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Sigma models of the AdS/CFT correspondence

Submitted as an exercise for a Ph.D. degree

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Summary

The thesis is dedicated to the investigation of the properties of particular two-dimensional quantum field theories, i.e. sigma-models with target space of the form $AdS_5 \times S^5_\gamma$ and $AdS_4 \times \mathbb{CP}^3$.

The main results of the thesis are as follows:

1. The leading finite size correction to the dispersion relation of a soliton solution of the $AdS_5 \times S^5_\gamma$ sigma model has been calculated (γ is a parameter describing the deformation of the S^5 metric). Nontrivial dependence on γ has been found.

2. The sigma model with target space $AdS_4 \times \mathbb{CP}^3$ has been analyzed in the light-cone gauge. In particular, in the classical limit a central extension of the global symmetry algebra has been found. It depends on the worldsheet momentum and is identical to the one of the $AdS_5 \times S^5$ model.

3. The quantization of the Green-Schwarz superstring in the background of the "spinning string" solution has been carried out for the $AdS_4 \times \mathbb{CP}^3$ model. The spectrum of excitations around this solution has been found, and the one-loop correction to the energy of this configuration has been calculated. The "spinning string" solution has two parameters, which describe two independent rotations of the string. In the limit, when one of the parameters is significantly greater than the other, some of the excitations become massless. The full Lagrangian describing the

low-energy dynamics of these massless modes has been found — it is a $\mathbb{CP}^3$ sigma model interacting with a Dirac fermion.

In the thesis a range of methods of modern mathematical physics has been used: the Green-Schwarz model for the superstring, the coset construction of the action, the M2 brane action, as well as elements of supergravity and the theory of integrable models. Some analytic calculations have been carried out using the Wolfram Mathematica program.

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Contents

Introduction

Quantum gauge theory is the mathematical foundation of modern elementary particle physics, and, as such, it lies at the heart of the modern understanding of the fundamental laws of nature. Maxwell's classical theory of electrodynamics and later its quantum version constructed by Feynman, Schwinger etc. is described by gauge theory with a simplest abelian gauge group $U(1)$. The mathematical structure of this theory has been studied well, and the calculations produce amazingly sharp experimental predictions thanks to the fact that the so called coupling constant of the theory, which characterizes the strength of interactions of electrons and positrons with the electromagnetic field, is rather small (as is well-known, the fine structure constant at normal energies is approximately equal to $1/137$). Apart from electrodynamics, the Standard Model of elementary particle physics also describes weak and strong interactions, the weak and electromagnetic interactions being united into a *nonabelian* gauge group $SU(2) \times U(1)$, while the strong interactions are also described by a *nonabelian* group $SU(3)$ (which is frequently called the group of «color») [1]. To put it differently, the physical phenomena related to weak and strong interactions are described with the help of nonabelian gauge symmetry. Despite such similarity, there is a principal difference between the weak and strong interactions. Consistently with the names, the coupling constant in weak interactions is small, and it is

[1] The theory of strong interactions based on the gauge group $SU(3)$ is called quantum chromodynamics.

large for the strong interactions. This means that perturbation theory in the coupling constant is not applicable to the description of strong interactions at normal energies. There is also a physical confirmation of the non-applicability of perturbation theory in the real situation, namely the mismatch between the asymptotical spectrum of particles at zero coupling constant and the one in the real world. Indeed, quantum chromodynamics at zero coupling would predict the existence of massless gluons and quarks, either massless or massive. However, none of these particles are observed in nature. The particles that are observed are the bound states of quarks (mesons in the case of two quarks and baryons for three quarks), and the hypothetical bound states of gluons — the so-called glueballs — have not been discovered yet. As should be clear from the previous discussion, such disagreement between theory and experiment is usually imputed to the fact that the coupling constant is large, or, in other words, that perturbation theory is not applicable. It means that the leading approximation adopted in perturbation theory — the free field approximation — does not suit this problem. Hence, the problem of utmost importance is to find a different leading approximation which would be more suitable and which would allow to calculate various parameters of the elementary particles as a perturbation theory around this leading approximation. This problem, also known as the quark confinement problem, and the problem of mass gap in non-Abelian Yang-Mills theory, are challenges of paramount importance in quantum gauge theory for more than 40 years now, but they still remain unsolved [2]. It is worth mentioning that the solution of these problems will most likely be related to a fully mathematically consistent construction of the complete theory of quantum

[2]The problem of explaining the mass gap in Yang-Mills theory is one of the «millennium» problems, offered by the Clay Institute in the USA. A one million dollar reward has been announced for the solution of any one of these problems.

gauge fields (it is also known as the problem of «existence of the Yang-Mills theory in four-dimensional spacetime»). In particular, it will be necessary to treat the divergent perturbation series in a mathematically consistent way. In spite of the fact that so far there is no description of the dynamics of non-Abelian Yang-Mills theory «from the first principles», there is a satisfactory qualitative picture.

An important new method of the description of non-Abelian gauge theories at large coupling constants was put forward in the work of J.Maldacena [84]. Maldacena considered Yang-Mills theory in 3+1 dimensions with maximal possible supersymmetry — the so-called $\mathcal{N} = 4$ super-Yang-Mills theory. In Chapter 1 we will give a more complete description of this theory, but for the moment it is sufficient to mention that the Lagrangian of the theory is invariant under four Majorana supercharges. A theory with this number of supersymmetries is unique (in contrast to the theories with less supersymmetries), or, more precisely, there is a two-parametric family of such theories, parameterized by the coupling constant g and the gauge group «rank» N (for example, in the case of the group $U(M)$ one has $N = M$). The unexpected conclusion of Maldacena was that the $\mathcal{N} = 4$ SYM theory is equivalent in a very precise way to the theory of type IIB supertrings propagating in the space $AdS_5 \times S^5$ (here AdS is anti-de Sitter space and S is the sphere with its standard metric). This is a general correspondence between two physical theories, which look very different at first sight. This equivalence, or duality, was elaborated in the papers [67,108]. In particular, it was shown that the four-dimensional Minkowski space arises at the projective «infinity» of AdS. The equivalence discovered in [84] was called the AdS/CFT correspondence. More than twenty years before the work of J.Maldacena G.'t Hooft pointed out certain (at least qualitative) simplifications of the perturbation series, which arise in the limit $N \to \infty$,

$g \to 0$, $\lambda \equiv g^2N = \text{const}$. Since then this limit is called the 't Hooft limit, and the corresponding constant λ – 't Hooft's constant or parameter. Maldacena's result reduces in the 't Hooft limit to a weaker statement, which is nevertheless very interesting: the planar limit of the $\mathcal{N} = 4$ super-Yang-Mills theory is equivalent to the $AdS_5 \times S^5$ string sigma model (i.e. the free string propagating in this curved space).

We would like to emphasize that although formally the Lagrangian of the theory with $\mathcal{N} = 4$ supersymmetries is just a small complication of usual gauge theory, related to the presence of «matter fields», in reality the maximally symmetric theory has one important peculiarity, which makes it quite different from the less symmetric analogues. The fact is that this theory is invariant under conformal transformations at the quantum level. At the classical level any gauge theory is conformally invariant, if the matter fields are massless, but at the quantum level in the absolute majority of cases this is no longer so, since renormalizations of field, mass and charge are necessary. It is precisely in the maximally symmetric theory that the beta-function of the coupling constant turns out to be zero. A negative consequence of this observation is that the conformal theory cannot aspire to the role of a theory of elementary particles, since all phenomena in such a theory look analogous at all spacetime scales. In particular, it is not possible to define the scattering states.

It is worth mentioning that an important feature of the AdS/CFT correspondence is that the coupling constants on the two sides of the correspondence are inversely proportional to each other. Indeed, the coupling constant of the string sigma-model is the square of the radius of AdS: $R^2 \propto \sqrt{\lambda}$. The small coupling limit for the sigma model corresponds to large radius, i.e. to large values of 't Hooft's parameter. It is precisely this property which allows to investigate the strong coupling limit of gauge theory using the AdS/CFT correspondence.

Since the advent of the paper [84] significant progress has been achieved in the study of the AdS/CFT correspondence. First of all, the Green-Schwarz action of the string sigma-model with target space $AdS_5 \times S^5$ was built [90] (we remind the reader that in standard textbooks the case usually considered is the one of flat target-space $\mathbb{R}^{10}$). The limit of large 't Hooft coupling constant corresponds to the classical limit of the sigma model, therefore the knowledge of the Green Schwarz action allowed in particular to calculate the semiclassical corrections to the various classical solutions of the sigma model. One can choose the light-like (null) geodesic as such solution [34]. In this case the corresponding limit is called the «plane wave» limit — from a geometrical viewpoint it corresponds to the expansion of the metric around a null geodesic (the so-called Penrose limit).On the other hand, in field theory this limit arises if one considers long operators of the form $\mathrm{tr}(Y^k Z^J)$, where Y and Z are complex scalar fields, k is fixed and $J \to \infty, \lambda \to \infty, J/\sqrt{\lambda} = const$. Another interesting classical solution, which is in particular extensively studied in the present thesis, is the so-called «spinning string» [68], in other words the string rotating in AdS around its center-of-mass. In gauge theory the operators dual to this solution are $\mathrm{tr}(\phi D^S \phi)$, where $S \to \infty$, that is the operators with a large number of derivatives. It is worth noting that the leading semiclassical correction to the energy of such configuration was calculated in [55].

However, the main achievement of the investigations was in the discovery of integrability properties of the maximally supersymmetric Yang-Mills theory, as well as of the $AdS_5 \times S^5$ string sigma-model. The term «integrability» here is used in the sense of quantum integrable systems and it means that there is an infinite set of mutually commuting operators, which leads to exact answers (for all values of the coupling constants) for certain quantities. Nevertheless, integrability manifests itself in gauge theory and in the string sigma-model in different ways. It was shown in [33] that the classical e.o.m. of the string sigma-model may be written in Lax form, that is as a zero curvature condition for a one-parametric family of connections. From the point of view of classical mechanics of the system this leads to the existence of an infinite number of functionally independent integrals of motion in involution. In the $\mathcal{N} = 4$ Yang-Mills theory one usually considers composite gauge-invariant operators, i.e. local (depending on a single point x of Minkowski space) gauge-invariant functions of elementary operators A_μ, ϕ, ψ (here A_μ is the gauge field, ϕ is a scalar field and ψ is the spinor field). Despite the conformality of the theory, these operators can have nonzero anomalous dimensions, which are functions of the 't Hooft parameter λ. It turns out [92] that the calculation of these dimensions (at least in the one-loop approximation) reduces to the diagonalization of the Hamiltonian of a particular spin chain. The Hamiltonian of this spin chain is integrable — it means that there is a large number of operators, acting in the Hilbert space of the spin chain, commuting with each other and with the Hamiltonian. This property is by no means trivial and it allows, in particular, to encode the spectrum of the Hamiltonian into a solution of a particular set of algebraic equations (the Bethe equations). Nevertheless, the work [92] is dedicated to the investigation of the one-loop approximation, and the spin chain which arises in this case — Heisenberg's XXX spin chain — describes the interaction of neighboring spins. However, if one takes into account the two-loop corrections, then the spin chain is modified by the one with interactions of the three neighboring spins, then in the three-loop approximation four neighboring spins start interacting etc. Thus, it was necessary to generalize the Bethe equations valid in the one-loop approximation. The final answer to this question was given in the work [31], where the full set of Bethe equations was written down, valid to all orders of perturbation theory with the important qualification that the operators under consideration have large length. This result was coined in the literature the «all-loop asymptotic Bethe ansatz» (ABA). The word «asymptotic» here refers precisely to the fact that the operators, whose anomalous dimensions are given by these equations, have large length. As was shown for the first time in [7], in the case of finite-length operators there generally arise corrections to the ABA solutions in a specific order of perturbation theory. The fact that the ABA is not exact is related to the fact that in a certain order of perturbation theory the interaction radius of the spins in the spin chain starts exceeding the length of the spin chain. Such «long-range» interactions were called wrapping interactions. It is worth noting that so far a consistent method of taking these interactions into account does not exist at the spin chain side. Nevertheless, it is possible to take them into account from the point of view of the string sigma model — it turns out that they coincide with the worldsheet finite-volume corrections (infinitely long operators in gauge theory correspond to an infinitely long string, or, which is equivalent, to the decompactified worldsheet). In principle, there exists [83] a general (Luescher) method of calculating the leading finite-size correction in a generic relativistically invariant theory (the string sigma model in the light-cone gauge is not Lorentz-invariant, so for this case a generalization of Luescher's method was built [76]). In particular, the calculation of just this leading correction in the string sigma model enabled to find the four-loop anomalous dimension of the so-called «Konishi operator» ($\mathrm{tr}(\phi_i^2)$) — the shortest nontrivial operator in $\mathcal{N} = 4$ super-Yang-Mills theory [24]. For comparison, calculation of this quantity with the help of the usual methods would require computation of hundreds of Feynman supergraphs [46]. However, Luescher's method only gives the leading correction, and then the natural question arises, if and how it is possible to calculate the higher corrections or, in other words, how to obtain the spectrum of the theory in finite volume using the infinite volume data (the S-matrix, the spectrum, etc.). This is in fact possible in the case of an integrable theory. For the study of relativistic two-dimensional integrable theories at the end of 80-s — beginning of 90-s a new method was built, called the «thermodynamic Bethe ansatz» [110]- [111]. According to this method, the study of a relativistic theory in finite volume is equivalent to the study of the same theory in infinite volume but finite temperature. The solution of the latter problem does not pose serious difficulties, since in an integrable model the spectrum of the Hamiltonian H is known in the infinite volume (since the solution boils down to the calculation of the partition function $\mathrm{tr}(e^{-\beta H})$). In the non-relativistic theory, that is in the case under consideration, it is necessary to calculate the finite temperature partition function of a *dual* (or *mirror*) theory with Hamiltonian $\widetilde{H}$, which is uniquely determined by the Hamiltonian H. The papers [12]- [18] are dedicated to this problem, but it has not been completely solved yet.

In the above we described the results related to the study of the AdS/CFT correspondence for the case of $\mathcal{N} = 4$ super-Yang-Mills theory. Since the appearance of the original paper [84] other variants of AdS/CFT correspondences have been put forward. In particular, in the paper [82] the so-called TsT deformation of $AdS_5 \times S^5$ was considered. More precisely, in this case the metric of the sphere S^5 is subject to deformation determined by the parameters $\gamma_1, \gamma_2, \gamma_3$. This theory is conjectured to be dual to the nonsupersymmetric Yang-Mills theory with matter fields in $d = 3 + 1$ (if the deformation parameters are the same, then the theory possesses $\mathcal{N} = 1$ supersymmetry). For these theories several integrability features were found as well. For instance, the Lax pair has been built for the equations of motion of the string sigma model [49]. Chapter 2 of the current thesis is dedicated to the study of the TsT-deformed theory.

There also exist theories which (according to the Maldacena conjecture) have AdS-duals, but which differ from $AdS_5 \times S^5$ in a more substantial way, than just by a simple deformation. Target spaces of all such sigma models may be written in the form $AdS_{D+1} \times \mathcal{M}$ (plus fermions), where M is some compact space. In some cases these supersymmetric spaces admit a large group of isometries and, as a result, they may be regarded as quotient spaces of these supergroups. In the case where the corresponding symmetry algebra admits a Z_4 grading the Lax pair of a classical sigma model is built in a standard fashion [33]. All such theories (sigma models with Z_4-grading and central charge $c = 26$) were classified in [113]. One of such interesting examples is the sigma model for strings propagating in $AdS_4 \times CP^3$ [1]. The dual theory in that case is the Chern-Simons theory with $\mathcal{N} = 6$ supersymmetries in three-dimensional spacetime — this theory, as well as $\mathcal{N} = 4$ super-Yang-Mills, is conformal. Chapters 3, 4 and 5 are dedicated to the study of this example of the AdS/CFT correspondence.

As we discussed above, the $\mathcal{N} = 4$ super-Yang-Mills theory cannot aspire to the role of a genuine theory of elementary particles. However, one can hope that more physical theories can be obtained from it with the help of deformations, or perturbations. So far there has not been much progress along this way, and of course it would be very much desirable. At first sight it might seem that deformed theories, as well as other variants of the AdS/CFT correspondence in other space-time dimensions, are even less physical. From our point of view, the principal interest to study different variants of the AdS/CFT correspondence is to find out its range of validity in the framework of quantum field theory. In particular, the only justification for the AdS/CFT correspondence known at present is the one described in the original work [84], based on the study of the low-energy action for a large number of parallel four-dimensional branes in string theory (this argumentation is reviewed in Chapter 1). Since at the end of the day AdS/CFT theory reduces to a duality between two quantum field theories (the two-dimensional worldsheet theory and three or four-dimensional field theory, although gauge theories in other dimensions are also possible), the natural question arises whether it is possible to «see» the advent of this duality directly in quantum field theory, without any appellation to string theory whatsoever. In spite of the fact that at the qualitative level such an analogy seems rather plausible, moreover it was first put forward back in the 70-s [73], there still does not exist any quantitative description of this phenomenon. We hope that the study of different variants of the AdS/CFT correspondence can help advance in this direction.

The thesis consists of five chapters.

Chapter 1 is dedicated to the basics of the AdS/CFT correspondence. In this chapter we describe the origins of the idea that conformal quantum field theories are dual to theories of strings, propagating in AdS. In this Chapter we mainly follow the papers of J.Maldacena [84], [85].

In Chapter 2 we go beyond the most widespread example of AdS/CFT correspondence and we consider the so-called γ-deformed spaces $AdS_5 \times S^5_\gamma$. In Section 2.1 we give a definition of the γ deformation and we point out, how the deformed sphere may be obtained from the usual one by means of a sequence of TsT-trasformations. Section 2.2 is dedicated to finding a soliton solution for the e.o.m. of the deformed model (this solution is called the «giant magnon»). In Section 2.3 we obtain the main result of the Chapter – the dispersion relation of the giant magnon at large but finite J.

Chapters 3, 4 and 5 are dedicated to the investigation of the string sigma-model in the space $AdS_4 \times CP^3$, which is related by the AdS/CFT correspondence to the planar limit of the conformal super-Chern-Simons theory in three-dimensional spacetime.

In the beginning of Chapter 3 we give a definition of the sigma model under consideration, as well as the gauge theory dual to it. In Section 1.5.1 we consider CP^3 as a space of orthogonal complex structures in six-dimensional Euclidean space. In particular, we show that such approach provides a convenient parameterization of the coset $SO(6)/U(3)$. In Section 3.2 we build the light-cone gauge for the current string sigma-model. Besides, it is shown that in the light-cone gauge there is a residual symmetry algebra $su(2|2) \oplus u(1)$. Section 3.3 is dedicated to the study of transformation properties of the bosonic and fermionic fields of the model under the residual symmetry algebra. In Section 3.4 we introduce a convenient kappa-gauge. At last, in Section 3.5 the central extension of the symmetry algebra is calculated in the classical limit. In particular, it is shown that it coincides with an analogous central extension for the case $AdS_5 \times S^5$.

The problem, which is solved in Chapters 4 and 5, is based on the classical solution of the string sigma model e.o.m., which describes string rotation in AdS.

In Chapter 4 we expand the full Green-Schwarz action around the «spinning string» solution and we obtain the fluctuation spectrum, i.e. the dispersion relations for the fermions and bosons on the worldsheet. We notice that in the limit when the string rotations gets infinitely fast the string stretches to infinity (the so-called long string limit), several particles become massless: these are 6 bosons from CP^3 and 2 fermions. In Section 4.2 we describe the canonical construction of the Green-Schwarz action using a Z_4-graded coset. Section 4.3 is dedicated to the solution of the sigma model, which describes string rotation. In Section 4.4 we obtain a Lagrangian for the quadratic fluctuations around this solution and we obtain the dispersion relations for the bosonic and fermionic degrees of freedom. In Section 4.5 we present the main the result of that Chapter — the one-loop correction to the energy of the rotating string.

Part 5 is dedicated to the dynamics of massless modes, introduced in Chapter 4. In Section 5.3 we describe the strategy for the construction of an expansion around the two-spin solution. Special attention is devoted to the reasons why the action built with the help of the coset is not suitable for analyzing the massless limit. In Section 5.4 we turn to the consideration of the massless limit and we explain its importance using the well-studied example of $AdS_5 \times S^5$. In Section 5.5 the full Green-Schwarz action with 32 fermions is built. This construction consists of several stages, described in Sections 5.5.1 and 5.5.2. In Section 5.5.1 it is explained, how the type IIA string action may be obtained from the (three-dimensional) action of the M2 brane, and in Section 5.5.2 we provide for completeness a full description of the Hopf bundle, used in the dimensional reduction of Section 5.5.1. In Section 5.6 it is shown how the long-string limit is related to the worldsheet decompactification limit, and we also describe the technical details of the expansion around the «spinning string» solution. In particular, in this Section we describe the choice of the kappa-symmetry gauge. At last, in Section 5.7 we build a low-energy limit of the string worldsheet and discuss the subtleties connected with the kappa-symmetry and the independence of the final result on the chosen kappa-symmetry gauge.

The Appendix consists of five parts (A, B, C, D, E). In Section A one can find the explicit form of the matrices encountered in the main text, as well as the notations used throughout the thesis. Sections B-E correspond to Chapters 2-5.

The results reported in the dissertation were obtained by the author (and his collaborators) in the following papers:

1. D.V.Bykov, S.A.Frolov, *Giant magnons in TsT-transformed $AdS_5 \times S^5$*, JHEP0807:071 (2008), arXiv:0805.1070. The leading finite-size correction to the dispersion relation of the giant magnon in the γ-deformed sigma model was obtained. It explicitly depends on γ, but in the limit $\gamma \to 0$ it reduces to the one obtained previously in [19]. This result is reported in Chapter 2.

2. D.V.Bykov, *Off-shell symmetry algebra of the $AdS_4 \times \mathbb{CP}^3$ superstring*, Theor.Math.Phys. 163:1 (2010), pp. 114-131, arXiv:0904.0208. In the limit when the string is infinitely long one can relax the level-matching condition. In this case the global symmetry algebra of the superstring sigma-model in the light-cone gauge (called the off-shell algebra) acquires a central extension, which depends on the worldsheet momentum. We calculated this central extension for the $AdS_4 \times \mathbb{CP}^3$ superstring and showed that it is the same as the one in the $AdS_5 \times S^5$ case, the latter calculated in [15]. This result is reported in Chapter 3.

3. L.F.Alday, G.E.Arutyunov, D.V.Bykov, *Semiclassical Quantization of Spinning Strings in $AdS_4 \times \mathbb{CP}^3$*, JHEP0811:089 (2008), arXiv:0807.4400. The one-loop correction to the energy of the so-called spinning string (which has nonzero momenta in the AdS and $S^1 \subset \mathbb{CP}^3$ directions) was obtained. This result is reported in Chapter 4.

4. D.V.Bykov, *The worldsheet low-energy limit of the $AdS_4 \times \mathbb{CP}^3$ superstring*, Nuclear Physics, Section B 838 (2010), pp. 47-74, arXiv:1003.2199. We considered the $AdS_4 \times \mathbb{CP}^3$ IIA superstring sigma-model in the background of the spinning string classical solution, as

in the previous paper. In the limit when one of the spins is infinite there are massless excitations, which govern the infrared worldsheet properties of the model. We obtained a sigma-model of $\mathbb{CP}^3$ with fermions, which describes the dynamics of these massless modes. This result is reported in Chapter 5.

Chapter 1

Basics of the AdS/CFT correspondence

This Chapter is essentially a brief review — an introduction to the AdS/CFT correspondence. We will only consider the most well-studied example — the duality between $\mathcal{N} = 4$ Yang-Mills theory with $SU(N)$ gauge group in the large N limit and the $AdS_5 \times S^5$ string sigma model. In the beginning of the Chapter we provide a short description of the maximally supersymmetric Yang-Mills theory (with four Majorana supercharges, the so-called $\mathcal{N} = 4$ case) in four-dimensional spacetime. Later we turn to the discussion of one of the main properties of this theory, namely conformal invariance. We describe in detail the anti-de Sitter space, which plays a role of paramount importance in the AdS/CFT correspondence and whose physical meaning is related to the fact that it provides a vivid expression of the conformal invariance of Yang-Mills theory. In the second part of the Chapter we describe Maldacena's ideas, which led him to the formulation of the AdS/CFT duality conjecture.

1.1 $\mathcal{N}=4$ Yang-Mills theory

In the $\mathcal{N}=4$ supermultiplet there is one field of spin 1, four fields of spin 1/2 and 6 fields of spin 0. The R-symmetry group is $SU(4)$, which acts on the fields in the following fashion: the spin 1 fields are invariant, fields of spin 1/2 are in the $\mathbf{4}\oplus\bar{\mathbf{4}}$ representation, and the 6 scalar fields are in the vector representation (that is, the defining representation of $SO(6)$).

It follows from the Appendix 1.1.1 that the scalar fields ϕ_i of the vector representation $\mathbf{6}$ can be represented with the help of skew-symmetric matrices ϕ^{ij} satisfying an additional reality condition $(\phi^{ij})^* = \frac{1}{2}\epsilon_{ijkl}\phi^{kl}$.

1.1.1 $\mathcal{N}=1$ superspace

Recall that the $\mathcal{N}=1$ multiplet of the gauge superfield consists of one field of spin 1, that is the gauge field itself, and one spinor field. The chiral multiplet contains one complex scalar and one spinor field. In order to obtain $\mathcal{N}=4$ supersymmetric theory we should take one gauge multiplet (which is described in superspace by the spinor superfield W) and three chiral multiplets Φ_i, $i=1,2,3$. This is confirmed, first of all, by the fact that the elementary field counting gives a correct answer. The Lagrangian density has the form:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\mathrm{tr}\big(\int d^2\theta\,\bar{W}W\big) + \frac{1}{2}\mathrm{tr}\big(\int d^4\theta\,\Phi_i^\dagger\Gamma\Phi_i\Gamma^{-1}\big) + \\ &+ \left[\mathrm{tr}\int d^2\theta\,\big(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2\big) \,+\, \mathrm{c.c.}\right] \end{aligned} \tag{1.1}$$

The first term here is the action of the gauge superfield V, which is related to the «field strength» W by known, but rather special formulas, which we do not provide here — one can find them, for instance, in the book [106]. The second term contains interaction terms of the scalar superfields with the gauge superfield ($\Gamma = e^{-V}$). The last term is the superpotential $F =$

$\epsilon_{ijk}\Phi_i\Phi_j\Phi_k$ — it is precisely this form of superpotential which provides the $\mathcal{N} = 4$ invariance of the theory. The gauge transformations have the form:

$$\Phi_i \to \Omega\Phi_i\Omega^{-1} \tag{1.2}$$

$$\Gamma \to (\Omega^{-1})^\dagger\Gamma\Omega^{-1}, \tag{1.3}$$

where $\Omega = \exp(\omega)$ and ω is a chiral scalar field taking values in the Lie algebra of the gauge group.

1.1.2 Lagrangian in terms of component fields

The Lagrangian of Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry in terms of the component fields has the following form [106]:

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}(D_\mu\phi^{ij}D^\mu\phi_{ij}) + \frac{1}{2}\mathrm{tr}(\psi_R^T\epsilon \not{D}\psi_L) - \frac{1}{2}\mathrm{tr}(\psi_L^T\epsilon \not{D}\psi_R) -$$
$$-2\sqrt{2}\mathrm{Re}\,\mathrm{tr}(\phi^{ij}\psi_i^T\epsilon\psi_j) - \frac{1}{8}\mathrm{tr}([\phi^{ij},\phi^{kl}][\phi^*_{ij},\phi^*_{kl}]) - \frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

In each term tr means the trace over the «color» indices , that is the gauge group indices. Besides, $\psi_{L,R} = \frac{1\pm\gamma_5}{2}\psi$ are the left- and right-hand parts of the Majorana spinor ψ. Note, that in the formula 1.4 the following reality condition is understood[1]: $\psi^* = \beta\psi$, where $\beta = e\otimes e$. The matrix ϵ entering 1.4 is defined as $\epsilon = I_2 \otimes e$.

