa(T) \in S_0'^{+}(G \times G).$$

$\mathcal{H}_T$ is the completion of the subspace $\{\overline{T}\phi : \phi \in S_0(G)\}$ of $S_0'(G)$, with respect to the norm

$$\|\overline{T}\phi\| = \sqrt{\langle \phi \otimes \overline{\phi}, \kappa(T) \rangle},$$

equipped with the inner product

$$\langle \overline{T}\phi | \overline{T}\psi \rangle_T = \overline{\langle \phi \otimes \overline{\psi}, \kappa(T) \rangle},$$

where the antilinear operator $\overline{T} \in \overline{\mathcal{L}}(S_0(G), S_0'(G))$ is defined by

$$\langle \psi, \overline{T}\phi \rangle = \langle \overline{\phi} \otimes \psi, \kappa(T) \rangle.$$

The identification

$$\kappa(T) \leftrightarrow \mathcal{H}_T$$

of a Hilbert space with its reproducing kernel is an isomorphism of the cone of positive definite kernels $S_0'^{+}(G \times G)$ and the cone $Hilb(S_0'(G))$ of Hilbert subspaces of $S_0'(G)$.

Definition 27. A kernel

$$\kappa(T) \in S_0(G \times G)$$

is *diagonally invariant* if it is invariant under the action of the diagonal subgroup Δ_G of $G \times G$ in the sense that

$$\langle L_x\phi \otimes L_x\overline{\psi}, \kappa(T) \rangle = \langle \phi \otimes \overline{\psi}, \kappa(T) \rangle, \forall \phi, \psi \in S_0(G), \forall x \in G.$$

It is clear that the representation

$$L_x : \overline{T}\phi \mapsto \overline{T}L_x\phi$$

of the group G in the Hilbert space $\mathcal{H}_T$ is unitary if and only if the Schwartz reproducing kernel $\kappa(T)$ is diagonally invariant.

Definition 28. A Hilbert subspace $\mathcal{H}_T$ of $S_0'(G)$ is *translation-invariant* if $\kappa(T)$ is diagonally-invariant.

The set of diagonally-invariant kernels is obviously w*-closed in $S_0'(G \times G)$ and hence the set $Hilb_{\Delta_G}(S_0'(G))$ of translation-invariant Hilbert subspaces of $S_0'(G)$ is obviously a closed convex cone in $Hilb(S_0'(G))$.

The simplest example of a translation-invariant Hilbert subspace of $S_0'(G)$ is the one-dimensional Hilbert space $\mathcal{H}_{\hat{x}}$ with Schwartz reproducing kernel $\chi_{\hat{x}} \otimes \overline{\chi}_{\hat{x}}$, which is the Fourier image of the Hilbert subspace $L^2(\widehat{G}, \delta_{\hat{x}})$ of $S_0'(\widehat{G})$. The content of the following theorem is that every element in $Hilb_{\Delta G}(S_0'(G))$ is obtained as a direct integral of such spaces.

Theorem 11. *(The Bochner-Schwartz-Godement Theorem)*

$\mathcal{H}_T \in Hilb_{\Delta G}(S_0'(G))$ *if and only if there exists* $\mu_T \in T^+(\widehat{G})$ *such that for* $\phi, \psi \in S_0(G)$,

$$\langle \overline{T}\phi | \overline{T}\psi \rangle = \int_{\widehat{G}} \overline{\widehat{\phi}}(\widehat{x})\widehat{\psi}(\widehat{x})d\mu_T(\widehat{x}),$$

and hence

$$\overline{T}\phi \mapsto \widehat{\phi}, \phi \in S_0(G)$$

extends to a unitary antilinear isomorphism of $\mathcal{H}_T$ *onto* $L^2(\widehat{G}, d\mu_T)$.

Proof. For any $\mu_T \in T^+(\widehat{G})$, it is clear that $L^2(\widehat{G}, d\mu_T)$ is Δ_G-invariant with respect to modulation and hence that the Fourier image of $L^2(\widehat{G}, d\mu_T)$ is translation-invariant.

Conversely, let $\mathcal{H}_T \in Hilb_{\Delta_G}(S_0'(G))$. Then, since $\kappa(T)$ is diagonally-invariant, $\overline{T}$ is a Fourier multiplier and there exists $\sigma_T \in S_0'(G)$ such that

$$\begin{aligned} \langle \overline{T}\phi | \overline{T}\psi \rangle &= \langle \sigma_T \star \overline{\phi} | \sigma_T \star \overline{\psi} \rangle \\ &= \langle \psi, \sigma_T \star \overline{\phi} \rangle \\ &= \langle \psi \star \phi^\star, \sigma_T \rangle. \end{aligned}$$

Since

$$\langle \phi \star \phi^\star, \sigma_T \rangle = \|\overline{T}\phi\|^2,$$

we have that σ_T is a distribution of positive type and hence, by the Bochner theorem, there exists a positive translation-bounded measure $\mu_T \in T^+(\widehat{G})$ such that

$$\langle \overline{T}\phi | \overline{T}\psi \rangle = \int_{\widehat{G}} \overline{\widehat{\phi}}(\widehat{x})\widehat{\psi}(\widehat{x})d\mu_T(\widehat{x}).$$

Since μ_T is a translation-bounded measure, we have for each $\phi \in W(\widehat{G})$ that

$$\int_{\widehat{G}} |\phi(\widehat{x})|^2 d\mu_T(\widehat{x}) < \infty$$

and since $\mathcal{K}(\widehat{G})$ is dense in $W(\widehat{G})$, $L^2(\widehat{G}, d\mu_T)$ is the completion of $W(\widehat{G})$ with respect to the norm thus defined. Since $S_0(\widehat{G})$ is continuously and densely embedded into $W(\widehat{G})$, it is dense in $L^2(\widehat{G}, d\mu_T)$ and hence the mapping

$$\overline{T}\phi \mapsto \widehat{\phi}$$

extends to a unitary antilinear isomorphism of $\mathcal{H}_T$ onto $L^2(\widehat{G}, d\mu_T)$. □

An important application of the above result is the decomposition

$$L^2(G) = \int_{\widehat{G}}^{\oplus} \mathcal{H}_{\widehat{x}} \, d\widehat{x}$$

of $L^2(G)$ into minimal translation-invariant Hilbert subspaces of $S_0'(G)$; or, equivalently the decomposition of the representation of G through translation into its irreducible components. The Schwartz reproducing kernel of $L^2(G)$ is the convolution kernel δ_0; the corresponding operator is the linear embedding of $S_0(G)$ into its dual.

Chapter 2

Second Order Generalised Stochastic Processes on Locally Compact Abelian Groups

2.1 Introduction and general concepts

The use of distributional techniques in the harmonic analysis of stochastic processes is not a new idea. The classical spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ and other, less familiar, spaces have been used as test function spaces in order to define generalised stochastic processes [23, 24, 25]. This approach bypasses the need for vector integration and simplifies the proof of some of the classical results of the theory of second order stochastic processes.

The representation of stochastic processes on locally compact Abelian groups as the Fourier transform of stochastic measures is considered in [27]. In this case, a lot of preparatory work is required in order to define a stochastic integral and the results are only valid for bounded stochastic measures. Furthermore, in order to justify integral representations of harmonisable processes, the theory of bimeasure integration must be considered. However, the extension of the distributional approach to stochastic processes defined on a locally compact Abelian group G is not particularly straightforward since the definition of the test spaces $\mathcal{S}(G)$ and $\mathcal{D}(G)$ is somewhat involved and their use very unwieldy.

An ideal test function space for the harmonic analysis of stochastic processes on locally compact Abelian groups would combine the desirable properties of the Schwartz-Bruhat space $\mathcal{S}(G)$ - in particular its Fourier invariance - with structural simplicity. The Feichtinger algebra $S_0(G)$ is such a space. Its use as a test function space for generalised stochastic processes was first considered by Hörmann [20], who presented very concise proofs of classical results and extended results obtained by Niemi [27] using stochastic integration. However, the proof of the spectral representation theorem presented in [20] rests on two assumptions which have not been justified: "..any positive $g \in S_0(G)$ can be written in the form $g = f\overline{f}$, where $f \in S_0(G)$" (proof of theorem 1, part c) and, secondly, given $\sigma \in S_0(G)'$, ".....that $\widehat{\sigma}$ is positive and this is equivalent to σ positive definite" (proof of theorem 8). The second assumption is, in fact, the Bochner theorem for $S_0(G)'$, which has not been stated in [20]. Indeed, the notion of *positive definite* has not been defined in [20] for elements of $S_0(G)'$. The unjustified use of the Bochner theorem for $S_0(G)'$ does however illustrate its importance in the proof of the spectral representation theorem for generalised stochastic processes. In the work which follows, the spectral representation theorem for generalised stochastic processes is shown essentially to be a corollary of the Bochner-Schwartz-Godement theorem, which was proved for the first time for $S_0(G)$ in chapter 1. Furthermore, it is shown in this present work, using the fact that the time domain of a generalised stochastic process is isomorphic to a Hilbert subspace of $S_0(G)'$, that any generalised stochastic process with a bounded covariance bimeasure may be identified with a stochastic process. This bypasses theorem 11 of [20], the proof of which was quite technical.

We consider initially the basic concepts and results in the theory of second order stochastic processes on locally compact Abelian groups and state some of the classical results including the spectral representation theorem. This information and the notation is drawn essentially from [27].

2.1.1 Second order stochastic processes

In this work, we are interested in a particular class of processes defined by complex valued random variables ξ with zero *expectation* $\mathcal{E}\xi$ and finite *variance* $\mathcal{E}|\xi|^2$ with respect to a Borel probability measure P on the locally compact space Ω; in other words, elements of the centred probability space

$$L_0^2(\Omega, P) = \{\xi : \mathcal{E}\xi = \int_\Omega \xi(\omega)dP(\omega) = 0 \text{ and } \mathcal{E}|\xi|^2 < \infty\}. \qquad (2.1.1)$$

Definition 29. A *second order stochastic process* X on a locally compact Abelian group G is a function

$$X : G \longrightarrow L_0^2(\Omega, P).$$

Definition 30. The *realisation* of a second order stochastic process X on a locally compact Abelian group G corresponding to $\omega \in \Omega$ is the function X_ω on G, defined by

$$X_\omega(x) = X(x, \omega), x \in G.$$

Definition 31. The *correlation* of two second order stochastic processes X and Y on the locally compact Abelian group G is the function σ_{XY} on $G \times G$ defined by

$$\sigma_{XY}(x, y) = \mathcal{E}X(x)\overline{Y}(y).$$

The *autocorrelation* σ_{XX} of X is written σ_X.

Definition 32. The *time domain* $\mathcal{H}_X$ of a second order stochastic process X is the completion of the linear span of the set of random variables

$$\{X(x) : x \in G\} \subset L_0^2(\Omega, P).$$

2.1.2 Stochastic measures

Definition 33. A *stochastic measure* on the locally compact Abelian group G is a continuous linear mapping Z from $\mathcal{K}(G)$ into the space of random variables $L_0^2(\Omega, P)$. Given $\phi \in \mathcal{K}(G)$, $Z\phi$, denoted

$$\int_G \phi(x) dZ(\dot{x}),$$

is a *stochastic integral.*

Definition 34. The dual space

$$M(G, L_0^2(\Omega, P)) = \mathcal{L}(C_0(G), L_0^2(\Omega, P))$$

of $C_0(G)\widehat{\otimes}L_0^2(\Omega, P)$ is the space of *bounded stochastic measures.*

Definition 35. A stochastic measure Z is *orthogonally scattered* if

$$\langle Zf|Zg\rangle_{L_0^2(\Omega,P)} = 0,$$

for disjointly supported pairs of functions $f, g \in \mathcal{K}(G)$.

2.1.3 Stationary second order stochastic processes

Definition 36. A second order stochastic process X is *stationary* if there exists a continuous function γ_X on G such that

$$\sigma_X(x, y) = \gamma_X(x - y).$$

The following theorem is the most fundamental result in the theory of second order stochastic processes. Roughly speaking, the theorem states that a stochastic process on a locally compact Abelian group G which is stationary may be represented by something deterministic - a positive, bounded measure on the dual group $\widehat{G}$.

Theorem 12. *(The spectral representation theorem for stationary stochastic processes)*

A second order stochastic process X on a locally compact Abelian group G is stationary if and only if there exists a positive bounded measure μ_X on the dual group $\widehat{G}$ such that for $x, y \in G$,

$$\sigma_X(x, y) = \int_{\widehat{G}} \langle x - y, \widehat{x}\rangle d\mu_X(\widehat{x}). \tag{2.1.2}$$

Let X be a stationary stochastic process on the locally compact Abelian group G. Then there exists an orthogonally scattered bounded stochastic measure Z_X on the dual group such that

$$X(x) = \int_{\widehat{G}} \langle x, \widehat{x} \rangle dZ_X(\widehat{x}). \tag{2.1.3}$$

In addition,

$$\|Z_X \phi\|_{L_0^2(\Omega)} = \|\phi\|_{L^2(\widehat{G}, \mu_X)}, \forall \phi \in \mathcal{K}(\widehat{G})$$

and hence Z_X extends to a unitary isomorphism of the spectral domain $L^2(\widehat{G}, \mu_X)$ onto the time domain $\mathcal{H}_X$.

We shall not prove the theorem since we will later prove a more general version, but will comment briefly. The autocorrelation kernel of a stationary stochastic process X is a diagonally invariant positive-definite function and hence the representation 3.7.1 is a consequence of the classical Bochner theorem. The positive bounded measure μ_X defined by equation 3.7.1 is the *spectral measure* of X. The stochastic measure Z_X defined by equation 2.1.3 is the *Fourier transform* or *spectral process* of X and it is clear that Z_X is a bounded measure and

$$\|Z_X\| \leq \|\mu_X\|.$$

However, if the process G is not discrete, $\widehat{G}$ is not compact and, since the characters are not compactly supported, the integral 2.1.3 has yet to be defined. We shall not dwell on the vagaries of vector valued integration at the moment as we intend to introduce an environment that bypasses the need for this.

2.1.4 Harmonisable and V-bounded stochastic processes

A stationary stochastic process is the Fourier transform of a bounded, orthogonally scattered stochastic measure. A natural extension of the notion of stationarity is to consider stochastic processes which are the Fourier transform of bounded stochastic measures which are not necessarily orthogonally scattered.

Definition 37. Stochastic processes with a representation 2.1.3 as the Fourier transform of a stochastic measure are called *harmonisable.*

Definition 38. Associated to a harmonisable process X is its *covariance bimeasure* B_X, defined by

$$\langle \phi \otimes \overline{\psi}, B_X \rangle = \langle Z_X\phi | Z_X\psi \rangle, \phi, \psi \in C_0(\widehat{G}).$$

B_X is a bounded bimeasure and positive definite in the sense that

$$\begin{aligned} \langle \phi \otimes \overline{\phi}, B_X \rangle &= \|Z_X\phi\|^2 \\ &\geq 0, \forall \phi \in C_0(\widehat{G}). \end{aligned}$$

The autocorrelation of a harmonisable process may be expressed as the bimeasure integral

$$\sigma_X(x, y) = \int_{\widehat{G}\times\widehat{G}} \langle x, \widehat{x} \rangle \overline{\langle y, \widehat{y} \rangle} dB_X(\widehat{x}, \widehat{y}). \tag{2.1.4}$$

Conversely, any second order stochastic process X where the autocorrelation has the representation (2.1.4) can be represented as the Fourier transform of a bounded stochastic measure and is hence harmonisable [22]. The definition of harmonisible stochastic processes used in this work corresponds to the definition of *weakly harmonisable* processes used by other authors [22], [27]. *Strongly harmonisable processes* are defined by having a covariance bimeasure which is a bounded measure on $\widehat{G} \times \widehat{G}$. Strongly harmonisable processes are clearly weakly harmonisable and, since we do not intend to work with strongly harmonisable processes here, we refer to weakly harmonisable processes simply as *harmonisable.*

Definition 39. A weakly continuous second order stochastic process X is *V-bounded* if the range of X and

$$\mathcal{C} = \{\int_G X(x)\phi(x)dx : \|\widehat{\phi}\|_\infty \leq 1, \phi \in L^1(G)\}$$

are bounded subsets of $L_0^2(\Omega)$, where

$$\int_G X(x)\phi(x)dx$$

is a Bochner integral.

The following theorem, which is theorem 4.2 of [29], states that harmonisability and V-boundedness are equivalent.

Theorem 13. *A stochastic process X on a locally compact Abelian Group G is harmonisable if and only if it is V-bounded.*

We would like to stress at this point that the validity of the representation (2.1.4) will not be justified here; we refer instead to some of the treatises on the subject of bimeasure integration. It should be noted that another shortcoming of the classical theory is that it deals only with bounded stochastic measures. We will attempt to remedy this by employing a suitable space of random variable valued test functions.

2.2 Second order generalised stochastic processes

Definition 40. The dual of the Banach space of stochastic processes, $S_0(G)\widehat{\otimes}L_0^2(\Omega)$, is the space

$$(S_0(G)\widehat{\otimes}L_0^2(\Omega))' = \mathcal{L}(S_0(G), L_0^2(\Omega, P))$$

of *generalised stochastic processes* on the locally compact Abelian group G.

The definition of generalised stochastic process used here mimics that of a generalised function, or distribution, which is an element of the dual of a space of (deterministic) test functions. Previously, generalised stochastic processes have been defined to be continuous linear mappings from a function space such as $\mathcal{S}(\mathbb{R})$ or $\mathcal{D}(\mathbb{R})$ into a space of random variables [23, 24, 25]. Use of the test function space $S_0(G)$ in this work considerably simplifies proof of some of the classical theorems.

We note that the space of generalised stochastic processes is the space of translation-bounded stochastic quasimeasures and contains the space of *translation-bounded stochastic measures*,

$$(W(G)\widehat{\otimes}L_0^2(\Omega))' = \mathcal{L}(W(G), L_0^2(\Omega, P)).$$

In addition, since $S_0(G)\widehat{\otimes}L_0^2(\Omega)$ is dense in the Hilbert space

$$L^2(G, L_0^2(\Omega, P)) = L^2(G) \otimes L_0^2(\Omega, P),$$

the triple

$$(S_0(G)\widehat{\otimes}L_0^2(\Omega), L^2(G, L_0^2(\Omega, P)), \mathcal{L}(S_0(G), L_0^2(\Omega, P)))$$

is a Banach Gelfand triple of stochastic processes.

Since

$$|\langle X'\phi|Y'\psi\rangle_{L_0^2(\Omega,P)}| \leq \|X'\|\|Y'\|\|\phi\|_{S_0(G)}\|\psi\|_{S_0(G)},$$

it is clear that

$$\phi\otimes\overline{\psi} \mapsto \langle X'\phi|Y'\psi\rangle_{L_0^2(\Omega,P)}$$

extends to a continuous linear functional on $S_0(G)\widehat{\otimes}S_0(G) = S_0(G\times G)$, which we define to be the correlation kernel of X' and Y':

Definition 41. The *correlation* of two generalised stochastic processes X', Y' on the locally compact Abelian group G is the kernel

$$\sigma_{X'Y'} \in S_0'(G\times G)$$

defined by

$$\langle \phi\otimes\overline{\psi}, \sigma_{X'Y'}\rangle = \langle X'\phi|Y'\psi\rangle_{L_0^2(\Omega,P)}, \phi, \psi \in S_0(G). \tag{2.2.1}$$

Definition 42. The *time domain* of a generalised stochastic process X' on the locally compact Abelian group G is the Hilbert space $\mathcal{H}_{X'} \subset L_0^2(\Omega, P)$ obtained by completion of the space of random variables

$$\{X'\phi : \phi \in S_0(G)\}.$$

It is clear that the autocorrelation $\sigma_{X'} = \sigma_{X'X'}$ of any generalised stochastic process X' is a positive definite kernel and, from equation (2.2.1), that $\mathcal{H}_{X'}$ is unitarily antilinearly isomorphic to the Hilbert subspace of $S_0'(G)$ with reproducing kernel $\sigma_{X'}$.

2.3 The spectral process

Given a stochastic process X, we would like to define its Fourier transform $\widehat{X}$ in a way which is consistent with the (deterministic) Fourier transform - roughly speaking, we would like the Fourier transform of the realisations of the process to be equal to the realisations of the Fourier transform and which enables us to extend the definition of the Fourier transform to generalised stochastic processes. The following definition satisfies these conditions and is also consistent with the classical definition of the spectral process, defined by equation (2.1.3).

Definition 43. Given a stochastic process

$$X \in S_0(G)\widehat{\otimes}L_0^2(\Omega, P)$$

with a representation

$$X = \sum_n \phi_n \otimes \overline{\xi}_n,$$

the *spectral process* or *Fourier transform* of X is the stochastic process

$$\widehat{X} = \sum_n \widehat{\phi}_n \otimes \overline{\xi}_n$$

on the dual group $\widehat{G}$.

From the Fourier invariance of the Feichtinger algebra, it is clear that the operator

$$\mathcal{F}_G : X \mapsto \widehat{X}$$

is an isomorphism of the spaces $S_0(G)\widehat{\otimes}L_0^2(\Omega, P)$ and $S_0(\widehat{G})\widehat{\otimes}L_0^2(\Omega, P)$ and hence that the Fourier transform extends by duality to an isomorphism of the respective spaces of generalised stochastic processes on G and $\widehat{G}$. In fact, it is clear that we can say somewhat more.

Theorem 14. *(The Plancherel Theorem) The Fourier transform is a unitary Gelfand triple isomorphism of*

$$(S_0(G)\widehat{\otimes}L_0^2(\Omega), L^2(G, L_0^2(\Omega, P)), \mathcal{L}(S_0(G), L_0^2(\Omega, P)))$$

onto

$$(S_0(\widehat{G})\widehat{\otimes}L_0^2(\Omega), L^2(\widehat{G}, L_0^2(\Omega, P)), \mathcal{L}(S_0(\widehat{G}), L_0^2(\Omega, P))).$$

2.4 Stationary second order generalised stochastic processes and the spectral representation theorem

Definition 44. A second order generalised stochastic process X' on a locally compact Abelian group G is *stationary* if its autocorrelation $\sigma_{X'}$ is diagonally invariant, ie

$$\langle L_x\phi \otimes L_x\overline{\psi}, \sigma_{X'}\rangle = \langle \phi \otimes \overline{\psi}, \sigma_{X'}\rangle, \forall x \in G, \forall \phi, \psi \in S_0(G). \tag{2.4.1}$$

Definition 45. A generalised stochastic process X' is *orthogonally scattered* if

$$X'\phi \perp X'\psi,$$

for any pair of functions $\phi, \psi \in S_0(G)$ which are disjointly supported.

The following proposition shows that an orthogonally scattered generalised stochastic process may be characterised by the support of its autocorrelation. The proof is adapted from [20]

Proposition 17. *Let X' be a generalised stochastic process on the locally compact Abelian group G.*

$$X' \text{ is orthogonally scattered} \iff \text{supp}\ \sigma_{X'} \subset \Delta_G.$$

Proof. Since

$$\text{supp}\ f \cap \text{supp}\ g = \emptyset \iff \text{supp}(f \otimes g) \cap \Delta_G = \emptyset,$$

it is obvious that a generalised stochastic process with an autocorrelation supported on the diagonal is orthogonally scattered.

Conversely, let X' be orthogonally scattered. Given $\sigma \in S_0(G \times G)$ such that *supp* σ is compact and *supp* $\sigma \cap \Delta_G = \emptyset$, pick a compact neighbourhood U of the identity in $G \times G$ such that

$$(K + U) \cap \Delta_G = \emptyset$$

and a compact neighbourhood V of the identity in G such that

$$V \times V \subset U.$$

Then, given any V-BUPU $\{\psi_i\}_{i \in I}$, there exist finite subsets $I_\sigma, J_\sigma \subset I$ such that

$$\sigma = \sum_{i \in I_\sigma, j \in I_\sigma} \psi_i \otimes \psi_j \sigma$$

and

$$supp(\psi_i \otimes \psi_j) \cap \Delta_G = \emptyset, \forall\ (i,j) \in I_\sigma \times J_\sigma.$$

Hence, given any representation

$$\sigma = \sum_n f_n \otimes g_n \in S_0(G) \widehat{\otimes} S_0(G),$$

we have

$$\sigma = \sum_{i \in I_\sigma, j \in J_\sigma, n} \psi_i f_n \otimes \psi_j g_n,$$

with

$$supp(\psi_i f_n \otimes \psi_j g_n) \cap \Delta_G = \emptyset, \forall i \in I_\sigma, j \in J_\sigma, n.$$

Hence

$$\langle \sigma, \sigma_{X'} \rangle = 0.$$

□

We are now in a position to state and prove the spectral representation theorem.

Theorem 15. *(The spectral representation theorem for stationary second order stochastic processes on locally compact Abelian groups)*

Let X' be a second order generalised stochastic process on a locally compact Abelian group G.

1. *X' is stationary if and only if there exists a unique positive, translation-bounded measure $\mu_{X'} \in T^+(\widehat{G})$ such that $\forall \phi, \psi \in S_0(G)$,*

$$\langle \phi \otimes \overline{\psi}, \sigma_{X'} \rangle = \int_{\widehat{G}} \widehat{\phi}(\widehat{x}) \overline{\widehat{\psi}}(\widehat{x}) d\mu_{X'}(\widehat{x}). \quad (2.4.2)$$

2. *X' is stationary if and only if the spectral process $\widehat{X'}$ is an orthogonally scattered translation-bounded stochastic measure.*

3. *If X' is stationary, then*

$$\|\widehat{X'}\phi\| = (\int_{\widehat{G}} |\phi(\widehat{x})|^2 d\mu_{X'}(\widehat{x}))^{1/2}, \forall \phi \in S_0(\widehat{G})$$

and hence $\widehat{X'}$ extends to a unitary isomorphism of $L^2(\widehat{G}, \mu_{X'})$ and the time domain $\mathcal{H}_{X'}$.

Proof. 1. A generalised stochastic process whose autocorrelation kernel can be represented as in (2.4.2) is obviously stationary. Conversely, it is clear that the autocorrelation of a stationary generalised stochastic process is a positive-definite diagonally-invariant kernel and the representation 2.4.2 follows from the Bochner-Schwartz-Godement theorem.

2. The chain of equivalences

$$\begin{aligned} X' \text{ is stationary} &\iff \sigma_{X'} \text{ is } \Delta_G - \text{invariant} \\ &\iff supp\mathcal{F}_{G\times G}\sigma_{X'} \subset \Delta_{\widehat{G}}^{\perp} \\ &\iff supp\mathcal{F}_G \otimes \overline{\mathcal{F}}_G\sigma_{X'} \subset \Delta_{\widehat{G}} \\ &\iff supp\sigma_{\widehat{X'}} \subset \Delta_{\widehat{G}} \end{aligned}$$

shows that that $\widehat{X'}$ is orthogonally scattered if and only if X' is stationary. Given X' is stationary, for any $\phi \in S_0(\widehat{G})$, we have

$$\|\widehat{X'}\phi\|^2_{L^2_0(\Omega)} \leq \|\mu_{X'}\|_T \|\phi\|^2_{W(\widehat{G})},$$

and hence, from the density of $S_0(\widehat{G})$ in $W(\widehat{G})$, it is clear that $\widehat{X'}$ extends to an orthogonally scattered translation-bounded stochastic measure.

3. Since $\mathcal{H}_{X'}$ is antilinearly unitarily isomorphic to the Hilbert subspace of $S_0'(G)$ with reproducing kernel $\sigma_{X'}$, the final part of the theorem follows from the Bochner-Schwartz-Godement theorem.

□

Definition 46. The measure $\mu_{X'} \in T^+(\widehat{G})$ associated to a stationary generalised stochastic process X' is called the *spectral measure.* The Hilbert space $L^2(\widehat{G}, \mu_{X'})$ is the *spectral domain* of the process X'.

We consider now the relationship between generalised stochastic processes and stochastic processes. In particular, we would like to derive the classical spectral representation theorem for stationary stochastic processes from the equivalent theorem for generalised stochastic processes. In order to do so, we must decide when a stochastic process and a generalised stochastic process can be identified. The following two propositions, which are original, show that a stationary generalised stochastic process can be identified with a (classical) stationary stochastic process if and only if its spectral measure is bounded. This could be considered to be a special case of theorem 11 of [20].

The next proposition shows that every stationary stochastic process generates a generalised stochastic process with the same time domain.

Proposition 18. *Let X be a stationary stochastic process on a locally compact Abelian group G with spectral measure μ_X. For each $\phi \in S_0(G)$, let*

$$X'\phi = \int_G Xx\phi(x)dx,$$

where the integral is interpreted weakly. Then,

1. *X' is a stationary generalised stochastic process with spectral measure μ_X.*
2. *$\mathcal{H}_X = \mathcal{H}_{X'}$.*

Proof. 1. The estimate

$$\begin{aligned} \|X'\phi\|_{L^2_0(\Omega,P)} &= sup\{|\int_G \langle Xx|\xi\rangle\phi(x)dx| : \|\xi\|_{L^2_0(\Omega,P)} \leq 1\} \\ &\leq \|Xx\| \int_G |\phi(x)|dx \\ &\leq \|\mu_X\|^{1/2}_{M(\widehat{G})} \|\phi\|_{S_0(G)} \end{aligned}$$

shows that X' is a generalised stochastic process. Furthermore, since for $\phi, \psi \in$

$S_0(G)$,

$$\begin{aligned}\langle X'\phi | X'\psi\rangle &= \langle \int_G Xx\phi(x)dx | \int_G Xx\psi(x)dx\rangle \\ &= \int_{G\times G} \langle Xx|Xy\rangle \phi(x)\overline{\psi}(y)dxdy \\ &= \langle \phi\otimes\overline{\psi}, \sigma_X\rangle,\end{aligned}$$

X' is stationary with spectral measure μ_X.

2. We show that, for every $\phi \in S_0(G)$, $X'\phi$ may be norm approximated by an absolutely convergent sum of elements of $\mathcal{H}_X$ and is hence in $\mathcal{H}_X$.