1.1.3 Superconformal invariance

Conformal transformations of flat space $\mathbb{R}^{1,3}$

Recall that by definition conformal transformations are the ones which preserve angles bertween vectors[2]. If $g_{\mu\nu}(x)$ is the metric, then the angle

[1] Here we use the notation $e = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

[2] In the following by conformal transformations we mean those diffeomorphisms, which preserve the angles (between tangent vectors).

between two vectors $v^\mu(x)$ and $w_\nu(x)$ is defined according to the formula

$$\text{Cosine of the angle between } v \text{ and } w = \frac{(v,w)}{\sqrt{|(v,v)(w,w)|}}, \tag{1.4}$$

where the scalar product is defined using the metric: $(v,w) = v^\mu g_{\mu\nu} w^\nu$. Thus, it is clear that the conformal transformations are those, under which the metric tensor is multiplied by a scalar function: $g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x)$. In other words, suppose that the transformation in a particular set of coordinates has the form $x^a \to x'^a(x)$. The metric transforms as follows:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}\, dx^\alpha dx^\beta = g'_{\alpha\beta}(x')\, dx^\alpha dx^\beta, \tag{1.5}$$

where $g'_{\alpha\beta}(x') = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}$. The condition that the given transformation is conformal is:

$$g'_{\alpha\beta}(x') = \Omega(x) g_{\alpha\beta}(x). \tag{1.6}$$

It is clear that conformal transformations form a group. Let us pass to the case which is most interesting for us, namely the flat space with a Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$. In this case the equation 1.6 takes the form:

$$\eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu = \Omega\, \eta_{\alpha\beta}. \tag{1.7}$$

Multiplying by $\eta^{\alpha\beta}$ and contracting the indices, we find $\Omega = \frac{1}{4} \eta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu$, i.e. 1.7 can be rewritten as

$$\eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu - \left(\frac{1}{4} \eta_{\mu\nu} \eta^{ab} \partial_a x^\mu \partial_b x^\nu \right) \eta_{\alpha\beta} = 0 \tag{1.8}$$

We will limit ourselves by considering only those conformal transformations that are connected in a continuous way to the identical transformation (in other words, those which lie in the connected component of the identity of the conformal group). Such transformations may be uniquely restored if one knows their infinitesimal form (this is analogous to the procedure of exponentiation of a Lie algebra element, which leads to a group element). Therefore let us look for such transformations in the infinitesimal form:

$$x^\mu(x') = x'^\mu + \epsilon\, \tau^\mu(x') + o(\epsilon). \tag{1.9}$$

$\tau^\mu(x)$ are components of the vector field $\tau = \tau^\mu(x)\frac{\partial}{\partial x^\mu}$, which gives a full conformal transformation after exponentiation. This means that upon finding the field τ we will need to solve the equation $\frac{dx^\mu}{d\epsilon} = \tau^\mu(x)$ and then set in the solution $\epsilon = 1$.

Substituting the expansion 1.9 into the equation 1.8 and setting to zero the coefficient of ϵ, we obtain:

$$\partial_a \tau_b + \partial_b \tau_a - \frac{1}{2}\eta_{ab}\partial_c \tau_c = 0. \tag{1.10}$$

The solution of this equation that we are interested in has the following form:

$$\tau_\mu = \epsilon_\mu + q\, x_\mu + \omega_\mu^{\ \nu} x_\nu + (a_\mu x^2 - 2(ax)x_\mu), \tag{1.11}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, and the other parameters ϵ_μ, q, a_μ are independent. It is obvious that there are 15 independent parameters, 10 of them corresponding to isometric transformations ($\Omega = 1$), forming the Poincare group: ϵ_μ are the shifts, and $\omega_{\mu\nu}$ are the Lorentz rotations. Thus, we have found 5 new transformations, which in full (not infinitesimal) form look as follows:

Dilation: $x^\mu \to \lambda x^\mu$

Special conformal transformations: $x^\mu \to \dfrac{x^\mu + a^\mu x^2}{1 + 2(ax) + x^2 a^2}$

There is a simple description of the special conformal transformation. Denote by j the inversion $j(x) = y, y^\mu = \frac{x_\mu}{x^2}$ and by s_a the shift $s_a(x) = y, y^\mu = x^\mu + a^\mu$. Then the special conformal transformation may be written as a superposition $j \circ s_a \circ j$. This means that the full conformal group may be obtained from the Poincare group by adding the transformations of dilation and inversion.

Conformal algebra

Let us find out, what the Lie algebra of infinitesimal conformal transformations is. To do this it is sufficient to calculate the commutators of vector fields, generating the transformations 1.11. These vector fields have the form

$$P_\mu = \frac{\partial}{\partial x^\mu} \tag{1.12}$$

$$D = x^\mu \frac{\partial}{\partial x^\mu} \tag{1.13}$$

$$K_\mu = \frac{1}{2} x^2 \frac{\partial}{\partial x^\mu} - x_\mu x_\nu \frac{\partial}{\partial x^\nu} \tag{1.14}$$

$$L_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \tag{1.15}$$

Their commutators are as follows:

$$[D, L_{\mu\nu}] = 0,\ [P_\mu, P_\nu] = 0,\ [K_\mu, K_\nu] = 0,\ [D, L_{\mu\nu}] = 0 \tag{1.16}$$

$$[L_{\mu\nu}, P_\chi] = \eta_{\chi\nu} P_\mu - \eta_{\chi\mu} P_\nu \tag{1.17}$$

$$[L_{\mu\nu}, K_\xi] = \eta_{\xi\nu} K_\mu - \eta_{\xi\mu} K_\nu \tag{1.18}$$

$$[P_\mu, K_\nu] = L_{\mu\nu} - \eta_{\mu\nu} D \tag{1.19}$$

$$[L_{\mu\nu}, L_{\alpha\beta}] = \eta_{\mu\alpha} L_{\beta\nu} - \eta_{\nu\alpha} L_{\beta\mu} - \eta_{\mu\beta} L_{\alpha\nu} + \eta_{\nu\beta} L_{\alpha\mu} \tag{1.20}$$

$$[D, P_\mu] = -P_\mu,\ [D, K_\mu] = K_\mu \tag{1.21}$$

Supersymmetry

The superconformal algebra is an extension of the super-Poincare algebra (i.e. the usual supersymmetry algebra), which includes conformal transformations.

A remarkable property that we will need in the following is that the conformal group $SO(2,4)$ of the Minkowski space $\mathbb{R}^{1,3}$ coincides with the isometry group of five-dimensional anti-de Sitter space AdS_5. Moreover, the superconformal group $PSU(2,2|4)$ coincides with the isometry supergroup of the superspace $AdS_5 \times S^5$, which is, among other things, a solution of the equations of motion of the (ten-dimensional) type IIA supergravity.

1.2 Anti-de Sitter space AdS

The AdS space is a direct generalization of the Lobachevsky space to the case where the metric has Lorentz (and not Euclidean) signature. In particular, it has constant negative curvature. It is handy to recall what is the standard Lobachvesky plane and its multi-dimensional generalizations.

1.2.1 The Lobachevsky plane

Let us consider the surface

$$-x^2 + y^2 + z^2 = -1 \tag{1.22}$$

It is a two-sheeted hyperboloid which is «stretched» along the x axis. Let us consider just one of its sheets, for instance $x > 0$. If the space R^3 with coordinates x, y, z is equipped with a Lorentzian metric

$$ds^2 = -dx^2 + dy^2 + dz^2, \tag{1.23}$$

then the corresponding metric space is called the Lobachevsky plane $\mathbb{H}$. From 1.22 and 1.23 it is clear that on $\mathbb{H}$ there is an isometric action of the group $SO^+(1,2)$, in other words the ortochronous Lorentz group in 2+1 dimensions (the group $SO(1,2)$ has two connected components, one of them being the connected component of the identity $SO^+(1,2)$. Note that these two connected components switch between each other under the transformation $-1 = PT$, where P and T are the space and time parities correspondingly). One can show that the group $SO^+(1,2)$ acts transitively on $\mathbb{H}$. In this case according to the general theory the stabilizers of all points are isomorphic, so we can choose for convenience a particular point $\mathcal{O} = (1,0,0) \in \mathrm{R}^3$. Arbitrary rotations and reflections in the plane (y,z) leave the point $\mathcal{O}$ fixed, however reflections are not part of $SO^+(1,2)$. Thus, the group $SO(2)$ is a stabilizer of an arbitrary point on the hyperboloid. In this fashion we get a representation of the Lobachevsky plane as a quotient space (we will often loosely call it a «coset», since the more correct term «quotient space» seems too long) $\mathbb{H} \simeq SO^+(1,2)/SO(2)$. We have focused here on such a detailed description of this coset because similar cosets for anti-de Sitter spaces of higher dimensions play a paramount role in this thesis.

The equation 1.22 may be rewritten in a different way. To do it we first introduce the «light-cone» coordinates

$$x_{\pm} = y \pm x, \tag{1.24}$$

and then we make yet another change of variables [3]

$$u = \frac{1}{x_+},\ v = \frac{z}{x_+},\ s = x_-, \tag{1.25}$$

then the equation 1.22 can be written as

$$\frac{s}{u} + \frac{v^2}{u^2} + 1 = 0, \tag{1.26}$$

and after multiplication by u^2 as

$$su + v^2 + u^2 = 0. \tag{1.27}$$

Let us note that the original condition $x > 0$ (we considered just one sheet of a two-sheeted hyperboloid) is equivalent to

$$x_+ > 0 \Rightarrow u > 0. \tag{1.28}$$

One can rewrite 1.27 in one more way, finding the diagonal form for the quadratic form in 1.27 ($\widetilde{u} = u + \frac{s}{2}, \widetilde{s} = \frac{s}{2}$):

$$v^2 + \widetilde{u}^2 = \widetilde{s}^2. \tag{1.29}$$

[3]The inverse looks as follows: $x_+ = \frac{1}{u}, z = \frac{v}{u}, x_- = s$.

Let us write the metric 1.23 $ds^2 = dx_+dx_- + dz^2$ in the coordinates 1.25, using 1.27:

$$ds^2 = \frac{du^2 + dv^2}{u^2}. \tag{1.30}$$

Thus, we have obtained the standard Poincare metric on the upper half-plane ($u > 0$). It is well-known that another model for the Lobachevsky plane is provided by Poincare's unit disc. After the introduction of the complex coordinate $z' = v + iu$ it can be obtained by a conformal transformation of the upper half-plane [4]:

$$z' = -2i\frac{z-i}{z+i} \tag{1.31}$$

An elementary calculation allows to rewrite the metric 1.30 using the coordinate z on the unit disc:

$$ds^2 \propto \frac{dz d\bar{z}}{(1 - z\bar{z})^2}\,. \tag{1.32}$$

1.2.2 AdS_2

As we saw in the previous Section, on the Lobachevsky plane there is a positive-definite (Riemannian) metric of constant negative curvature. After a direct generalization we will get the space AdS_2 of constant negative curvature, but this time also with Lorentzian signature. To do this let us consider the hyperboloid

$$x^2 + y^2 - z^2 = 1, \tag{1.33}$$

embedded into Minkowski space with the metric

$$ds^2 = dx^2 + dy^2 - dz^2. \tag{1.34}$$

We introduce the parameterization

$$x = \mathrm{ch}(\chi)\cos(\phi),\ y = \mathrm{ch}(\chi)\sin(\phi),\ z = \mathrm{sh}(\chi) \tag{1.35}$$

[4]To make it clear, $|z| \leq 1$, i.e. z is in the unit disc, whereas $\mathrm{Im}(z') > 0$, i.e. z' is in the upper half-plane. The inverse transformation has the form $z = -i\frac{z'-2i}{z'+2i}$.

The coordinates χ, ϕ are called global coordinates of the anti-de Sitter space, since they cover the whole hyperboloid. To cover with these coordinates the whole hyperboloid one should choose the following range: $\chi \in (-\infty, \infty)$, $\phi \in [0, 2\pi)$. The metric in these coordinates looks like

$$ds^2 = \mathrm{ch}^2(\chi) d\phi^2 - d\chi^2 \tag{1.36}$$

According to Penrose, we can write the metric in the following way:

$$ds^2 = \mathrm{ch}^2(\chi) \left(d\phi^2 - \left(\frac{d\chi}{\mathrm{ch}(\chi)} \right)^2 \right) \tag{1.37}$$

The expression in brackets can be transformed to flat-space form by the introduction of the coordinate $w = \arctan(e^\chi)$ (which satisfies the equation $\frac{dw}{d\chi} = \frac{1}{\mathrm{ch}(\chi)}$), changing in the limits $w \in (0, \frac{\pi}{2})$:

$$ds^2 = \sin^2(w) \cos^2(w)(d\phi^2 - dw^2) \tag{1.38}$$

To put it differently, the metric ds^2 is conformally equivalent to the flat metric $\widetilde{ds}^2 = d\phi^2 - dw^2$. As we will see in the following, it is natural to consider the variable ϕ as time (it is not clear in this two-dimensional example, but in three and more dimensions it is precisely this variable which is timelike). It follows from the requirement of causality that there should be no closed timelike geodesics, therefore one usually considers the infinite covering of the hyperboloid 1.33, which can be obtained from the covering $\mathbb{R} \to S^1$ of the circle, parameterized by the angle ϕ. In other words, in the following we will assume that the variable $\phi \equiv t$ takes values in the whole real axis and we will no longer identify ϕ mod 2π. Then the area under consideration is a strip in the (w, ϕ) plane — it is the so-called Penrose diagram for the space AdS_2. The lines $w = 0$ and $w = \frac{\pi}{2}$ form the «conformal boundary» of AdS_2, whose multi-dimensional generalization plays a role of utmost importance in the AdS/CFT correspondence. Let us mention that in terms of the original parameters the conformal boundary corresponds to $\chi \to \pm\infty$, i.e. the infinitely distant points of the hyperboloid.

1.2.3 AdS_3

We dedicate a separate subsection to the space AdS_3, because it has a remarkable property — it is the group manifold of $SL(2, \mathbb{R})$. This fact may be interpreted in different ways, for instance we may relate it to the isomorphism $SO(2,2) \simeq SO(1,2)^2$ — this is a direct analogue of the isomorphism $SO(4) \simeq SO(3)^2$ in the case of Lorentz signature. Indeed, the manifolds AdS_D for all D are quotient spaces $SO(2, D-1)/SO(1, D-1)$, so at $D = 3$ we have $AdS_3 \simeq SO(2,2)/SO(1,2)$.

AdS_3 is the universal cover of the hyperboloid defined by the equation

$$x^2 + y^2 - z_1^2 - z_2^2 = 1 \tag{1.39}$$

and embedded into four-dimensional space with Lorentz metric

$$ds^2 = dx^2 + dy^2 - dz_1^2 - dz_2^2 \tag{1.40}$$

Similarly to the way it was done in the previous Section one can introduce «global» coordinates, i.e. the ones which cover the whole hyperboloid (and in fact its universal cover as well):

$$x = \mathrm{ch}(r)\cos(t), \; y = \mathrm{ch}(r)\sin(t), \; z_1 = \mathrm{sh}(r)\cos(\varphi), \; z_2 = \mathrm{sh}(r)\sin(\varphi).$$

Then the metric takes the form

$$ds^2 = -\,\mathrm{ch}^2(r)dt^2 + dr^2 + \mathrm{sh}^2(r)d\varphi^2 \tag{1.41}$$

The ranges of the variables are: $r \in [0, \infty)$, $t \in (-\infty, \infty)$, $\varphi \in [0, 2\pi)$. The difference from the AdS_2 case is that in AdS_3 the variable r has a meaning of a genuine radial variable, i.e. it takes values only in $\mathbb{R}^+$.

One can make more manifest the causal structure of the space AdS_3, similarly to the way it was done for AdS_2 in the previous Section. In order to do it notice that the metric 1.41 can be written in the form $ds^2 = \mathrm{ch}^2(r)\widetilde{ds}^2$, where

$$\widetilde{ds}^2 = -dt^2 + \left(\frac{dr}{\mathrm{ch}(r)}\right)^2 + \tanh^2(r)d\varphi^2 \tag{1.42}$$

Let us make the change of variables $w = 2\arctan(e^r) - \frac{\pi}{2}$. Then the metric takes the form

$$\widetilde{ds}^2 = -dt^2 + dw^2 + \sin^2(w)d\varphi^2 \tag{1.43}$$

The last two terms coincide with the metric on the sphere S^2, but the variable w has a range $[0, \frac{\pi}{2})$ (rather than $[0, \pi]$, as a polar angle of the sphere does). As a result the coordinates (w, φ) cover only one hemisphere or in other words the disk D^2. Thus, 1.43 is the metric on a full cylinder $\mathbb{R} \times D^2$, which represents the Penrose diagram of the space AdS_3. We note that the boundary of the cylinder $\mathbb{R} \times S^1$ is the conformal boundary of the space AdS_3. In the original coordinates it corresponds to the limit $r \to \infty$, i.e. to the infinitely distant points of the hyperboloid.

1.2.4 AdS_5

The AdS_D space can be viewed as a hyperboloid, embedded into the space $\mathbb{R}^{2,D-1}$. For example, in the AdS_5 case we need to consider the surface

$$-X_0^2 - X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = -R^2 \tag{1.44}$$

embedded into the space $\mathbb{R}^{2,4}$, equipped with the metric $ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2$. The parameter R, called the radius of AdS, characterizes its curvature. Setting $X_0 = \cosh(\rho)\cos(T)$, $X_1 = \cosh(\rho)\sin(T)$ and introducing spherical coordinates for X_2, X_3, X_4, X_5 (we will take $\sinh(\rho)$ as the radius of the sphere), we obtain the global parameterization of AdS_5. The metric then looks as follows:

$$(ds^2)_{AdS_5} \equiv G_{\mu\nu}dY^\mu dY^\nu = R^2\left(-\cosh^2(\rho)dT^2 + d\rho^2 + \sinh^2(\rho)d\Omega_3\right). \tag{1.45}$$

1.3 Supergravity description of parallel branes in string theory

In this Section we review the main facts which led J.Maldacena in 1997 to the AdS/CFT correspondence. We will mainly follow the review [2]. It is worthwhile to point out that this argumentation is qualitative, or physical — up to the present day a more rigorous justification of the results described below does not exist (except for direct comparisons of various quantities in the string sigma-model and in field theory, which do not explain however the *reasons*, why these quantities turn out to be equal).

The idea of the AdS/CFT correspondence comes from the consideration of a system of N parallel D3-branes in type IIB superstring theory. A «D3-brane» is a 3+1-dimensional plane in a ten-dimensional Minkowski space. All the N branes are situated close to each other and even coincide in the limit. Such a system has two descriptions from the point of view of string theory. Let us discuss them in more detail.

The first description arises in the usual perturbative quantization of the two-dimensional worldsheet — this means that the string coupling constant g_s is close to zero. If it is exactly zero, we can obtain in this way only the spectrum of particles. In the absence of branes the massless particles are those which enter the gravity multiplet (the graviton, gravitino, dilaton and the NS-NS 2-form). In the presence of branes there are additional massless modes, which correspond to open strings ending on branes — in this way one obtains the $\mathcal{N}=4$ supermultiplet, i.e. the necessary number (N^2) of gauge fields, as well as the spinors and scalars related to them by supersymmetry. We should emphasize that these fields are massless only in the limit where the branes coincide. Separation of branes is equivalent to the scalar fields acquiring nonzero v.e.v.'s, which leads to the Higgs effect.

The second description comes from considering the D-brane as a classical supergravity solution of a «black hole» type. Such solutions were found for the first time in the work of G.Horowitz and A.Strominger [74]. The solution, which corresponds to a D3-brane, has the form:

$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \qquad (1.46)$$

$$F_5 = (1 + *)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d(f^{-1}) \qquad (1.47)$$

$$\text{where } f = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_s \alpha'^2 N. \qquad (1.48)$$

Here α' is the inverse string tension and it has dimensionality $1/(\text{mass})^2$. Besides, $d\Omega_5^2$ is a line element on the sphere S^5, and F_5 is a self-dual RR 5-form, which enters the IIB supergravity e.o.m. (self-duality means that $*F_5 = F_5$, and this property is clear from the construction, since $F_5 \propto (1+*)$, and the star has the property $*^2 = 1$). It is useful to notice that R is a transversal size of the brane in the sense that at distances $r > R$ the presence of the brane is almost imperceptible.

Let us now pass to the low-energy limit, when we can neglect the massive excitations of the string. All masses in string theory are proportional to $1/\sqrt{\alpha'}$, therefore the massive states decouple in the limit $\alpha' \to 0$. The idea is to pass to the low-energy limit in both descriptions of the D3-branes.

We start from the former description. Interactions of massless excitations of IIB string theory in the presence of D3 branes are described using a low-energy action. This action consists of the ten-dimensional IIB supergravity action, the four-dimensional action of the $\mathcal{N}=4$ super-Yang-Mills theory, as well as the interaction terms of the supergravity fields with the super-Yang-Mills fields.

An important observation is that in the low-energy limit $\alpha' \to 0$ the gravitational field becomes free and, thus, it completely decouples from the $\mathcal{N} = 4$ super-Yang-Mills theory. To see this notice that the Einstein-Hilbert action has the following form:

$$S_{\text{grav}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{\det g}\, \mathcal{R}, \tag{1.49}$$

where $\mathcal{R}$ is the Ricci scalar, and the coupling constant κ is from the point of view of string theory equal to $\kappa = g_s \alpha'^2$. Perturbation theory is constructed around the flat solution:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{1.50}$$

and as a result the action has the form

$$S_{\text{grav}} \sim \int d^{10}x\, (\partial h)^2 + \kappa(\partial h)^2 h + \ldots, \tag{1.51}$$

in other words all interaction terms of the gravitational field are proportional to powers of κ. Precisely from this it follows that the interactions turn to zero when $\alpha' \to 0$. Thus, when we take the low-energy limit in this description we are left with free gravity and an interacting $\mathcal{N} = 4$ super-Yang-Mills theory (since, in contrast to gravity, the coupling constant is dimensionless and proportional to g_s rather than to $g_s \alpha'^2$).

Let us now consider the same low-energy limit, but from the point of view of the supergravity solution (1.46). It means the following: we need to consider fields propagating in a space with the metric (1.46) and to find out, which of them have small energy. For gravitons at finite r it suffices to simply let $\alpha' \to 0$ in this solution — in this way we get free gravitons in flat space. However, in the vicinity of the brane (that is when $r < \sqrt{\alpha'}$) there is a surface layer, which is in fact most interesting for us. The fact is that all the particles in the proximity of the brane, including the massive ones, have effectively low energy because of the redshift of their energies: $E = f^{-1/4}E_r \underset{r\to 0}{\to} 0$. To take into account this phenomenon in the metric (1.46) we need to consider the limit $r \to 0$, or, more precisely, $r < R$. This boils down to neglecting the unity in the function f:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5 \tag{1.52}$$

— this is nothing but the metric of $AdS_5 \times S^5$. Thus, the second analysis of the same situation has led us again to two systems, which do not interact with each other: the gravitons in ten-dimensional space and a full spectrum of string excitations of the space $AdS_5 \times S^5$. The results which we obtained along two different paths of deduction, have one common feature — it is the free ten-dimensional gravity. Since on the other hand these results have to be completely identical, we are led to the conjecture that the string theory in $AdS_5 \times S^5$ is in some sense equivalent to $\mathcal{N} = 4$ super-Yang-Mills theory. This is the Maldacena conjecture, or the AdS/CFT correspondence.

1.4 The γ-deformation of the $AdS_5 \times S^5$ theory

We start with the following general sigma model action describing propagation of a fermionic closed string in a background with several U(1) isometries

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi} \quad & \left[\gamma^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j\, G^0_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j\, B^0_{ij}\right. \\ & \left. +2\partial_\alpha\phi^i\left(\gamma^{\alpha\beta}U^0_{\beta,i} - \epsilon^{\alpha\beta}V^0_{\beta,i}\right) + \mathcal{L}^0_{\text{rest}}\right] . \end{aligned} \tag{1.53}$$

Here $\frac{\sqrt{\lambda}}{2\pi}$ is the effective string tension, $\epsilon^{01} \equiv \epsilon^{\tau\sigma} = 1$ and $\gamma^{\alpha\beta} \equiv \sqrt{-h}\, h^{\alpha\beta}$, where $h^{\alpha\beta}$ is a world-sheet metric with Minkowski signature. In the conformal gauge $\gamma^{\alpha\beta} = \text{diag}(-1,1)$ although in the following we will not attempt to fix any gauge. We assume that the action is invariant under U(1) isometry transformations geometrically realized as shifts of the angle variables ϕ_i, $i = 1, 2, \ldots, d$. That means that the string background contains a d-dimensional torus T^d. We show explicitly the dependence of the action on ϕ_i, and their coupling to the background fields G^0_{ij}, B^0_{ij} and $U^0_{\beta,i}, V^0_{\beta,i}$ which generalizes the usual coupling of bosons to the target space metric and B-field. These background fields are independent of ϕ^i but can depend on other bosonic and fermionic string coordinates which are neutral under the U(1) isometry transformations. By $\mathcal{L}^0_{\rm rest}$ we denote the part of the Lagrangian which depends on these other fields of the theory. The Green-Schwarz action for superstrings on $AdS_5 \times S^5$ can be cast to the form (1.53).

The action has d global symmetries corresponding to constant shifts of $\phi' s$. The corresponding Noether currents are

$$J^\alpha_i(\phi) = -\sqrt{\lambda}\Big(\gamma^{\alpha\beta}\partial_\beta\phi^j\, G^0_{ij} - \epsilon^{\alpha\beta}\partial_\beta\phi^j\, B^0_{ij} + \gamma^{\alpha\beta}U^0_{\beta,i} - \epsilon^{\alpha\beta}V^0_{\beta,i}\Big)\,, \quad (1.54)$$

and they are conserved, $\partial_\alpha J^\alpha_i = 0$, as a consequence of the equations of motion.

In the following two sections our exposition follows the paper [5].

1.4.1 T-duality.

To explain what T-duality is we perform this transformation on a circle parametrized by ϕ_1. To find the T-duality rules it is useful to represent the action (1.53) in the following equivalent form

$$S = -\sqrt{\lambda}\int d\tau\frac{d\sigma}{2\pi}\Bigg[p^\alpha\left(\partial_\alpha\phi_1 + \frac{\widehat{U}^0_{\alpha,1}}{G^0_{11}} - \gamma_{\alpha\beta}\epsilon^{\beta\rho}\frac{\widehat{V}^0_{\rho,1}}{G^0_{11}}\right) - \frac{1}{2\,G^0_{11}}\gamma_{\alpha\beta}\,p^\alpha p^\beta \quad (1.55)$$
$$-\frac{1}{2}\gamma^{\alpha\beta}\frac{\widehat{U}^0_{\alpha,1}\widehat{U}^0_{\beta,1} - \widehat{V}^0_{\alpha,1}\widehat{V}^0_{\beta,1}}{G^0_{11}} + \frac{1}{2}\epsilon^{\alpha\beta}\frac{\widehat{U}^0_{\alpha,1}\widehat{V}^0_{\beta,1} - \widehat{U}^0_{\beta,1}\widehat{V}^0_{\alpha,1}}{G^0_{11}} + \mathcal{L}'_{\rm rest}\Bigg]\,,$$

where

$$\widehat{U}^0_{\alpha,1} \equiv U^0_{\alpha,1} + \partial_\alpha \phi^j \, G^0_{1j} \,, \qquad \widehat{V}^0_{\alpha,1} \equiv V^0_{\alpha,1} + \partial_\alpha \phi^j \, B^0_{1j} \,, \tag{1.56}$$

and $\mathcal{L}'_{\rm rest}$ denotes the part of the Lagrangian which does not depends on ϕ_1. Indeed, varying with respect to p^α, one gets the following equation of motion for p^α

$$p^\alpha = \gamma^{\alpha\beta}\partial_\beta\phi_1 G^0_{11} + \gamma^{\alpha\beta}\widehat{U}^0_{\beta,1} - \epsilon^{\alpha\beta}\widehat{V}^0_{\beta,1} \,. \tag{1.57}$$

Substituting (1.57) into (1.55) and using the identity $\epsilon^{\alpha\gamma}\gamma_{\gamma\rho}\epsilon^{\rho\beta} = \gamma^{\alpha\beta}$, we reproduce the action (1.53). Let us also mention that up to an unessential multiplier p^α coincides with the U(1) current corresponding to the shifts of ϕ_1:

$$p^\alpha \sim J^\alpha_1 \,.$$

On the other hand, varying (1.55) with respect to ϕ_1 gives

$$\partial_\alpha \, p^\alpha = 0 \,. \tag{1.58}$$

The general solution to this equation can be written in the form

$$p^\alpha = \epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}_1 \,, \tag{1.59}$$

where $\tilde{\phi}_1$ is the scalar T-dual to ϕ_1. Substituting (1.59) into the action (1.55), we obtain the following T-dual action

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi} \quad & \Big[\gamma^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j \, \widetilde{G}_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j \, \widetilde{B}_{ij} \\ & +2\partial_\alpha\tilde{\phi}^i\left(\gamma^{\alpha\beta}\widetilde{U}_{\beta,i} - \epsilon^{\alpha\beta}\widetilde{V}_{\beta,i}\right) + \widetilde{\mathcal{L}}_{\rm rest}\Big] \,. \end{aligned} \tag{1.60}$$

with the new fields $\widetilde{G}_{ij}$, etc. given in terms of the original ones.

$$\widetilde{G}_{11} = \frac{1}{G^0_{11}}, \quad \widetilde{G}_{ij} = G^0_{ij} - \frac{G^0_{1i}G^0_{1j} - B^0_{1i}B^0_{1j}}{G^0_{11}}, \quad \widetilde{G}_{1i} = \frac{B^0_{1i}}{G^0_{11}}, \tag{1.61}$$

$$\widetilde{B}_{ij} = B^0_{ij} - \frac{G^0_{1i}B^0_{1j} - B^0_{1i}G^0_{1j}}{G^0_{11}}, \quad \widetilde{B}_{1i} = \frac{G^0_{1i}}{G^0_{11}},$$

$$\widetilde{U}_{\alpha,1} = \frac{V^0_{\alpha,1}}{G^0_{11}}, \quad \widetilde{V}_{\alpha,1} = \frac{U^0_{\alpha,1}}{G^0_{11}},$$

$$\widetilde{U}_{\alpha,i} = U^0_{\alpha,i} - \frac{G^0_{1i}U^0_{\alpha,1} - B^0_{1i}V^0_{\alpha,1}}{G^0_{11}},$$

$$\tilde{V}_{\alpha,i} = V^0_{\alpha,i} - \frac{G^0_{1i}V^0_{\alpha,1} - B^0_{1i}U^0_{\alpha,1}}{G^0_{11}},$$

$$\widetilde{\mathcal{L}}_{\text{rest}} = \mathcal{L}^0_{\text{rest}} - \gamma^{\alpha\beta}\frac{U^0_{\alpha,1}U^0_{\beta,1} - V^0_{\alpha,1}V^0_{\beta,1}}{G^0_{11}} + \epsilon^{\alpha\beta}\frac{U^0_{\alpha,1}V^0_{\beta,1} - V^0_{\alpha,1}U^0_{\beta,1}}{G^0_{11}},$$

$$\epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}^1 = \gamma^{\alpha\beta}\partial_\beta\phi^i G^0_{1i} - \epsilon^{\alpha\beta}\partial_\beta\phi^i B^0_{1i} + \gamma^{\alpha\beta}U^0_{\beta,1} - \epsilon^{\alpha\beta}V^0_{\beta,1}\,, \tag{1.62}$$

$$\tilde{\phi}^i = \phi^i\,, \quad i \geq 2\,.$$

In principle these formulas can be used to find the T-duality transformed NS-NS and RR fields of the background in which the strings propagate.

It is important to note, however, that for the spaces under consideration the T-duality we have been talking about is not a genuine symmetry of the full string theory (as it is in the familiar case of $\mathbb{R}^9 \times S^1$), but it is rather a convenient method for generating supergravity solutions. This is a consequence of the fact that the tori on which we perform the T-transformations are in fact contractible inside the space they are embedded into and therefore the string winding modes are not well defined in this case. A useful example to have in mind is the plane in polar coordinates $ds^2 = d\rho^2 + \rho^2 d\phi^2$, where the circle parametrized by ϕ is not a genuine circle, i.e. it is contractible.

1.4.2 The TsT transformation

Now we perform a TsT transformation of the angle variables. To this end we pick up a two-torus, for instance, the one, generated by ϕ_1 and ϕ_2. The TsT transformation consists in dualizing the variable ϕ_1 with the further shift $\phi_2 \to \phi_2 + \hat{\gamma}\phi_1$ and dualizing ϕ_1 back. Application of the TsT transformation can be symbolically expressed as the change of variables

$$(\phi_1, \phi_2) \overset{\text{TsT}}{\to} (\tilde{\phi}_1, \tilde{\phi}_2)\,. \tag{1.63}$$

The procedure to construct the TsT-transformed action has been explained above. The corresponding action can be written in the same fashion as the original one

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi} \quad & \Big[\gamma^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\, G_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\, B_{ij} \\ & +2\partial_\alpha\tilde{\phi}^i\left(\gamma^{\alpha\beta}U_{\beta,i} - \epsilon^{\alpha\beta}V_{\beta,i}\right) + \mathcal{L}_{\text{rest}}\Big] \end{aligned} \tag{1.64}$$

with the new fields G_{ij}, etc given in terms of the original ones. Clearly, the new action also has the same number of symmetries related to the constant shifts of the variables $\tilde{\phi}^i$. The conserved Noether currents have the form

$$\tilde{J}_i^\alpha(\tilde{\phi}) = -\sqrt{\lambda}\Big(\gamma^{\alpha\beta}\partial_\beta\tilde{\phi}^j\, G_{ij} - \epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}^j\, B_{ij} + \gamma^{\alpha\beta}U_{\beta,i} - \epsilon^{\alpha\beta}V_{\beta,i}\Big)\,. \tag{1.65}$$

The relation between the dual variables $\tilde{\phi}$ and the original ones ϕ is given by

$$\begin{aligned} \partial_\alpha\tilde{\phi}^1 &= \partial_\alpha\phi^1 - \hat{\gamma}\epsilon_{\alpha\beta}\gamma^{\beta\tilde{\beta}}\partial_{\tilde{\beta}}\phi^i G_{i2} + \hat{\gamma}\partial_\alpha\phi^i B_{i2} - \hat{\gamma}\epsilon_{\alpha\beta}\gamma^{\beta\tilde{\beta}}U_{\tilde{\beta}2} - \hat{\gamma}V_{\alpha 2} \\ \partial_\alpha\tilde{\phi}^2 &= \partial_\alpha\phi^2 + \hat{\gamma}\epsilon_{\alpha\beta}\gamma^{\beta\tilde{\beta}}\partial_{\tilde{\beta}}\phi^i G_{i1} - \hat{\gamma}\partial_\alpha\phi^i B_{i1} + \hat{\gamma}\epsilon_{\alpha\beta}\gamma^{\beta\tilde{\beta}}U_{\tilde{\beta}1} + \hat{\gamma}V_{\alpha 1} \\ \partial_\alpha\tilde{\phi}^i &= \partial_\alpha\phi^i\,, \quad i \geq 3 \end{aligned} \tag{1.66}$$

Using these transformation rules, one can check that the following relation holds

$$\tilde{J}_i^\alpha(\tilde{\phi}) = J_i^\alpha(\phi)\,. \tag{1.67}$$

It shows that independently of the form of the action (1.53) and the presence of fermions the TsT transformation preserves the U(1) isometry currents corresponding to the angles ϕ_i.

The relation (1.67) allows one to find a relation between the σ-derivatives of the original and the transformed angles

$$
\begin{aligned}
\tilde{\phi}'_1 - \phi'_1 &= -\gamma J^\tau_2 \,, \qquad \hat{\gamma} = \sqrt{\lambda}\gamma \\
\tilde{\phi}'_2 - \phi'_2 &= \gamma J^\tau_1 \,, \\
\tilde{\phi}'_i - \phi'_i &= 0 \,, \quad i \geq 3 \,.
\end{aligned}
\tag{1.68}
$$

Here J^τ means the τ-component of the conserved current. This is the *same* relation as was found in the bosonic case [49].

Since we are considering closed strings on the γ-deformed background the angles $\tilde{\phi}_i$ have the following periodicity conditions

$$
\tilde{\phi}_i(2\pi) - \tilde{\phi}_i(0) = 2\pi n_i \,, \qquad n_i \in \mathbb{Z} \,. \tag{1.69}
$$

Then integrating eqs.(1.68) we obtain the twisted boundary conditions for the original angles ϕ_1 and ϕ_2, and the usual periodicity conditions (1.69) for the other $d-2$ angles

$$
\begin{aligned}
\phi_1(2\pi) - \phi_1(0) &= 2\pi(n_1 + \gamma J_2) \,, \\
\phi_2(2\pi) - \phi_2(0) &= 2\pi(n_2 - \gamma J_1) \,,
\end{aligned}
\tag{1.70}
$$

where

$$
J_i = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} J^\tau_i
$$

is the corresponding Noether charge. We see that the twisted boundary conditions are universal and do not depend on the details of the background and the presence of fermions. They depend only on the angles involved in the TsT transformation, and the total U(1) charges.

To understand better the meaning of the relations (1.67) and (1.68) we notice that the time components of the U(1) currents coincide with the momenta canonically conjugated to the angles ϕ_i: $J_i^\tau = p_i = \delta S/\delta\dot{\phi}_i$. Therefore, (1.67) and (1.68) can be written in the form

$$\tilde{p}_i = p_i \,, \qquad \tilde{\phi}_i' = \phi_i' - \gamma_{ij} p_j \,, \qquad i,j = 1,2,\ldots,d \,, \tag{1.71}$$

where we take summation over j, and γ_{ij} is skew-symmetric, $\gamma_{ij} = -\gamma_{ji}$, with just one nonvanishing component equal to the deformation parameter: $\gamma_{12} = \gamma$.

It is obvious from the relations (1.71) that up to the twisted boundary conditions a TsT transformation is just a simple linear canonical transformation of the U(1) isometry variables. It is the twist that makes the original and TsT-transformed theories inequivalent. It is also clear that the most general multi-parameter TsT-transformed background obtained by applying TsT transformations successively, many times, each time picking up a new torus and a new deformation parameter, is completely characterized by the relations (1.71) with an arbitrary skew-symmetric matrix γ_{ij}. Therefore, a background containing a d-dimensional torus admits a $d(d-1)/2$-parameter TsT deformation. In particular, the most general TsT-transformed $AdS_5 \times S^5$ background with TsT transformations applied only to the five-sphere S^5 (to preserve the isometry group of AdS_5) has three independent parameters, and, therefore, is the one found in [49]. The twisted boundary conditions for the original angles ϕ_i in the case of the most general deformation take the form

$$\phi_i(2\pi) - \phi_i(0) \;=\; 2\pi\,(n_i - \nu_i) \,, \qquad \nu_i = -\gamma_{ik}\, J_k \,. \tag{1.72}$$

Notice, that the twists ν_i always satisfy the restriction $\nu_i\, J_i = 0$.

The general three-parameter γ-deformed background is obtained by applying the TsT transformation three times. We express the corresponding procedure as

$$(\phi_1, \phi_2, \phi_3) \stackrel{\gamma_3}{\to} (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) \stackrel{\gamma_1}{\to} (\tilde{\tilde{\phi}}_1, \tilde{\tilde{\phi}}_2, \tilde{\tilde{\phi}}_3) \stackrel{\gamma_2}{\to} (\check{\phi}_1, \check{\phi}_2, \check{\phi}_3)\,. \tag{1.73}$$

Since under every step the corresponding Noether currents remain the same we can summarize relation between the angles in the following table

$$\begin{array}{lll} \tilde{\phi}'_1 - \phi'_1 = -\gamma_3 J_2^T & \tilde{\tilde{\phi}}'_1 - \tilde{\phi}'_1 = 0 & \check{\phi}_1 - \tilde{\tilde{\phi}}'_1 = \gamma_2 J_3^T \\ \tilde{\phi}'_2 - \phi'_2 = \gamma_3 J_1^T & \tilde{\tilde{\phi}}'_2 - \tilde{\phi}'_2 = -\gamma_1 J_3^T & \check{\phi}'_2 - \tilde{\tilde{\phi}}'_2 = 0 \\ \tilde{\phi}'_3 - \phi'_3 = 0 & \tilde{\tilde{\phi}}'_3 - \tilde{\phi}'_3 = \gamma_1 J_2^T & \check{\phi}'_3 - \tilde{\tilde{\phi}}'_3 = -\gamma_2 J_1^T \end{array} \tag{1.74}$$

From here we straightforwardly find the relation between the derivatives of the angles ϕ_i and the derivatives of $\check{\phi}_i$, the latter being attributed to string on the γ-deformed background:

$$\check{\phi}'_i - \phi'_i = \epsilon_{ijk}\gamma_j J_k^T\,. \tag{1.75}$$

We see from the formula that $\gamma_{ik} = -\epsilon_{ijk}\gamma_j$. Integrating eq.(1.75) and taking into account that $\check{\phi}_i(2\pi) - \check{\phi}_i(0) = 2\pi n_i\,, n_i \in \mathbb{Z}$, we obtain the twisted boundary conditions for the original angles

$$\phi_i(2\pi) - \phi_i(0) = 2\pi(n_i - \nu_i)\,, \qquad \nu_i = \epsilon_{ijk}\gamma_j J_k\,. \tag{1.76}$$

1.4.3 The dual gauge theory

In the beginning of this Chapter we described the maximally supersymmetric gauge theory in four-dimensional space-time, which is dual, by the AdS/CFT correspondence, to the string theory on $AdS_5 \times S^5$. One can ask the reasonable question, which dual gauge theory will arise when we consider the γ-deformed background $AdS_5 \times S^5_\gamma$. In the previous Sections we discussed the most general three-parametric γ-deformation, and in this generic case the dual theory cannot be presented in compact form. Nevertheless, when all three γ-parameters coincide, the theory possesses $\mathcal{N} = 1$ supersymmetry (breaking the $\mathcal{N} = 4$ supersymmetry of the maximally supersymmetric theory). In this case the Lagrangian may be written in terms of the superfields, in particular, the only part of it which does not follow on general grounds and which we are free to specify is the superpotential. It looks as follows:

$$W = h(\gamma, g, N)\,\mathrm{tr}(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2) \qquad (1.77)$$

In this way the dual gauge theory is completely defined. We should emphasize, however, that for a generic function $h(\gamma, g, N)$ this theory is not conformal, and in fact it is conformal only for a very particular choice of $h(\gamma, g, N)$ [81]. This function is unknown a priori, but it can be defined, for instance, term by term in perturbation theory, if one requires the absence of the conformal anomaly (i.e. vanishing of the beta-function).

1.5 The $AdS_4 \times \mathbb{CP}^3$ theory

Relatively recently a new example of the AdS/CFT correspondence [84], [68], [108] was put forward — the so-called ABJM model [1]. On the string theory side one deals with an $\mathrm{AdS}_4 \times S^7/Z_k$ near-horizon limit of a solution in 11-dimensional supergravity describing a stack of coincident M2-branes at a Z_k-orbifold singularity. The Z_k acts on the S^7 in a peculiar way: namely, if one considers the Hopf fiber bundle $\pi : S^7 \to \mathbb{CP}^3$ with fiber S^1, the Z_k reduces the circumference of the circle by k times, so in the limit $k \to \infty$ one gets rid of the circle completely, and we are left with the projective space $\mathbb{CP}^3$. The gauge theory dual to this $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background is the $N = 6$ supersymmetric Chern-Simons theory in three space-time dimensions (supersymmetry implies that the theory contains matter fields and is not topological for this reason).

Quite similar to the $\mathrm{AdS}_5 \times S^5$ model, various signs of integrability have been discovered in this case, too. Namely, on the gauge theory side, integrability of the two-loop Hamiltonian (the one-loop Hamiltonian vanishes due to a discrete symmetry) was found in [92] by direct check. Soon after this the algebraic curve for corresponding classical solutions and the all-loop asymptotic Bethe-ansatz were proposed [64], [65]. Under the assumption of $su(2|2) \oplus u(1)$ symmetry algebra, the exact factorizable S-matrix was found in [3]. In the same paper the authors diagonalized the S-matrix and derived the Bethe ansatz equations, which agreed with those of [64].

On the string theory side, from the fact that the target space under consideration is maximally (super)symmetric, it follows that the string sigma model can be formulated as a coset model [11]. Using the coset formulation, one finds a Lax representation [11], from which classical integrability follows.

1.5.1 Quotient space

The $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background (which we denote by $\mathcal{M}$ in what follows) is a ten-dimensional manifold, which admits the action of a topological group $\mathrm{G} = \mathrm{OSP}(6|2,2)$. The latter is a supergroup, which has $\mathrm{O}(6) \times \mathrm{USP}(2,2)$ as its maximal bosonic subgroup. The supergroup acts transitively on the manifold, the stabilizer of an arbitrary point x_0 in $\mathcal{M}$ being $\mathrm{H} = \mathrm{U}(3) \times \mathrm{O}(3,1)$. Thus, $\mathcal{M}$ is homeomorphic to G/H, the latter equipped with quotient topology.

Action of group G on the manifold $\mathcal{M}$ means that for a point x_0 in $\mathcal{M}$ and g in G corresponds another point $x_1 \equiv g(x_0)$, and this correspondence is compatible with the group structure. Below we find this transformation law in suitable coordinates on $\mathcal{M}$.

$\mathbb{CP}^3$ may be viewed as the space of orthogonal complex structures in R^6.

Indeed, $U(3) \subset O(6)$ is the subgroup preserving a given complex structure, which we denote K_6 and, following [11], choose in the form $K_6 = I_3 \otimes i\sigma_2$ (I_3 is the 3×3 identity matrix). Then the Lie subalgebra $u(3) \subset o(6)$ is described by 6×6 matrices, commuting with K_6. In other words, as vector spaces, $o(6) = u(3) \oplus V_\perp$.

The quotient vector space W, which describes the tangent space $T_x\mathcal{M}$ (tangent spaces are isomorphic for all x, since $\mathcal{M}$ is a manifold), is described by skew-symmetric matrices (elements of $o(6)$, that is) which anticommute with the complex structure. Indeed, we notice that for any $\omega \in O(6)$ the adjoint action $\omega K_6 \omega^{-1}$ is again a complex structure. For ω sufficiently close to unity $\omega = 1 + \epsilon + ...$, thus, $(K_6 + [\epsilon, K_6])^2 + O(\epsilon^2) = -I_6$. Linear order in ϵ gives $\{K_6, [\epsilon, K_6]\} = 0$. Define a map $f : o(6) \to o(6)$ by $f(a) = [a, K_6]$. Since $\mathrm{Ker}(f) = \mathrm{u}(3)$, W is isomorphic to $\mathrm{Im}(f)$. One can also check that if $g(b) \equiv \{K_6, b\} = 0$, then $b \in \mathrm{Im}(f) = W$. [5] Let us note in passing that all of the above can be summarized by the following exact sequence of vector space homomorphisms (i being inclusion):

$$0 \to u(3) \xrightarrow{i} o(6) \xrightarrow{f} o(6) \xrightarrow{g} \mathbb{R}^N, \tag{1.78}$$

$\mathbb{R}^N$ being the vector space of symmetric matrices.

It is easy to construct a basis in this linear space explicitly. Denoting by J_1, J_2, J_3 the three generators of $O(3)$ in the vector 3 representation (see Appendix for an explicit form), we get:

$$V_\perp = \mathrm{Span}\{J_i \otimes \sigma_1; J_i \otimes \sigma_3\} \tag{1.79}$$

To make contact with the notations of [11] we will write out the T_i generators

[5] In fact, this choice of representatives in the quotient space becomes canonical once we adopt the Killing scalar product (since f is skew-symmetric with respect to this scalar product $\mathrm{tr}(a, f(c)) = -\mathrm{tr}(f(a), c)$). Indeed, for $a \in u(3)$ and $b \in \mathrm{Im}(f)$ we have $\mathrm{tr}(ab) = \mathrm{tr}(a[c, K_6]) = \mathrm{tr}(acK_6 - aK_6c) = 0$, since $[a, K_6] = 0$). This justifies the use of the symbol $V_\perp$ for W.

used in their paper in terms of the basis introduced above:

$$T_{1,3,5} = J_{1,2,3} \otimes \sigma_3, \; T_{2,4,6} = J_{1,2,3} \otimes \sigma_1. \tag{1.80}$$

The main property which these generators exhibit and which will be important for us is the following:

$$\{T_1, T_2\} = \{T_3, T_4\} = \{T_5, T_6\} = 0. \tag{1.81}$$

For the following it is convenient to introduce the complex combinations

$$\mathcal{T}_1 = \frac{1}{2}(T_1 - iT_2), \;\; \mathcal{T}_2 = \frac{1}{2}(T_3 - iT_4). \tag{1.82}$$

$\bar{\mathcal{T}}_1$ and $\bar{\mathcal{T}}_2$ will denote the conjugate combinations.

1.5.2 The sigma model action

The coset construction of the Green-Schwarz superstring action is rather simple [11]. The key to this simplicity lies in the fact that the $OSP(6|4)$ superalgebra possesses a Z_4 group of automorphisms (this is a cyclic group, generated by an element which we call Ω). Then, suppose we take a representative $g(x, \theta)$ of the coset (θ here represent the fermions, and x the worldsheet coordinates), and build the left-invariant current

$$J = -g^{-1}\, dg(x, \theta). \tag{1.83}$$

Then one can determine the 4 components of this current, which lie in the eigenspaces of the Ω transformation. Let us denote them by $J^{(k)}$, $k = 0, 1, 2, 3$, and their characteristic property is $\Omega J^{(k)} \Omega^{-1} = i^k\, J^{(k)}$. Then the action invariant under the Ω automorphism (or, equivalently, under the Z_4 group of automorphisms) is built uniquely in the following way:

$$S = \frac{g}{2} \int d\sigma\, d\tau\, \left(\gamma^{\alpha\beta}\, \mathrm{Str}(J^{(2)}_\alpha\, J^{(2)}_\beta) + \varkappa\, \epsilon^{\alpha\beta}\, \mathrm{Str}(J^{(1)}_\alpha\, J^{(3)}_\beta)\right). \tag{1.84}$$

When $\varkappa = \pm 1$ this action possesses an important gauge symmetry, called kappa-symmetry, which is a fermionic gauge symmetry in the sense that the gauge parameters are anticommuting.

We emphasize that throughout the present thesis, depending on the particular needs and aims, we use different parameterizations for the coset element g, and in each case the corresponding choice is elaborated separately.

Chapter 2

AdS/CFT correspondence for γ-deformed theories

2.1 The γ-deformed theories

An interesting example of the AdS/CFT duality [84] between gauge and string theory models with reduced supersymmetry is provided by an exactly marginal deformation of $\mathcal{N} = 4$ super Yang-Mills theory [81] and string theory on a deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background suggested in [82]. The deformed models depend on a continuous complex parameter β, and are often called β-deformed. If $\beta \equiv \gamma$ is real the deformed string background can be derived from $\mathrm{AdS}_5 \times \mathrm{S}^5$ by using a TsT transformation which is a combination of a T-duality on one angle variable, a shift of another isometry variable, followed by the second T-duality on the first angle [49, 82]. Moreover, since S^5 has three commuting Killing vectors, a chain of TsT transformations can be used to construct a regular three-parameter deformation of $\mathrm{AdS}_5 \times \mathrm{S}^5$ dual to a non-supersymmetric deformation of $\mathcal{N} = 4$ SYM [49]. The Lagrangian of the γ_i-deformed gauge theory can be obtained from the undeformed one by replacing the usual product by the associative $*$-product [30, 49, 82]. The resulting model is conformal in the planar limit to any order of perturbation theory [8].

Another important property of a TsT transformation is that it preserves the classical integrability of string theory on $AdS_5 \times S^5$ [49]. In particular the Lax pair for strings on $AdS_5 \times S^5$ [33] and a TsT transformation can be used to find a Lax pair for strings on a deformed background [5, 49]. Moreover, the Green-Schwarz action for strings on $AdS_5 \times S^5$ is mapped under a TsT transformation to a string action on the γ-deformed background providing a nontrivial example of non-supersymmetric Green-Schwarz action for strings on RR backgrounds [5]. In fact in the Hamiltonian (first-order) formalism the Green-Schwarz action for strings on the γ-deformed background is canonically equivalent to the action for strings on $AdS_5 \times S^5$ satisfying quasi-periodic or twisted boundary conditions [5, 49]. The twists however are quite unusual because they depend on charges carried by a string and are given by linear combinations of products of the deformation parameters and $\mathfrak{su}(4)$ charges.

This also implies that in the light-cone gauges of [9, 50] the string dynamics on both the γ-deformed background and $AdS_5 \times S^5$ is described by the same Hamiltonian density. The γ-dependence enters only through the twisted boundary conditions and the level-matching condition which is modified because a closed string in the deformed background in general corresponds to an open string in $AdS_5 \times S^5$. Correspondingly, in the decompactification limit where one of the $\mathfrak{su}(4)$ charges, say J, is sent to infinity while the string tension and the deformation parameters are kept fixed the dependence of the light-cone Hamiltonian on the deformation parameters disappears because in this limit all physical fields must vanish at the space infinity[1]. As a result, if one considers the light-cone gauge-

[1] A γ-dependence remains in the pp-wave [34] and spinning string [56] limits because in these limits

fixed string sigma model off-shell, that is if one does not impose the level-matching condition then the deformed string model is indistinguishable from the undeformed one, and they share the same magnon dispersion relation [27], the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$-invariant world-sheet S-matrix [20, 25, 101] and the dressing factor [17, 28, 29, 32, 47, 71]. Therefore, the γ-dependence in the decompactification limit is only due to the level-matching condition.

Thus, to see the dependence of the off-shell spectrum of the model on the deformation parameters one should analyze it for finite values of the $\mathfrak{su}(4)$ charges. The leading dependence can then be captured by the asymptotic Bethe ansatz which would differ from the usual one [31] only by the twists reflecting the non-periodic boundary conditions for finite J. This conclusion is also confirmed by the one-loop considerations in the γ-deformed gauge theory [30,35,97] where it is shown that the one-loop integrability of $\mathcal{N} = 4$ SYM [92] is preserved by the deformation, and the corresponding one-loop Bethe ansatz involves the same twists that appear in string theory [30]. In the asymptotic approximation the dispersion relation is not modified and the twists lead to a very mild modification of the string spectrum which basically reduces to γ-dependent shifts of string mode numbers, see [51,52,82] for some examples.

The asymptotic Bethe ansatz is not exact and for finite J one expects to find a non-trivial γ-dependence already in the large string tension limit where classical string considerations[2] can be used. In particular, it is interesting to determine how the dispersion relation for a giant magnon [72] depends on the deformation parameters. In the infinite J limit a giant magnon is

the effective length $J/\sqrt{\lambda}$ and the twists $\sim \gamma_i J_k$ are kept fixed, and therefore the string sigma model is defined on a circle with fields obeying quasi-periodic boundary conditions. The pp-wave limits of the deformed backgrounds were discussed in [86, 89, 95], and the finite-gap integral equations [78] describing spinning strings in the γ-deformed $\mathfrak{su}(2)$ sector were derived in [51].

[2]Throughout the thesis classical refers to zero worldsheet coupling, i.e. the limit $\lambda \to \infty$.

dual to a gauge theory spin chain magnon, and in the conformal gauge it can be identified with an open string solution of the sigma model reduced to $R \times S^2$. The end-points of the open string move along the equator of S^2 parametrized by an angle ϕ, and the momentum p carried by the dual spin chain magnon is equal to the difference in the angle ϕ between the two end-points of the string [72]. On the other hand in a light-cone gauge a giant magnon is identified with a world-sheet soliton and the momentum p is equal to the world-sheet momentum $p_{\rm ws}$ of the soliton [19]. For finite J the equality between p and $p_{\rm ws}$ holds only in the light-cone gauge $t = \tau$, $p_\phi = 1$ [19].

In this Chapter we determine the leading γ-dependence of the dispersion relation for a finite J giant magnon. We use the conformal gauge and the string sigma model reduced to $R \times S^3$ which in the deformed case is the smallest consistent reduction due to the twisted boundary conditions. Even for the three-parameter deformation the reduced model depends only on one of the parameters which we denote γ. Since there are two isometry angles ϕ_1 and ϕ_2 a solution of the reduced model can have two non-vanishing charges J_1 and J_2. A giant magnon is then an open string solution of the model which carries only one charge $J \equiv J_1$. The momentum p of the magnon is correspondingly identified with the difference in the angle ϕ_1 between the two end-points of the open string because in the light-cone gauge $t = \tau$, $p_{\phi_1} = 1$ it is equal to the world-sheet momentum of a soliton. The second angle ϕ_2 satisfies a twisted boundary condition which can be found by using the general formulas from [49]

$$\Delta\phi_2 = 2\pi(n_2 - \gamma J) , \quad n_2 \in \mathbf{Z} ,$$

where n_2 is an integer winding number of the string in the second isometry direction of the deformed sphere S^3_γ. Collecting all the requirements together, we conclude that a γ-deformed giant magnon can be identified with an open string in $R \times S^3$ satisfying the following conditions

$$\Delta\phi_1 = p\,, \quad \Delta\phi_2 = 2\pi(n_2 - \gamma J)\,, \quad J_1 = J\,, \quad J_2 = 0\,.$$

We analyze the equations of motion and find that a solution exists only for one integer n_2 which obeys the condition $|n_2 - \gamma J| \leq \frac{1}{2}$, and therefore there is only one deformation of a giant magnon solution in $R \times S^2$. Then, the leading correction to the dispersion relation in the large J limit has the following form

$$E - J = 2g \sin\frac{p}{2}\left(1 - \frac{4}{e^2}\sin^2\frac{p}{2}\cos\Phi\, e^{-\frac{\mathcal{J}}{\sin p/2}} + \ldots\right)\,, \qquad \Phi = \frac{2\pi(n_2 - \gamma J)}{2^{3/2}\cos^3\frac{p}{4}}\,,$$

where $g = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension, and $\mathcal{J} = J/g$. The formula reduces in the limit $\gamma \to 0$ (or $\Phi \to 0$) to the one obtained in [19]. In the large J limit the γ-dependence disappears in agreement with the discussion above, and if γ is kept fixed then the winding number n_2 goes to infinity too.

The deformed theory has less supersymmetry, and one expects that the energy of a γ-deformed magnon would be higher than the energy of the undeformed one with the same momentum and charge. It is indeed the case because $\cos\Phi < 1$.

It would be interesting to understand how to reproduce the dispersion relation by using Lüscher's approach [83]. This would generalize the computation performed in [76] to the deformed case. The dispersion relation has a peculiar γ-dependence for finite J, and it is not quite clear how such a dependence follows from the S-matrix approach. This would require to generalize Lüscher's formulas to the case of the nontrivial twisted boundary conditions.