   Let U be a compact neighbourhood of the identity in G and $\{\psi_i\}_{i\in I}$ be a U-BUPU in $S_0(G)$. Then

$$\begin{aligned}\|X'\phi - \sum_{i\in I}\int_{x_i+U}\phi(x)\psi_i(x)dxXx_i\| &= \|\sum_{i\in I}\int_{x_i+U}(Xx - Xx_i)\phi(x)\psi_i(x)dx\| \\ &\le \sum_{i\in I}\int_{x_i+U}\|Xx - Xx_i\||\phi(x)|\psi_i(x)dx\end{aligned}$$

   Now, for $x\in x_i+U$,

$$\begin{aligned}\|Xx - Xx_i\|^2 &= \langle Xx|Xx\rangle + \langle Xx_i|Xx_i\rangle - \langle Xx|Xx_i\rangle - \langle Xx_i|Xx\rangle \\ &\le 2sup_{x\in(x_i+U)}|\tilde{\sigma}_X(0) - \tilde{\sigma}_X(x-x_i)| \\ &= 2sup_{x\in U}|\tilde{\sigma}_X(0) - \tilde{\sigma}_X(x)|.\end{aligned}$$

   Since $\tilde{\sigma}_X$ is the Fourier transform of a positive bounded measure, it is uniformly continuous and hence

$$lim_{U\to e}\|X'\phi - \sum_{i\in I}\int_{x_i+U}\phi(x)\psi_i(x)dxXx_i\| = 0.$$

   Since

$$\begin{aligned}\sum_{i\in I}\|\int_{x_i+U}\phi(x)\psi_i(x)dxXx_i\| &\le \sum_{i\in I}\int_{x_i+U}|\phi(x)|\psi_i(x)dx\|Xx_i\|. \\ &\le \|\phi\|_{S_0(G)}\|\mu_X\|^{1/2}_{M(\widehat{G})},\end{aligned}$$

   we hence have

$$X'\phi \in \mathcal{H}_X.$$

To complete the proof, we show that $\mathcal{H}_{X'}$ is total in $\mathcal{H}_X$. For $x \in G$ and $\phi \in S_0(G)$,

$$\langle Xx|X'\phi\rangle = \int_G \sigma_X(x,y)\overline{\phi}(y)dy)dy$$

and hence we have that Xx is orthogonal to $\mathcal{H}_{X'}$ if and only if the function

$$y \mapsto \sigma_X(x,y)$$

is the zero element in $S_0'(G)$. Equivalently, since $\mathcal{H}_{X'}$ is unitarily isomorphic to the dual of a Hilbert subspace of $S_0'(G)$, we have that Xx is orthogonal to $\mathcal{H}_{X'}$ if and only if $Xx = 0$.

□

Definition 47. A generalised stochastic process X' may be *identified with a stochastic process* X if X generates X' in the sense of proposition 18.

We now show the converse of proposition 18 - that every stationary generalised stochastic process with a bounded spectral measure can, in fact, be identified with a stochastic process.

Proposition 19. *Let X' be a stationary generalised stochastic process with bounded spectral measure $\mu_{X'}$. Then, $\widehat{X'}$ is a bounded measure and X' may be identified with a stochastic process with range in $\mathcal{H}_{X'}$.*

Proof. Given $\phi \in S_0(\widehat{G})$,

$$\begin{aligned} \|\widehat{X'}\phi\| &= (\int_{\widehat{G}} |\phi(\widehat{x})|^2 d\mu_{X'}(\widehat{x}))^{1/2} \\ &\leq \|\phi\|_\infty \|\mu_{X'}\|^{1/2}_{M(\widehat{G})} \end{aligned}$$

and hence by the density of $S_0(\widehat{G})$ in $C_0(\widehat{G})$, $\widehat{X'}$ extends to a bounded measure on $\widehat{G}$.

Given $\phi \in S_0(G)$, the estimate

$$\begin{aligned} |\int_{\widehat{G}} \widehat{\phi}(\widehat{x})\langle x, \widehat{x}\rangle d\mu_{X'}(\widehat{x})| &\leq (\int_{\widehat{G}} |\widehat{\phi}(\widehat{x})|^2 d\mu_{X'}(\widehat{x}))^{1/2} (\int_{\widehat{G}} d\mu_{X'}(\widehat{x}))^{1/2} \\ &= \|X'\phi\| \|\mu_{X'}\|^{1/2}_{M(\widehat{G})} \end{aligned}$$

shows that

$$X'\phi \mapsto \int_{\widehat{G}} \widehat{\phi}(\widehat{x})\langle x, \widehat{x}\rangle d\mu_{X'}(\widehat{x})$$

is an element of $\mathcal{H}'_{X'}$ and hence that $Xx \in \mathcal{H}_{X'}$, where

$$\langle X'\phi | Xx \rangle = \int_{\widehat{G}} \widehat{\phi}(\widehat{x}) \langle x, \widehat{x} \rangle d\mu_{X'}(\widehat{x}).$$

Given $\phi, \psi \in S_0(G)$,

$$\begin{aligned} \langle X'\phi | \int_G Xx\psi(x)dx \rangle &= \int_G \int_{\widehat{G}} \widehat{\phi}(\widehat{x}) \langle x, \widehat{x} \rangle d\mu_{X'}(\widehat{x}) \overline{\psi}(x) dx \\ &= \int_{\widehat{G}} \widehat{\phi}(\widehat{x}) \int_G \overline{\psi}(x) \langle x, \widehat{x} \rangle dx d\mu_{X'}(\widehat{x}) \\ &= \int_{\widehat{G}} \widehat{\phi}(\widehat{x}) \overline{\widehat{\psi}}(\widehat{x}) d\mu_{X'}(\widehat{x}) \\ &= \langle X'\phi | X'\psi \rangle \end{aligned}$$

and hence we have that

$$X'\phi = \int_G Xx\phi(x)dx, \forall \phi \in S_0(G),$$

as required. □

2.5 White noise

Definition 48. The stationary generalised stochastic process whose spectral measure is $1_{\widehat{G}}$, the Haar measure of the dual group, is called *white noise* and is denoted by W.

For $\phi, \psi \in S_0(G)$, we have

$$\langle W\phi | W\psi \rangle = \int_{\widehat{G}} \widehat{\phi}(\widehat{x}) \overline{\widehat{\psi}}(\widehat{x}) d\widehat{x}$$

and hence the associated spectral domain is $L^2(\widehat{G})$. The spectral process $\widehat{W}$ is also a stationary stochastic process with spectral measure 1_G - in other words, $\widehat{W}$ is white noise on the dual group $\widehat{G}$. White noise is important in the synthesis of stochastic processes through filtering. We will consider this at a later stage.

2.6 Harmonisable and V-bounded generalised stochastic processes

The following definition is a new generalisation of the definition of a harmonisable stochastic process.

Definition 49. A generalised stochastic process X' is *Harmonisable* if $\widehat{X'}$ is a translation-bounded stochastic measure.

The covariance bimeasure of a harmonisable stochastic process is bounded which implies that the spectral process is a bounded stochastic measure. The spectral process of a generalised stochastic process is, however, not necessarily bounded, but the covariance bimeasure is translation bounded, which is the content of the next proposition.

Proposition 20. *A generalised stochastic process X' is harmonisable if and only if the autocorrelation kernel $\sigma_{X'}$ has the representation*

$$\langle \widehat{\phi} \otimes \overline{\widehat{\psi}}, \sigma_{X'} \rangle = \langle \phi \otimes \overline{\psi}, B_{X'} \rangle, \phi, \psi \in S_0(\widehat{G}), \tag{2.6.1}$$

where $B_{X'}$ is a translation-bounded bimeasure.

Proof. Let the generalised stochastic process X' be harmonisable. Then $\widehat{X'}$ is a translation-bounded stochastic measure, and it is clear that

$$B_{X'} : \phi \otimes \overline{\psi} \mapsto \langle \widehat{X'}\phi | \widehat{X'}\psi \rangle, \phi, \psi \in W(\widehat{G}),$$

defines a translation-bounded bimeasure. The representation 2.6.1 then follows.

Conversely, let the autocorrelation of the generalised stochastic process X' be represented as in equation 2.6.1, where $B_{X'}$ is a translation-bounded bimeasure. Then, for $\phi \in S_0(\widehat{G})$,

$$\|\widehat{X'}\phi\|^2 = \langle \phi \otimes \overline{\phi}, B_{X'} \rangle.$$

Hence

$$\|\widehat{X'}\phi\| \leq \|\phi\|_{W(\widehat{G})} \|B_{X'}\|_{TBM}^{1/2}, \forall \phi \in S_0(\widehat{G}),$$

and, since $S_0(\widehat{G})$ is dense in $W(\widehat{G})$, it is clear that $\widehat{X'}$ extends to a translation-bounded stochastic measure on $\widehat{G}$. □

We now consider the relationship between harmonisable stochastic processes and harmonisable generalised stochastic processes. We show in the next two propositions, using original methods that bypass the use of theorem 11 of [20], that a harmonisable generalised stochastic process, is a (harmonisable) stochastic process if and only if its covariance bimeasure is bounded.

Proposition 21. *Let X be a harmonisable stochastic process on a locally compact Abelian group G. For $\phi \in S_0(G)$, let*

$$X'\phi = \int_G Xx\phi(x)dx,$$

where the integral is interpreted weakly. Then

1. *X' is a harmonisable generalised stochastic process with covariance bimeasure B_X.*
2. *$\mathcal{H}_{X'} = \mathcal{H}_X$.*

Proof. 1. For $\phi \in S_0(G)$,

$$\begin{aligned} \|X'\phi\| &\leq \int_G \|Xx\||\phi(x)|dx \\ &\leq sup_{x\in G}\|Xx\| \int_G |\phi(x)|dx \\ &\leq \|\sigma_X\|_\infty^{1/2}\|\phi\|_{S_0(G)} \\ &\leq \|B_X\|_{BM}^{1/2}\|\phi\|_{S_0(G)}, \end{aligned}$$

which shows that X' is a generalised stochastic process.
For $\widehat{\phi}, \widehat{\psi} \in S_0(G)$,

$$\begin{aligned} \langle X'\widehat{\phi}|X'\widehat{\psi}\rangle &= \langle \int_G Xx\widehat{\phi}(x)dx| \int_G Xx\widehat{\psi}(x)dx\rangle \\ &= \int_{G\times G} \langle Xx|Xy\rangle \widehat{\phi}(x)\overline{\widehat{\psi}}(y)dxdy \\ &= \langle \widehat{\phi}\otimes\overline{\widehat{\psi}}, \sigma_X\rangle \\ &= \langle \phi\otimes\overline{\psi}, B_X\rangle, \end{aligned}$$

as required.

2. We show that $\mathcal{H}_X$ and $\mathcal{H}_{X'}$ are total in each other. For $x \in G$ and $\widehat{\phi} \in S_0(G)$,

$$\begin{aligned} \langle Xx | X'\widehat{\phi} \rangle &= \int_G \sigma_X(x,y)\overline{\phi}(y)dy \\ &= \mathcal{F}_{\widehat{G}} B_X \overline{\phi}(x) \end{aligned}$$

and hence

$$\begin{aligned} X'\widehat{\phi} \perp Xx, \forall x \in G &\Rightarrow \mathcal{F}_{\widehat{G}} B_X \overline{\phi}(x) = 0, \forall x \in G \\ &\Rightarrow B_X \overline{\phi} = 0 \\ &\Rightarrow X'\phi = 0, \end{aligned}$$

since the time domain $\mathcal{H}_{X'}$ is unitarily isomorphic to $\mathcal{H}_{B_X}$, whose dual is a Hilbert subspace of $S_0'(G)$.

Conversely,

$$Xx \perp X'\widehat{\phi}, \forall \widehat{\phi} \in S_0(G)$$

implies that the function

$$y \mapsto \sigma_X(x,y)$$

is the zero element in $S_0'(G)$ and hence that $Xx = 0$.

□

We now show a converse to the above proposition.

Proposition 22. *Let X' be a harmonisable generalised stochastic process on a locally compact Abelian group G. If the covariance bimeasure $B_{X'}$ is bounded, then $\widehat{X'}$ is a bounded measure and X' can be identified with a (harmonisable) stochastic process with range in $\mathcal{H}_{X'}$.*

Proof. For $\phi \in S_0(\widehat{G})$,

$$\|\widehat{X'}\phi\|^2 = \langle \phi \otimes \overline{\phi}, B_{X'} \rangle$$

and hence, if $B_{X'}$ is bounded,

$$\|\widehat{X'}\phi\| \leq \|\phi\|_\infty \|B_{X'}\|_{BM}^{1/2},$$

and, since $S_0(\widehat{G})$ is dense in $C_0(\widehat{G})$, $\widehat{X'}$ extends to a bounded measure on $\widehat{G}$.

Given $\widehat{\phi} \in S_0(G)$, the estimate

$$
\begin{aligned}
|\int_{\widehat{G}\times\widehat{G}} \phi(\widehat{x})\overline{\langle x,\widehat{y}\rangle} dB_{X'}(\widehat{x},\widehat{y})| &= |\int_{\widehat{G}} \overline{\langle x,\widehat{y}\rangle} dB_{X'}(\phi)(\widehat{y})| \\
&= |\mathcal{F}_{\widehat{G}} B_{X'}(\phi)(x)| \\
&\leq \|B_{X'}(\phi)\|_{M(\widehat{G})} \\
&= sup\{|\langle \overline{\psi}, B_{X'}(\phi)\rangle| : \psi \in S_0(\widehat{G}), \|\psi\|_\infty \leq 1\} \\
&= sup\{|\langle \phi \otimes \overline{\psi}, B_{X'}\rangle| : \psi \in S_0(\widehat{G}), \|\psi\|_\infty \leq 1\} \\
&= sup\{|\langle \widehat{X'}\phi | \widehat{X'}\psi\rangle| : \psi \in S_0(\widehat{G}), \|\psi\|_\infty \leq 1\} \\
&\leq \|\widehat{X'}\phi\| \|\widehat{X'}\| \\
&\leq \|X'\widehat{\phi}\| \|\widehat{X'}\|,
\end{aligned}
$$

where $\|\widehat{X'}\|$ is the bounded stochastic measure norm of $\widehat{X'}$, shows that

$$X'\widehat{\phi} \mapsto \int_{\widehat{G}\times\widehat{G}} \phi(\widehat{x})\overline{\langle x,\widehat{y}\rangle} dB_{X'}(\widehat{x},\widehat{y})$$

is an element of $\mathcal{H}'_{X'}$. Given $x \in G$, $Xx \in \mathcal{H}_{X'}$ defined by

$$\langle X'\widehat{\phi}|Xx\rangle = \int_{\widehat{G}\times\widehat{G}} \phi(\widehat{x})\overline{\langle x,\widehat{y}\rangle} dB_{X'}(\widehat{x},\widehat{y}), \forall\widehat{\phi} \in S_0(G),$$

determines a stochastic process X on G with range in $\mathcal{H}_{X'}$.

We show now that X generates X'. Given $\widehat{\phi}, \widehat{\psi} \in S_0(G)$,

$$
\begin{aligned}
\langle X'\widehat{\phi}| \int_G Xx\widehat{\psi}(x)dx\rangle &= \int_G \langle X'\widehat{\phi}|Xx\rangle \overline{\widehat{\psi}}(x)dx \\
&= \int_G \int_{\widehat{G}\times\widehat{G}} \phi(\widehat{x})\overline{\langle x,\widehat{y}\rangle} dB_{X'}(\widehat{x},\widehat{y})\overline{\widehat{\psi}}(x)dx \\
&= \int_G \int_{\widehat{G}} \overline{\langle x,\widehat{y}\rangle} dB_{X'}(\phi)(\widehat{y})\overline{\widehat{\psi}}(x)dx \\
&= \int_{\widehat{G}} \int_G \overline{\widehat{\psi}}(x)\overline{\langle x,\widehat{y}\rangle} dx dB_{X'}(\phi)(\widehat{y}) \\
&= \int_{\widehat{G}} \overline{\psi}(\widehat{y}) dB_{X'}(\phi)(\widehat{y}) \\
&= \langle \phi \otimes \overline{\psi}, B_{X'}\rangle \\
&= \langle X'\widehat{\phi}|X'\widehat{\psi}\rangle
\end{aligned}
$$

and hence

$$X'\phi = \int_G Xx\phi(x)dx, \forall\phi \in S_0(G),$$

as required.

We show finally that X is harmonisable.

$$\begin{aligned}\langle X'\widehat{\phi}|X'\widehat{\psi}\rangle &= \langle \int_G Xx\widehat{\phi}(x)dx| \int_G Xx\widehat{\psi}(x)dx\rangle \\ &= \int_{G\times G} \widehat{\phi}(x)\overline{\widehat{\psi}}(y)\langle Xx|Xy\rangle dxdy\end{aligned}$$

Hence,

$$\langle \widehat{\phi}\otimes\overline{\widehat{\psi}}, \sigma_X\rangle = \langle \phi\otimes\overline{\psi}, B_{X'}\rangle,$$

which shows that X is harmonisable. □

Definition 50. A generalised stochastic process X' on a locally compact Abelian group G is *V-bounded* if the set

$$\mathcal{C}_{X'} = \{X'\phi : \phi \in S_0(G), \|\widehat{\phi}\|_\infty \leq 1\} \tag{2.6.2}$$

is bounded.

Proposition 23. *Let X' be a generalised stochastic process on a locally compact Abelian group G.*

$$X' \textit{is V-bounded} \iff X \textit{is a harmonisable stochastic process.}$$

Proof. Since $S_0(\widehat{G})$ is dense in $C_0(G)$, it is clear that that the set of random variables $\mathcal{C}_{X'}$ is bounded if and only if $\widehat{X'}$ is a bounded stochastic measure. □

In fact, if we replace the Bochner integral by a weak integral in the definition of a V-bounded stochastic process as in definition 3.1.1 of [27], it is clear from the density of $S_0(G)$ in $L^1(G)$, that every V-bounded stochastic process may be identified with a a V-bounded generalised stochastic process and vice-versa. Hence, we have proved the following proposition, which is theorem 3.2.1 of [27].

Proposition 24. *Let X be a stochastic process on a locally compact Abelian group. Then*

$$X \textit{is V-bounded} \iff \widehat{X} \textit{is a harmonisable stochastic process.}$$

2.7 Filtered generalised stochastic processes

The space of generalised stochastic processes $\mathcal{L}(S_0(G), L_0^2(\Omega, P))$ on the locally compact Abelian group G is a right Banach $\mathcal{L}(S_0(G))$-module. Given a generalised stochastic process X' and $T \in \mathcal{L}(S_0(G))$, we denote by $\tilde{T}X'$ the generalised stochastic process

$$\phi \mapsto X'T\phi$$

and call this process X' *filtered by* T.

Two subalgebras of $\mathcal{L}(S_0(G))$ are of particular interest as regards the filtering of generalised stochastic processes - the modulation spaces

$$M_{1,\infty}(G) = W(A, l^\infty)(G)$$

and

$$M_{\infty,1}(G) = W(A', l^1)(G).$$

The convolution algebra $M_{\infty,1}(G)$ is the space of *time-invariant filters.* We define the convolution of $\mu \in M_{\infty,1}(G)$ with the generalised stochastic process X' by transposition in the sense that

$$\mu \star X'\phi = X'\tilde{\mu} \star \phi.$$

$M_{\infty,1}(G)$ contains $M(G)$ and, in particular, it contains the Dirac measures. For $x \in G$, we have

$$\delta_{-x} \star X'\phi = X'L_x\phi.$$

The function algebra $M_{1,\infty}(G)$ is the space of *frequency invariant* filters and is the Fourier image of $M_{\infty,1}(\widehat{G})$; in particular, it contains the characters. The generalised stochastic process X', filtered by $\psi \in M_{1,\infty}(G)$ is the generalised stochastic process defined by

$$\psi X'\phi = X'\psi\phi.$$

The Fourier transform $\widehat{T}$ of an operator $T \in \mathcal{L}(S_0(G))$ is defined by transposition in the sense that, for $\widehat{\phi} \in S_0(G)$,

$$\widehat{T}\phi = T\widehat{\phi}.$$

It is the clear that the following is true for any $T \in \mathcal{L}(S_0(G))$ and any generalised stochastic process X':

$$\widehat{TX'} = \widehat{T}\widehat{X'}.$$

In particular, for any $\mu \in M_{\infty,1}(G)$ and any generalised stochastic process X', we have that

$$\widehat{\mu \star X'} = \widehat{\mu}\widehat{X'}.$$

Proposition 25. *Let W denote white noise on a locally compact Abelian group G. Then, the mapping*

$$T \mapsto TW$$

is a unitary isomorphism of $\mathcal{HS}(L^2(G))$ onto $L^2(G, L_0^2(\Omega, P))$.

Proof. Let S, T be elements of the operator algebra $\mathcal{B}$. Then, if we denote the identity operator corresponding to white noise by I, we have

$$\begin{aligned} \langle SW|TW\rangle &= trSIT^\star \\ &= \langle S|T\rangle \end{aligned}$$

and hence, from the density of $\mathcal{B}$ in $\mathcal{HS}(L^2(G))$, the mapping $T \mapsto TW$ extends to a unitary isomorphism of $\mathcal{HS}(L^2(G)$ into $L^2(G, L_0^2(\Omega, P))$.

We complete the proof by showing that the mapping is a surjection. If we write

$$L^2(G, L_0^2(\Omega, P)) = L^2(G) \otimes L_0^2(\Omega, P),$$

then it is clear that we have the following identification:

$$L^2(G, L_0^2(\Omega, P)) = \mathcal{HS}(L^2(G), L_0^2(\Omega, P)).$$

Hence, given $X' \in L^2(G, L_0^2(\Omega, P))$, $X'^\star X'$ is a positive, trace class operator in $L^2(G)$. The positive square root of $X'^\star X'$ is an element of $\mathcal{HS}(L^2(G))$ which we denote by $P_{X'}$. Then

$$\begin{aligned} \langle P_{X'}W | P_{X'}W \rangle &= tr P_{X'} P_{X'}^\star \\ &= tr X'^\star X', \end{aligned}$$

which completes the proof.

□

White noise is used to synthesise stationary stochastic processes through time-invariant filtering. The proof of the following proposition is a matter of routine.

Proposition 26. *Given $\mu \in M_{\infty,1}(G)$, the filtered white noise process $\mu \star W$ is stationary with spectral measure $|\widehat{\mu}|^2 1_{\widehat{G}}$.*

Chapter 3

Multidimensional Second Order Generalised Stochastic Processes on Locally Compact Abelian Groups

3.1 Introduction

We extend the results of the theory of scalar generalised stochastic processes on locally compact Abelian groups to infinite dimensional processes - ie processes which are defined using Hilbert space valued random variables. This results in much shorter and more transparent proofs of results obtained by Kakihara in [22]. In particular, the theory of stochastic measures and operator valued bimeasures developed by Kakihara is bypassed by again using $S_0(G)$ as a test function space and exploiting its remarkable properties. In this present work, an extended spectral representation theorem for multidimensional generalised stochastic processes on locally compact Abelian groups is proved. This requires a Bochner theorem for $\mathcal{L}(\mathcal{H})$-valued distributions, where $\mathcal{H}$ is an separable, infinite dimensional Hilbert space. In order to prove the Bochner theorem, an operator valued Fourier transform is defined and some preparatory results proved. Conditions are defined for a stationary multidimensional generalised stochastic process to be identified with a stochastic process. Harmonisable multidimensional generalised stochastic processes are investigated and representation theorems proved. All material in this chapter, except where indicated, is original. We begin with some background material taken from [22] on normal Hilbert $\mathcal{L}(\mathcal{H})$-modules, which are very useful in describing infinite dimensional stochastic processes.

3.2 Normal Hilbert $\mathcal{L}(\mathcal{H})$-Modules

Let $\mathcal{H}$ be a separable Hilbert space.

Definition 51. A *normal pre-Hilbert $\mathcal{L}(\mathcal{H})$-module* is a left $\mathcal{L}(\mathcal{H})$-module $\mathcal{Y}$, equipped with a mapping,

$$[\ ,\ ] : \mathcal{Y} \times \mathcal{Y} \longrightarrow \tau(\mathcal{H}),$$

the *grammian*, which satisfies the following conditions: for $X, Y, Z \in \mathcal{Y}$ and $T \in L(\mathcal{H})$,

1. $[X, X] \geq 0$ and $[X, X] = 0 \iff X = 0$;
2. $[X + Y, Z] = [X, Z] + [Y, Z]$;
3. $[TX, Y] = T[X, Y]$;
4. $[Y, X] = [X, Y]^\star$

A normal pre-Hilbert $\mathcal{L}(\mathcal{H})$-module which is complete with respect to the norm

$$\|X\| = \sqrt{\|[X, X]\|_{\tau(\mathcal{H})}},$$

is a *normal Hilbert $\mathcal{L}(\mathcal{H})$-module.*

We will use the following version of the Schwartz inequality which is lemma 2, p.18 of [22].

Proposition 27. *Let $\mathcal{Y}$ be a normal $\mathcal{L}(\mathcal{H})$-module. Then, for $X, Y \in \mathcal{Y}$,*

$$\|[X, Y]\|_{\tau(\mathcal{H})} \leq \|X\|_{\mathcal{Y}} \|Y\|_{\mathcal{Y}}.$$

Definition 52. Let $\mathcal{Y}$ and $\mathcal{Z}$ be normal $\mathcal{L}(\mathcal{H})$-modules.

1. A bounded linear operator
$$T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$$
which commutes with the module action of $\mathcal{L}(\mathcal{H})$ is a *module homomorphism* from $\mathcal{Y}$ into $\mathcal{Z}$.

2. A module homomorphism which preserves the grammian is a *grammian unitary isomorphism.*

3. Let $\mathcal{Y}, \mathcal{Z}$ be normal Hilbert $\mathcal{H}$ modules. $\mathcal{Y}$ and $\mathcal{Z}$ are *grammian unitarily isomorphic* if there exists a grammian unitary isomorphism of $\mathcal{Y}$ onto $\mathcal{Z}$.

4. A module homomorphism from $\mathcal{Y}$ into $\tau(\mathcal{H})$ is a *bounded linear functional* on $\mathcal{Y}$.

It is clear from the Schwartz inequality that every element Z of a normal $\mathcal{L}(\mathcal{H})$-module $\mathcal{Y}$ defines a bounded linear functional

$$\Lambda_Z : Y \mapsto [Y, Z].$$

The converse is the Riesz representation theorem, which is proposition 6, p.25 of [22].

Proposition 28. *(The Riesz Representation Theorem) Let Z' be a bounded linear functional on a normal Hilbert $\mathcal{L}(\mathcal{H})$-module $\mathcal{Y}$. Then there exists a unique $Z \in \mathcal{Y}$ such that*
$$Z'(Y) = [Y, Z], \forall\, Y \in \mathcal{Y}.$$
and
$$\|Z'\| = \|Z\|.$$

Definition 53. A *trace class operator valued measure* on a locally compact Abelian group G is a continuous linear mapping
$$\mu : \mathcal{K}(G) \longrightarrow \tau(\mathcal{H}).$$
For $\phi \in \mathcal{K}(G)$, we denote the trace class operator $\mu(\phi)$ by
$$\int_G \phi(x)d\mu(x)$$
and say μ is *positive* if, for all positive $\phi \in \mathcal{K}(G)$,
$$\int_G \phi(x)d\mu(x) \in \tau(\mathcal{H})^+,$$
where $\tau(\mathcal{H})^+$ denotes the positive trace class operators.

Example 1. *Let μ be a positive trace class operator valued Radon measure on the locally compact Abelian group G. Then, for each $\phi \in \mathcal{K}(G)$, we have that $\phi\mu$ is a trace class operator valued measure, where, for any $\psi \in \mathcal{K}(G)$,*

$$\int_G \psi(x) d\phi\mu(x) = \int_G \psi(x)\phi(x) d\mu(x).$$

If, for $T \in \mathcal{L}(\mathcal{H})$ and $\phi, \psi \in \mathcal{K}(G)$,

$$\int_G \psi(x) dT\phi\mu(x) = T \int_G \psi(x)\phi(x) d\mu(x),$$

then the set of trace class operator valued Radon measures

$$\{T\phi\mu : \phi \in \mathcal{K}(G), T \in \mathcal{L}(\mathcal{H})\}$$

is an $\mathcal{L}(\mathcal{H})$-module which becomes a normal pre-Hilbert $\mathcal{L}(\mathcal{H})$-module when equipped with the grammian

$$[S\phi\mu, T\psi\mu] = S \int_G \phi(x)\overline{\psi}(x) d\mu(x) T^\star.$$

The normal Hilbert $\mathcal{L}(\mathcal{H})$-module obtained through completion is denoted $L^2(G, \mu)$.

For further information on normal Hilbert $\mathcal{L}(\mathcal{H})$-modules we refer to [22].

3.3 Multidimensional second order stochastic processes

We review some of the basic definitions and results on multidimensional stochastic processes from [22].

Let Ω be a locally compact space equipped with a Radon probability measure P. Second order processes are defined by the space of Hilbert space valued random variables

$$L^2(\Omega, \mathcal{H}, P) = \{\xi : \Omega \longrightarrow \mathcal{H}; \int_\Omega \|\xi(\omega)\|^2_{\mathcal{H}} dP(w) < \infty\}.$$

Zero mean stochastic processes take values in the space of centred Hilbert space valued random variables

$$L^2_0(\Omega, \mathcal{H}, P) = \{\xi \in L^2(\Omega, \mathcal{H}, P) : \mathcal{E}\xi = \int_\Omega \xi(\omega) dP(\omega) = 0\},$$

which we denote by $\mathcal{X}$, for brevity. $\mathcal{X}$, equipped with the grammian

$$[\xi_1, \xi_2] = \int_\Omega \xi_1(\omega) \otimes \xi_2(\omega)^\star dP(\omega),$$

is a normal Hilbert $\mathcal{L}(\mathcal{H})$-module.