Our consideration can be generalized to solutions carrying several spins, see [69, 79, 91] for recent discussions of the undeformed model. It would be also interesting to compute the one-loop quantum correction generalizing the considerations in [63, 70].

In section 2 we discuss possible giant magnon solutions in the deformed background and explain how they can be mapped to open strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$. In section 3 we sketch the derivation of the leading correction to the dispersion relation in the large J limit and discuss its structure. The details of the derivation can be found in Appendix.

2.2 The γ-deformed giant magnon

The bosonic part of the Green-Schwarz action for strings on the γ-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background [5] reduced to $R \times S^5_\gamma$ can be written in the following form

$$S = -\frac{g}{2}\int_{-r}^{r} d\sigma d\tau \Big[\gamma^{\alpha\beta}\left(-\partial_\alpha t \partial_\beta t + \partial_\alpha \rho_i \partial_\beta \rho_i + G\,\rho_i^2 \partial_\alpha \varphi_i \partial_\beta \varphi_i + G\,\rho_1^2\rho_2^2\rho_3^2 (\hat{\gamma}_i \partial_\alpha \varphi_i)(\hat{\gamma}_j \partial_\beta \varphi_j)\right)$$
$$-2\,G\,\epsilon^{\alpha\beta}\left(\hat{\gamma}_3 \rho_1^2\rho_2^2 \partial_\alpha\varphi_1 \partial_\beta\varphi_2 + \hat{\gamma}_1 \rho_2^2\rho_3^2 \partial_\alpha\varphi_2 \partial_\beta\varphi_3 + \hat{\gamma}_2 \rho_3^2\rho_1^2 \partial_\alpha\varphi_3 \partial_\beta\varphi_1\right)\Big]\,. \qquad (2.1)$$

Here $g = \frac{R^2}{\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension, and $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$ where $h^{\alpha\beta}$ is a world-sheet metric with Minkowski signature. The function G is defined as follows

$$G^{-1} = 1 + \hat{\gamma}_3^2\rho_1^2\rho_2^2 + \hat{\gamma}_1^2\rho_2^2\rho_3^2 + \hat{\gamma}_2^2\rho_1^2\rho_3^2\,, \qquad \sum_{i=1}^{3}\rho_i^2 = 1\,, \qquad (2.2)$$

and φ_i are the three isometry angles of the deformed S^5_γ. The deformation parameters $\hat{\gamma}_i$ are kept fixed in the string sigma model perturbation theory, and are related to the parameters γ_i which appear in the dual gauge theory as $\hat{\gamma}_i = 2\pi g \gamma_i = \sqrt{\lambda}\gamma_i$. The standard $\mathrm{AdS}_5 \times \mathrm{S}^5$ background is recovered after setting the deformation parameters $\hat{\gamma}_i$ to zero. For equal $\hat{\gamma}_i = \hat{\gamma}$ this becomes the supersymmetric background of [82], and the deformation parameter γ enters the $\mathcal{N} = 1$ SYM superpotential as follows $W = h\,\mathrm{tr}(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2)$.

The TsT transformations that map the $AdS_5 \times S^5$ string theory to the γ_i-deformed string theory allow one to relate the angle variables ϕ_i of S^5 to the angle variables φ_i of the γ-deformed geometry. The relations take their simplest form being expressed in terms of the momenta p_i, π_i conjugate to ϕ_i, φ_i, respectively[3] [49]

$$p_i = \pi_i \,, \tag{2.3}$$

$$\rho_i^2 \phi_i' = \rho_i^2(\varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k) \,, \qquad i = 1, 2, 3 \,, \tag{2.4}$$

where in (2.4) we sum only in j, k. The relation (2.3) implies that the $U(1)$ charges $J_i = \int d\sigma p_i$ are invariant under a TsT transformation.

Assuming that none of the "radii" ρ_i vanish on a string solution, we get

$$\phi_i' = \varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k \,. \tag{2.5}$$

Integrating eq.(2.5) and taking into account that

$$\Delta\varphi_i = \varphi_i(r) - \varphi_i(-r) = 2\pi n_i \,, n_i \in \mathbb{Z} \tag{2.6}$$

for a closed string in the γ-deformed background, we obtain the twisted boundary conditions for the angle variables ϕ_i of the original S^5 space

$$\Delta\phi_i = \phi_i(r) - \phi_i(-r) = 2\pi(n_i - \nu_i) \,, \; \nu_i = \epsilon_{ijk}\gamma_j J_k \,, \; J_i = \int_{-r}^{r} d\sigma \; p_i. \tag{2.7}$$

It is clear that if the twists ν_i are not integer then a closed string in the deformed geometry is mapped to an open string in $AdS_5 \times S^5$. A giant magnon solution in this respect does not differ essentially from a closed string in $AdS_5 \times S^5_\gamma$. It corresponds to an open string in the deformed geometry, and its image in $AdS_5 \times S^5$ is an open string too. The only difference is that not all of the winding numbers n_i are integer for a giant magnon solution. In fact one linear combination of the winding numbers should be identified with the momentum p carried by the giant magnon.

[3]Here we use definitions of momenta p_i, which differ by a factor of 2π from those of [49], therefore we have an extra 2π in (2.4).

To determine the linear combination we notice that in the infinite $J \equiv J_1 + J_2 + J_3$ limit the end-points of a giant magnon should move with the speed of light along a null geodesic of the background [72]. In the undeformed case any geodesics is just a big circle of S^5, and the solution is described by a soliton of the string sigma model reduced to $R \times S^2$. The momentum carried by the soliton is identified with the difference in the angle ϕ between the two end-points of the string where ϕ parametrizes the equator of S^2 [72]. In the light cone gauge $t = \tau$, $p_\phi = 1$ the momentum p is equal to the world-sheet momentum of the giant magnon solution and because of that the identification can be also used for finite J [19].

In the γ-deformed background there are infinitely many inequivalent geodesics which correspond to solutions of the Neumann-Rosochatius integrable system [52] (which also describes multi-spin string solutions [16, 21]), and one should choose only those which give the minimum energy satisfying the BPS condition $E = J$. These geodesics were described in [52] where it was shown that for generic values of γ_i there are three BPS states which have only one of the three charges J_i nonvanishing. Choosing for definiteness the nonvanishing charge to be $J_1 = J$, the BPS state corresponds to the geodesics parametrized by the angle φ_1 and having $\rho_1 = 1$, $\rho_2 = \rho_3 = 0$. An infinite J giant magnon with the end-points moving along the geodesics is then a solution of the string sigma model reduced to $R \times S^3_\gamma$ where S^3_γ is obtained from the deformed S^5_γ by setting $\rho_3 = 0$. The momentum p carried by the soliton is identified with the difference $\Delta\varphi_1 = \varphi_1(r) - \varphi_1(-r)$. In fact it is easy to see that the TsT transformation maps the infinite J giant magnon solution of the undeformed model to the γ-deformed giant magnon, and therefore the infinite J dispersion relation is not modified, and has no γ dependence. For finite J however the dispersion relation gets a nontrivial γ-dependence which we determine in the next section. This follows from the fact that for the magnon solution $J_2 = J_3 = 0$, and therefore the twist $\nu_1 = 0$, and the corresponding angles ϕ_1 and ϕ_2 of the undeformed S^3 satisfy the following twisted boundary conditions

$$\Delta\phi_1 = \phi_1(r) - \phi_1(-r) = p\,,\ \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = 2\pi(n_2 - \gamma J)\,, \tag{2.8}$$

where $\gamma \equiv \gamma_3\,,\ J \equiv J_1$. As a result the dispersion relation for the finite J γ-deformed giant magnon depends on p, J and $\delta \equiv 2\pi(n_2 - \gamma J)$. To find the dispersion relation one can either use the conformal gauge [72] or the light-cone gauge [19].

Let us also mention that in the case where the deformation parameters satisfy the relations $\gamma_i = c\,k_i$ where c is any real number and k_i are arbitrary integers, there is another family of BPS states with the following charges [52]

$$J_i = k\,k_i \sim \gamma_i\,, \tag{2.9}$$

where (in quantum theory) k is any integer. In particular, in the supersymmetric case $\gamma_i = \gamma$ the BPS states are the states $(J/3, J/3, J/3)$ with three equal charges. Since $J_i \sim \gamma_i$ for these BPS states the twists ν_i vanish and both the γ-deformed giant magnon and its TsT image satisfy the same twisted boundary conditions which take the simplest form in terms of the following new angle variables and their conjugate momenta

$$\psi_1 = k_1\phi_1 + k_2\phi_2 + k_3\phi_3\,, \qquad \pi_1 = \frac{p_1+p_2+p_3}{k_1+k_2+k_3}\,, \tag{2.10}$$

$$\psi_2 = k_1\phi_1 - (k_1+k_3)\phi_2 + k_3\phi_3\,, \qquad \pi_2 = \frac{k_2p_1 - k_1p_2}{k_1(k_1+k_2+k_3)}\,, \tag{2.11}$$

$$\psi_3 = k_1\phi_1 + k_2\phi_2 - (k_1+k_2)\phi_3\,, \qquad \pi_3 = \frac{k_3p_1 - k_1p_3}{k_1(k_1+k_2+k_3)}\,. \tag{2.12}$$

Then, the giant magnon solution with the charges satisfying (2.9) satisfies the following boundary conditions

$$\Delta\psi_1 = p\,,\ \Delta\psi_2 = 0\,,\ \Delta\psi_3 = 0\,. \tag{2.13}$$

Since the boundary conditions do not depend on γ_i in the classical theory the dispersion relation for the giant magnon does not depend on the deformation parameters either. A disadvantage of this giant magnon solution is that the corresponding Bethe ansatz is not known.

2.3 Finite J dispersion relation

To determine the dispersion relation we impose the conformal gauge $\gamma^{\alpha\beta} = \mathrm{diag}(-1, 1)$, set $t = \tau$, and use the following parametrization of S^3

$$x_i^2 = 1\,,\; x_1 + ix_2 = \rho_1 e^{i\phi_1}\,,\; x_3 + ix_4 = \rho_2 e^{i\phi_2}\,,\; \rho_2^2 = 1 - \rho_1^2 = \chi\,. \quad (2.14)$$

Then the sigma model action for strings on $R \times S^3$ takes the following form

$$S = -\frac{g}{2}\int_{-r}^{r} d\sigma d\tau \left(\frac{\partial_\alpha\chi\partial^\alpha\chi}{4\chi(1-\chi)} + (1-\chi)\partial_\alpha\phi_1\partial^\alpha\phi_1 + \chi\partial_\alpha\phi_2\partial^\alpha\phi_2 \right) .$$

and solutions of the equations of motion should also satisfy the Virasoro constraints

$$\frac{\dot\chi^2 + \chi'^2}{4\chi(1-\chi)} + (1-\chi)\left(\dot\phi_1^2 + \phi_1'^2\right) + \chi\left(\dot\phi_2^2 + \phi_2'^2\right) = 1\,, \quad (2.15)$$

$$\frac{\dot\chi\chi'}{4\chi(1-\chi)} + (1-\chi)\dot\phi_1\phi_1' + \chi\dot\phi_2\phi_2' = 0\,. \quad (2.16)$$

Since $t = \tau$ the range of σ is related to the space-time energy E of a solution as follows

$$2r = \frac{E}{g} \equiv \mathcal{E}\,. \quad (2.17)$$

The two charges $J_1 \equiv J$ and J_2 corresponding to shifts of ϕ_1 and ϕ_2 are

$$J = g\int_{-r}^{r} \mathrm{d}\sigma\,(1-\chi)\,\dot\phi_1\,, \quad J_2 = g\int_{-r}^{r} \mathrm{d}\sigma\,\chi\,\dot\phi_2\,. \quad (2.18)$$

As was discussed in the previous section, the γ-deformed giant magnon solution has only one nonvanishing charge J, and the angles ϕ_1 and ϕ_2 satisfy the following twisted boundary conditions

$$\Delta\phi_1 = \phi_1(r) - \phi_1(-r) = p\,,\ \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = \delta\,, \tag{2.19}$$

where $\delta = 2\pi(n_2 - \gamma J)$, $\gamma = \gamma_3$ and n_2 is the winding number in the φ_2 direction of the deformed S^5_γ . It is worth mentioning that the dependence on γ and n_2 comes only through their linear combination δ which in fact plays the role of the deformation parameter.

The problem of finding a finite J giant magnon solution is thus basically equivalent to the problem of finding a two-spin giant magnon solution discussed in appendix C of [19], and can be solved by using a similar ansatz

$$\begin{aligned}
\phi_1(\sigma,\tau) &= \omega\tau + \frac{p}{2r}(\sigma - v\tau) + \phi(\sigma - v\tau)\,, && (2.20)\\
\phi_2(\sigma,\tau) &= \nu\tau + \frac{\delta}{2r}(\sigma - v\tau) + \alpha(\sigma - v\tau)\,, && (2.21)\\
\chi(\sigma,\tau) &= \chi(\sigma - v\tau)\,, && (2.22)
\end{aligned}$$

where $\chi(\sigma)$, $\phi(\sigma)$ and $\alpha(\sigma)$ satisfy the periodic boundary conditions.

Substituting the ansatz into the equations of motion, integrating the equations for ϕ and α once, and using the Virasoro constraint (2.15) , we get the following three equations

$$\begin{aligned}
\phi' &= f_0 + \frac{f_1}{1-\chi}\,, \quad \alpha' = a_0 + \frac{a_1}{\chi}\,, && (2.23)\\
\kappa^2\chi'^2 &= (\chi - \chi_{\text{neg}})(\chi - \chi_{\text{min}})(\chi_{\text{max}} - \chi)\,, && (2.24)
\end{aligned}$$

where the constants in the equations are functions of $\omega, \nu, v, p, \delta$, and $\chi_{\text{neg}}, \chi_{\text{min}}, \chi_{\text{max}}$ are ordered as $\chi_{\text{neg}} \le 0 \le \chi_{\text{min}} < \chi_{\text{max}}$. Moreover, giant magnon solutions exist only if $\chi_{\text{max}} \le 1$ and for these solutions $\chi_{\text{min}} \le \chi \le \chi_{\text{max}}$, see Appendix for detail.

If the deformation parameter δ goes to 0 then χ_{neg} , a_0 , a_1 approach 0 too, and we recover the equations of motion for a finite J undeformed giant magnon [19].

For any value of δ we can always choose the initial conditions so that $\chi(\sigma)$ is an even function and $\phi(\sigma)$ and $\alpha(\sigma)$ are odd functions of σ, and since they are also periodic functions, we can always look for a solution satisfying the following boundary conditions

$$\chi(-r) = \chi(r) = \chi_{\min}\,,\ \chi(0) = \chi_{\max}\,,\ \chi(-\sigma) = \chi(\sigma)\,, \tag{2.25}$$
$$\phi(-r) = \phi(0) = \alpha(-r) = \alpha(0) = 0\,,\ \phi(-\sigma) = -\phi(\sigma)\,,\ \alpha(-\sigma) = -\alpha(\sigma)\,.$$

Due to the conditions we can restrict our attention to the half of the string from $-r$ to 0, and since χ is an increasing function on this interval we can also replace integrals over σ by integrals over χ from $\chi_{\min}$ to $\chi_{\max}$. Then a solution is completely determined by the following five equations which are analyzed in detail in Appendix

$$\text{Periodicity of } \phi: \quad r\, f_0 + f_1 \int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{(1-\chi)|\chi'|} = 0\,,$$

$$\text{Periodicity of } \alpha: \quad r\, a_0 + a_1 \int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{\chi|\chi'|} = 0\,,$$

$$\text{Charge } \mathcal{J} \equiv \frac{J_1}{g}: \quad \mathcal{J} = -2r\, v\, f_1 + \frac{\omega}{1-v^2} \int\limits_{\chi_{\min}}^{\chi_{\max}} d\chi\, \frac{1-\chi}{|\chi'|}\,,$$

$$\text{Charge } J_2 = 0: \quad 0 = -2r\, v\, a_1 + \frac{\nu}{1-v^2} \int\limits_{\chi_{\min}}^{\chi_{\max}} d\chi\, \frac{\chi}{|\chi'|}\,,$$

$$\text{Length of string:} \quad \int_{-r}^{0} d\sigma = r = \int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{|\chi'|}\,,$$

where all constants should be expressed in terms of the charge $\mathcal{J}$, the soliton momentum p and the deformation parameter δ.

The dispersion relation can be found in the large $\mathcal{J}$ limit as an expansion in $e^{-\frac{\mathcal{J}}{\sin(p/2)}}$, and up to the first correction it has the following form ($0 \leq p \leq \pi$)

$$E - J = 2g \sin\frac{p}{2}\left(1 - \frac{4}{e^2}\sin^2\frac{p}{2}\cos\Phi\, e^{-\frac{\mathcal{J}}{\sin p/2}} + ...\right), \tag{2.26}$$

where

$$\Phi = \frac{\delta}{2^{3/2}\cos^3\frac{p}{4}} = \frac{2\pi(n_2 - \gamma J)}{2^{3/2}\cos^3\frac{p}{4}}. \tag{2.27}$$

The dispersion relation in the γ-deformed model reduces in the limit $\delta \to 0$ (or $\Phi \to 0$) to the one obtained in [19].

Some remarks are in order.

1. We see that in the limit $\mathcal{J} \to \infty$ the dispersion relation is independent of the deformation parameter. This is contrary to papers [37, 44] where it was claimed that the momentum is shifted by the deformation parameter $2\pi\gamma$. As was discussed in the previous section, $2\pi\gamma$ is identified with $\hat{\gamma}/g$, and therefore the shift by γ cannot be seen in classical theory in any case. It would be a one-loop effect, and the discussion in the Introduction indicates that the momentum p is not shifted at one loop at all but one should take into account that in quantum theory magnons carry other charges of order one, and therefore $p = \Delta\phi_1$ is not equal to $p_{\rm ws} = \Delta\varphi_1$. According to (2.7), if we have several (or just one) magnons with the total charges J_2, J_3 then the momenta are related as $p = p_{\rm ws} + 2\pi\gamma_3 J_2 - 2\pi\gamma_2 J_3$. If the state is physical then the total world-sheet momentum $p_{\rm ws}$ should vanish leading to the condition $p = 2\pi\gamma_3 J_2 - 2\pi\gamma_2 J_3$ (up to an integer multiple of 2π). This condition is equivalent to the cyclicity constraint in the twisted Bethe ansatz [30].

2. Since $\cos\Phi < 1$ the energy of a γ-deformed magnon is higher than the energy of the undeformed one with the same momentum and charge.

3. The derivation of the dispersion relation performed in Appendix shows that a giant magnon solution exists if Φ satisfies the restriction

$$-\pi \leq \Phi \leq \pi\,, \tag{2.28}$$

and therefore if we require a solution to exist for all values of p from $-\pi$ to π the parameter δ must also satisfy the same restriction

$$-\pi \leq \delta \leq \pi \quad \Longleftrightarrow \quad |n_2 - \gamma J| \leq \frac{1}{2}\,. \tag{2.29}$$

This means that n_2 is the integer closest to γJ. We see that for any γJ there is only one integer n_2 which satisfies the condition, and therefore there is only one deformation of a giant magnon solution in $R \times S^2$. If the fractional part of γJ is less than $1/2$ then n_2 is equal to the integer part of γJ, and if the fractional part of γJ is greater than $1/2$ then n_2 is equal to the integer part of $\gamma J + 1$.

4. For small enough values of p however the first-order perturbation theory in $e^{-\frac{J}{\sin(p/2)}}$ allows one to have two or three integers satisfying the restriction (2.28): n_2 satisfying (2.29), and $n_2 \pm 1$. We expect that the latter possibilities will be ruled out at higher orders of the perturbation theory. Anyway, according to (2.26) their energies would be higher than the energy of the main solution.

Chapter 3

Symmetry algebra of the $AdS_4 \times \mathbb{CP}^3$ superstring

3.1 Introduction

The issue of integrability in the $AdS_4 \times \mathbb{CP}^3$ model, which was introduced in Chapter 1, has not been fully resolved so far. First of all, in a string theory calculation of one-loop correction to the spinning string (that is, a string with two charges J and S) energy a mismatch was found with the Bethe ansatz prediction [4]. Subsequently this result was confirmed by a calculation of the energy correction to a different string configuration — the so-called circular string, which is a rational classical solution of the sigma-model [88]. One of the explanations relies on the possible modification of $h(\lambda)$ (effective string tension, or coupling constant) due to loop corrections. Since $h(\lambda)$ enters the dispersion relation of the giant magnon, which in turn can be derived from the centrally-extended supersymmetry algebra as a BPS (multiplet-shortening) condition, the calculation of loop corrections to the central extension would prove useful and could help finally settle the issue.

Another puzzle in the integrability program is that of the so-called 'heavy modes' in the BMN expansion on the string theory side. Indeed, the quadratic (leading) order of the BMN expansion was found in [96] and confirmed in [11], and it follows from these papers that, apart from a multiplet of light particles of mass $\frac{1}{2}$ there is also a multiplet of heavy particles of mass 1. The heavy excitations are not among the elementary excitations of the spin chain, namely, they contain two elementary momentum-carrying Bethe roots, which suggests that they are some sort of 'bound pair' of elementary magnons.

A possible resolution of this problem has been recently put forward in [112]. The idea is that loop corrections remove the heavy particles from the spectrum. Namely, the pole of the corresponding heavy-particle propagator disappears, once, say, a one-loop correction is taken into account. This is related to the fact that the mass of the heavy particles lies precisely at the two-particle production threshold of light-particles and, of course, on the interactions of the theory, which are non-relativistic, since we move away from the strict BMN limit.

The central extension of the supersymmetry algebra $psu(2|2) \oplus \mathfrak{psu}(2,2|4)(2|2) \oplus u(1)$ in the $AdS_5 \times S^5$ case was introduced in [25] [1]. If the symmetry algebra of the $AdS_4 \times \mathbb{CP}^3$ superstring were altered as compared to the $\mathrm{AdS}_5 \times S^5$ case, this could perhaps give some clues to the solution of the massive modes problem. However, as we explain below, the central extension is the same.

Other aspects of integrability of the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ have been studied [102], namely near-BMN corrections to the energies of states in certain sectors were calculated therein.

The Chapter is organized as follows. In section 3.2, we proceed to impose

[1]In the worldsheet approach used throughout the thesis this symmetry algebra is the subalgebra of the full symmetry algebra, which is realized linearly in the light-cone gauge.

the light-cone gauge. Next, in section 3.3 we discuss the transformation properties of all the physical fields of the sigma-model under the residual light-cone global symmetry group. In section 3.4 we describe the kappa-symmetry gauge, which respects the global bosonic symmetries of the light-cone gauge. In section 3.5, we derive the central extension through the calculation of Poisson brackets. In carrying out the calculation we closely follow [15], for instance, we use the so-called "hybrid" expansion introduced therein. In the Appendix the reader will find the explicit form of the necessary matrices, all global charges written out in terms of the fields, the Poisson brackets of these fields, as well as a general discussion of geodesics in $\mathbb{CP}^3$.

3.2 Light-cone gauge

An extensive review of the light-cone gauge quantization of the $\mathrm{AdS}_5 \times S^5$ superstring (which can be generalized to other maximally symmetric spaces), among many other things, can be found in the review [10]. We introduce the light-cone coordinates:

$$x_+ = \frac{1}{2}(\varphi + t), \quad x_- = \varphi - t \tag{3.1}$$

The corresponding canonical momenta p_+ and p_- are conjugate to x_- and x_+ respectively. Recall that the light-cone gauge comprises two conditions: $x_+ = \tau$, $p_+ = \text{const}$. We would like our string Lagrangian (and, consequently, Hamiltonian) not to depend on time τ even after the light-cone gauge is imposed. This requirement leads us to the following choice of parameterization for the coset element:

$$g = g_O g_\chi g_B, \tag{3.2}$$

where $g_O = \exp\left(\frac{i}{2}t\Gamma_0 + \frac{\varphi}{2}T_6\right)$, $g_\chi = \exp\chi$, $g_B = \exp\left(\frac{\alpha}{2}T_5\right) g_{\mathbb{CP}}\, g_{\mathrm{AdS}}$. We have chosen the coset representative g_{AdS} for AdS space in a way similar to the one in [50]:

$$g_{\mathrm{AdS}} = \frac{1}{\sqrt{1+\frac{z^2}{4}}}\left(1+\frac{i}{2}\sum_{i=1}^{3} z_i\Gamma_i\right), \tag{3.3}$$

where $z^2 = -\sum_{i=1}^{3} z_i^2$. The matrix $g_{\mathbb{CP}}$ gives, in turn, a parametrization of $\mathbb{CP}^2$ and is an obvious reduction of the coset element from [11]:

$$g_{\mathbb{CP}} = I + \frac{1}{\sqrt{1+|w|^2}}\left(W+\bar{W}\right) + \frac{\sqrt{1+|w|^2}-1}{|\omega|^2\sqrt{1+|w|^2}}\left(W\bar{W}+\bar{W}W\right), \tag{3.4}$$

where

$$W = w_1\mathcal{T}_1 + w_2\mathcal{T}_2,\ \ \bar{W} = \bar{w}_1\bar{\mathcal{T}}_1 + \bar{w}_2\bar{\mathcal{T}}_2. \tag{3.5}$$

χ is the fermionic matrix of the following form:

$$\chi = \begin{bmatrix} 0 & \theta \\ \eta & 0 \end{bmatrix},\ \theta = \begin{bmatrix} n_{11} & \cdots & n_{16} \\ \vdots & & \vdots \\ n_{41} & \cdots & n_{46} \end{bmatrix},\ \eta = -\theta^T C_4.$$

The reality condition of the algebra also ensures that the two lower lines of θ are complex conjugates of the upper lines (we refer the interested reader to [11] for more information regarding this and other properties of the coset):

$$n_{3j} = -n_{2j}^*,\ n_{4j} = n_{1j}^*. \tag{3.6}$$

It is now easy to see that with this choice the current $A \equiv g^{-1}dg$, out of which the Lagrangian is built, does not explicitly depend on world-sheet time τ. To make this property even more obvious, we rewrite the first exponent g_O in terms of the light-cone coordinates:

$$g_O = \exp\left(\frac{i}{2}x_+\Sigma_+ + \frac{i}{4}x_-\Sigma_-\right), \tag{3.7}$$

where we have introduced $\Sigma_\pm = \pm\Gamma_0 - iT_6 = \mathrm{diag}\{\pm\Gamma_0; -iT_6\}$.

As is usual for gauge fixing procedures, after fixing the gauge we lose a certain amount of symmetry. The next problem we are going to tackle is to define the symmetry subgroup of G which is left after imposition of the light-cone gauge. The subgroup of such transformations will be denoted by G_{lc}. Its Lie algebra consists of matrices which commute with the light-cone direction Σ_+. The block-diagonal bosonic subalgebra g_{lc}^{Bose} is furnished by matrices which commute with both Γ_0 and T_6 (i.e. the precise combination Σ_+ is only important for the fermionic part of the algebra, that is, for the supercharges). One can explicitly check that $g_{lc}^{Bose} = \mathrm{su}(2) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1)$. One of these su(2)s comes from the requirement of commutation with Γ_0, whereas $\mathrm{su}(2) \oplus \mathrm{u}(1)$ is the subalgebra of matrices, which commute with T_6. One can vaguely refer to the former as the su(2) coming from AdS_4 whereas the latter is the algebra originating from $\mathbb{CP}^3$. Schematically the position of the embeddings of the corresponding matrices looks as follows:

$$\omega = \begin{pmatrix} su(2)|_{4\times 4}^{\mathrm{AdS}} & 0 & 0 \\ 0 & u(1)|_{2\times 2}^{\mathbb{CP}} & 0 \\ 0 & 0 & su(2)|_{4\times 4}^{\mathbb{CP}} \end{pmatrix} \tag{3.8}$$

For a precise description of these matrices see Appendix.

Suppose we now want to calculate the full algebra g_{lc}, which is left after the light-cone condition has been imposed. This means, that we will include supersymmetry transformations, and will no longer limit ourselves to the bosonic part g_{lc}^{Bose}. Then, as one can explicitly check, the full algebra turns out to be $g_{lc} = \mathrm{su}(2|2) \oplus \mathrm{u}(1)$. It is precisely this algebra that acquires a central extension after quantization. We leave a more elaborate discussion of this point until section 6.