Definition 54. A *multidimensional second order stochastic process* on a locally compact Abelian group G is a function $X : G \longrightarrow \mathcal{X}$.

Definition 55.

1. The *operator correlation* of two multidimensional second order stochastic processes X and Y on a locally compact Abelian group G is the function
$$\Gamma_{XY} : G \times G \longrightarrow \tau(\mathcal{H}); \Gamma_{XY}(x, y) = [Xx, Yy].$$

2. The corresponding *scalar correlation* is the function
$$\sigma_{XY} = tr\Gamma_{XY}.$$

3. The *operator autocorrelation* of the multidimensional second order stochastic process X on the locally compact Abelian group G is the trace class operator valued function
$$\Gamma_X = \Gamma_{XX}.$$
The corresponding *scalar autocorrelation* is the function
$$\sigma_X = tr\Gamma_X.$$

Definition 56. Let X be a multidimensional second order stochastic process on a locally compact Abelian group G.

1. X is *stationary* if the scalar autocorrelation σ_X is diagonally invariant - ie if
$$\sigma_X(y - x, z - x) = \sigma_X(y, z), \forall x, y, z \in G.$$

2. X is *scalarly stationary* if, for each $h \in \mathcal{H}$, the scalar process X_h is stationary, where
$$X_h x(\omega) = \langle Xx(\omega) | h \rangle, x \in G, \omega \in \Omega.$$

3. X is *operator stationary* if the operator autocorrelation Γ_X is diagonally invariant - ie
$$\Gamma_X(y - x, z - x) = \sigma_X(y, z), \forall x, y, z \in G.$$

Theorem 16. *(The spectral representation theorem for operator stationary multidimensional stochastic processes on locally compact Abelian groups)*

Let X be a multidimensional second order stochastic process on a locally compact Abelian group G.

1. *X is operator stationary if and only if the operator autocorrelation Γ_X has the representation*
$$\Gamma_X(x,y) = \int_{\widehat{G}} \langle y - x, \widehat{x}\rangle d\mu_X(\widehat{x}),$$
where μ_X is a positive, regular, countably additive, trace class operator valued Borel measure on the dual group $\widehat{G}$.

2. *X is operator stationary if and only if there exists a regular, countably additive, grammian orthogonally scattered Borel measure Z on the dual group $\widehat{G}$ such that*
$$Xx = \int_{\widehat{G}} \langle x, \widehat{x}\rangle dZ(\widehat{x}).$$

The measures Z_X and μ_X are related by
$$[\int_U dZ_X(\widehat{x}), \int_U dZ_X(\widehat{x})] = \int_U d\mu_X(\widehat{x}), U \in \mathcal{B}_{\widehat{G}}.$$

The measure theoretical terms used in the theorem will not be defined here and, as in the scalar case, no attempt will be made to justify the integral representations. The details may be found in [22].

The following proposition shows that operator and scalar stationarity are one and the same thing.

Proposition 29. *[22] Let X be multidimensional second order stochastic process on a locally compact Abelian group. Then*

$$X \text{ is scalarly stationary} \iff X \text{ is operator stationary.}$$

As in the scalar case, we may extend the notion of stationarity by defining harmonisable processes.

Definition 57. Let X be a multidimensional second order stochastic process on a locally compact Abelian group G.

1. X is *harmonisable* if its scalar autocorrelation has the representation

$$\sigma_X(x,y) = \int_{\widehat{G}\times\widehat{G}} \langle x, \widehat{x}\rangle \overline{\langle y, \widehat{y}\rangle} dB_X^{sc}(\widehat{x}, \widehat{y}),$$

where B_X^{sc} is a bounded bimeasure.

2. X is *scalarly harmonisable* if, for each $h \in \mathcal{H}$, the scalar process X_h is harmonisable.

3. X is *operator harmonisable* if its operator autocorrelation has the representation

$$\Gamma_X(x,y) = \int_{\widehat{G}\times\widehat{G}} \langle x, \widehat{x}\rangle \overline{\langle y, \widehat{y}\rangle} dB_X(\widehat{x}, \widehat{y}),$$

where B_X is an operator bimeasure of bounded operator semivariation.

Proposition 30. *[22] Let X be a multidimensional second order stochastic process on a locally compact Abelian group G. X is operator harmonisable if and only if there exists a regular, countably additive Borel measure Z_X of bounded operator semivariation on the dual group $\widehat{G}$ such that*

$$Xx = \int_{\widehat{G}} \langle x, \widehat{x}\rangle dZ_X(\widehat{x}).$$

Again, we do not attempt to define the terms used or to justify the integral representation.

3.4 Multidimensional second order generalised stochastic processes

The following definition differs from that of one dimensional generalised stochastic processes only in so far as the random variables are Hilbert space valued. As far as can be ascertained from the literature, the definition is original.

Definition 58. A *multidimensional second order generalised stochastic process* on a locally compact Abelian group G is an element of the Banach space

$$(S_0(G)\widehat{\otimes}\mathcal{X})' = \mathcal{L}(S_0(G), \mathcal{X}).$$

Given a multidimensional second order generalised stochastic process X' on a locally compact Abelian group G, each $h \in \mathcal{H}$ defines a scalar process, X'_h, where

$$X'_h\phi(\omega) = \langle X'\phi(\omega)|h\rangle, \omega \in \Omega.$$

Proposition 31. *Let X', Y' be two multidimensional second order generalised stochastic processes on a locally compact Abelian group G. Then the mapping $\Gamma_{X'Y'}$, defined by*

$$\begin{aligned}[\phi \otimes \overline{\psi}, \Gamma_{X'Y'}] &= [X'\phi, Y'\psi] \\ &= \int_\Omega X'\phi(\omega) \otimes Y'\psi(\omega)^\star dP(\omega), \phi, \psi \in S_0(G),\end{aligned}$$

where the integral is interpreted weakly, extends to a bounded linear operator from $S_0(G \times G)$ into $\tau(\mathcal{H})$ and

$$\|\Gamma_{X'Y'}\| \leq \|X'\|\|Y'\|.$$

Proof. Let

$$\sum_{i \in I} \phi_i \otimes \overline{\psi}_i \in S_0(G)\widehat{\otimes}S_0(G)$$

be a representation of $F \in S_0(G \times G)$. Using the Schwartz inequality, we have

$$\begin{aligned}\|[F, \Gamma_{X'Y'}]\|_{\tau(\mathcal{H})} &\leq \sum_{i \in I} \|[\phi_i \otimes \overline{\psi}_i, \Gamma_{X'Y'}]\|_{\tau(\mathcal{H})} \\ &= \sum_{i \in I} \|[X'\phi_i, Y'\overline{\psi}_i]\|_{\tau(\mathcal{H})} \\ &\leq \sum_{i \in I} \|X'\phi_i\|\|Y'\overline{\psi}_i\| \\ &\leq \|X'\|\|Y'\| \sum_{i \in I} \|\phi_i\|\|\psi_i\|\end{aligned}$$

The norm estimate

$$\|\Gamma_{X'Y'}\| \leq \|X'\|\|Y'\|$$

then follows by taking the infimum of the right hand side over all possible representations of F. □

Definition 59. Let X', Y' be multidimensional second order stochastic processes on a locally compact Abelian group G.

1.
$$\Gamma_{X'Y'} \in \mathcal{L}(S_0(G \times G), \tau(\mathcal{H}))$$
is the *operator correlation* of X' and Y'.

2. $\Gamma_{X'X'}$, written $\Gamma_{X'}$, is the *operator autocorrelation* of X'.

3. The *scalar autocorrelation* of X' is the kernel $\sigma_{X'} \in S_0'(G \times G)$ defined by
$$\langle \phi \otimes \overline{\psi}, \sigma_{X'} \rangle = tr[\phi \otimes \overline{\psi}, \Gamma_{X'}], \phi, \psi \in S_0(G).$$

Proposition 32. *Let X' be a multidimensional second order generalised stochastic process on a locally compact Abelian group G. Then, given any ONB $\{e_i\}_{i \in I}$ of $\mathcal{H}$,*
$$\sigma_{X'} = \sum_{i \in I} \sigma_{X'_{e_i}}.$$

Proof. Since $\{e_i \otimes e_j^\star\}_{i,j \in I}$ is an ONB for $\mathcal{HS}(\mathcal{H})$, we have
$$\begin{aligned} tr[\phi \otimes \overline{\psi}, \Gamma_{X'}] &= tr \sum_{i,j \in I} \langle \phi \otimes \overline{\psi}, \sigma_{X'_{e_i} X'_{e_j}} \rangle e_i \otimes e_j^\star \\ &= \sum_{i \in I} \langle \phi \otimes \overline{\psi}, \sigma_{X'_{e_i}} \rangle \end{aligned}$$
and hence
$$\sigma_{X'} = \sum_{i \in I} \sigma_{X'_{e_i}}.$$
□

Definition 60. 1. Let G be a locally compact Abelian group. The Banach space
$$\mathcal{L}(S_0(G \times G), \tau(\mathcal{H}))$$
is the space of *trace class operator valued kernels.*

2. A trace class operator valued kernel Γ is *positive definite* if
$$[\phi \otimes \overline{\phi}, \Gamma] \in \tau(\mathcal{H})^+, \forall \phi \in S_0(G).$$

We note that the operator autocorrelation of any multidimensional second order generalised stochastic process is positive-definite.

Definition 61. Let X' be a multidimensional second order generalised stochastic process on a locally compact Abelian group G. The completion of the normal pre-Hilbert $\mathcal{L}(\mathcal{H})$-module
$$\{X'\phi : \phi \in S_0(G)\}$$
in $L_0^2(\Omega, \mathcal{H}, P)$ is the *modular time domain* of X'.

3.5 Reproducing kernel Hilbert $\mathcal{L}(\mathcal{H})$-modules

To every trace class operator valued kernel Γ we can associate the operator

$$T_\Gamma \in \mathcal{L}(S_0(G), \mathcal{L}(S_0(G), \tau(\mathcal{H}))),$$

where the identification between the operator and kernel is defined by

$$T_\Gamma\phi(\overline{\psi}) = [\phi \otimes \overline{\psi}, \Gamma].$$

Definition 62. 1. The Banach $\mathcal{L}(\mathcal{H})$-module

$$\mathcal{L}(S_0(G), \tau(\mathcal{H}))$$

is the space of *trace class operator valued distributions.*

2. The Banach $\mathcal{L}(\mathcal{H})$-module

$$\mathcal{L}(W(G), \tau(\mathcal{H}))$$

is the space of *trace class operator valued translation-bounded measures.*

If Γ is positive definite, then the set

$$\{TT_\Gamma\phi : \phi \in S_0(G), T \in \mathcal{L}(\mathcal{H})\},$$

equipped with the grammian

$$[ST_\Gamma\phi, TT_\Gamma\psi] = S[\phi \otimes \overline{\psi}, \Gamma]T^\star,$$

is a normal pre-Hilbert $\mathcal{L}(\mathcal{H})$-module. We denote the Hilbert $\mathcal{L}(\mathcal{H})$-module obtained through completion by $\mathcal{H}_\Gamma$ and call Γ the *reproducing kernel* of $\mathcal{H}_\Gamma$.

We have already noted that the operator covariance kernel $\Gamma_{X'}$ of a multidimensional generalised stochastic process X' is positive definite. It is clear that, for $\phi, \psi \in S_0(G)$, we have

$$[X'\phi X'\psi] = [T_{\Gamma_{X'}}\phi, T_{\Gamma_{X'}}\psi]$$

and hence that the modular time domain $\mathcal{H}_{X'}$ and the normal Hilbert $\mathcal{L}(\mathcal{H})$-module $\mathcal{H}_{\Gamma_{X'}}$ are grammian unitarily isomorphic.

We denote, as before, the algebra obtained by the projective tensor product of the Wiener algebra with itself by W_0 - ie

$$W_0(G \times G) = W(G)\widehat{\otimes}W(G),$$

Definition 63. The space of kernels

$$\mathcal{L}(W_0(G \times G), \tau(\mathcal{H}))$$

is the space of *trace class operator valued translation-bounded bimeasures.*

Given a positive definite trace class operator valued bimeasure B, we write the operator T_B simply as B and, for each $\phi \in W(G)$, we have that $B\phi$ is a trace class operator valued translation-bounded measure. The grammian corresponding to a positive definite translation-bounded bimeasure is written

$$\begin{aligned} [B\phi, B\psi] &= \int_{G\times G} \phi(x)\overline{\psi}(y)dB(x,y) \\ &= \int_G \overline{\psi}(y)dB\phi(y) \\ &= B\phi(\overline{\psi}). \end{aligned}$$

If the positive definite trace class operator valued translation-bounded bimeasure B is generated by a trace class operator valued Radon measure μ in the sense that for each $\phi \in S_0(G)$

$$B\phi = \mu\phi,$$

then we have simply

$$[\mu\phi, \mu\psi] = \int_G \phi(x)\overline{\psi}(x)d\mu(x).$$

Definition 64. A trace class operator valued kernel

$$\Gamma \in \mathcal{L}(S_0(G \times G), \tau(\mathcal{H}))$$

is *diagonally invariant* if

$$[L_x\phi \otimes L_x\overline{\psi}, \Gamma] = [\phi \otimes \overline{\psi}, \Gamma], \forall x \in G, \forall \phi, \psi \in S_0(G).$$

We note that where a positive definite operator valued kernel is diagonally invariant, for each $x \in G$, the translation operator L_x, defined for each $\phi \in S_0(G)$ by

$$L_x T_\Gamma \phi = T_\Gamma L_x \phi,$$

is grammian unitary in $\mathcal{H}_\Gamma$ and hence that

$$x \mapsto L_x$$

is a *grammian unitary representation* of G in $\mathcal{H}_\Gamma$.

3.6 Operator valued Fourier transform and the Bochner theorem

Let

$$\sum_i \phi_i \otimes T_i \in S_0(G) \widehat{\otimes} \tau(\mathcal{H}).$$

Then, for $h, k \in \mathcal{H}$, we have

$$\langle \sum_i \phi_i \otimes T_i h | k \rangle \in S_0(G).$$

The Fourier transform $\mathcal{F}_G \sum_i \phi_i \otimes T_i$ of $\sum_i \phi_i \otimes T_i$ is the operator valued function on $\widehat{G}$ defined for $h, k \in \mathcal{H}$ and $\widehat{x} \in \widehat{G}$ by

$$\langle \mathcal{F}_G \sum_i \phi_i \otimes T_i(\widehat{x}) h | k \rangle = \int_G \langle \sum_i \phi_i \otimes T_i h | k \rangle (x) \overline{\langle x, \widehat{x} \rangle} dx, h, k \in \mathcal{H}.$$

It is clear that

$$\mathcal{F}_G \sum_i \phi_i \otimes T_i = \sum_i \widehat{\phi}_i \otimes T_i$$

and hence that $\mathcal{F}_G$ is an isomorphism of the $S_0(G)\widehat{\otimes}\tau(\mathcal{H})$ onto $S_0(\widehat{G})\widehat{\otimes}\tau(\mathcal{H})$, which extends to an isomorphism of the dual spaces via the relation

$$\widehat{\mu}(\phi) = \mu(\widehat{\phi}),$$

where $\mu \in \mathcal{L}(S_0(G), \mathcal{L}(\mathcal{H})), \phi \in S_0(\widehat{G})$.