3.3 Transformation properties of the fields

3.3.1 Bosons

In this section we will find out the transformation properties of the fields, both bosonic and fermionic, under G_{lc}^{Bose} (or g_{lc}^{Bose} in infinitesimal form). It is important to notice that $G_{lc}^{Bose} \subset H$. Let us act on the coset element (3.2) from the left by a bosonic group element from G_{lc}^{Bose}, which we denote by $\exp a$, assuming that a is in the Lie algebra g_{lc}:

$$g \to e^a g = g_O e^{ad_a}(g_\chi) e^{ad_a}(g_B) e^a, \tag{3.9}$$

where we have taken into account that $[a, \Gamma_0] = [a, T_6] = 0$. The exponent at the very right is irrelevant, since it belongs to the stabilizer, and, as such, does not change the corresponding conjugacy class. The above formula shows, then, that the bosons and fermions are in the adjoint representation of G_{lc}^{Bose}.To be absolutely clear, we will describe the transformation properties of the fields even more explicitly. We work in the basis of γ-matrices described in the appendix, from which it follows that for $k = 1, 2, 3$ we have $\gamma_k = i\sigma_2 \otimes \sigma_k$, so that the AdS part of the coset is written in the following form:

$$g_{\text{AdS}} = \frac{1}{\sqrt{1+\frac{z^2}{4}}} \begin{pmatrix} I_2 & \sum\limits_{i=1}^{3} z_i\sigma_i \\ -\sum\limits_{i=1}^{3} z_i\sigma_i & I_2 \end{pmatrix} \tag{3.10}$$

As described above, under the action of the $SU(2)$ group from AdS this element transforms in the adjoint. Thus, introducing notation $Z \equiv \sum\limits_{i=1}^{3} z_i\sigma_i$, we get ($\Delta$ is the diagonal embedding defined in Appendix):

$$g_{\text{AdS}} \to \Delta(\omega) g_{\text{AdS}} \Delta(\omega^\dagger) = \frac{1}{\sqrt{1+\frac{z^2}{4}}} \begin{pmatrix} I_2 & \omega Z\omega^\dagger \\ -\omega Z \omega^\dagger & I_2 \end{pmatrix} \tag{3.11}$$

Since Z is a traceless Hermitian matrix, $Z \to \omega Z \omega^\dagger$ with $\omega \in SU(2)$ defines a vector representation. In order to single out other irreducible representations, we introduce three complex combinations of the T_i generators:

$$\tau_1 = T_1 + iT_2;\ \tau_2 = T_3 + iT_4;\ \tau_3 = T_5 + iT_6. \tag{3.12}$$

Then we can rewrite $\sum_{i=1}^{4} \beta_i T_i = \beta_1^+ \tau_1 + \beta_2^+ \tau_2 + \text{C.c.}$, where $\beta_{1,2}^+$ are a set of new (complex) coordinates. It is now easy to check that under su(2) $\oplus$ u(1) from $\mathbb{CP}^3$ these coordinates form a complex doublet and, consequently, β_i transform as $2^1 + 2^{-1}$, where the exponent refers to u(1) charge. Indeed, let us denote the su(2)s from AdS$_4$ and $\mathbb{CP}^3$ as su$(2)_R$ and su$(2)_L$ respectively. Let

$$\omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \widetilde{\omega} \end{pmatrix} \text{ with } \widetilde{\omega} = \begin{pmatrix} \omega_u & 0 \\ 0 & \omega_2 \end{pmatrix} \tag{3.13}$$

be a generic transformation matrix. Using the fact that for compact groups the exponential map is surjective, we introduce an explicit parameterization for these matrices: $\omega_u = \exp(-\alpha\, i\sigma_2)$ and $\omega_2 = \exp(\sum_{i=1}^{3} \delta_i\, s_i)$. Now, the non-zero part of the top line of the matrix W (see Appendix) can be written in the form

$$\widehat{\omega} = (\omega_1, \omega_2) \otimes \frac{1}{2}(1, -i). \tag{3.14}$$

Acting on it by ω_2^{-1} from the right, we obtain the transformation law:

$$(\omega_1, \omega_2) \to (\omega_1, \omega_2) \left(\exp(\pm i \sum_{j=1}^{3} \delta_j \sigma_j) \right)^{\mathrm{T}}, \tag{3.15}$$

which is the canonical $SU(2)$ action (defined on row-vectors, rather than on column-vectors).

One can actually propose an even stronger statement, namely that under the adjoint action of $H = U(3)$ the τ_i are in the 3 irrep, that is, they transform as a complex triplet. This means that T_i are in the $3 + \bar{3}$ representation. One of the consequences of this fact is the following interesting property: those skew-symmetric 6×6 matrices (that is, the ones in so(6)) which commute with T_6 simultaneously commute with T_5. Thus, transformations which leave T_6 invariant also leave T_5 invariant.

3.3.2 Fermions

Next we turn to the transformation properties of the fermions χ. They form a representation of $\mathrm{su}(2) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1)$, and we need to decompose it into irreducibles. We will proceed in analogous way to what was done for the bosonic case. The fermionic matrix θ undergoes the following transformation

$$\theta \to \omega_1 \, \theta \, \widetilde{\omega}^{-1} \tag{3.16}$$

As is described in Appendix, the matrix basis of the Lie algebra of $su(2)|^{\mathbb{CP}}_{4\times 4}$ looks as $A \otimes B$, where B is either the identity matrix I_2 or the skew-symmetric matrix $i\sigma_2$. Moreover, the $u(1)$ generator is simply $i\sigma_2$. Thus, it makes sense to single out the components of the fermionic matrix, which correspond to eigenvalues of the σ_2 matrix. In this way we obtain:

$$\theta = \theta^{(+1)} + \theta^{(-1)} + \theta^{(0)}_{+} + \theta^{(0)}_{-}, \tag{3.17}$$

where $\theta^{(\pm 1)}$ have non-zero columns 1 and 2, $\theta^{(0)}_{\pm}$ have zero columns 1 and 2. To simplify some expressions below we will for the moment cut off the zero columns from all of these matrices, namely, we will regard $\theta^{(\pm 1)}$ as 2×4 matrix and $\theta^{(0)}_{\pm}$ as 4×4 matrix. It should be clear from the context, if these matrices should be embedded into bigger ones. Then these matrices can be defined as follows:

$$\theta^{(+1)} = \kappa^{(+1)} \otimes (1, -i), \; \theta^{(-1)} = \kappa^{(-1)} \otimes (1, i), \tag{3.18}$$

$$\theta^{(0)}_{+} = \chi^{+0} \otimes (1, -i), \; \theta^{(0)}_{-} = \chi^{-0} \otimes (1, i). \tag{3.19}$$

One can consult the Appendix for an explicit form of the matrix θ in terms of all of these components.

Being multiplied by ω_u^{-1} from the right, obviously $\chi^{\pm 0}$ remain unaltered, whereas $\kappa^{(\pm 1)}$ transform as follows:

$$\kappa^{(\pm 1)} \to e^{\pm i\alpha}\kappa^{(\pm 1)}. \tag{3.20}$$

On the other hand, being multiplied by ω_2^{-1} from the right, κ is unaltered, but $\chi^{(\pm 0)}$ transform as follows:

$$\chi^{(\pm 0)} \to \chi^{(\pm 0)} \left(\exp\left(\pm i \sum_{j=1}^{3} \delta_j \sigma_j\right) \right)^{\mathrm{T}}, \tag{3.21}$$

which is the same transformation law as (3.15). We have written out the components of the matrices $\kappa^{(+1)}$ and χ^{+0} explicitly in the Appendix (see (C.4)). In terms of these components the transformation properties of the fermions are as follows:

$$\begin{aligned} \kappa^{a,+1} &\to e^{i\phi}(\omega_1)^{a}_{\ b}\,\kappa^{b,+1}, \\ (\chi^{+0})^{a}_{\alpha} &\to (\omega_1)^{\ a}_{\ b}\,(\omega_2)^{\ \beta}_{\ \alpha}(\chi^{+0})^{b}_{\beta}. \end{aligned} \tag{3.22}$$

In other words, if regarded as a matrix, $\chi = \{\chi^a_\alpha\}$ transforms as $\chi \to \omega_1 \chi (\omega_2)^T$. Of course, we also need to know how combinations like $n_{11} - in_{12}$ (which comprise $\kappa^{(-1)}$ and χ^{-0}) transform. It turns out that they are in the conjugate representation with respect to the $\mathbb{CP}^3$ part of the algebra, and in the same representation of the AdS part of the algebra. It will be useful to give χ^- the transformation properties identical to those of χ^+ and to convert κ^{-1} to the representation conjugate to the one of κ^{+1}. This is convenient, because $\chi^{\pm}$'s are not charged with respect to the $U(1)$, whereas $\kappa^{\pm}$ have opposite $U(1)$ charges. Since κ's carry opposite $U(1)$ charges, it is also natural to give them opposite transformation properties with respect to the $SU(2)$, which comes from AdS (they are uncharged with respect to the $SU(2)$ which comes from $\mathbb{CP}^3$).

It is always possible to change the transformation properties of the fields in this fashion, since the fundamental and conjugate-fundamental representations of $SU(2)$ are equivalent, which means that there is a matrix $C \in SU(2)$, securing a relation

$$\omega^* = C\omega C^{-1} \,\text{for}\, \omega \in SU(2). \tag{3.23}$$

In fact, $C = i\sigma_2$. Again, one can find the explicit form of the relevant combinations in (C.5): they transform as in (3.22), apart from the fact that the exponent $e^{i\phi}$ needs to be replaced with $e^{-i\phi}$ to account for the opposite transformation property with respect to the $U(1)$.

The indices have been chosen to suggest, what the representations of the fermions are. One can summarize the transformation properties described above as follows: $\kappa^{\pm 1}_{1,2}$ are in the fundamental of $\mathrm{su}(2)_R$, the $\pm$ carrying opposite charges with respect to the $\mathrm{u}(1)$, whereas $\chi^{\pm 0}_{\alpha\dot\beta}$ are in the bifundamental of $\mathrm{su}(2)_R \oplus \mathrm{su}(2)_L$ and neutral under $\mathrm{u}(1)$. From (3.6) it follows that the two lower lines of the matrix θ transform in an analogous way. In total we have 12 complex fermion fields, which have been grouped into irreps as $\kappa^{\pm 1}_{\alpha}$, $\chi^{\pm 0}_{\alpha\dot\beta}$.

3.4 κ-symmetry gauge

As is well-known, string sigma-models in the Green-Schwarz formulation possess, besides diffeomorphism and Weyl invariance, another sort of gauge invariance — the κ-symmetry, which is fermionic in the sense that the gauge (=local) parameters are fermionic (denoted by ϵ in what follows). Existence of such transformations was first observed by Green and Schwarz for the flat background, however, it was also discovered for string models in other backgrounds, including the $\mathrm{AdS}_5 \times S^5$ case. It is of course a remarkable property that the same sort of invariance also holds in the $AdS_4 \times \mathbb{CP}^3$ case under consideration [11].

Once the existence of κ-symmetry is established, one needs to choose a gauge (a representative in every gauge orbit). This can be done in various ways, however, one aims at preserving as much global symmetry as possible during this process, since global symmetry allows for a better classification of field multiplets and ultimately leads to a simpler formulation of the theory. In our case this requirement means that the whole G_{lc}^{Bose} should be preserved. Let us now elaborate on how this can be done. As found in [11], the leading order in ϵ of the θ field variation is

$$\delta\theta = \begin{bmatrix} 0 & 0 & \epsilon_1 & \epsilon_2 & -i\epsilon_2 & -i\epsilon_1 \\ 0 & 0 & \epsilon_3 & \epsilon_4 & -i\epsilon_4 & -i\epsilon_3 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \end{bmatrix}, \tag{3.24}$$

stars denoting complex conjugated variables, totally parallel to (3.6). One might observe that (3.24) is the upper right block of a generic fermionic matrix ϑ, which has the property $f_1(\vartheta) \equiv [\vartheta, \Sigma_+] = 0$ (not to mention reality conditions discussed numerous times above).

Let W be the full fermionic vector space. Factorizing over the gauge-equivalent combinations, we thus get $W/\mathrm{Ker} f_1 \sim \mathrm{Im}\, f_1$. Let us check that $\mathrm{Im}\, f_1$ is invariant under the action of G_{lc}^{Bose}. If a belongs to g_{lc}^{Bose} (which means that $[a, \Sigma_+] = 0$) and $c = [d, \Sigma_+] \in \mathrm{Im} f_1$, then $e^{\mathrm{ad}_a}(c) = [e^{\mathrm{ad}_a}(d), \Sigma_+] \in \mathrm{Im}\, f_1$. Restricting the fermion to $\mathrm{Im}\, f_1$ corresponds to setting

$$n_{15} = in_{14},\ n_{16} = in_{13},\ n_{25} = in_{24},\ n_{26} = in_{23}, \tag{3.25}$$

which is the explicit form of the gauge we will be using in what follows. Then, as one can easily see from (C.4), (C.5), $\chi^{+0} = \chi^{-0} \equiv \chi^0$, so we effectively get rid of one of the multiplets. As a result, we are left with 8 complex fermions, which is the correct number for supersymmetry. We want to emphasize that this choice of kappa-symmetry gauge is equivalent to requiring that for any χ there is a matrix ξ such that $\chi = [\Sigma_+, \xi]$. Obviously, this matrix is not unique.

3.5 Central extension

In section 4 we discussed the representation of the fermionic fields under the action of the bosonic part of the symmetry algebra. The odd part of the symmetry algebra may be realized in a way very similar to the fermionic fields — that is, as odd elements of a $4|6 \times 4|6$ matrix. This means, that the representation of supercharges is a subrepresentation of the one the fermions transform under. Indeed, as compared to the fermions, there is an extra condition on the supercharges, namely, the requirement that they must commute with Σ_+. This leaves only four complex independent supercharges, as expected for an $su(2|2)$ algebra.

In the context of the $su(2) \oplus su(2)$ algebra we use Latin indices for the AdS part and Greek indices for the $\mathbb{CP}^3$ part. The generators of su(2|2) can be conveniently described by two traceless bosonic operator-valued matrices R_α^β and L_a^b, and an operator-valued (complex) fermionic matrix $\mathcal{Q}_\alpha^a$. It should thus be clear that L and R describe AdS and $\mathbb{CP}^3$ rotations, respectively. With respect to the Poisson bracket, the entries of these matrices form the following Lie superalgebra:

$$\begin{aligned}
[\mathrm{R}_\alpha^\beta, \mathrm{R}_\gamma^\delta] &= \delta_\alpha^\delta \mathrm{R}_\gamma^\beta - \delta_\gamma^\beta \mathrm{R}_\alpha^\delta \\
[\mathrm{L}_a^b, \mathrm{L}_c^d] &= \delta_a^d \mathrm{L}_c^b - \delta_c^b \mathrm{L}_a^d \\
\{\mathcal{Q}_\alpha^a, \bar{\mathcal{Q}}_b^\beta\} &= \delta_\alpha^\beta \mathrm{L}_b^a - \delta_b^a \mathrm{R}_\alpha^\beta + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathrm{H} \\
\{\mathcal{Q}_\alpha^a, \mathcal{Q}_\beta^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathrm{P}_1
\end{aligned} \qquad (3.26)$$

$$\{\bar{\mathcal{Q}}_a^\alpha, \bar{\mathcal{Q}}_b^\beta\} = \epsilon_{ab}\epsilon^{\alpha\beta}\mathrm{P}_2.$$

Obviously, $P_2 = \bar{P}_1$. Besides, one might check that the bosonic part (the first two lines) is precisely su(2) $\oplus$ su(2) if one makes the following identifications: $R_1^1 = \frac{1}{2}\sigma_3,\ R_1^2 = \sigma_-,\ R_2^1 = \sigma_+$.

Using Noether's theorem, one can find the matrix of supercharges, that is, a divergence-free vector field with values in the Lie algebra $osp(6|2,2)$:

$$J^\alpha = g\left(\gamma^{\alpha\beta}A_\beta^{(2)} + \frac{\kappa}{2}\epsilon^{\alpha\beta}(A_\beta^{(3)} - A_\beta^{(1)})\right)g^{-1} \tag{3.27}$$

Here, as usual, the upper indices in brackets denote the corresponding component of the current A_α under the Z_4 grading. The components of J_α under the decomposition over the Lie algebra basis are conserved currents corresponding to various charges, both bosonic and fermionic.

We proceed by imposing the light-cone gauge. To do that, we will use the first-order formalism, as described in [50]. This is not necessary, but simplifies the calculations. Thus, we rewrite the Lagrangian in the following form:

$$\frac{2\pi}{\sqrt{\lambda}}\mathcal{L} = \frac{1}{2\gamma^{00}}\,\mathrm{Str}((\mathcal{P}_0)^2) - \mathrm{Str}(\mathcal{P}_0^{(2)}(A_0 + \frac{\gamma^{01}}{\gamma^{00}}A_1)) + \frac{1}{2\gamma^{00}}\,\mathrm{Str}((A_1^{(2)})^2) - \frac{\kappa}{2}\epsilon^{\alpha\beta}\,\mathrm{Str}(A_\alpha^{(1)}A_\beta^{(3)}). \tag{3.28}$$

In the first term we could have written $\mathcal{P}_0^{(2)}$, but all other terms decouple anyway, so they may only contribute to the normalization of the path integral, which is irrelevant so far. In fact, the physical meaning of $\mathcal{P}_0$ is that it provides for a decomposition of the momentum over the local (super)vielbein (at least when the Wess-Zumino term is neglected). Indeed, denoting by X_μ the set of all possible fields, $A_0^{(2)} \equiv -E_\mu^a\dot{X}_\mu T_a$, so, if one neglects the Wess-Zumino term, $p_{X_\mu} = E_\mu^a\,\mathrm{Str}(\mathcal{P}_0T_a) = E_\mu^a\mathcal{P}_{0,a}$. Thus, in this way we effectively avoid the complicated contributions to the explicit expressions of momenta, which come from the vielbein. The possibility of dropping the Wess-Zumino term in our case is justified by the fact that it does not contribute to any variables entering the algebra in the leading order. Indeed, for the calculation of the algebra we need the term in the supercharges, linear in the fermions, and the term in the bosonic charges, quadratic in the bosons, whereas the Wess-Zumino term is (at least) quadratic in the fermions.

From (3.28) one immediately reads off the Virasoro conditions:

$$V_1 \equiv \mathrm{Str}(\mathcal{P}_0^{(2)} A_1) = 0, \tag{3.29}$$

$$V_2 \equiv \mathrm{Str}((\mathcal{P}_0^{(2)})^2 + (A_1^2)^2) = 0 \tag{3.30}$$

It is important to note that, once the kappa-gauge has been imposed, the action of supersymmetry transformations on physical fields is given by $g \to e^{\epsilon} g e^{\widetilde{\epsilon}}$, where $\widetilde{\epsilon}$ is a compensating kappa transformation (and it is uniquely determined by ϵ). This is in contrast to the action $g \to e^{\epsilon} g$, which (as described above) one has before imposing the kappa-symmetry gauge. This is not special to the case under consideration, but rather is a general property of superstring theories — for instance, it is also present in the flat case [60]. This is very similar, for instance, to the Wess-Zumino gauge in supersymmetric theories: it manifestly breaks supersymmetry, but there is a symmetry of the gauge-fixed action, which is a combination of the supersymmetry transformation and a gauge transformation [107].

In order to calculate the central elements $\mathrm{P}_1, \mathrm{P}_2$ entering the algebra (3.26) we need to know the explicit expressions of the various charges entering the algebra in terms of the canonical fields on the worldsheet. The next three subsections are dedicated to writing out these expressions. The notations that will be used are explained in the Appendix 3.1.

3.5.1 Fermionic charges

The fermionic charges look the following way, when written in a manifestly covariant form:

$$
\begin{aligned}
\mathcal{Q}^a_\alpha &= \frac{i}{4}\int d\sigma\, e^{-i\frac{x_-}{2}}\,\big(2p_y\,\chi^a_\alpha + 2\epsilon^{ab}(Z^*)_b^c(\epsilon_{\alpha\beta}\bar{\chi}^\beta_c + i\epsilon_{cd}\chi'^d_\alpha) - \\
&- 2i\epsilon^{ab}(P^*_z)_b^c\epsilon_{\alpha\beta}\bar{\chi}^\beta_c - i\epsilon_{\alpha\beta}\bar{w}^\beta(\kappa^{a,+1} - 2i(\bar{\kappa}')^{a,-1}) - i\epsilon^{ab}w_\alpha(\kappa^{-1}_b - 2i(\bar{\kappa}')^{+1}_b) + \\
&+ 2\epsilon^{ab}P_{w,\alpha}\kappa^{-1}_b + 2\epsilon_{\alpha\beta}\bar{P}^\beta_w\kappa^{a,+1} - 2i\,y\,(\chi^a_\alpha + i\epsilon^{ab}\epsilon_{\alpha\beta}(\bar{\chi}')^\beta_b)\big) \\
\bar{\mathcal{Q}}^\alpha_a &= -\frac{i}{4}\int d\sigma\, e^{i\frac{x_-}{2}}\,\big(2p_y\,\bar{\chi}^\alpha_a + 2\epsilon_{ab}(Z)^b_{\ c}(\epsilon^{\alpha\beta}\chi^c_\beta - i\epsilon^{cd}(\bar{\chi}')^\alpha_d) + \\
&+ 2i\epsilon_{ab}(P_z)^b_{\ c}\epsilon^{\alpha\beta}\chi^c_\beta + i\epsilon^{\alpha\beta}w_\beta(\bar{\kappa}^{+1}_a + 2i(\kappa')^{-1}_a) + i\epsilon_{ab}\bar{w}^\alpha(\bar{\kappa}^{b,-1} + 2i(\kappa')^{b,+1}) + \\
&+ 2\epsilon_{ab}\bar{P}^\alpha_w\bar{\kappa}^{b,-1} + 2\epsilon^{\alpha\beta}P_{w,\beta}\bar{\kappa}^{+1}_a + 2i\,y\,(\bar{\chi}^\alpha_a - i\epsilon_{ab}\epsilon^{\alpha\beta}(\chi')^b_\beta)\big)
\end{aligned} \tag{3.31}
$$

One can see that these charges are complex conjugate. They would be hermitian conjugate with respect to the Hilbert space scalar product in quantum theory.

3.5.2 Bosonic charges

Once written in covariant notation, the part of the bosonic charges quadratic in bosons looks as follows:

$$L^b_a = \frac{i}{4}\int d\sigma\ \left((P_z)_a^{\ c}Z_c^{\ b} - Z_a^{\ c}(P_z)_c^{\ b}\right) \tag{3.32}$$

$$R^b_a = \frac{i}{4}\int d\sigma\ \left(\bar{w}_b p_{w_a} - \bar{p}_{w_b}w_a + \frac{1}{2}\delta_{ab}\sum_{i=1}^{2}(w_i\bar{p}_{w_i} - \bar{w}_i p_{w_i})\right) \tag{3.33}$$

$$
\begin{aligned}
H &= \frac{1}{2}\int d\sigma\ \Big(\frac{1}{2}\mathrm{Tr}\,(P_z^2 + Z'^2 + Z^2) + p_y^2 + y'^2 + y^2 + \\
&+ \sum_{i=1}^{2}(p_{w_i}\bar{p}_{w_i} + w_i'\bar{w}_i' + \frac{1}{4}w_i\bar{w}_i)\Big)
\end{aligned} \tag{3.34}
$$

The $U(1)$ charge is

$$U = \frac{i}{2}\int d\sigma\ (\bar{w}_1 p_{w_1} + \bar{w}_2 p_{w_2} - w_1\bar{p}_{w_1} - w_2\bar{p}_{w_2}) \tag{3.35}$$

The worldsheet momentum is

$$
\begin{aligned}
p_{ws} \equiv p &= \int d\sigma\, x'_- = -\int d\sigma \left(\frac{1}{2}\mathrm{Tr}\,(P_z Z') + \frac{1}{2}\sum_{i=1}^{2}(p_{w_i}\bar{w}'_i + \bar{p}_{w_i} w'_i) + p_y y' \right.\\
&\left. - \; i\bar{\chi}^{\alpha}_{a}\chi^{a'}_{\alpha} - i\bar{\kappa}^{+1}_{a}\kappa^{+1'}_{a} - i\bar{\kappa}^{-1}_{a}\kappa^{-1'}_{a}\right)
\end{aligned} \tag{3.36}
$$

The following comments regarding the expressions for the charges are in order:

- One may notice that the expressions written above are quadratic in the fields. This is a consequence of the fact that we are sticking to the first nontrivial order of perturbation theory in the worldsheet coupling $1/g$ (g is supposed to be large). Indeed, g is the quantity that entered the original Lagrangian only is a factor in front of it. Therefore, after imposing the light-cone gauge we have rescaled the "transverse" fields (the ones orthogonal to the light-cone directions) as $x \to \frac{1}{\sqrt{g}}x$ to get rid of the g factor in front of the Lagrangian. Having done this we get canonical normalization of the fields, i.e. the quadratic part of the Lagrangian does not depend on g anymore. However, this also means that inverse powers of g enter the interaction vertices, for example the x^4 interaction is accompanied with a factor of $1/g$. Such interaction vertices have been omitted in the formulas written above.

- There is an important qualification to what we have just said. Namely, one of the Virasoro conditions has schematically the following form before the light-cone gauge has been imposed: $p_- x'_+ + p_+ x'_- + \sum_{i=1}^{8} p_i x'_i = 0$. The light-cone gauge corresponds to setting $x_+ = \tau, p_+ = 1$, therefore after rescaling $x_i \to \frac{1}{\sqrt{g}}x_i$ the Virasoro condition takes the form $x'_- + \frac{1}{g}\sum_{i=1}^{8} p_i x'_i = 0$. This means that the variable x_-[2], which

[2]Or, equivalently, the worldsheet momentum $p = \int d\sigma\, x'_-$

also enters our formulas (3.31), is of order $\frac{1}{g}$ in perturbation theory. However, in order to perform the calculation we treat x_- as a rigid object, i.e. we do not expand the exponents $e^{\pm i\frac{x_-}{2}}$ in powers of $1/g$. This is the essence of the so-called "hybrid" expansion of ref. [15] and it corresponds to treating the worldsheet momentum p as fixed (independent of g) in the classical limit $g \to \infty$.

3.5.3 Poisson brackets

The Poisson structure can be read off, for example, from the expression for p (3.36). We obtain:

$$\begin{gathered}\{Z_a{}^b, (P_z)_c{}^d\}_P = 2\delta_a^d\delta_c^b - \delta_a^b\delta_c^d \qquad (3.37)\\ \{w_\alpha, \bar{p}_{w_\beta}\}_P = 2\delta_{\alpha\beta},\ \{\bar{w}_\alpha, p_{w_\beta}\}_P = 2\delta_{\alpha\beta},\\ \{\chi_a^\alpha, i\bar{\chi}_\beta^b\}_P = \delta_a^b\delta_\beta^\alpha,\ \{\kappa_a^{+1}, i\bar{\kappa}_b^{+1}\}_P = \delta_{ab},\ \{\kappa_a^{-1}, i\bar{\kappa}_b^{-1}\}_P = \delta_{ab},\end{gathered}$$

all other brackets being zero.

In terms of the components of $Z \equiv z_i\sigma_i$ and $P_z \equiv P_{z_i}\sigma_i$ one can express the Poisson bracket of the z_i with p_{z_i} in the canonical form:

$$\{z_i, p_{z_j}\}_P = \delta_{ij},\ \{y, p_y\}_P = 1. \qquad (3.38)$$

Please note the convention of the Poisson bracket for complex fields. It is not canonical, strictly speaking, but it has been chosen in such a way that, once we write out the complex fields in terms of the real components as $w = a + ib$ and $p_w = p_a + ip_b$, then a, b, p_a, p_b have canonical brackets $\{a, p_a\} = \{b, p_b\} = 1$, $\{a, b\} = \{p_a, p_b\} = 0$. This makes it easy, for instance, to check the masses of the corresponding fields, once we plug these decompositions into the Hamiltonian.

To perform the calculation of the Poisson bracket we extensively use the formulas obtained above. A straightforward calculation gives the following result for the central extensions entering formulas (3.26):

$$P_1 = -\frac{i}{2}\int d\sigma\, e^{-ix_-}\, x'_- = \frac{1}{2}e^{-ix_-(-\infty)}(e^{-ip}-1) = \frac{\xi}{2}(e^{-ip}-1), \quad (3.39)$$

$$P_2 = \frac{i}{2}\int d\sigma\, e^{ix_-}\, x'_- = \frac{1}{2}e^{ix_-(-\infty)}(e^{ip}-1) = \frac{\xi^*}{2}(e^{ip}-1), \quad (3.40)$$

where we have introduced $\xi \equiv e^{-ix_-(-\infty)}$.

It is interesting to mention, that this sort of algebra may well be called worldsheet supersymmetry algebra, since it includes worldsheet charges p and H, as well as the supersymmetry generators and other target-space charges. In fact, the only difference from the usual supersymmetry algebra is the fact that p and H are central. However, in the ordinary supersymmetry algebra, once one omits the Lorentz generators, the corresponding energy-momentum becomes central, too. Of course, there is no Lorentz algebra in the light-cone worldsheet theory, since it is not Lorentz-invariant. Even if it were, Lorentz symmetry is quite simple in two dimensions. Nevertherless, in this case there is a much more interesting counterpart, namely the $SL(2)$ group of outer automorphisms of the algebra [26]. It acts as the three-dimensional rotation group (or Lorentz group) [72]. These automorphisms are outer, since they do not preserve the reality properties of the fermionic charges.