The following theorem is an obvious consequence of the definition of the Fourier transform.

Theorem 17. *(The Plancherel Theorem)*
The Fourier transform is a unitary Gelfand triple isomorphism of

$$(S_0(G)\widehat{\otimes}\tau(\mathcal{H}), L^2(G, \mathcal{HS}(\mathcal{H})), \mathcal{L}(S_0(G), \mathcal{L}(\mathcal{H})))$$

onto

$$(S_0(\widehat{G})\widehat{\otimes}\tau(\mathcal{H}), L^2(\widehat{G}, \mathcal{HS}(\mathcal{H})), \mathcal{L}(S_0(\widehat{G}), \mathcal{L}(\mathcal{H}))).$$

Definition 65. Let $\mu \in \mathcal{L}(S_0(\widehat{G}), \mathcal{L}(\mathcal{H}))$.

1. μ is *positive* if, for each positive function $\phi \in S_0(G)$,

$$\mu(\phi) \in \mathcal{L}(\mathcal{H})^+,$$

   where $\mathcal{L}(\mathcal{H})^+$ denotes the positive operators in $\mathcal{L}(\mathcal{H})$.

2. μ *is of positive type* if

$$\mu(\phi \star \phi^\star) \in \mathcal{L}(\mathcal{H})^+, \forall \phi \in S_0(G).$$

Proposition 33. *Let $\mu \in \mathcal{L}(S_0(G), \mathcal{L}(\mathcal{H}))$ be positive. Then*

1. $\mu \in \mathcal{L}(W(G), \mathcal{L}(\mathcal{H}))$ *- ie every positive $\mathcal{L}(\mathcal{H})$-valued distribution is an $\mathcal{L}(\mathcal{H})$-valued translation-bounded measure.*

2. *The $\mathcal{L}(S_0(G), \mathcal{L}(\mathcal{H}))$ and $\mathcal{L}(W(G), \mathcal{L}(\mathcal{H}))$ norms of μ are equivalent; both norms are equivalent to*

$$sup_{\|h\|\leq 1}\|\mu_h\|_{T(G)}.$$

3. *If $\mu \in \mathcal{L}(S_0(G), \tau(\mathcal{H}))$, then $\mu \in \mathcal{L}(W(G), \tau(\mathcal{H}))$.*

Proof. 1. For $h, k \in \mathcal{H}$, it is clear that $\mu_{hk} \in S_0'(G)$, where, for $\phi \in S_0(G)$,

$$\langle \phi, \mu_{hk} \rangle = \langle \mu(\phi) h | k \rangle,$$

and, for $h = k$,

$$\mu_{hh} = \mu_h \in T^+(G).$$

For any positive $\phi \in S_0(G)$,

$$(h, k) \mapsto \langle \phi, \mu_{hk} \rangle$$

is a sesquilinear form on $\mathcal{H}$ and hence, for $h, k \in \mathcal{H}$,

$$|\langle \phi, \mu_{hk} \rangle| \leq \langle \phi, \mu_h \rangle^{1/2} \langle \phi, \mu_k \rangle^{1/2}.$$

Hence, given $\phi \in S_0(G)$, positive and compactly supported, we have, for $x \in G$,

$$|\langle L_x\phi, \mu_{hk} \rangle| \leq \langle L_x\phi, \mu_h \rangle^{1/2} \langle L_x\phi, \mu_k \rangle^{1/2}$$

and hence, taking the sup of both sides over all $x \in G$, we have

$$\mu_{hk} \in T(G),$$

with

$$\|\mu_{hk}\|_{T(G)} \leq \|\mu_h\|_{T(G)}^{1/2} \|\mu_k\|_{T(G)}^{1/2}.$$

Hence, for any $\phi \in S_0(G)$ and $h, k \in \mathcal{H}$ with $\|h\|, \|k\| \leq 1$, we have, using the positivity of each μ_h, that

$$\begin{aligned} |\langle \mu(\phi) h | k \rangle| &= |\int_G \phi(x) d\mu_{hk}(x)| \\ &\leq \|\phi\|_{W(G)} sup_{\|h\| \leq 1} \|\mu_h\|_{T(G)} \\ &= \|\phi\|_{W(G)} sup_{\|h\| \leq 1} \|\mu_h\|_{S_0'(G)} \\ &\leq \|\phi\|_{W(G)} sup_{\|h\| \leq 1} sup_{\|\psi\|_{S_0(G)} \leq 1} |\langle \psi, \mu_h \rangle| \\ &= \|\phi\|_{W(G)} sup_{\|h\| \leq 1} sup_{\|\psi\|_{S_0(G)} \leq 1} |\langle \mu(\psi) h | h \rangle| \\ &\leq \|\phi\|_{W(G)} \|\mu\|, \end{aligned}$$

where $\|\mu\|$ denotes the $\mathcal{L}(S_0(G), \mathcal{L}(\mathcal{H}))$ norm. Hence, from the density of $S_0(G)$ in $W(G)$, we have that μ extends to an $\mathcal{L}(\mathcal{H})$-valued translation-bounded measure.

2. From the inequalities

$$\begin{aligned} sup_{\|h\| \leq 1} \|\mu_h\|_{T(G)} &= sup_{\|h\| \leq 1} sup_{\|\phi\|_{W(G)} \leq 1} |\langle \mu(\phi) h | h \rangle| \\ &\leq sup_{\|\phi\|_{W(G)} \leq 1} \|\mu(\phi)\|_{\mathcal{L}(\mathcal{H})} \end{aligned}$$

and

$$\begin{aligned}
\|\mu(\phi)\|_{\mathcal{L}(\mathcal{H})} &= sup_{\|h\|,\|k\|\leq 1}|\langle\mu(\phi)h|k\rangle| \\
&= sup_{\|h\|,\|k\|\leq 1}|\int_G \phi(x)d\mu_{hk}(x)| \\
&\leq \|\phi\|_{W(G)} sup_{\|h\|,\|k\|\leq 1}\|\mu_{hk}\|_{T(G)} \\
&= \|\phi\|_{W(G)} sup_{\|h\|\leq 1}\|\mu_h\|_{T(G)},
\end{aligned}$$

we see that the $\mathcal{L}(W(G), \mathcal{L}(\mathcal{H}))$ norm of μ is equivalent to $sup_{\|h\|\leq 1}\|\mu_h\|_{T(G)}$. To show the same is true for the $\mathcal{L}(S_0(G), \mathcal{L}(\mathcal{H}))$ norm, we note from [12] that there exists a constant C such that any $\phi \in S_0(G)$ can be written in the form

$$\phi = \sum_{0\leq l\leq 3} i^l \phi_l,$$

where ϕ_l is a non-negative function in $S_0(G)$ and

$$\|\phi_l\|_{S_0(G)} \leq C\|\phi\|_{S_0(G)}, 0 \leq l \leq 3.$$

Hence, for any $\phi \in S_0(G)$, we have, using the positivity of the operator $\mu(\phi_l)$ for each l,

$$\begin{aligned}
\|\mu(\phi)\|_{\mathcal{L}(\mathcal{H})} &\leq \sum_{0\leq l\leq 3} \|\mu(\phi_l)\|_{\mathcal{L}(\mathcal{H})} \\
&= \sum_{0\leq l\leq 3} sup_{\|h\|\leq 1}|\langle\mu(\phi_l)h|h\rangle| \\
&= \sum_{0\leq l\leq 3} sup_{\|h\|\leq 1}|\langle\phi_l, \mu_h\rangle| \\
&\leq \sum_{0\leq l\leq 3} \|\phi_l\|_{S_0(G)} sup_{\|h\|\leq 1}\|\mu_h\|_{S_0'(G)} \\
&\leq 4C\|\phi\|_{S_0(G)} sup_{\|h\|\leq 1}\|\mu_h\|_{T(G)}.
\end{aligned}$$

The proof is completed by the inequality

$$\begin{aligned}
sup_{\|h\|\leq 1}\|\mu_h\|_{T(G)} &= sup_{\|h\|\leq 1}\|\mu_h\|_{S_0'(G)} \\
&= sup_{\|h\|\leq 1} sup_{\|\phi\|_{S_0(G)}\leq 1}|\langle\phi, \mu_h\rangle| \\
&= sup_{\|h\|\leq 1} sup_{\|\phi\|_{S_0(G)}\leq 1}|\langle\mu(\phi)h|h\rangle| \\
&\leq sup_{\|\phi\|_{S_0(G)}\leq 1}\|\mu(\phi)\|_{\mathcal{L}(\mathcal{H})}.
\end{aligned}$$

3. Since $S_0(G)$ is dense in $W(G)$, given any $\phi \in W(G)$, we have that $\mu(\phi)$ can be expressed as the limit in $\mathcal{L}(\mathcal{H})$ of a net $\{\mu(\phi_\alpha) : \phi_\alpha \in S_0(G)\}$ of trace class operators and is hence compact. For any ONB $\{e_i\}_{i\in I}$ of $\mathcal{H}$,

$$tr\mu(\phi) = \sum_{i\in I}\langle \phi, \mu_{e_i}\rangle$$

and since the linear functional

$$\phi \mapsto tr\mu(\phi)$$

is an element of $S_0'(G)$, it is clear that

$$\sum_{i\in I}\mu_{e_i} \in S_0(G)'.$$

Since each μ_{e_i} is positive, we have that

$$\sum_{i\in I}\mu_{e_i} \in T(G)$$

and

$$\begin{aligned}\sum_{i\in I}|\langle\mu(\phi)e_i|e_i\rangle| &= \sum_{i\in I}|\int_G \phi(x)d\mu_{e_i}| \\ &\leq \|\phi\|_{W(G)}\sum_{i\in I}\|\mu_{e_i}\|_{T(G)} \\ &= \|\phi\|_{W(G)}\|\sum_{i\in I}\mu_{e_i}\|_{T(G)} \\ &< \infty.\end{aligned}$$

Hence, $\mu(\phi)$ is a trace class operator and since we have from the first part of the theorem that $\mu \in \mathcal{L}(W(G), \mathcal{L}(\mathcal{H}))$, a routine application of the closed graph theorem then shows that

$$\mu \in \mathcal{L}(W(G), \tau(\mathcal{H})).$$

□

An easy consequence of the above proposition is an $\mathcal{L}(\mathcal{H})$-valued Bochner's theorem.

Theorem 18. *(Bochner's theorem)*
$\mu \in \mathcal{L}(S_0(G), \mathcal{L}(\mathcal{H}))$ *is of positive type if and only if* $\widehat{\mu}$ *is a positive,* $\mathcal{L}(\mathcal{H})$*-valued, translation-bounded measure.*

Proof. The theorem follows from the previous proposition and the chain of equivalences:

$$\begin{aligned} \mu \text{ is of positive type} &\iff \mu_h \text{ is of positive type}, \forall h \in \mathcal{H} \\ &\iff \widehat{\mu}_h \text{ is positive}, \forall h \in \mathcal{H} \\ &\iff \widehat{\mu} \text{ is positive.} \end{aligned}$$

□

Proposition 34. *Let* μ *be a positive* $\mathcal{L}(\mathcal{H})$*-valued measure on a locally compact Abelian group* G*. If* μ *is bounded - ie*

$$\mu \in \mathcal{L}(C_0(G), \mathcal{L}(\mathcal{H})),$$

then, for $h, k \in \mathcal{H}$*, the measure* μ_{hk} *is bounded and, furthermore,*

$$\|\mu\| = sup_{\|h\| \leq 1} \|\mu_h\|_{M(G)},$$

where $\|\mu\|$ *denotes the* $\mathcal{L}(C_0(G), \mathcal{L}(\mathcal{H}))$ *norm of* μ.

Proof. If μ is a positive, bounded, $\mathcal{L}(\mathcal{H})$-valued measure on G, then, for each $h \in \mathcal{H}$ and $\phi \in \mathcal{K}(G)$, we have

$$\begin{aligned} \left|\int_G \phi(x)\mu_h(x)\right| &= |\langle \mu(\phi)h|h\rangle| \\ &\leq \|\phi\|_\infty \|\mu\| \|h\|^2. \end{aligned}$$

Hence, μ_h is bounded and

$$sup_{\|h\| \leq 1} \|\mu_h\|_{M(G)} \leq \|\mu\|.$$

Since μ is positive,

$$((\phi, h), (\psi, k)) \mapsto \int_G \phi(x)\overline{\psi}(x) d\mu_{hk}(x)$$

is a sesquilinear form on $S_0(G) \times \mathcal{H}$. For any $\phi \in \mathcal{K}(G)$, pick $\psi \in \mathcal{K}(G)$ identically equal to 1 on *supp* ϕ. Then, from the Schwartz inequality,

$$\begin{aligned} \left|\int_G \phi(x) d\mu_{hk}(x)\right| &\leq \left(\int_G |\phi(x)|^2 d\mu_h(x)\right)^{1/2} \left(\int_G |\psi(x)|^2 d\mu_k(x)\right)^{1/2} \\ &\leq \|\phi\|_\infty \|\mu_h\|_{M(G)}^{1/2} \|\mu_k\|_{M(G)}^{1/2} \\ &\leq \|\phi\|_\infty \|\mu\| \|h\| \|k\|. \end{aligned}$$

Hence, μ_{hk} is a bounded measure and

$$\|\mu_{hk}\|_{M(G)} \leq \|\mu_h\|_{M(G)}^{1/2} \|\mu_k\|_{M(G)}^{1/2}.$$

The proof is completed by the inequality

$$\begin{aligned} \|\mu\| &= sup\{sup\{|\int_G \phi(x) d\mu_{hk}(x)| : \|h\|, \|k\| \leq 1\} : \phi \in \mathcal{K}(G), \|\phi\|_\infty \leq 1\} \\ &\leq sup\{\|\mu_{hk}\|_{M(G)} : \|h\|, \|k\| \leq 1\} \\ &= sup_{\|h\| \leq 1} \|\mu_h\|_{M(G)}. \end{aligned}$$

□

The previous proposition facilitates a pointwise definition of the Fourier transform $\widehat{\mu}$ of a positive, bounded $\mathcal{L}(\mathcal{H})$-valued measure μ.

Definition 66. Let μ be a positive, bounded $\mathcal{L}(\mathcal{H})$-valued measure on a locally compact Abelian group G. The Fourier transform of μ is the $\mathcal{L}(\mathcal{H})$-valued function $\widehat{\mu}$ on $\widehat{G}$ defined by

$$\langle \widehat{\mu}(\widehat{x}) h | k \rangle = \int_G \overline{\langle x, \widehat{x} \rangle} d\mu_{hk}(x), \forall h, k \in \mathcal{H}.$$

The following proposition is reminiscent of the scalar case.