3.6 Conclusion

In the first part of this Chapter we proposed a kappa-symmetry gauge, compatible with the bosonic $su(2) \oplus su(2) \oplus u(1)$ symmetries. The second part was devoted to the classical calculation of the central extension to the supersymmetry algebra in the framework of the so-called hybrid expansion. Calculation of the corresponding Poisson brackets between the supercharges led to the same result, as had been previously obtained for the $\mathrm{AdS}_5 \times S^5$ case. As a slight deviation from the main line of the text, in the appendix we present a general scheme for the analysis of geodesics in $\mathbb{CP}^3$ (which is of course also suitable for any other symmetric space).

Chapter 4

Spectrum of a string rotating in $\mathrm{AdS}_4 \times \mathbb{CP}^3$

4.1 Introduction and summary

An important tool to investigate the question about quantum integrability is provided by semiclassical quantization of rigid string solutions [55, 56]. Starting from a simple classical string configuration, one finds the spectrum of fluctuations around it. Summing up the fluctuation energies gives the one-loop correction to the classical energy of the spinning string which can be then compared to the value predicted by the Bethe ansatz. A particularly convenient $\mathrm{AdS}_5 \times \mathrm{S}^5$ solution allowing for an explicit evaluation of the one-loop energy correction is given by a rigid folded string carrying Lorentz spin S along AdS_5. In the long string approximation the corresponding correction to the energy scales logarithmically with S and is found [55] to be

$$\delta E = -\frac{3\log 2}{\pi}\log S\,.$$

On the gauge theory side, this string solution corresponds to twist two operators with large Lorentz spin S, for which the difference between the scaling dimension Δ and spin S behaves as

$$\Delta - S = f(\lambda)\log S\,,$$

where the function $f(\lambda)$ is the universal scaling function, also known as the cusp anomaly [68]. The quantity δE provides the first correction to the strong coupling value of the cusp anomaly. The one-loop correction to the long (S, J)-string, which in addition to the Lorentz spin S also carries angular momentum J along a big circle of five-sphere, has been obtained in [54] and it provides the leading strong coupling correction to the so-called generalized scaling function $f(\lambda, J/\log S)$ [48].

The purpose of the present Chapter is to perform a similar semiclassical quantization of a rigid string spinning in the $AdS_4 \times \mathbb{CP}^3$ space-time and obtain the corresponding one-loop energy shift. Namely, we consider a rigid folded string with Lorentz spin S and angular momentum J along a circle $S^1 \subset \mathbb{CP}^3$. The gauge theory operators dual to this string solution are made of two bi-fundamental scalars with S light-cone derivatives distributed among them, and they transform in the irrep $[J, 0, J]$ of $\mathfrak{su}(4)$. By finding the fluctuation spectrum around the classical solution in the long string approximation, we obtain the corresponding one-loop energy shift as a function of S and J. In particular, in the limit of "slow" rotation, $J \ll \log S$, we find that the corresponding one-loop correction is given by

$$\delta E = -\frac{5\log 2}{2\pi}\log S\,.$$

Apparently, this result appears to be in contradiction with the one conjectured in [65]. According to [65], the energy correction should be half of that for the corresponding string solution on $AdS_5 \times S^5$, *i.e.* it should be equal to $-\frac{3\log 2}{2\pi}\log S$. The conjecture of [65] was based on the assumption that an unknown function $h(\lambda)$ entering the all-loop Bethe ansatz has a vanishing subleading (constant) term at strong coupling. Provided we adopt the same definition of the cusp anomaly, we see that it is not the case. Clearly, further investigations are needed to clarify this important issue.

We would like to stress that our computation is a genuine field-theoretic one and it does not rely on the knowledge of the algebraic curve. It is done exclusively in the framework of the coset sigma model. As observed in [11], strings which carry only AdS spin provide an example of a singular string background, because the corresponding κ-symmetry transformations instead of generic rank 8 have higher rank equal to 12. Thus, to properly treat the fluctuation spectrum around this singular solution, we keep throughout the calculations a non-vanishing angular momentum J which can be then regarded as the regularization parameter. We find that the resulting expression for the one-loop energy shift admits a smooth limit $J \to 0$, which allows us to obtain the above stated result for the cusp anomalous dimension of high spin operators. Finally, we notice that in the limit $J \ll \log S$ the fluctuation spectrum contains 6 massless bosons and 2 massless fermions. Thus, in opposite to what happens in the $AdS_5 \times S^5$ case [6] where only 5 bosonic massless excitations are present, a would be "quantum bosonic $\mathbb{CP}^3$ model" is not splitting off in this limit.

The Chapter is organized as follows. In the next section we discuss the coset sigma model which captures the physics of type IIA strings on $AdS_4 \times \mathbb{CP}^3$. In section 3 we present the (S, J)-solution in terms of a coset element. Section 4 is devoted to the analysis of the fluctuation spectrum around the (S, J)-solution. Finally, in section 5 we compute the corresponding one-loop energy shift. Appendix A contains the details on the description of the coset space $AdS_4 \times \mathbb{CP}^3$. In appendix B we provide a detailed treatment of κ-symmetry transformations around the (S, J)-solution.

4.2 String action

The general strategy for the construction of the Green-Schwarz action was outlined in Section 1.5.2. To proceed, one has to choose an explicit parametrization of the coset element g. We will pick up

$$g = g_{\mathrm{O}} g_{\chi} g_{\mathrm{B}}, \tag{4.1}$$

where $g = e^{\chi}$ depends on the odd matrix χ comprising the 24 fermionic degrees of freedom of the model. The element g_{O} can be chosen in different ways depending on which commuting isometries we would like to be realized linearly. For instance, one can take[1]

$$g_{\mathrm{O}} = \begin{pmatrix} e^{\frac{i}{2}t\Gamma_0} & 0 \\ 0 & e^{-\frac{\varphi}{2}(T_{34}+T_{56})} \end{pmatrix}, \tag{4.2}$$

where t and φ are the global AdS time and one of the angles of $\mathbb{CP}^3$, respectively[2]. Since the global symmetry group $\mathrm{OSP}(2,2|4)$ acts on g from the left the isometries corresponding to constants shifts of t and ϕ will be realized linearly and they do not act on the fermionic variables, *i.e.*, fermions are unchanged under the corresponding $\mathrm{U}(1)$ transformations. We note that such a parametrization will be suitable for imposition of the uniform light-cone gauge [9].

Finally, the element g_{B} comprises all the coordinates parametrizing AdS_4 and $\mathbb{CP}^3$ except those which parametrize the element g_{O}. Explicitly,

$$g_{\mathrm{B}} = \begin{pmatrix} g_{\mathrm{AdS}} & 0 \\ 0 & g_{\mathbb{CP}} \end{pmatrix}. \tag{4.3}$$

[1] For the definition of gamma-matrices, the $\mathfrak{so}(6)$ Lie algebra generators T_{ij}, the matrices C_4, K_4, K, Υ and T_i appearing throughout the Chapter see appendix A, and [11].

[2] In particular, the algebra $\mathfrak{so}(3,2)$ has rank two, so that one can choose the diagonal matrices $\frac{i}{2}\Gamma_0$ and $\frac{1}{4}[\Gamma_1,\Gamma_2]$ as the generators of commuting isometries.

The matrix $g_{\rm AdS}$ have the following characteristic properties

$$g^t_{\rm AdS} C_4 g_{\rm AdS} C_4^{-1} = \mathbb{I}\,, \quad K_4 g_{\rm AdS} = g^t_{\rm AdS} K_4\,, \qquad \Gamma_5 g_{\rm AdS} = g^{-1}_{\rm AdS}\Gamma_5\,,$$

Analogously, $g_{\mathbb{CP}}$ obeys the following requirements

$$g^t_{\mathbb{CP}} g_{\mathbb{CP}} = \mathbb{I}\,, \quad K_6 g_{\mathbb{CP}} = g^t_{\mathbb{CP}} K_6\,.$$

As the consequence, the element $g_{\rm B}$ satisfies the following identity

$$\Upsilon g_{\rm B} \Upsilon^{-1} = g_{\rm B}^{-1}\,, \tag{4.4}$$

where Υ defines an inner automorphism Ω of the complexified algebra $\mathfrak{osp}(6|4)$. It is worth to point out that the matrix $g_{\rm o}$ is orthosymplectic but it does not obey eq.(4.4) satisfied by the element $g_{\rm B}$.

As was explained in [5], a convenient and compact representation of the sigma model Lagrangian can be constructed in terms of the following matrix G

$$G = \begin{pmatrix} g_{\rm AdS} K_4 g^t_{\rm AdS} & 0 \\ 0 & g_{\mathbb{CP}} K_6 g^t_{\mathbb{CP}} \end{pmatrix} = g_{\rm B} K g^t_{\rm B}\,. \tag{4.5}$$

By construction, this matrix is skew-symmetric: $G^t = -G$. Introducing the split

$$g_\chi^{-1} g_{\rm o}^{-1} d(g_{\rm o} g_\chi) = {\rm F} + {\rm B}\,, \tag{4.6}$$

where F and B are odd and even superalgebra elements, respectively, the Lagrangian in the action (1.84) can be cast in the form [5, 11]

$$\begin{aligned} \mathscr{L} &= \frac{1}{4}{\rm str}\Big[\gamma^{\alpha\beta}(\partial_\alpha G G^{-1}\partial_\beta G G^{-1} + 4{\rm B}_\alpha \partial_\beta {\rm G}{\rm G}^{-1} + 2{\rm B}_\alpha {\rm B}_\beta + 2{\rm B}_\alpha {\rm G}{\rm B}^t_\beta {\rm G}^{-1}) \\ &+ 2i\kappa\,\epsilon^{\alpha\beta}{\rm F}_\alpha G {\rm F}^{st}_\beta G^{-1}\Big]\,. \end{aligned} \tag{4.7}$$

The Lagrangian (4.7) provides a convenient starting point for studying the fluctuation spectrum around classical solutions of the string sigma model.

4.3 The (S, J)-string

We choose as the background solution string spinning in the directions ϕ and φ of AdS_4 and $\mathbb{CP}^3$ spaces, respectively. This naturally suggests to pick up as g_o the following matrix

$$g_o = \begin{pmatrix} e^{\frac{i}{2}t\Gamma_0 - \frac{1}{4}\phi[\Gamma_1,\Gamma_2]} & 0 \\ 0 & e^{-\frac{\varphi}{2}(T_{34}+T_{56})} \end{pmatrix} . \tag{4.8}$$

Then, the AdS part of the element g_B can be chosen as follows

$$g_{AdS} = e^{\frac{i}{2}\rho\sin\psi\,\Gamma_3 - \frac{i}{2}\rho\cos\psi\,\Gamma_2} . \tag{4.9}$$

Hence, in addition to the global time t, the space AdS_4 is parametrized by the non-negative variable ρ and by two angles, ϕ and ψ. As to g_{CP}, since we distinguish the angle ϕ, it is convenient to choose the remaining five coordinates on $\mathbb{CP}^3$ in the same way as was done in [11], namely, we parametrize g_{CP} by one real coordinate x_4 and by two complex variables y_1 and y_2, see [11] for details. In order to keep the present discussion clear, we refer the reader to appendix A for the full details concerning the parametrization of g_{CP}.

The background solution corresponding to the (S, J)-string can be now obtained by putting to zero the AdS angle ψ together with the $\mathbb{CP}^3$ coordinates x_4 and y_1, y_2, and picking up the rotating string ansatz for the remaining variables

$$t = \varkappa\tau , \qquad \phi = \omega_1\tau , \qquad \varphi = \omega_2\tau , \qquad \rho \equiv \rho(\sigma) . \tag{4.10}$$

Of course, the spinning string ansatz is embedded in the subspace $AdS_3 \times S^1$ of $AdS_4 \times \mathbb{CP}^3$, and, for this reason, the corresponding solution must coincide with the one obtained in [55]. We see that for the rotating ansatz the components of g_B reduce to

$$g_{AdS} = e^{-\frac{i}{2}\rho\Gamma_2} , \qquad g_{\mathbb{CP}} = e^{\frac{\pi}{4}T_5} \tag{4.11}$$

and, as the consequence the coset element underlying the (S, J)-string solution is of the form

$$g = \begin{pmatrix} e^{\frac{i}{2}\varkappa\tau\Gamma_0 - \frac{1}{2}\omega_1\tau\Gamma_1\Gamma_2} e^{-\frac{i}{2}\rho\Gamma_2} & 0 \\ 0 & e^{-\frac{1}{2}\omega_2\tau(T_{34}+T_{56})} e^{\frac{\pi}{4}T_5} \end{pmatrix} . \tag{4.12}$$

In the next section we will use this representation to find the Lagrangian for fluctuation modes.

Finally, we note that the parameters of the solution $(\varkappa, \omega_1, \omega_2)$ are related to the Noether charges of the model which are the space-time energy E, the AdS spin S, and the $\mathbb{CP}^3$ spin J as follows

$$E = \sqrt{\lambda}\,\varkappa \int \tfrac{d\sigma}{2\pi} \cosh^2\rho\,, \;\; S = \sqrt{\lambda}\,\omega_1 \int \tfrac{d\sigma}{2\pi} \sinh^2\rho\,, \;\; J = \sqrt{\lambda}\,\omega_2\,. \tag{4.13}$$

They are, of course, the same as for the (S, J)-string spinning in $\mathrm{AdS}_3 \times \mathrm{S}^1$. Here the parameter λ is related to the AdS radius[3] as $\sqrt{\lambda} = \frac{R^2}{\alpha'}$.

In this Chapter we are mostly interested in the so-called long string limit corresponding to $\omega_1, \omega_2 \to \infty$ with the ratio $u = \frac{\omega_2}{\omega_1} = \frac{1}{\sqrt{1+x^2}}$ fixed. In this limit,

$$\kappa \approx \omega_1 \quad \text{and} \quad x = \frac{\sqrt{\lambda}}{\pi J} \ln \frac{S}{J} \quad \text{fixed}\,. \tag{4.14}$$

The energy of the long string is then

$$E = S + J\sqrt{1+x^2} + \ldots \tag{4.15}$$

and it can be further approximated by assuming the "fast" or "slow" limits which correspond to taking $x \ll 1$ or $x \gg 1$, respectively [54].

Let us make some comments regarding the formula (4.15), which will hopefully clarify the essence of what we will be doing next. The formula (4.15) is a classical result, i.e. the one valid for $\lambda \to \infty$ when the ratios $\frac{J}{\sqrt{\lambda}}, \frac{S}{\sqrt{\lambda}}$

[3]Note that λ is related to the gauge theory parameters k and N as $\lambda = 2\pi^2 N/k$.

are kept fixed (this is the typical semiclassical situation). In other words, all terms in this formula are of order $\sqrt{\lambda}$. One can calculate quantum (from the point of view of the sigma model, i.e. $\frac{1}{\sqrt{\lambda}}$) corrections to this result. In the next section we focus on the calculation of the first quantum correction, which is of order one and which is the one-loop correction. This can be done in the following way. One may expand the action around the given classical solution up two quadratic order in the fluctuations and calculate the determinant of the quadratic form which has emerged. Treating in this way a generic classical solution is very complicated, since one would need to calculate the determinant of a second-order differential operator which depends nontrivially on the worldsheet coordinates (through the dependence on the background solution on these coordinates). However, in the long-string limit which we have just described, this dependence disappears and one can easily calculate the determinant. In this case the coefficients of the quadratic form are constant and, therefore, the problem is simply a problem of diagonalizing a Hamiltonian of a system of harmonic oscillators. Calculating the one-loop correction to the energy is the same as calculating the ground state energy of this system of oscillators, i.e. $\sum_{i,n}(-1)^{q_i}\frac{\omega_{i,n}}{2}$, where $q_i = \pm$, depending on whether a given oscillator is bosonic or fermionic. Here the index i refers to the oscillator type (or the field in the Lagrangian that it corresponds to), and n is the string Fourier mode. There are infinitely many Fourier modes, so in a generic situation one would get an infinite answer, however in our case the answer turns out to be finite due to a cancellation between the bosons and the fermions of the quadratic and logarithmic divergences.

4.4 Lagrangian for quadratic fluctuations

4.4.1 Spectrum of bosonic fluctuations

The Lagrangian for the quadratic fluctuations follows straightforwardly from the bosonic part of the action (4.7). In the conformal gauge we find

$$\begin{aligned}\mathscr{L}_B^{(2)} = & -\cosh^2\rho\,\partial_\alpha\tilde{t}\partial^\alpha\tilde{t} + \sinh^2\rho\,\partial_\alpha\tilde{\phi}\partial^\alpha\tilde{\phi} + 2\sinh 2\rho\,\tilde{\rho}(\varkappa\partial_\tau\tilde{t} - \omega_1\partial_\tau\tilde{\phi}) + \partial_\alpha\tilde{\rho}\partial^\alpha\tilde{\rho} + \\ & +(\varkappa^2 - \omega_1^2)\cosh 2\rho\,\tilde{\rho}^2 + \sinh^2\rho\,(\partial_\alpha\psi\partial^\alpha\psi + \omega_1^2\psi^2) + \partial_\alpha\tilde{\varphi}\partial^\alpha\tilde{\varphi} \\ & +\partial_\alpha x\partial^\alpha x + \omega_2^2 x^2 + \partial_\alpha v_r\partial^\alpha\bar{v}_r + \frac{\omega_2^2}{4}v_r\bar{v}_r\,.\end{aligned}$$

We see that the part of the action for the AdS_4 fields $\tilde{t}, \tilde{\rho}, \tilde{\phi}, \psi$ and the angular $\mathbb{CP}^3$ variable $\tilde{\varphi}$, shown in the first two lines, exactly agree with those in equation (5.10) of [55]. In addition we have five $\mathbb{CP}^3$ fields, x and two complex fields v_r. Furthermore, the linearized Virasoro constraints read

$$\frac{1}{2}(\omega_1^2 - \varkappa^2)\sinh 2\rho\tilde{\rho} - \varkappa\cosh^2\rho\partial_\tau\tilde{t} + \omega_2\partial_\tau\tilde{\varphi} + \omega_1\sinh^2\rho\partial_\tau\tilde{\phi} + \rho'\,\partial_\sigma\tilde{\rho} \approx 0\,, \tag{4.16}$$

$$\rho'\,\partial_\tau\tilde{\rho} - \varkappa\cosh^2\rho\partial_\sigma\tilde{t} + \omega_2\partial_\sigma\tilde{\varphi} + \omega_1\sinh^2\rho\partial_\sigma\tilde{\phi} \approx 0\,. \tag{4.17}$$

Obviously, the linearized Virasoro constraints are identical to equations (5.11) and (5.12) of [55]. We should also add that, although in principle there are also fermionic contributions to the Virasoro constraints, they can be dropped at this order, since they are suppressed by $\frac{1}{\lambda^{1/4}}$. Therefore they give nontrivial contributions only at the two-loop level and beyond.

The physical fields from $\mathbb{CP}^3$ decouple completely from the rest. As it can be seen from the Lagrangian and also as shown in [11], these are five massive fields. In units of ω_2, one of these fields have mass $m = 1$ and the other four $m = 1/2$.

As for the other fields, in [54] it was shown how to compute the spectrum, in the long string limit, around the solution we are interested it. It turns out that from the AdS_4 fields and $\tilde{\varphi}$ we get three physical fields. Let us elaborate on why this happens. First, one can use the Virasoro conditions (4.16)-(4.17) to eliminate one of the fields, for instance $\tilde{\rho}$, i.e. we can express the derivatives $\partial_{\sigma,\tau}\tilde{\rho}$ from these constraints and plug them into the Lagrangian. Once this has been done, one can calculate the spectrum of the nine remaining excitations, and the statement is that one more of these excitations will always be massless. This is the field which we consider as "unphysical", meaning that its contribution is actually cancelled by a contribution of a massless ghost particle, present in the conformal gauge. There is another way to look at this situation, namely to impose a light-cone gauge along a light-cone direction, which lies completely in AdS (this is different from the light-cone gauge we used in the previous chapter, since there the 'time' was take from AdS and the space coordinate was taken from $\mathbb{CP}^3$). In this case the extra massless particle is simply absent in the spectrum, which confirms the correctness of our interpretation. One is ψ, with mass $m_\psi^2 = 2\varkappa^2 - \omega_2^2$, while the other two modes have frequencies

$$\Omega^B_{\pm n} = \sqrt{n^2 + 2\varkappa^2 \pm 2\sqrt{\varkappa^4 + n^2\omega_2^2}}, \qquad n = 0, \pm 1, \pm 2, ... \tag{4.18}$$

In the long string approximation $\varkappa \approx \omega_1$. Notice that in the limit $\omega_1 \gg \omega_2$, we get one field with mass (in units of ω_1) $m^2 = 4$, one field with mass $m^2 = 2$ and six massless fields, as opposed to the situation in $\mathrm{AdS}_5 \times \mathrm{S}^5$, where we get two fields of $m^2 = 2$ and five massless fields. It is these five massless fields which give rise to the O(6) sigma model in this special limit [6]. As we will see in the next section, in the present case we will also find two massless fermions in this limit. Hence the situation is pretty different to what happens in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

4.4.2 Spectrum of fermionic fluctuations

Here we will work out the spectrum of fermionic fluctuations around the (S, J)-string. The relevant part of the Lagrangian (4.7) contributing the quadratic action for fermions is

$$\mathscr{L}_F^{(2)} = \mathrm{str}\Big[\frac{1}{2}\gamma^{\alpha\beta}\mathrm{B}_\alpha(\mathrm{B}_\beta + G\mathrm{B}_\beta^{\mathrm{t}}\mathrm{G}^{-1}) + \gamma^{\alpha\beta}\mathrm{B}_\alpha\partial_\beta \mathrm{G}\mathrm{G}^{-1} + \frac{\mathrm{i}}{2}\kappa\,\epsilon^{\alpha\beta}\,\mathrm{F}_\alpha\,\mathrm{G}\,\mathrm{F}_\beta^{\mathrm{st}}\mathrm{G}^{-1}\Big]. \tag{4.19}$$

According to the formula (4.6), up to terms quadratic in fermions, we have

$$\mathrm{F} \;=\; D\chi\,, \qquad \mathrm{B} = g_{\mathrm{o}}^{-1}dg_{\mathrm{o}} + \frac{1}{2}(D\chi\chi - \chi D\chi)\,, \tag{4.20}$$

where we have introduced the covariant differential $D\chi = d\chi + [g_{\mathrm{o}}^{-1}dg_{\mathrm{o}}, \chi]$. In the conformal gauge[4] we, therefore, find the following quadratic action

$$\begin{aligned}
\mathscr{L}_F^{(2)} &= \frac{1}{2}\mathrm{str}\Big[-(g_{\mathrm{o}}^{-1}\partial_\tau g_{\mathrm{o}} + G(g_{\mathrm{o}}^{-1}\partial_\tau g_{\mathrm{o}})^t G^{-1})(D_\tau\chi\chi - \chi D_\tau\chi) + \\
&+ \partial_\sigma G G^{-1}(D_\sigma\chi\chi - \chi D_\sigma\chi)\Big] \\
&+ \frac{i}{2}\kappa\,\mathrm{str}\Big[D_\tau\chi G(D_\sigma\chi)^{st}G^{-1} - D_\sigma\chi G(D_\tau\chi)^{st}G^{-1}\Big]\,,
\end{aligned} \tag{4.21}$$

where we made use of the fact that g_{o} and G do not depend on σ and τ, respectively. Explicitly,

$$g_{\mathrm{o}}^{-1}\partial_\tau g_{\mathrm{o}} = \begin{pmatrix} \frac{i}{2}\varkappa\Gamma_0 - \frac{1}{2}\omega_1\Gamma_1\Gamma_2 & 0 \\ 0 & -\frac{1}{2}\omega_2(T_{34}+T_{56}) \end{pmatrix} \tag{4.22}$$

and $\partial_\sigma G G^{-1} = \mathrm{diag}(-i\rho'\Gamma_2, 0)$.

It is clear that, in general, fermion masses will depend non-trivially on the non-constant function $\rho(\sigma)$ and its derivative which enter in the above Lagrangian through the matrix G. However, in the long string limit we are most interested in here, one can approximate $\rho'(\sigma) \approx$ const. Thus, in this limit one can attempt to redefine fermions as

$$\chi \to W\chi W^{-1}\,, \tag{4.23}$$

where the role of W would be to remove the ρ-dependence from G. The matrix $W \in \mathrm{OSP}(2,2|6)$ must satisfy a few natural requirements. First,

[4] We take $\gamma^{\tau\tau} = -1 = -\gamma^{\sigma\sigma}$ and $\gamma^{\tau\sigma} = 0$.

under redefinition (4.23) the covariant differential $D\chi$ undergoes the following transformation

$$D\chi \to W\big(d\chi + [W^{-1}g_o^{-1}dg_oW,\chi] + [W^{-1}dW,\chi]\big)W^{-1}\,. \qquad (4.24)$$

Thus, if we do not want to introduce an extra dependence on ρ, the matrix $W^{-1}dW$ should depend on the derivatives of ρ' only and, when being restricted to its AdS block, it should commute in the long string limit with the corresponding block of $g_o^{-1}\partial_\tau g_o$. The last requirement also guarantees that the kinetic term in eq.(4.28) will not receive an extra ρ-dependence under such redefinition of fermions. Second, W must commute with $\partial_\sigma GG^{-1}$, which is equivalent to the requirement of commuting with Γ_2 (naturally embedded into 10×10-matrices). This will ensure that the term in the Lagrangian containing $\partial_\sigma GG^{-1}$ will not receive new ρ-dependent terms. Finally, W must be capable to remove ρ from G, i.e the element $W^{-1}G(W^t)^{-1}$ should be independent of ρ. The conditions on W stated above can be satisfied *in the long string limit only* and they fix W essentially uniquely. To construct W, we note that in the long string limit $\varkappa \approx w_1$, so that the AdS part of $g_o^{-1}\partial_\tau g_o$ becomes proportional to $i\Gamma_0 - \Gamma_1\Gamma_2$. Thus, we have to find an $\mathfrak{so}(3,2)$ Lie algebra element, such that it commutes with two matrices

$$i\Gamma_0 - \Gamma_1\Gamma_2 \qquad \text{and} \qquad \Gamma_2\,.$$

One can easily see that the corresponding element is given by

$$i\Gamma_2 - \Gamma_0\Gamma_1\,.$$

Here Γ_2 is the Lie algebra coset representative, while $[\Gamma_0,\Gamma_1]$ belongs to the stability subalgebra $\mathfrak{so}(3,1)$. The last observation implies that taking W in the form

$$W = \begin{pmatrix} e^{-\frac{\rho}{2}(i\Gamma_2-\Gamma_0\Gamma_1)} & 0 \\ 0 & e^{\frac{\pi}{4}T_5} \end{pmatrix}, \qquad (4.25)$$

we satisfy all the requirements stated, getting, in particular,

$$W^{-1}G(W^{-1})^t = K\,,$$

where the matrix K is defined by eq.(D.5). Since $e^{-\frac{\pi}{4}T_5}(T_{34}+T_{56})e^{\frac{\pi}{4}T_5} = -T_6$, we see that after redefining the fermions by W, the covariant derivative (4.24) acquires the following form

$$D_\alpha = \partial_\alpha + [Q_\alpha, \ldots],$$

where the composite vector field Q_α has the components

$$Q_\tau = \mathrm{diag}\Big(Q_\tau^{\mathrm{AdS}},\ -\frac{1}{2}\omega_2 T_6\Big)\,, \qquad Q_\sigma = W^{-1}\partial_\sigma W\,, \tag{4.26}$$

where

$$\begin{aligned} Q_\tau^{\mathrm{AdS}} &= \frac{1}{4}(\varkappa+\omega_1)\Big(i\Gamma_0 - \Gamma_1\Gamma_2\Big) + \\ &+ \frac{1}{4}(\varkappa-\omega_1)\Big[\cosh 2\rho\Big(i\Gamma_0+\Gamma_1\Gamma_2\Big) + \sinh 2\rho\Big(i\Gamma_1+\Gamma_0\Gamma_2\Big)\Big]\,. \end{aligned} \tag{4.27}$$

In the long string limit $\varkappa \approx \omega_1$ the function ρ drops out of Q_τ as it should be. Also, by construction, in the long string limit the commutator $[Q_\alpha, Q_\beta]$ vanishes, *i.e.* the connection D_α becomes flat.

We further note that since $\chi \in \mathfrak{osp}(6|4)$ its supertranspose is $\chi^{st} = -C\chi C^{-1}$ and, therefore, after the redefinition of fermions has been done, the action (4.21) can be cast in the form

$$\begin{aligned} \mathscr{L}_F^{(2)} &= \frac{1}{2}\,\mathrm{str}\Big[-(Q_\tau - \Upsilon Q_\tau\Upsilon^{-1})(D_\tau\chi\chi - \chi D_\tau\chi) + \\ &+ \partial_\sigma G G^{-1}(D_\sigma\chi\chi - \chi D_\sigma\chi)\Big] + \\ &+ \frac{i}{2}\kappa\,\mathrm{str}\Big[D_\tau\chi\Upsilon D_\sigma\chi\Upsilon^{-1} - D_\sigma\chi\Upsilon D_\tau\chi\Upsilon^{-1}\Big]\,. \end{aligned} \tag{4.28}$$

Note that the kinetic term for fermions is projected on the space $\mathcal{A}^{(2)}$ as $Q_\tau - \Upsilon Q_\tau \Upsilon^{-1} \in \mathcal{A}^{(2)}$. In particular, in the long string limit

$$Q_\tau - \Upsilon Q_\tau\Upsilon^{-1} \approx i\varkappa\Gamma_0 + \omega_2 T_6\,.$$

One can check that for a generic χ the kinetic term of the Lagrangian (4.28) is degenerate and it has rank 16. This is a manifestation of κ-symmetry which allows one to remove 8 unphysical fermions out of 24 making thereby the kinetic term non-degenerate [11]. As is shown in appendix B, an admissible and convenient κ-symmetry gauge choice is

$$\theta\, T_{56} = 0\,, \tag{4.29}$$

which removes the fermions from the fifth and the sixth column of χ.

Introducing a 4 by 4 matrix ϑ made of non-vanishing entries of θ, we can write the quadratic κ-gauge fixed Lagrangian in the long string approximation as follows

$$\begin{aligned} \mathscr{L}_F^{(2)} &= -\varkappa\,\mathrm{tr}(\vartheta^t\Gamma_2\dot{\vartheta}) + \frac{\varkappa^2}{2}\mathrm{tr}\Big[\vartheta^t C_4(\mathbb{I} + i\Gamma - 0\Gamma_1\Gamma_2)\vartheta\Big] + \frac{\omega_2^2}{4}\mathrm{tr}(\mathbb{I} - \Gamma_0)\vartheta^t C_4\vartheta \\ &- \rho'\mathrm{tr}(\vartheta^t\Gamma_0\vartheta') + \frac{\rho'^2}{2}\mathrm{tr}\Big[\vartheta^t C_4(\mathbb{I} - i\Gamma_0\Gamma_1\Gamma_2)\vartheta\Big] \\ &+ i\kappa\rho'\,\mathrm{tr}(\vartheta^t\Gamma_0\Gamma_5\dot{\vartheta}K_4) + i\kappa\varkappa\,\mathrm{tr}(\vartheta^t\Gamma_2\Gamma_5\vartheta' K_4) + \\ &+ \kappa\varkappa\rho'\,\mathrm{tr}\Big[\vartheta^t(\mathbb{I} - i\Gamma_0\Gamma_1\Gamma_2)\Gamma_5\vartheta K_4\Big]\,. \end{aligned} \tag{4.30}$$

One can check that this action is hermitian provided the fermions satisfy the reality condition $\vartheta^\dagger = i\vartheta^t\Gamma_2$. Introducing the Dirac conjugate $\bar{\vartheta} = \vartheta^\dagger\Gamma_0 = i\vartheta^t\Gamma_0\Gamma_2 = -\vartheta^t C_4$, we recognize that the reality condition is nothing else but the Majorana condition.

To compute the one-loop energy shift, one has first to determine the spectrum of fermion frequencies from the quadratic action (4.30). This is essentially the same as to solve the corresponding equations of motion. Every solution will be characterized by the energy k_0 and the momentum k_1. Then, every zero eigenvalue of the quadratic form defining (4.30) will give a (dispersion) relation between k_0 and k_1, while the corresponding eigenstate will be a solution of the equations of motion. Thus, we may look for the spectrum of the model by requiring that the determinant of the corresponding quadratic form is zero. There are as many particles in the theory as there are linearly independent solutions.

This is precisely the strategy we would like to follow in this Chapter, therefore let us discuss in more detail some subtleties which we encounter on our way. Combining the fermions ϑ in one 16-dimensional vector, the action implied by (4.30) can be schematically written as

$$S = -\frac{R^2}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\,(\vartheta_i K_\tau^{ij}\partial_\tau\vartheta_j + \vartheta_i K_\sigma^{ij}\partial_\sigma\vartheta_j + \vartheta_i M^{ij}\vartheta_j)\,. \tag{4.31}$$

In our treatment we will impose the reality condition on ϑ only at the end of calculation, i.e. we prefer to start with the action above, where in each term we have two ϑ's rather than ϑ and ϑ^*. Varying the action, we get

$$\delta S \;=\; -\frac{R^2}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\,\left(\delta\vartheta_i \widehat{K}_\tau^{ij}\partial_\tau\vartheta_j + \delta\vartheta_i \widehat{K}_\sigma^{ij}\partial_\sigma\vartheta_j + \delta\vartheta_i \widehat{M}^{ij}\vartheta_j\right),$$

where we have used anti-commutativity of fermions and integration by parts. Here

$$\begin{aligned}\widehat{K}_\tau &= K_\tau + K_\tau^t\,,\\ \widehat{K}_\sigma &= K_\sigma + K_\sigma^t\,,\\ \widehat{M} &= M - M^t.\end{aligned}$$

Thus, equations for motion look as

$$(\widehat{K}_\tau\partial_\tau + \widehat{K}_\sigma\partial_\sigma + \widehat{M})\theta = 0. \tag{4.32}$$

In momentum space the equation above yields

$$(i\widehat{K}_\tau\,k_0 + i\widehat{K}_\sigma\,k_1 + \widehat{M})\theta = 0\,. \tag{4.33}$$

As it follows from the discussion above, the spectrum of the model is determined by the condition

$$\mathscr{D} = \det\Big[i\widehat{K}_\tau\,k_0 + i\widehat{K}_\sigma\,k_1 + \widehat{M}\Big] = 0, \tag{4.34}$$

where $\widehat{K}_\tau$ and $\widehat{K}_\sigma$ are symmetric matrices, and $\widehat{M}$ is antisymmetric. We view (4.34) as an algebraic equation for k_0, and its roots (as functions of k_1) give us the dispersion relations for all particles in the theory.

Computing the determinant, we find

$$\begin{aligned} \mathscr{D} &= 2^8\omega_2^{16}\big[(2k_0-\omega_2)^2-4(k_1^2+\varkappa^2)\big]^2\big[(2k_0+\omega_2)^2-4(k_1^2+\varkappa^2)\big]^2\times \\ &\quad\times\big[k_0^4-k_0^2(2k_1^2+\varkappa^2)+k_1^2(k_1^2-\omega_2^2+\varkappa^2)\big]^2 . \end{aligned} \tag{4.35}$$

Setting $k_1 \equiv n \in \mathbb{Z}$, due to the fact that this momentum corresponds to the compact σ direction of the string world-sheet, yields the following result for the fermionic frequencies (counting given in terms of elementary fermions, rather than Majorana sets of fermions):

- 2 fermions with frequency $\frac{\omega_2}{2}+\sqrt{n^2+\varkappa^2}$
- 2 fermions with frequency $-\frac{\omega_2}{2}+\sqrt{n^2+\varkappa^2}$
- 2 fermions with frequency $\sqrt{n^2+\frac{1}{2}\varkappa^2+\frac{1}{2}\sqrt{\varkappa^4+4\omega_2^2n^2}}$
- 2 fermions with frequency $\sqrt{n^2+\frac{1}{2}\varkappa^2-\frac{1}{2}\sqrt{\varkappa^4+4\omega_2^2n^2}}$

plus the other eight fermions whose frequencies are equal to the above with negative sign. The reality condition then implies that these negative frequency fermions are nothing else but the conjugate momenta for the positive frequency ones. The constant shifts by $\pm\omega_2/2$ in the first four frequencies can be removed by an extra time-dependent redefinition of fermions, similar to that done in [11]. The resulting dispersion relation is the same as for relativistic fermions with the mass $m^2=\varkappa^2$. In any case, even without doing this field redefinition, the shifts by $\pm\omega_2/2$ are cancelled out in the one-loop energy correction. In the special limit $\varkappa \approx \omega_1 \gg \omega_2$ the spectrum above will contain two massless fermions.

4.5 One-loop energy shift

Having found the frequency modes of bosons and fermions, we can readily compute the one-loop correction to the energy of the long (S, J)-string. This computation is very similar to that of [54]. The one-loop correction to the energy is given by the following sum

$$\delta E = \frac{1}{\omega_1}\sum_{n=1}^{\infty}\Bigg[\Big(\Omega^B_{+,n}+\Omega^B_{-,n}+\sqrt{n^2+2\omega_1^2-\omega_2^2}+\sqrt{n^2+\omega_2^2}+4\sqrt{n^2+\frac{\omega_2^2}{4}}\Big)- \\ -\Big(2\Omega^F_{+n}+2\Omega^F_{-n}+4\sqrt{n^2+\omega_1^2}\Big)\Bigg]\,, \quad (4.36)$$

where

$$\Omega^F_{\pm,n} = \sqrt{n^2+\frac{\omega_1^2}{2}\pm\frac{1}{2}\sqrt{\omega_1^4+4\omega_2^2n^2}}\,, \quad (4.37)$$

are the non-trivial fermionic frequencies found in the previous section. It is gratifying to see that the divergencies of bosons cancel against those of fermions, so that the sum (4.36) is convergent.

We are most interested in the value of the sum in the scaling limit, $\omega_1, \omega_2 \to \infty$ with $u = \frac{\omega_2}{\omega_1}$ fixed. Following [54], in this limit, the sum can be replaced by an integral

$$\delta E = \omega_1\int_0^{\infty}\mathrm{d}p\,\Bigg[\Big(\Omega^B_+(p)+\Omega^B_-(p)+\sqrt{p^2+2-u^2}+\sqrt{p^2+u^2}+4\sqrt{p^2+\frac{u^2}{4}}\Big)- \\ -\Big(2\Omega^F_+(p)+2\Omega^F_-(p)+4\sqrt{p^2+1}\Big)\Bigg]\,, \quad (4.38)$$

where

$$\begin{aligned}\Omega^B_\pm(p) &= \sqrt{p^2+2\pm2\sqrt{1+p^2u^2}}, \\ \Omega^F_\pm(p) &= \sqrt{p^2+\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4p^2u^2}}\,.\end{aligned} \quad (4.39)$$

The simplest way to compute the integral is to impose a cut-off and send it to infinity at the end of the computation. The integrals involving Ω^B and Ω^F can be simplified by using the identities

$$\begin{aligned}\Omega^B_+(p)+\Omega^B_-(p) &= \sqrt{4u^2+(p+\sqrt{p^2+4-4u^2})^2}\,, \\ \Omega^F_+(p)+\Omega^F_-(p) &= \frac{1}{2}\sqrt{4u^2+(2p+\sqrt{4p^2+4-4u^2})^2}\,.\end{aligned}$$

Notice that these two expressions are related as

$$\Omega_+^B(p) + \Omega_-^B(p) = 2\Omega_+^F(p/2) + 2\Omega_-^F(p/2).$$

The integrals can then be easily performed by making the change of variables $p \to q = p + \sqrt{p^2 + 4 - 4u^2}$. Finally, we obtain

$$\delta E = \frac{\omega_1}{4}\Big[- 2u^2 \log u^2 - 2\log(8 - 4u^2) + u^2 \log(16(2 - u^2)) + \\ +2\left(-1 + u^2 + \sqrt{1 - u^2} + (u^2 - 2)\log(1 + \sqrt{1 - u^2})\right)\Big] . \quad (4.40)$$

This formula describing the one-loop correction to the classical energy of the long (S, J)-string is our main result. It has to be compared with a corresponding result for the (S, J)-string spinning in $\mathrm{AdS}_5 \times \mathrm{S}^5$ given by eq.(2.29) in [54]. Curiously enough, we find that

$$\delta E^{\mathrm{AdS}_4 \times \mathbb{CP}^3} - \frac{1}{2}\delta E^{\mathrm{AdS}_5 \times \mathrm{S}^5} = \omega_1(u^2 - 1)\log 2 . \quad (4.41)$$

This result is with an apparent contradiction to the conjecture made in [64]. According to their claim, the r.h.s. of eq.(4.41) should vanish. In the $u \to 0$ limit we obtain

$$\delta E = -\frac{5}{2}\omega_2 x \log 2 = -\frac{5\log 2}{2\pi}\log\frac{S}{J} . \quad (4.42)$$

The coefficient in front of $\log\frac{S}{J}$ should be interpreted as the one loop correction to the cusp anomalous dimension.

Chapter 5

Dynamics of worldsheet massless modes of the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ string

5.1 Introduction

It has been known for a long time [66] that the anomalous dimensions of operators of the form $\mathrm{tr}(\Psi D_+^S \Psi)$ with $S \to \infty$ (that is, with large spin S) are of the form $\Delta = f(\lambda)\log(S)$, where $f(\lambda)$ is some function of the 't Hooft coupling constant λ, which in principle can be determined order by order in perturbation theory. In the context of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory the spin S enters the conformal $SO(2,4)$ part of the superconformal algebra. Hence, after the advent of the AdS/CFT correspondence it was realized by [68] that the characteristic $\log S$ behaviour can be reproduced on the sigma model side of the correspondence by a particular classical solution of the worldsheet string theory in the AdS background (which we will call the GKP solution in what follows). Indeed, provided one takes the correspondence for granted, their solution can even explain this behaviour.

If one quantizes the Green-Schwarz action of the superstring in the background of such a spinning string solution, one can find the spectrum of masses of the worldsheet particles (this problem was first solved in the $AdS_5 \times S^5$ setup in [54, 55]). Some of them are massive, whereas others are massless. As emphasized by Alday and Maldacena [6] for the case of $AdS_5 \times S^5$, there're certain quantities in the sigma model which receive the greatest contribution from the massless particles and their interactions. This is what we call "the worldsheet low-energy limit". Thus, it is an interesting question, what these massless particles are and what their interactions are in the $AdS_4 \times \mathbb{C}P^3$ case. The first part of the question was answered in [4,80,87], and here we will answer the second half. It turns out that there're 6 massless bosons and one Dirac fermion, their dynamics being described by the following Lagrangian:

$$\mathcal{L} = \eta^{\alpha\beta}\,\overline{\mathcal{D}_\alpha z^j}\,\mathcal{D}_\beta z^j + i\overline{\Psi}\gamma^\alpha\widehat{\mathcal{D}}_\alpha\Psi + \frac{1}{4}(\overline{\Psi}\gamma^\alpha\Psi)^2, \tag{5.1}$$

where index j runs from 1 to 4, $\mathcal{D}_\alpha = \partial_\alpha - i\,\mathcal{A}_\alpha$, $\widehat{\mathcal{D}}_\alpha = \partial_\alpha + 2\,i\,\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha$ is a $U(1)$ gauge field without a kinetic term — it can be integrated out to provide the conventional Fubini-Study form of the action. Besides, in (5.1) the z^j fields are restricted to lie on the $S^7 \subset \mathbb{C}^4$:

$$\sum_{j=1}^{4} |z^j|^2 = 1 \tag{5.2}$$

The Chapter is organized as follows. In Section 5.2 we describe the spinning string classical solution of the sigma-model. In Section 5.4 we focus on the Alday-Maldacena limit in the familiar case of $AdS_5 \times S^5$. The discussion of Section 5.4 emphasizes the reason, why the coset construction does not suffice for the consideration of an analogous limit in the present case. In Section 5.5 we elaborate on the construction of the full GS action with 32 fermions for the present case, following [57, 58]. In Section 5.6 we explain various properties of the expansion around the spinning string solution. Finally, in Section 5.7 we find the low-energy limit of the expanded action, and find the sought for Lagrangian of the $\mathbb{C}P^3$ sigma model with fermions (5.1). In the Appendix the interested reader will find details of the calculations.

5.2 The spinning string

The GKP solution [68] describes a string moving in the AdS_3 subspace of the entire space. It is due to this reason that it is meaningful in the $AdS_4 \times \mathbb{C}P^3$ case as much as in the $AdS_5 \times S^5$ case. The solution is most easily described in the "global" coordinates, that is the coordinates which cover the whole of AdS space. It is well-known that the AdS space can be described as a hyperboloid embedded in the $\mathbb{R}^{2,D-1}$ space, namely (for the AdS_4 case) the surface

$$-X_0^2 - X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2 \tag{5.3}$$

embedded into $\mathbb{R}^{2,3}$ with metric $ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2$. The parameter R is the "radius" of the AdS space and describes its curvature. If one writes $X_0 = \cosh(\rho)\cos(T)$, $X_1 = \cosh(\rho)\sin(T)$ and introduces spherical coordinates for X_2, X_3, X_4, the radius of the sphere being $\sinh(\rho)$, one obtains the global parametrization of the AdS_4 space. The metric then obtains the following form:

$$(ds^2)_{AdS_4} \equiv G_{\mu\nu} dY^\mu dY^\nu = R^2 \left(-\cosh^2(\rho) dT^2 + d\rho^2 + \sinh^2(\rho) d\Omega_2\right). \tag{5.4}$$

To consider the AdS_3 space one simply needs to change $d\Omega_2 \to d\phi^2$. In these coordinates the spinning string ansatz may be written in the following form:

$$T = \kappa\tau,\ \phi = \omega\tau,\ \rho = \rho(\sigma) \tag{5.5}$$

If one denotes by $G_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ the pull-back of the target-space metric, the Virasoro conditions are:

$$G_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}G_{cd} = 0. \tag{5.6}$$

As is customary, only two of them are independent. Let us impose the conformal gauge, then the Virasoro conditions are $G_{00} + G_{11} = 0,\ G_{01} = 0$. The latter is trivially satisfied by the ansatz, whereas the former produces an equation

$$\rho'(\sigma)^2 - \cosh^2(\rho)\,\kappa^2 + \sinh^2(\rho)\,\omega^2 = 0. \tag{5.7}$$

One can check that the equations of motion for T and ϕ are satisfied identically, whereas the one for $\rho(\sigma)$ coincides with the σ derivative of (5.7). The general solution $\rho(\sigma;\kappa,\omega)$ of (5.7) can be written in terms of elliptic functions. One should recall that the solution for the closed string is also subject to the periodicity condition

$$\rho(\sigma + 2\pi;\kappa,\omega) = \rho(\sigma;\kappa,\omega),$$

(which can be satisfied if one assumes that the string is folded) and this condition relates κ to ω [55, 68]. In the limit $\kappa \to \infty$, however, it turns out that $\omega = \kappa + \ldots$, so the solution simplifies drastically:

$$T = \kappa\tau,\ \phi = \kappa\tau,\ \rho = \pm\kappa\sigma + \rho_0 \tag{5.8}$$

The limit $\kappa \to \infty$ is called the long string limit, since, as one can see from the solution above, the extent of the string in AdS space becomes infinitely large.

Solutions of the equations of motion can be classified according to the values of their Cartan charges, for example in this case one has[1]:

$$E = g \int_0^{2\pi} \cosh^2(\rho)\, \dot{T}\, d\sigma = \kappa\, g \int_0^{2\pi} \cosh^2(\rho)\, d\sigma, \tag{5.9}$$

$$S = g \int_0^{2\pi} \sinh^2(\rho)\, \dot{\phi}\, d\sigma = \kappa\, g \int_0^{2\pi} \sinh^2(\rho)\, d\sigma. \tag{5.10}$$

For every nontrivial solution the function $T(\tau, \sigma)$ is non-constant, since otherwise the Virasoro condition $G_{00} + G_{11} = 0$ would require all target-space coordinates to be constant, thus virtually every solution possesses the E charge. Due to this, one usually refers to a solution with a nonzero S charge as a one-spin solution.

Clearly, the solution (5.8) has one parameter κ, but it is convenient to use $S(\kappa)$ as a genuine parameter, since it has a more clear meaning. Then the target-space energy E becomes a function of S, so let us calculate this function. First of all, $E - S = 2\pi\kappa g$. Besides,

$$S \underset{\kappa\to\infty}{\to} \frac{g}{2} e^{\pi\kappa} \tag{5.11}$$

It follows that

$$\kappa \approx \frac{1}{\pi} \log(\frac{S}{g}), \tag{5.12}$$

or in other words

$$E - S \approx 2g \log(\frac{S}{g}), \tag{5.13}$$

so indeed the $\log(S)$ behaviour is reproduced. Besides, we immediately get a prediction that in the strong coupling limit $f(\lambda) \approx 2g$. What we are interested in are the quantum corrections to this function — these are the $1/g$ corrections in the sigma model setup.

[1] g is the string tension, and it is related to the 't Hooft coupling constant λ via $g \sim \sqrt{\lambda}$.

5.2.1 The two-spin solution

The two-spin solution is a generalization of the spinning string solution described above, which possesses two nonzero Noether charges instead of one. One of these charges is the same S charge, which originates from the motion in AdS space, whereas the second one, called J, describes motion in the $\mathbb{C}P^3$ part of the space. This motion is in fact very simple and is just rotation around a big circle S^1 (which we will parametrize by φ) inside of $\mathbb{C}P^3$. Thus, the ansatz looks as follows:

$$T = \kappa\tau,\ \phi = \omega_1\tau,\ \varphi = \omega_2\tau,\ \rho = \rho(\sigma) \tag{5.14}$$

Then the nontrivial Virasoro condition gives, instead of (5.7):

$$\rho'(\sigma)^2 - \cosh^2(\rho)\,\kappa^2 + \sinh^2(\rho)\,\omega_1^2 + \omega_2^2 = 0. \tag{5.15}$$

Once again, the solution simplifies tremendously in the limit $\kappa \to \infty$, when $\omega_1 \approx \kappa$, so $|\rho'(\sigma)| = \sqrt{\kappa^2 - \omega_2^2}$.

The charge J has the following value:

$$J = g\int_0^{2\pi} \dot{\varphi}\, d\sigma = 2\pi g\omega_2. \tag{5.16}$$

It will be convenient for further use to introduce a parameter

$$u = \frac{2\omega_2}{\kappa} = \frac{J}{g\,\log(S)} \tag{5.17}$$

In this more general case the relation between energy and spin is $E - S = f(g,u)\log(S)$, where $f(g,u)$ is called the generalized scaling function [48].

5.3 Expansion around the two-spin solution

One can impose the conformal gauge and expand the coset action constructed in 1.5.2 around the two-spin solution (5.14) and determine the spectrum of the fluctuation fields. This was done in [4,87], where the following result was obtained: when $\omega_2 = 0$, all the bosonic fields from $\mathbb{C}P^3$ are massless, two of the AdS fields have masses $m^2 = 2\kappa^2$ and $m^2 = 4\kappa^2$, and the remaining two AdS fields are massless, but their contribution is supposed to cancel against the ghost contribution (in other words, they do not contribute to the cohomology of the BRST operator). As for the fermions, for a generic value of ω_2 the determinant of the fermionic quadratic form in the action was found to be:

$$\begin{aligned}\mathscr{D} &= 2^8\omega_2^{16}\big[(2k_0-\omega_2)^2-4(k_1^2+\varkappa^2)\big]^2\big[(2k_0+\omega_2)^2-4(k_1^2+\varkappa^2)\big]^2\times\\ &\quad\times\big[k_0^4-k_0^2(2k_1^2+\varkappa^2)+k_1^2(k_1^2-\omega_2^2+\varkappa^2)\big]^2\,. \qquad (5.18)\end{aligned}$$

It follows from this expression that there are the following fermionic excitations in the model (counting given in terms of complex Weyl fermions, and we're using the parameter u introduced in (5.17)):

- 2 fermions with frequency $\frac{1}{4}u\,\kappa+\sqrt{n^2+\kappa^2}$
- 2 fermions with frequency $-\frac{1}{4}u\,\kappa+\sqrt{n^2+\kappa^2}$
- 2 fermions with frequency $\sqrt{n^2+\frac{\kappa^2}{2}(1+\sqrt{1+u^2n^2})}$
- 2 fermions with frequency $\sqrt{n^2+\frac{\kappa^2}{2}(1-\sqrt{1+u^2n^2})}$

The spectrum in the background of the *spinning string* solution is obtained when $u\to 0$ (or $\omega_2\to 0$), and we see that the spectrum becomes relativistic in this limit: there are 6 massive and 2 massless complex fermions, i.e. 12 massive and 4 massless real ones. However, one can then see from (5.18) that the quantization in the background of the *spinning string* solution, that is in the limit $u\to 0$, is no longer well-defined, since the determinant is zero. This corresponds to the fact that, as noted for the first time in [11], the kappa symmetry transformations degenerate at the quadratic level, if the background configuration describes motion purely in the AdS space. We would like to stress that this is a peculiarity of the coset construction for the $AdS_4 \times \mathbb{C}P^3$ superstring. One can rephrase the statement above by saying that the quadratic part of the action will contain terms of the form $u^2\,\psi\partial_\pm\psi$ ($\partial_\pm$ being the light-cone derivatives), which will vanish when $u = 0$. One could be tempted to rescale the fermions $\psi \to \frac{1}{u}\psi$, however it turns out that in this case $1/u$ factors would appear in front of the interaction terms. Thus, in this way we would merely shift the problem to a different place, and this only confirms that the difficulty is intrinsic to the coset formulation.

In fact, as we have explained, the most interesting for us are the massless worldsheet modes, i.e. the ones which become massless in the limit $u \to 0$. We have checked that it is precisely these massless fermions ψ, which suffer from the problem described above, namely the quadratic action of these fermions becomes degenerate in the limit $u \to 0$. Hence, it seems to us that everything points out to the fact that the coset formulation is not well suited to this background. Therefore we will have to resort to the full Green-Schwarz action with 32 fermions, which was built in [57, 58] (see [42] for another application of that construction), in order to solve our problem.

The problem we have encountered is, in fact, connected with a singular gauge choice, and similar issues arise also in conventional gauge theories, as described in Appendix B.

5.4 The worldsheet low-energy limit of the $AdS_5 \times S^5$ superstring

Here we describe the limit introduced in [6][2]. In the previous section, after quantization of the superstring worldsheet action in the background of the

[2] I am grateful to Kostya Zarembo for clarifying to me some details of this limit.

spinning string solution we obtained a spectrum of masses of the worldsheet particles present in the theory. This spectrum becomes particularly simple in the limit $u \to 0$, namely it becomes relativistic, and some of the particles are massless. It is this limit which was emphasized in the paper [6], their main idea being as follows. One considers the partition function (or, equivalently, free energy) of the worldsheet theory in the background of the two-spin solution, regarding u as a variable parameter, which in fact is the chemical potential for one of the global charges of the model. The limit of small u, that is when $\omega_2 \ll \kappa$, is effectively the low-energy limit, and one expects the massless particles to give dominant contributions to the free energy, and therefore in this sense the massless particles "decouple". In the $AdS_5 \times S^5$ case the only massless particles are the 5 bosons coming from the S^5 part of the background, so their dynamics is described by the $SO(6)$ sigma model, which decouples from the rest of the theory in this limit. From the exact solution of the $SO(6)$ model it is known that, in fact, its spectrum consists of 6 massive particles rather than 5 massless ones, however the mass gap cannot be seen in perturbation theory ($m \sim e^{-ag}$), which is still applicable as long as $\omega_2 \gg m$.

To be more precise, the leading contributions to the free energy in the limit $u \to 0$ look as follows:

$$E \propto u^2 \left(a_0\, g + \sum_{n=1} \frac{1}{g^{n-1}} \left(a_n \left(\log u\right)^n + \ldots\right) \right) \log(S) \tag{5.19}$$

One can see that the power of the logarithm grows with the order of $\frac{1}{g}$, which is a common feature of perturbation theory. The main quantitative claim of [6] was that in the $AdS_5 \times S^5$ case the numbers a_n can be determined from the pure $SO(6)$ sigma model, and it was confirmed in [98] up to two loops.

In the $AdS_5 \times S^5$ case the only massless particles are bosons from S^5, and their interactions are determined by the $SO(6)$ symmetry, so the model is uniquely defined by these properties. In the $AdS_4 \times \mathbb{C}P^3$ case there are additional massless fermions, as we discussed above, and the amount of symmetry is not enough to determine their interactions uniquely. Because of that a genuine calculation is necessary, and it is carried out below.

5.5 The full IIA superstring action in the $AdS_4 \times \mathbb{C}P^3$ background

As explained in the previous sections, for the analysis of the action, expanded around the spinning string solution, one needs to use the complete type IIA Green-Schwarz superstring action (that is, with 32 real fermions), rather than the reduced coset formulation (in the $AdS_5 \times S^5$ case the coset action is the complete Green-Schwarz action — it contains 32 fermions — and it was first built in [90]). The construction of such action is no easy task and for generic supergravity backgrounds it has not been carried out. However, as we will explain, the $AdS_4 \times \mathbb{C}P^3$ case under consideration is special, since it can be obtained by a dimensional reduction of the $AdS_4 \times S^7$ solution of the eleven-dimensional supergravity equations of motion. One might wonder, why this would simplify anything. However, from the work [36] it is known that there exists a three-dimensional world-volume action of a membrane, coupled to an arbitrary eleven-dimensional supergravity background (usually this membrane is called the M2 brane). This action is quite similar to the superstring action in many ways, for example in the sense that it, too, possesses a local fermionic symmetry. It was argued in [45], that if one compactifies a target-space coordinate and a world-sheet coordinate of this action simultaneously, then one obtains the Green-Schwarz type IIA superstring action. Thus, the remaining question is whether it is easy or not to build the M2 brane worldvolume action. The answer to this question depends on the chosen supergravity background, but in our case the task simplifies, since the $AdS_4 \times S^7$ background can be described by a coset $OSP(8|4) \,/\, SO(7) \times SO(1,3)$. Indeed, its bosonic part is $SP(4) \,/\, SO(1,3) \times SO(8) \,/\, SO(7) \approx AdS_4 \times S^7$. This strategy was pursued in the papers [57, 58], where as a result the sought for type IIA Green-Schwarz action was built. Here we elaborate on this construction in a, hopefully, transparent way.