Proposition 35. *Let μ be positive, bounded $\mathcal{L}(\mathcal{H})$-valued measure. Then*

1. *$\widehat{\mu}$ is a bounded, weakly continuous, $\mathcal{L}(\mathcal{H})$-valued function on $\widehat{G}$ and*

$$sup_{\widehat{x} \in \widehat{G}} \|\widehat{\mu}(\widehat{x})\|_{\mathcal{L}(\mathcal{H})} = \|\widehat{\mu}(\widehat{0})\|_{\mathcal{L}(\mathcal{H})} = \|\mu\|,$$

where $\|\mu\|$ is the bounded $\mathcal{L}(\mathcal{H})$-valued measure norm of μ.

2. *If μ is a trace class operator valued measure, then $\widehat{\mu}$ is a trace class operator valued function on $\widehat{G}$ and*

$$sup_{\widehat{x} \in \widehat{G}} \|\widehat{\mu}(\widehat{x})\|_{\tau(\mathcal{H})} = \|\widehat{\mu}(\widehat{0})\|_{\tau(\mathcal{H})} = tr \int_G d\mu(x) = \|\mu\|,$$

where $\|\mu\|$ is the bounded $\tau(\mathcal{H})$-valued measure norm of μ.

Proof. 1. For $h, k \in \mathcal{H}$ and $\widehat{x} \in \widehat{G}$, we have

$$|\langle \widehat{\mu}(\widehat{x})h|k\rangle| \leq \|\mu_{hk}\|$$

and hence $\widehat{\mu}$ is an $\mathcal{L}(\mathcal{H})$-valued function with

$$sup_{\widehat{x}\in\widehat{G}}\|\widehat{\mu}(\widehat{x})\|_{\mathcal{L}(\mathcal{H})} \leq \|\mu\|.$$

$\widehat{\mu}(\widehat{0})$ is clearly a positive operator and hence

$$\begin{aligned} \|\widehat{\mu}(\widehat{0})\|_{\mathcal{L}(\mathcal{H})} &= sup_{\|h\|\leq 1}|\langle \widehat{\mu}(\widehat{0})h|h\rangle| \\ &= sup_{\|h\|\leq 1}\int_G d\mu_h(x) \\ &= sup_{\|h\|\leq 1}\|\mu_h\| \\ &= \|\mu\|. \end{aligned}$$

Since the Fourier transform of a bounded measure is continuous, it is clear that $\widehat{\mu}$ is weakly continuous.

2. Since μ is a trace class operator valued measure, the mapping

$$\phi \mapsto tr\int_G \phi(x)d\mu(x)$$

is a bounded measure on G. Given any ONB $\{e_i\}_{i\in I}$ of $\mathcal{H}$, we have

$$\begin{aligned} tr\int_G \phi(x)d\mu(x) &= \sum_{i\in I}\langle \int_G \phi(x)d\mu(x)e_i|e_i\rangle \\ &= \sum_{i\in I}\int_G \phi(x)d\mu_{e_i}(x) \end{aligned}$$

and hence that $\sum_{i\in I}\mu_{e_i}$ is a bounded measure. Furthermore, since each μ_{e_i} is a positive bounded measure, we have

$$\|\sum_{i\in I}\mu_{e_i}\|_{M(G)} = \sum_{i\in I}\|\mu_{e_i}\|_{M(G)}.$$

With $\{e_i\}_{i\in I}$ as above and $\widehat{x} \in \widehat{G}$, we have, for each $i \in I$,

$$\begin{aligned}
\|\widehat{\mu}(\widehat{x})e_i\|^2 &= \|\int_G \overline{\langle x, \widehat{x}\rangle} d\mu(x)e_i\|^2 \\
&= sup_{\|h\|\leq 1}|\langle \int_G \overline{\langle x, \widehat{x}\rangle} d\mu(x)e_i|h\rangle|^2 \\
&= sup_{\|h\|\leq 1}|\int_G \overline{\langle x, \widehat{x}\rangle} d\mu_{e_i h}(x)|^2 \\
&\leq sup_{\|h\|\leq 1}\|\mu_{e_i h}\|^2_{M(G)} \\
&\leq sup_{\|h\|\leq 1}\|\dot{\mu}_h\|_{M(G)}\|\mu_{e_i}\|_{M(G)} \\
&= \|\mu\|\|\mu_{e_i}\|_{M(G)}.
\end{aligned}$$

Hence,

$$\sum_{i\in I}\|\widehat{\mu}(\widehat{x})e_i\|^2 \leq \|\mu\|\sum_{i\in I}\|\mu_{e_i}\|_{M(G)}$$

and thus $\widehat{\mu}(\widehat{x})$ is a Hilbert-Schmidt operator.

Since $\widehat{\mu}(\widehat{x})$ is compact, the estimate

$$\begin{aligned}
\sum_{i\in I}|\langle\widehat{\mu}(\widehat{x})e_i|e_i\rangle| &= \sum_{i\in I}|\int_G \overline{\langle x, \widehat{x}\rangle} d\mu_{e_i}(x)| \\
&\leq \sum_{i\in I}\|\mu_{e_i}\|_{M(G)}
\end{aligned}$$

shows that it is a trace class operator.

Since μ is positive, we have

$$\begin{aligned}
\|\int_G d\mu(x)\|_{\tau(\mathcal{H})} &= tr\int_G d\mu(x) \\
&= \sum_{i\in I}\langle\int_G d\mu(x)e_i|e_i\rangle \\
&= \sum_{i\in I}\int_G d\mu_{e_i}(x) \\
&= \sum_{i\in I}\|\mu_{e_i}\|_{M(G)} \\
&= \|\sum_{i\in I}\mu_{e_i}\|_{M(G)}.
\end{aligned}$$

Given any $\phi \in C_0(G)$, we have that

$$\begin{aligned}
\|\int_G \phi(x)d\mu(x)\|_{\tau(\mathcal{H})} &= \|[\mu\phi, \mu]\|_{\tau(\mathcal{H})} \\
&\leq (tr\int_G |\phi(x)|^2 d\mu(x))^{1/2}(tr\int_G d\mu(x))^{1/2} \\
&= (\sum_{i\in I}\int_G |\phi(x)|^2 d\mu_{e_i}(x))^{1/2}\|\sum_{i\in I}\mu_{e_i}\|_{M(G)}^{1/2} \\
&\leq \|\phi\|_\infty \|\sum_{i\in I}\mu_{e_i}\|_{M(G)}
\end{aligned}$$

and hence we have

$$\|\mu\| \leq \|\sum_{i\in I}\mu_{e_i}\|_{M(G)}.$$

Conversely,

$$\begin{aligned}
\|\sum_{i\in I}\mu_{e_i}\|_{M(G)} &= sup_{\|\phi\|_{C_0(G)}\leq 1}|\sum_{i\in I}\int_G \phi(x)d\mu_{e_i}(x)| \\
&= sup_{\|\phi\|_{C_0(G)}\leq 1}|tr\int_G \phi(x)d\mu(x)|.
\end{aligned}$$

For any trace class operator T, we have

$$\|T\|_{\tau(\mathcal{H})} = sup\sum_{i,j\in I}|\langle Te_i|f_j\rangle|,$$

where the sup is taken over all pairs $(\{e_i\}_{i\in I}, \{f_j\}_{j\in I})$ of orthonormal bases of $\mathcal{H}$ and hence it is clear that

$$|trT| \leq \|T\|_{\tau(\mathcal{H})}.$$

Hence,

$$\begin{aligned}
\|\sum_{i\in I}\mu_{e_i}\|_{M(G)} &\leq sup_{\|\phi\|_{C_0(G)}\leq 1}\|\int_G \phi(x)d\mu(x)\|_{\tau(\mathcal{H})} \\
&= \|\mu\|.
\end{aligned}$$

□

We recall that every positive-definite trace class operator valued kernel Γ is the reproducing kernel of a Hilbert $\mathcal{L}(\mathcal{H})$-module $\mathcal{H}_\Gamma$. Where the reproducing kernel is a bounded, positive-definite bimeasure B, we note that, for each $\phi \in C_0(G)$, $B\phi$ is a bounded, $\tau(\mathcal{H})$-valued measure and hence the Fourier transform $\widehat{B\phi}$ defined by

$$\widehat{B\phi}(\widehat{x}) = \int_G \overline{\langle x, \widehat{x} \rangle} dB\phi(x), \widehat{x} \in \widehat{G},$$

is a $\tau(\mathcal{H})$-valued function on $\widehat{G}$.

Proposition 36. *Let B be a bounded, positive-definite $\tau(\mathcal{H})$-valued bimeasure on the locally compact Abelian group G. Then, for each $\widehat{x} \in \widehat{G}$, the linear mapping defined by*

$$B\phi \mapsto \int_G \overline{\langle x, \widehat{x} \rangle} dB\phi(x), \widehat{x} \in \widehat{G},$$

for each $\phi \in S_0(G)$ extends to a bounded linear functional on $\mathcal{H}_B$.

Proof. If we denote the bounded $\tau(\mathcal{H})$-valued measure norm of $B\phi$ by $\|B\phi\|$ and the bounded $\tau(\mathcal{H})$-valued bimeasure norm of B by $\|B\|$, then we have

$$\begin{aligned} \| \int_G \overline{\langle x, \widehat{x} \rangle} dB\phi(x) \|_{\tau(\mathcal{H})} &\leq \|B\phi\| \\ &= sup_{\|\psi\|_{C_0(G)} \leq 1} \| \int_G \overline{\psi}(x) dB\phi(x) \|_{\tau(\mathcal{H})} \\ &= sup_{\|\psi\|_{C_0(G)} \leq 1} \| [B\phi, B\psi] \|_{\tau(\mathcal{H})} \\ &\leq sup_{\|\psi\|_{C_0(G)} \leq 1} \|B\psi\|_{\mathcal{H}_B} \|B\phi\|_{\mathcal{H}_B} \\ &\leq \|B\| \|B\phi\|_{\mathcal{H}_B}. \end{aligned}$$

□

3.7 Stationary multidimensional second order generalised stochastic processes

Definition 67. Let X' be a multidimensional second order generalised stochastic process on a locally compact Abelian group G.

1. X' is *stationary* if the scalar autocorrelation $\sigma_{X'}$ is diagonally invariant.
2. X' is *scalarly stationary* if, for each $h \in \mathcal{H}$, the scalar process X'_h is stationary.
3. X' is *operator stationary* if the operator autocorrelation $\Gamma_{X'}$ is diagonally invariant - ie if

$$[L_x\phi \otimes L_x\overline{\psi}, \Gamma_{X'}] = [\phi \otimes \overline{\psi}, \Gamma_{X'}], \forall x \in G, \forall \phi, \psi \in S_0(G).$$

Definition 68. Let X' be a multidimensional second order generalised stochastic process on a locally compact Abelian group G.

1. X' is *grammian orthogonally scattered* if

$$[X'\phi, X'\psi] = 0,$$

for any disjointly supported pair of functions $\phi, \psi \in S_0(G)$.

2. X' is *scalarly orthogonally scattered* if, for each $h \in \mathcal{H}$, the scalar generalised stochastic process X'_h is orthogonally scattered.

Proposition 37. *Let X' be a multidimensional second order generalised stochastic process on a locally compact Abelian group G. Then X' is grammian orthogonally scattered if and only if X' is scalarly orthogonally scattered.*

Proof. Let X' be grammian orthogonally scattered. Then for each $h \in \mathcal{H}$ and disjointly supported pair of functions $\phi, \psi \in S_0(G)$,

$$\begin{aligned} \langle X'_h\phi | X'_h\psi\rangle &= tr[\phi \otimes \overline{\psi}, \Gamma_{X'}]h \otimes h^\star \\ &= 0 \end{aligned}$$

and hence X' is scalarly orthogonally scattered.

Conversely, any trace class operator T can be expressed as a sum

$$T = (T + T^\star)/2 + i(T - T^\star)/2i$$

of two self-adjoint trace class operators and it is hence easy to show that

$$\langle Th|h\rangle = 0, \forall h \in \mathcal{H} \Rightarrow T = 0.$$

Consequently, if X' is scalarly orthogonally scattered, then for any disjointly supported pair of functions $\phi, \psi \in S_0(G)$, we have that

$$[\phi \otimes \overline{\psi}, \Gamma_{X'}] = 0$$

and hence that X' is grammian orthogonally scattered. □

Theorem 19. *(The spectral representation theorem for operator stationary multidimensional generalised stochastic processes)*

Let X' be a multidimensional second order generalised stochastic process on a locally compact Abelian group G.

1. *X' is operator stationary if and only if its operator autocorrelation can be represented in the form*

$$[\phi \otimes \overline{\psi}, \Gamma_{X'}] = \int_{\widehat{G}} \widehat{\phi}(\widehat{x})\overline{\widehat{\psi}}(\widehat{x}) d\mu_{X'}(\widehat{x}), \phi, \psi \in S_0(G), \tag{3.7.1}$$

where $\mu_{X'}$ is a positive $\tau(\mathcal{H})$-valued translation-bounded measure.

2. *X' is operator stationary if and only if the spectral process $\widehat{X'}$ is a grammian orthogonally scattered translation-bounded measure.*

3. *If X' is operator stationary, then the modular time domain $\mathcal{H}_{X'}$ is grammian unitarily isomorphic to the modular spectral domain $L^2(G, \mu_{X'})$.*

Proof. 1. It is clear that $\Gamma_{X'}$ is diagonally invariant if it can be represented as in equation 3.7.1.

Conversely, the diagonal invariance of $\Gamma_{X'}$ implies that, for $h, k \in \mathcal{H}$, there exists $\mu_{hk} \in S_0'(G)$ such that for $\phi, \psi \in S_0(G)$,

$$\langle [\phi \otimes \overline{\psi}, \Gamma_{X'}] h | k \rangle = \langle \phi \star \psi^\star, \mu_{hk} \rangle.$$

Given $\varphi \in S_0(G)$, define $\tilde{\Gamma}_{X'}(\varphi)$, a linear operator on $\mathcal{H}$, according to

$$\langle \tilde{\Gamma}_{X'}(\varphi) h | k \rangle_{\mathcal{H}} = \sum_{i \in I} \langle \phi_i \star \psi_i^\star, \mu_{hk} \rangle, \text{ for } h, k \in \mathcal{H},$$

where $\sum_{i \in I} \phi_i \star \psi_i^\star$ is an admissible convolution tensor product representation of φ (see proposition 10). For $\|h\|, \|k\| \leq 1$, denoting the $\mathcal{L}(S_0(G \times G), \mathcal{L}(\mathcal{H}))$ norm of $\Gamma_{X'}$ by $\|\Gamma_{X'}\|$, we have

$$|\langle \tilde{\Gamma}_{X'}(\varphi) h | k \rangle_{\mathcal{H}}| \leq \|\Gamma_{X'}\| \sum_{i \in I} \|\phi\|_{S_0(G)} \|\psi\|_{S_0(G)},$$

and hence, taking the inf of the right hand side over all admissible representations of φ, we have

$$\|\tilde{\Gamma}_{X'}(\varphi)\|_{\mathcal{L}(\mathcal{H})} \leq \|\Gamma_{X'}\| \|\varphi\|_{S_0(G)}.$$

This yields $\tilde{\Gamma}_{X'}(\varphi) \in \mathcal{L}(S_0(G), \mathcal{L}(\mathcal{H}))$ and since $\tilde{\Gamma}_{X'}(\varphi) \in \tau(\mathcal{H})$ for each φ in $S_0(G)$, a routine application of the closed graph theorem gives

$$\tilde{\Gamma}_{X'}(\varphi) \in \mathcal{L}(S_0(G), \tau(\mathcal{H})).$$

For any $\phi \in S_0(G)$,

$$\tilde{\Gamma}_{X'}(\phi \star \phi^\star) = [\phi \otimes \overline{\phi}, \Gamma_{X'}],$$

and, since $\Gamma_{X'}$ is a positive definite kernel, $\tilde{\Gamma}_{X'}$ is of positive type, and the representation 3.7.1 follows from the Bochner theorem and the final part of proposition 33.

2. Let X' be operator stationary and $\mu_{X'}$ be the associated trace class operator valued spectral measure. Then, for any pair of functions $\phi, \psi \in S_0(\widehat{G})$ with disjoint support,

$$\begin{aligned}[\widehat{X'}\phi, \widehat{X'}\psi] &= \int_{\widehat{G}} \phi(\widehat{x})\overline{\psi}(\widehat{x})d\mu_{X'}(\widehat{x}) \\ &= 0\end{aligned}$$

which shows that $\widehat{X'}$ is grammian orthogonally-scattered.

For any $\phi \in S_0(\widehat{G})$, we have from the first part of the theorem,

$$\begin{aligned}\|\widehat{X'}\phi\|_{\mathcal{X}} &= \sqrt{\|[\widehat{X'}\phi, \widehat{X'}\phi]\|_{\tau(\mathcal{H})}} \\ &= \sqrt{\|\int_{\widehat{G}} |\phi(\widehat{x})|^2 d\mu_{X'}(\widehat{x})\|_{\tau(\mathcal{H})}} \\ &\leq \|\phi\|_{W(\widehat{G})} \|\mu_{X'}\|^{1/2}_{\mathcal{L}(W(\widehat{G}),\tau(\mathcal{H}))}.\end{aligned}$$

Hence, since $S_0(\widehat{G})$ is dense in $W(\widehat{G})$, $\widehat{X'}$ extends to a $\mathcal{X}$-valued translation-bounded measure on $\widehat{G}$.

Conversely, if $\widehat{X'}$ is a grammian orthogonally scattered translation-bounded measure, then, for each $h, k \in \mathcal{H}$, the translation-bounded bimeasure

$$B_{X'_h X'_k} : \phi \otimes \overline{\psi} \mapsto \langle [\widehat{X'}\phi, \widehat{X'}\psi]h|k\rangle$$

is supported in the diagonal subgroup $\Delta_{\widehat{G}}$ and hence the correlation kernel $\sigma_{X'_h X'_k}$ is diagonally invariant in each case. Thus, $\Gamma_{X'}$ is diagonally-invariant and X' is operator stationary.

3. The final part of the theorem follows from the representation 3.7.1.

□

Proposition 38. *Let X' be a multidimensional second order generalised stochastic process on the locally compact Abelian group G. Then*

$$X' \textit{ is scalarly stationary} \iff \widehat{X'} \textit{ is scalarly orthogonally scattered.}$$

Proof. The statement is an easy consequence of the fact that, for each $h \in \mathcal{H}$,

$$\widehat{X'}_h = \widehat{X'_h}.$$

□

We can now characterise scalarly stationary processes.

Proposition 39. *Let X' be a multidimensional second order generalised stochastic process on the locally compact Abelian group G. Then*

$$X' \text{ is operator stationary} \iff X' \text{ is scalarly stationary.}$$

Proof. The proposition is a simple consequence of proposition 37 and proposition 38. □

It is clear that operator stationarity implies stationarity and hence, from the previous proposition, that scalar stationarity implies stationarity. The converse is, however, untrue; for a counterexample, see example IV.6 of [22].

3.8 Multidimensional stationary stochastic processes and multidimensional stationary generalised stochastic processes

Let X be a multidimensional second order stochastic process on a locally compact group G which is bounded in the sense that

$$sup_{x\in G}\|Xx\|_{\mathcal{X}} < \infty.$$

If, for each $\phi \in S_0(G)$, we let

$$X\phi = \int_G Xx\phi(x)dx,$$

where the integral is defined weakly, we have

$$\|X\phi\|_{\mathcal{X}} \leq sup_{x\in G}\|Xx\|_{\mathcal{X}}\|\phi\|_{S_0(G)},$$

and hence that X defines a multidimensional generalised stochastic process.

A stationary stochastic process X is with spectral measure μ_X is certainly bounded and

$$\begin{aligned}
[X\phi, X\psi] &= \int_{G\times G} [Xx, Xy]\phi(x)\overline{\psi}(y)dxdy \\
&= [\phi \otimes \overline{\psi}, \Gamma_X] \\
&= \int_{\widehat{G}} \widehat{\phi}(\widehat{x})\overline{\widehat{\psi}}(\widehat{x})d\mu_X(\widehat{x}).
\end{aligned}$$

Hence the generalised stochastic process generated by a stationary process is stationary and has the same spectral measure as the original process. The following proposition describes a converse.

Proposition 40. *Let X' be a stationary multidimensional second order generalised stochastic process X' on a locally compact Abelian group G. If the spectral measure $\mu_{X'}$ of X' is bounded, ie*

$$\mu_{X'} \in \mathcal{L}(C_0(\widehat{G}), \tau(\mathcal{H})),$$

then the spectral process $\widehat{X'}$ is a bounded stochastic measure and X' may be identified with a stationary multidimensional second order stochastic process on G.

Proof. For any $\phi \in S_0(\widehat{G})$, we have

$$\begin{aligned}
\|\widehat{X'}\phi\|^2_{\mathcal{X}} &= tr[\widehat{X'}\phi, \widehat{X'}\phi] \\
&= tr \int_{\widehat{G}} |\phi(\widehat{x})|^2 d\mu_{X'}(\widehat{x}) \\
&= \sum_{i\in I} \int_{\widehat{G}} |\phi(\widehat{x})|^2 d\mu_{X'_{e_i}}(\widehat{x}) \\
&\leq \|\phi\|^2_\infty \|\sum_{i\in I} \mu_{X'_{e_i}}\|_{M(\widehat{G})}
\end{aligned}$$

and hence, from the density of $S_0(\widehat{G})$ in $C_0(\widehat{G})$, $\widehat{X'}$ extends to a bounded measure on $\widehat{G}$.

Using proposition 35 and the Schwartz inequality, we have

$$\begin{aligned}
\|\int_{\widehat{G}} \widehat{\phi}(\widehat{x})\langle x, \widehat{x}\rangle d\mu_{X'}(\widehat{x})\|_{\tau(\mathcal{H})} &\leq \|\int_{\widehat{G}} |\widehat{\phi}(\widehat{x})|^2 d\mu_{X'}(\widehat{x})\|^{1/2}_{\tau(\mathcal{H})} \|\int_G d\mu_{X'}(\widehat{x})\|^{1/2}_{\tau(\mathcal{H})} \\
&= \|X'\phi\|_{\mathcal{H}_{X'}} \|\mu_{X'}\|
\end{aligned}$$

Hence, from the Riesz representation theorem, the function X, defined by

$$[X'\phi, Xx] = \int_{\widehat{G}} \widehat{\phi}(\widehat{x})\langle x, \widehat{x}\rangle d\mu_{X'}(\widehat{x}), \phi \in S_0(G),$$

is a $\mathcal{H}_{X'}$-valued stochastic process on G.

We show X generates X'. Given $\phi, \psi \in S_0(G)$,

$$\begin{aligned}
[X'\phi, \int_G Xx\psi(x)dx] &= \int_G [X'\phi, Xx]\overline{\psi}(x)dx \\
&= \int_G \int_{\widehat{G}} \widehat{\phi}(\widehat{x})\langle x, \widehat{x}\rangle d\mu_{X'}(\widehat{x})\overline{\psi}(x)dx \\
&= \int_{\widehat{G}} \widehat{\phi}(\widehat{x}) \int_G \overline{\psi}(x)\langle x, \widehat{x}\rangle dx d\mu_{X'}(\widehat{x}) \\
&= \int_{\widehat{G}} \widehat{\phi}(\widehat{x})\widehat{\overline{\psi}}(\widehat{x}) d\mu_{X'}(\widehat{x}) \\
&= [X'\phi, X'\psi],
\end{aligned}$$

as required. □

3.9 Harmonisable multidimensional second order generalised stochastic processes

Definition 69. Let X' be a multidimensional second order generalised stochastic process on a locally compact Abelian group G.

1. X' is *harmonisable* if $\widehat{X'}$ is a translation-bounded measure.
2. X' is *scalarly harmonisable* if, for each $h \in \mathcal{H}$, the scalar process X'_h is harmonisable.
3. X' is *operator harmonisable* if the spectral process $\widehat{X'}$ is a translation-bounded $\mathcal{X}$-valued measure.

Proposition 41. *Let X' be a multidimensional second order generalised stochastic process on a locally compact Abelian group G. Then X' is operator harmonisable if and only if the operator autocorrelation $\Gamma_{X'}$ has the representation*

$$[\widehat{\phi \otimes \overline{\psi}}, \Gamma_{X'}] = [\phi \otimes \overline{\psi}, B_{X'}],$$

where $B_{X'}$ is a $\tau(\mathcal{H})$-valued translation-bounded bimeasure, ie

$$B_{X'} \in \mathcal{L}(W(\widehat{G})\widehat{\otimes}W(\widehat{G}), \tau(\mathcal{H})).$$

Proof. Let X' be harmonisable. Then $\widehat{X'}$ is a translation-bounded measure and, using the Schwartz inequality, the estimate

$$\begin{aligned} \|[\phi \otimes \overline{\psi}, \Gamma_{\widehat{X'}}]\|_{\tau(\mathcal{H})} &= \|[\widehat{X'}\phi, \widehat{X'}\psi]\|_{\tau(\mathcal{H})} \\ &\leq \|\widehat{X'}\psi\| \|\widehat{X'}\phi\| \\ &\leq \|\widehat{X'}\|^2 \|\psi\|_{W(\widehat{G})} \|\phi\|_{W(\widehat{G})} \end{aligned}$$

for any $\phi, \psi \in S_0(\widehat{G})$ and the density of $S_0(\widehat{G} \times \widehat{G})$ in $W(\widehat{G}) \widehat{\otimes} W(\widehat{G})$ show that $\Gamma_{\widehat{X'}}$ is a bounded $\tau(\mathcal{H})$-valued bimeasure.

Conversely, let $\Gamma_{\widehat{X'}}$ be a $\tau(\mathcal{H})$-valued translation-bounded bimeasure. It is clear that the kernel $tr\Gamma_{\widehat{X'}}$ in $S_0'(\widehat{G} \times \widehat{G})$ defined by

$$\langle \phi \otimes \overline{\psi}, tr\Gamma_{\widehat{X'}} \rangle = tr[\phi \otimes \overline{\psi}, \Gamma_{\widehat{X'}}]$$

is a translation-bounded bimeasure. Hence, given any ONB $\{e_i\}_{i \in I}$, we have that $\sum_{i \in I} \Gamma_{\widehat{X'}_{e_i}}$ is a translation-bounded bimeasure which is independent of the ONB. The estimate

$$\begin{aligned} \|\widehat{X'}\phi\|^2 &= tr[\phi \otimes \overline{\phi}, \Gamma_{\widehat{X'}}] \\ &= \sum_{i \in I} \langle \phi \otimes \overline{\phi}, \Gamma_{\widehat{X'}_{e_i}} \rangle \\ &\leq \|\phi\|^2_{W(\widehat{G})} \| \sum_{i \in I} \Gamma_{\widehat{X'}_{e_i}} \|_{TBM} \end{aligned}$$

for any $\phi \in S_0(\widehat{G})$ and the density of $S_0(\widehat{G})$ in $W(\widehat{G})$ show that $\widehat{X'}$ extends to a translation-bounded measure. □

3.10 Harmonisable multidimensional stochastic processes and harmonisable multidimensional generalised stochastic processes

Let X be an operator harmonisable multidimensional second order stochastic process on a locally compact Abelian group G. Since

$$\begin{aligned} sup_{x \in G} \|Xx\|^2 &= sup_{x \in G} \sigma_X(x, x) \\ &\leq \|B_X^{sc}\|_{BM}, \end{aligned}$$

where B_X^{sc} is the scalar covariance bimeasure of X, X is bounded and hence generates a generalised stochastic process via the weak integral

$$X'\phi = \int_G Xx\phi(x)dx.$$

It is easy to show that X' is operator harmonisable with operator covariance bimeasure B_X. We show now that the converse is true.

Proposition 42. *Let X' be an operator harmonisable multidimensional second order generalised stochastic process on a locally compact Abelian group G. If the operator covariance bimeasure $B_{X'}$ of X' is bounded, then $\widehat{X'}$ is a bounded measure and X' may be identified with a stochastic process on G.*

Proof. Let X' and $B_{X'}$ be as above. Then, for any $\phi \in S_0(G)$,

$$\begin{aligned} \|\widehat{X'}\phi\|^2 &= tr[\phi \otimes \overline{\phi}, B_{X'}] \\ &\leq \|\phi\|_\infty^2 \|B_{X'}^{sc}\|_{BM} \end{aligned}$$

and hence, from the density of $S_0(\widehat{G})$ in $C_0(\widehat{G})$, $\widehat{X'}$ extends to a bounded measure.

It is then clear from proposition 36 and the Riesz representation theorem that the equation

$$[X'\widehat{\phi}, Xx] = \int_{\widehat{G}\times\widehat{G}} \phi(\widehat{x})\overline{\langle x, \widehat{y}\rangle} dB_{X'}(\widehat{x}, \widehat{y})$$

defines a stochastic process X on G with range in $\mathcal{H}_{X'}$.

We show now that X generates X'. For $\widehat{\phi}, \widehat{\psi} \in S_0(G)$,

$$\begin{aligned} [X'\widehat{\phi}, \int_G Xx\widehat{\psi}(x)dx] &= \int_G [X'\widehat{\phi}, Xx]\overline{\widehat{\psi}}(x)dx \\ &= \int_G \int_{\widehat{G}\times\widehat{G}} \phi(\widehat{x})\overline{\langle x, \widehat{y}\rangle} dB_{X'}(\widehat{x}, \widehat{y})\overline{\widehat{\psi}}(x)dx \\ &= \int_G \int_{\widehat{G}} \overline{\langle x, \widehat{y}\rangle} dB_{X'}(\phi)(\widehat{y})\overline{\widehat{\psi}}(x)dx \\ &= \int_{\widehat{G}} \int_G \overline{\widehat{\psi}}(x)\overline{\langle x, \widehat{y}\rangle} dx \; dB_{X'}(\phi)(\widehat{y}) \\ &= \int_{\widehat{G}} \overline{\psi}(\widehat{y}) dB_{X'}(\phi)(\widehat{y}) \\ &= [\phi \otimes \overline{\psi}, B_{X'}] \\ &= [X'\widehat{\phi}, X'\widehat{\psi}] \end{aligned}$$

and hence

$$\int_G X x \widehat{\psi}(x) dx = X' \widehat{\psi},$$

as required. □

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