5.5.1 The M2 brane action and the coset $OSP(8|4) \,/\, SO(7) \times SO(1,3)$

In this section we describe the construction of the M2 brane three-dimensional worldvolume action for the case of the $AdS_4 \times S^7$ supergravity background. In what follows we will denote the membrane worldvolume coordinates as σ, τ and y. We remind the reader that the fields in the eleven-dimensional supergravity are the graviton $g_{\mu\nu}$, the gravitino ψ^α and the three-form potential $\mathcal{H}^{(3)}$ (the field strength being the four-form $\mathcal{F}^{(4)} = d\mathcal{H}^3$). In our construction we heavily exploit the $OSP(8|4) \,/\, SO(7) \times SO(1,3)$ coset structure of the space. The $OSP(8|4)$ supergroup is very similar to the $OSP(6|4)$ supergroup, and its matrix realization is described in the Appendix 5.2. To construct the explicit matrix realization of the coset one also needs to choose the embeddings of "the denominator" $SO(7) \times SO(1,3)$ into this supergroup. For instance, $SO(7)$ can be embedded into $SO(8)$ in different nonequivalent ways (here we mean, that the embeddings are in general not related by a similarity transformation). We will elaborate more on this in the Appendix 5.3, however for the moment let us give a clear

Example. One can embed $SO(7) \subset SO(8)$ diagonally (that is, as a 7×7 matrix inside of a 8×8 matrix), let us denote this embedding

$h_1 : SO(7) \to SO(8)$. There's another, "spinorial", embedding. Indeed, let γ^μ, $\mu = 1...7$ denote the real skew-symmetric seven-dimensional gamma-matrices ($\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$), which have dimensionality 8. Then the commutators $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ generate the $so(7)$ algebra inside of $so(8)$. We denote the corresponding group embedding as h_2. Now, h_1 and h_2 cannot be related by a similarity transformation. Indeed, let z be a fixed element of the $so(7)$ algebra, such that $h_2(z)$ is nondegenerate, for instance we can take $z = h_2^{-1}(\gamma^{12})$. If there were a similarity transformation relating the two embeddings, it would imply that $h_1(z) = \omega\, h_2(z)\, \omega^{-1}$ for some $\omega \in SO(8)$. However, since h_1 is the diagonal embedding, $\det(h_1(z)) = 0$, whereas $\det(h_2(z)) \neq 0$, which leads to a contradiction. $\triangleright$

We stress that the correct embedding for our purposes is the "spinorial" one, that is the $so(7)$ algebra is generated by $\gamma^{\mu\nu}$.

The M2 brane action can be built from the bosonic and fermionic vielbeins, denoted by $E^A = \{E^{\bar{a}},\, E^a\}$ and E^α respectively, which in turn can be obtained in a simple way from the $OSP(8|4) \,/\, SO(7) \times SO(1,3)$ coset. Here the indices $\{\bar{a} = 0...3,\, a = 1...7\}$ refer to the AdS_4 and S^7 spaces respectively, and the index $\alpha = 1...32$ numbers the fermionic directions. The number 32 comes from the fact that we're dealing with a single Majorana spinor in eleven dimensions, which has 32 real components. If considered from the point of view of the representation of the ten-dimensional Lorentz group, it splits into two Majorana-Weyl spinors — one left-handed and one right-handed. Let g be a representative of the coset. Then one can build the left-invariant current $J = -g^{-1}\, dg$ and find the vielbein and connection components of this current:

$$J = -g^{-1}\, dg = E^a T_a + E^\alpha Q_\alpha + A^{ab}\Omega_{ab}, \tag{5.20}$$

where Ω_{ab} are elements of the stabilizer (denominator of the coset),

Q_α furnish the fermionic basis of the $osp(8|4)$ algebra, and T_a are the complementary bosonic directions, namely the directions tangent to the manifold. The current is flat by construction, that is its curvature vanishes:

$$dJ - J \wedge J = 0. \tag{5.21}$$

The action, as in the superstring case, consists of two terms, which can be loosely called 'the metric part' and the 'Wess-Zumino part'. The action takes the following form:

$$S = \frac{g}{2\pi} \int d\sigma\, d\tau\, dy\, \left(\eta_{ab}\, E^a E^b + \varkappa\, \epsilon^{abc}\, \partial_a X^m\, \partial_b X^n\, \partial_c X^p\, \mathcal{H}_{mnp}\right), \tag{5.22}$$

the first term being the metric part and the second term — the Wess-Zumino part[3]. The second term is the pull-back to the worldvolume of a three-form $\mathcal{H} = \mathcal{H}_{mnp}\, dX^m \wedge dX^n \wedge dX^p$. Recall that this three-form can be found from its field-strength four-form $\mathcal{F}$, and the latter can be expressed in terms of the vielbeins in the following way[4]:

$$\mathcal{F} = \frac{1}{8}\epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} E^{\bar{a}} \wedge E^{\bar{b}} \wedge E^{\bar{c}} \wedge E^{\bar{d}} + \widehat{\lambda}\, E^\alpha \wedge [\mathbf{\Gamma}_A, \mathbf{\Gamma}_B]_\alpha^\beta\, E_\beta \wedge E^A \wedge E^B. \tag{5.23}$$

The first term here is the volume element of the AdS space, whereas the second term is intrinsically fermionic and manifestly $Spin(1, 10)$ invariant. Note that the fermionic indices (α, β) are raised and lowered using the eleven-dimensional charge conjugation matrix C_{11}. Using the Maurer-Cartan equation (5.21) one can check that the form $\mathcal{F}$ is closed for a suitable value of the constant $\widehat{\lambda}$. A closed form is locally exact, so one can find its potential $\mathcal{H}$ by a standard procedure. We will follow this route in the next sections to build the part of the Green-Schwarz action that we need.

[3]By the coordinates X^m we mean both bosonic and fermionic ones.

[4] $\widehat{\lambda}$ is a constant. Barred indices refer to the AdS space, that is they run from 0 to 3.

5.5.2 The Hopf fibration $S^7 \to \mathbb{C}P^3$ and the dimensional reduction

So far we have been dealing with the $AdS_4 \times S^7$ solution of the eleven-dimensional supergravity. However, our ultimate goal is to arrive at the $AdS_4 \times \mathbb{C}P^3$ solution of the IIA supergravity in ten dimensions. This is achieved through a compactification, based on the Hopf fibration, so we recall what the latter looks like. There are different variants of the Hopf fibration, most of them originating from the tautological fiber bundle $\mathbb{C}^{n+1} \to \mathbb{C}P^n$ (the one which arises naturally from the definition of the projective space). The most common version of the Hopf fibration arises when one restricts the total space $\mathbb{C}^{n+1}$ to the unit sphere $\sum\limits_{i=1}^{n+1} |z_i|^2 = 1$, then we get $\pi : S^{2n+1} \to \mathbb{C}P^n$. The tautological bundle is a line bundle (its fiber is $\mathbb{C}$), so after imposing the absolute value restriction the π fibration has the circle S^1 as a fiber. The most well-known case of the Hopf fiber bundle is explained in the following

Example. It is the case $n = 1$, i.e. $\pi : S^3 \to \mathbb{C}P^1 \approx S^2$. If we write the sphere S^3 as

$$|z_1|^2 + |z_2|^2 = 1, \tag{5.24}$$

then the map π can be written out explicitly as $\pi(z_1, z_2) = \frac{z_2}{z_1} \equiv Z$, and Z should be regarded as a stereographic coordinate on the sphere S^2. To see what is going on more clearly, let us solve (5.24) in the following way: $z_1 = \frac{y_1}{\sqrt{|y_1|^2+|y_2|^2}}$, $z_2 = \frac{y_2}{\sqrt{|y_1|^2+|y_2|^2}}$. The metric on the sphere is the metric, induced in flat space $ds^2 = |dz_1|^2 + |dz_2|^2$ by the embedding (5.24). Thus it is clear that it will be invariant under arbitrary rescalings $y_i \to \lambda y_i$, $\lambda \in \mathbb{R}$. In other words, when regarded as a metric on the 4-space parametrized by y_1, y_2, it is degenerate. This means, of course, that we can fix a "gauge", however we do not want to do it for the moment. We can rewrite the metric in the following way:

$$(ds^2)_{S^3} = \frac{dy_i d\bar{y}_i}{\rho^2} - \frac{|dy_i \bar{y}_i|^2}{\rho^4} + \left(i\frac{dy_i\bar{y}_i - y_i d\bar{y}_i}{2\rho^2}\right)^2, \text{ where } \rho^2 = |y_i|^2. \quad (5.25)$$

The first two terms constitute precisely the $\mathbb{C}P^1$ metric in homogeneous coordinates. Let us now choose the following inhomogeneous coordinates:

$$y_1 = e^{i\varphi},\ y_2 = e^{i\varphi} Z, \quad (5.26)$$

then the metric takes the canonical form:

$$(ds^2)_{S^3} = \frac{dZd\bar{Z}}{(1+Z\bar{Z})^2} + (d\varphi - A)^2, \text{ where } A = i\frac{dZ\,\bar{Z} - Z\,d\bar{Z}}{1+Z\bar{Z}}. \quad (5.27)$$

Note that from (5.26) it follows that, in accordance with our discussion, $Z = \frac{y_2}{y_1} = \frac{z_2}{z_1}$. $\triangleright$

Although the space which interests us in this Chapter is $\mathbb{C}P^3$ rather than $\mathbb{C}P^1$, it can be considered in a similar way to the example, and as a result one obtains a metric on S^7 in the form, which exhibits the Hopf bundle in a clear way. This discussion shows, that the dimensional reduction of the metric can be achieved by dropping the $(d - A)^2$ term in the metric, or getting rid of the corresponding einbein. Note also that, when fermions are turned on, A is interpreted as the R-R one-form $\mathcal{A}^{(1)}$ of the IIA background. As for the four-form $\mathcal{F}$, it can be written in the following way:

$$\mathcal{F} = \mathcal{K}^{(3)} \wedge d\varphi + \mathcal{G}^{(4)}, \quad (5.28)$$

where $\mathcal{K}^{(3)}$ and $\mathcal{G}^{(4)}$ do not depend on the fiber coordinate φ. Then the dimensional reduction (after setting $\varphi = y$, y being the third worldvolume coordinate [45]) boils down to leaving the $\mathcal{K}^{(3)}$ piece of the $\mathcal{F}$ form: dim.red.$(\mathcal{F}) = \mathcal{K}^{(3)}$. One can see that, since $d\mathcal{F} = 0$, it follows that $d\mathcal{K}^{(3)} = d\mathcal{G}^{(4)} = 0$. In fact, it follows from our discussion that $\mathcal{K}^{(3)}$ is the field-strength of the (locally defined) NS-NS two-form potential $\mathcal{B}^{(2)}$, which constitutes the Wess-Zumino term of the string action: $\mathcal{K}^{(3)} = d\mathcal{B}^{(2)}$. We note in passing that $\mathcal{D}^{(4)} = \mathcal{G}^{(4)} - \mathcal{A}^{(1)} \wedge \mathcal{K}^{(3)}$ is the R-R 4-form field-strength of the IIA background (we will not need it later). In this way we have determined all the ingredients of the ten-dimensional IIA supergravity solution from the eleven-dimensional supergravity solution.

Since the construction we will carry out heavily relies on the $OSP(8|4)$ coset, we need to parametrize the coset in such a way which would exhibit the Hopf fiber bundle, similarly to 5.27. Compared to the purely bosonic case, the main difference is that various reductions can be made, which preserve different amounts of supersymmetry. The correct one, i.e. the one we're looking for, should provide for a $OSP(6|4)$ symmetry group of the *reduced* background. The reductions differ by the embedding of $OSP(6|4) \subset OSP(8|4)$, so in the Appendix 5.3 we explain, following [57,58], that the correct one is the standard diagonal embedding.

5.6 Quantum corrections to the spinning string state

5.6.1 The decompactification

The Green-Schwarz action in its standard form does not allow for a simple quantization of the theory. This is due to the fact that the action does not contain a term quadratic in the fermions with no bosons, but rather terms, in which fermions are coupled to bosons. The quantization is possible in the background of a classical solution, if the classical solution makes the quadratic term of the fermions nondegenerate. As we discussed above, this requirement is fulfilled for the two-spin solution even in the $OSP(6|4)$ coset formulation, however in the limit when the J charge vanishes this is no longer the case. It is precisely to overcome this difficulty that we needed to invoke the full Green-Schwarz action with 32 fermions. Once we have built the full action, we may quantize in the background of the spinning string (one spin) solution (5.8) in the limit $\kappa \to \infty$. It will be convenient however to make a change of the worldsheet coordinates:

$$\sigma' = \kappa\sigma, \quad \tau' = \kappa\tau. \tag{5.29}$$

The important observation is that after such redefinition the only parameter which is changed is the length of the worldsheet circle, or the string length: $L = 2\pi\kappa$. In other words, κ enters only the integration limits in the action, but not anywhere else. Since (5.8) is a solution for $\kappa \to \infty$, in this limit we achieve the decompactification of the worldsheet.

5.6.2 The coset element for the spinning string background.

We take the following AdS_3 coset element:

$$g_{AdS_3} = e^{\frac{i}{2}t\Gamma^0 - \frac{1}{2}\phi\Gamma^1\Gamma^2}\, e^{-\frac{1}{2}\rho\,(i\Gamma^2 - \Gamma^0\Gamma^1)} \tag{5.30}$$

When we insert the spinning string solution (5.8), it simplifies as follows:

$$g_{spin} = e^{\frac{1}{2}(i\Gamma^0 - \Gamma^1\Gamma^2)\,\kappa\tau}\, e^{-\frac{1}{2}\rho(\sigma)\,(i\Gamma^2 - \Gamma^0\Gamma^1)} \tag{5.31}$$

The important property of this parametrization is that $[i\Gamma^0 - \Gamma^1\Gamma^2, i\Gamma^2 - \Gamma^0\Gamma^1] = 0$, and it is precisely thanks to this fact that the current $g^{-1}dg$ only includes the derivative $\rho' = \pm\kappa$, but not ρ (and hence not σ) itself. The latter fact is the reason why it is possible to determine the spectrum of quadratic fluctuations at all, and it is what makes the theory in this background tractable.

5.6.3 The expansion

The full coset element in the spinning string background looks as follows:

$$g = g_{spin}\, g_{AdS}\, g_{\mathbb{C}P}\, g_\theta\, e^{\varphi\epsilon}\, g_v, \tag{5.32}$$

where g_{AdS} describes fluctuations of the AdS fields, $g_{\mathbb{C}P}$ is the $\mathbb{C}P^3$ coset element, φ is the fiber coordinate, g_θ and g_v are the fermionic coset elements. In particular, g_θ only includes the 24 fermions which were present in the $OSP(6|4)$ coset (see (5.36) for θ), whereas $g_v = e^{v_\lambda Q^\lambda}$ contains the 8 additional fermions (see (A.3) for v). It will be convenient to take the coset elements g_{AdS} and $g_{\mathbb{C}P}$ in the following form:

$$g_{AdS} = e^{\frac{(z_+-z_-)}{2}(i\Gamma_0-\Gamma_1\Gamma_2)}\, e^{\frac{(z_++z_-)}{2}(i\Gamma_2-\Gamma_0\Gamma_1)}\, \frac{1+\frac{i}{2}(z_1\Gamma_1+z_2\Gamma_3)}{\sqrt{1-\frac{1}{4}(z_1^2+z_2^2)}} \tag{5.33}$$

$$g_{\mathbb{C}P} = 1 + \frac{W+\overline{W}}{\sqrt{1+|w_i|^2}} + \frac{\sqrt{1+|w_i|^2}-1}{|w_j|^2\sqrt{1+|w_i|^2}}\left(W\overline{W}+\overline{W}W\right), \tag{5.34}$$

where $W = w_i\mathcal{T}_i$, $\overline{W} = \overline{w}_i\overline{\mathcal{T}}_i$, and one can find definitions of $\mathcal{T}_i$, $\overline{\mathcal{T}}_i$ in Section 1.5.1. Note that the two exponents in (5.33) commute with each other, as well as with g_{spin} of (5.31), which is the reason why we have chosen the parametrization in this way (as a result, the current $J = -g^{-1}dg$ will depend on $z_\pm$ only through their derivatives).

We need to plug the coset element into the expression for the action of the 11D theory (5.22), fix the kappa symmetry gauge and to determine, which of the fermions are massive and which are massless. One might recall however that, as determined before, the spectrum consists of 12 massive and just four massless fermions. Thus, let us use the symmetry properties of the theory as a shortcut to the result. Indeed, if one multiplies the coset element (5.32) by an element $\omega \in SU(3)$ from the left, then we get the following:

$$\omega g = g_{spin}\, g_{AdS}\left(\omega\, g_{\mathbb{C}P}\,\omega^{-1}\right)\left(\omega\, g_\theta\,\omega^{-1}\right) e^{\varphi\epsilon}\, g_v\,\omega, \tag{5.35}$$

where ω at the very right can be dropped, as it belongs to the stabilizer of the coset (the $SU(3)$ generators are the L_i's of (E.15-E.17), and, since they're linear combinations of $\gamma^{\mu\nu}$, it follows that $SU(3) \subset SO(7)$). Clearly $\omega\, g_{\mathbb{C}P}\, \omega^{-1}$ simply transforms the bosonic fields of the model, whereas $\omega\, g_\theta\, \omega^{-1}$ transforms the fermionic θ fields. We see that the transformation does not affect the v-fermions. The masses of the theory are determined by the quadratic part of the Lagrangian, in which the bosonic and fermionic fields clearly decouple. In fact, we might forget about the bosonic fields for the moment, since we already know their spectrum. Let us remind the reader that the matrix of θ fermions can be written as follows:

$$\theta = \begin{bmatrix} \theta^1_1 & \theta^1_2 & \theta^1_3 & \theta^1_4 & \theta^1_5 & \theta^1_6 & 0 & 0 \\ \theta^2_1 & \theta^2_2 & \theta^2_3 & \theta^2_4 & \theta^2_5 & \theta^2_6 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 & 0 \end{bmatrix}, \tag{5.36}$$

where the stars stand for complex conjugated fermions.

The action $\omega\, g_\theta\, \omega^{-1}$ means, that the θ^1 and θ^2 fermions (6 complex in each line) furnish $3 + \bar{3}$ representations of $SU(3)$ each. The important point is that none of the θ fermions are invariant under the $SU(3)$, whereas all of the v fermions are. For the quadratic part of the action this means that the θ and v fermions decouple. Now, a 3 or $\bar{3}$ representation of $SU(3)$ involves 6 real fermions, so if the $SU(3)$ symmetry can be preserved by a choice of the kappa symmetry gauge, this means that every 6 θ-fermions entering a single multiplet have the same mass. A suitable kappa symmetry gauge is the one, which sets the second and third lines of the θ and v matrices to zero (from the Appendix 5.5 it is clear that such gauge is indeed admissible). This condition can be summarized as follows:

$$\frac{I - i\Gamma_0\Gamma_1\Gamma_2}{2}\,\theta = \frac{I - i\Gamma_0\Gamma_1\Gamma_2}{2}\,v = 0. \tag{5.37}$$

We know that there are just 4 massless fermions, so the $SU(3)$ multiplets are too big for that, and the θ fermions are destined to be *massive*. Thus, we're left with the 8 v-fermions. The residual kappa-symmetry should allow to eliminate 4 of them, and the remaining 4 ones should be *massless*. We have checked by a direct calculation of the quadratic part of the action that this is indeed true.

5.7 The worldsheet low-energy limit

In the previous section it was explained that, before the kappa symmetry gauge is imposed, the θ- and v-fermions decouple in the quadratic action. In fact, one can check that in the leading (linear) order the kappa symmetry transformations do not mix θ's and v's. This is the reason, why it is possible to choose the kappa-gauge in such a way, that θ's and v's remain decoupled in the gauge-fixed quadratic action. By now we know that the θ fermions (meaning the ones that remain after the kappa gauge is imposed) are all massive, so we can safely set them to zero. Then we are left with an action depending solely on the bosonic fields and the v-fermions, and there will be a residual kappa symmetry of rank 4 acting on these fields. One could certainly gauge-fix this residual symmetry as well from the beginning, however it is useful to check that the resulting low-energy action which we will obtain is independent of the kappa-symmetry gauge choice. For this reason we will prefer not to fix the gauge for the v-part of the kappa-symmetry.

First of all, let us find out, how exactly this v-part of the kappa-symmetry looks like in the leading order. For this purpose we write out the piece, which depends on v's, of the quadratic part of the string action[5]:

$$\mathcal{L}_v^{(2)} = i(\bar{\chi} - \bar{\xi})\partial_+(\chi - \xi) - i(\bar{\psi} - \bar{\eta})\partial_-(\psi - \eta) \tag{5.38}$$

[5]The notations for the fermions are explained in Appendix A, 'v-fermions'.

It follows that the kappa-transformations have a very simple form:

$$\delta\chi = \delta\xi = \epsilon_1, \quad \delta\psi = \delta\eta = \epsilon_2, \tag{5.39}$$

$\epsilon_{1,2}$ being two Weyl spinors. Suppose we now want to determine the low-energy limit of the worldsheet theory. What terms should be left from the full Green-Schwarz action in order to achieve this? Clearly, one should get rid of the massive fields. Besides, one should also drop certain interaction vertices of the massless fields (the ones that are suppressed by powers of some mass), since they, too, may be regarded as effectively reflecting the presence of massive modes. In other words, all terms in the low-energy Lagrangian should have dimension not greater than two. In 2D the bosons w have canonical dimension 0, the fermions have dimension 1/2 and the derivative, clearly, has dimension 1. In this way, for instance, terms of the form $\psi_1\,\psi_2\,\chi_1\,\partial_+\chi_1$ and $\psi_1\psi_2\partial_+ w\,\partial_-\bar{w}$ have dimension 3, terms $\psi_1\,\partial_-\psi_2\,\chi_1\,\partial_+\chi_2$ have dimension 4, so such terms should be dropped in the low-energy limit. Most of the terms which should be preserved, have dimension two: these are, for instance, $\partial_+\bar{w}\partial_- w$, $\bar{\chi}\partial_+\chi$, $\bar{w}\partial_- w\,\bar{\psi}\psi$, $\bar{\psi}\psi\bar{\chi}\chi$ etc. However, a very important fact which should not be overlooked, is that there are terms of dimensions 1 and 0 in the Lagrangian as well: the terms of dimension 1 are $z\,v^2$, z being the massive AdS fields, and the terms of dimension 0 are the mass terms z^2. Hence, there's a very important qualification to dropping the massive fields: they can be dropped everywhere, except for these terms. What is the meaning of these terms or, in other words, what is the reason for their being present in the final Lagrangian? It turns out that these terms are precisely what is needed to maintain the kappa-symmetry of the low-energy Lagrangian, that is to say they provide the independence of the low-energy Lagrangian of the chosen kappa-gauge. One can see this in the following way. In the lowest order the kappa-symmetry transformations look similar to the case of the flat target space:

$$\delta v = \epsilon, \quad \delta z = \epsilon v \tag{5.40}$$

We will not bother writing any indices or projectors here, since this schematic exposition is sufficient to convey the general idea. From (5.40) it follows that

$$\delta(v^4) \propto \epsilon v^3, \quad \delta(z^2) \propto \epsilon v z, \quad \delta(z\, v^2) \propto \epsilon v^3 + \epsilon v z. \tag{5.41}$$

Our claim is that the three terms z^2, $z\, v^2$ and v^4 enter the Lagrangian in a kappa-invariant combination, which is allowed by the rules (5.41). There's another way to make the same point, namely, let us integrate out the massive fields z. What we obtain as a result are terms of the form v^4 (v here can mean any of the 8 different v-fields, so there are many ways to build v^4 terms). Then, our statement can be reformulated by saying that all the v^4 terms should combine into one term

$$(\bar{\chi} - \bar{\xi})(\chi - \xi)(\bar{\psi} - \bar{\eta})(\psi - \eta) \tag{5.42}$$

which is the only quartic combination of the v-fields, invariant under the transformations (5.39.) We have checked that this is indeed the case, and the reader can find the details of the calculation in the Appendix 5.5.

Once we have checked the independence of the resulting Lagrangian of the choice of the kappa-symmetry gauge, we may impose the one which we find most convenient (5.37) to obtain the following Lagrangian (see Appendix A for fermion field notations):

$$\mathcal{L} = \frac{1}{2} G_{\bar{a}b}(z, \bar{z})\, (\partial_+ z_a \partial_- \bar{z}_b + \partial_- z_a \partial_+ \bar{z}_b) + i(\bar{\chi}\mathcal{D}_+\chi - \bar{\psi}\mathcal{D}_-\psi) + 3\, \bar{\chi}\chi\bar{\psi}\psi, \tag{5.43}$$

where $\chi = \psi_1 + i\chi_1$, $\psi = \psi_2 + i\chi_2$ and the "covariant" derivatives are:

$$\mathcal{D}_\pm = \partial_\pm + \frac{\bar{z}_i \partial_\pm z_i - z_i \partial_\pm \bar{z}_i}{|z_j|^2} \tag{5.44}$$

Besides, $G_{\bar{a}b}$ is the Hermitian ($G_{\bar{a}b} = G^*_{\bar{b}a}$) Fubini-Study metric on $\mathbb{C}P^3$:

$$G_{\bar{a}b} = \frac{\delta_{\bar{a}b}}{|z_j|^2} - \frac{\bar{z}_a\, z_b}{|z_j|^4} \tag{5.45}$$

Upon the introduction of a Dirac field $\Psi = \binom{\psi}{\chi}$ the Lagrangian we have obtained may be cast into the form announced in the Introduction [6]:

$$\mathcal{L} = \eta^{\alpha\beta}\, \overline{\mathcal{D}_\alpha z^j}\, \mathcal{D}_\beta z^j \; + i\overline{\Psi}\gamma^\alpha \widehat{\mathcal{D}}_\alpha \Psi + \frac{1}{4}(\overline{\Psi}\gamma^\alpha\Psi)^2, \tag{5.46}$$

where index j runs from 1 to 4, $\mathcal{D}_\alpha = \partial_\alpha - i\,\mathcal{A}_\alpha$, $\widehat{\mathcal{D}}_\alpha = \partial_\alpha + 2\,i\,\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha$ is a $U(1)$ gauge field without a kinetic term — it can be integrated out to provide the conventional Fubini-Study form of the action. Besides, in (5.46) the z^j fields are restricted to lie on the $S^7 \subset \mathbb{C}^4$:

$$\sum_{j=1}^{4} |z^j|^2 = 1 \tag{5.47}$$

Note that the Lagrangian in (5.46) is invariant under global $U(1) \times U(1)$ transformations of the fermions, that is $\Psi \to e^{i\alpha}\Psi$ and $\Psi \to e^{i\alpha\gamma^5}\Psi$.

As it had been expected, we have obtained a fermionic extension of the nonlinear sigma-model with target space $\mathbb{C}P^3$.

5.8 Open problems

We have obtained the Lagrangian (5.46) for a model describing the infrared limit of the worldsheet theory of the $AdS_4 \times \mathbb{C}P^3$ superstring, quantized in a certain background. This Lagrangian can be used to calculate certain quantitative characteristics of the model, as discussed in Section 5.4. The latter could be compared with the Bethe ansatz predictions (as it has been done in [62] for the $AdS_5 \times S^5$ case).

[6]The γ-matrices here are the 2D gamma-matrices, for instance, in our notations $\gamma^0 = -i\sigma^2, \gamma^1 = \sigma_1, \gamma^5 = \sigma_3$ and $\mathcal{D}_\pm = \mathcal{D}_0 \pm \mathcal{D}_1$. They should not be confused with the 7D γ matrices from the Appendix and other parts of the text. The conjugation is defined as $\overline{\Psi} = \Psi^{\,\dagger}\gamma^1$.

Apart from the string theory applications, the model we have obtained might be interesting in its own right. In the past a great deal of effort was devoted to the understanding of various $\mathbb{C}P^3$ models, with and without supersymmetry [104,109,114], since it was hoped they could give some insight into the infrared dynamics of QCD. These models are asymptotically free, however it is only the bosonic model that exhibits confinement, whereas models with fermions usually describe liberated $U(N)$ solitons. In most, if not all, cases the quantum S-matrix of such solitons can be computed exactly. It is an interesting question, whether the model we have obtained is integrable as well. One could also wonder, whether there is any fundamental explanation for the $6:2$ ratio of the bosonic vs. fermionic degrees of freedom. There are many questions which remain to be answered. We plan to address them in a subsequent publication.

Appendix A

Matrices and notations

1.1 Matrices and notations

Here we explicitly present the matrices, which appeared in the main text.

AdS Γ-matrices

The representation of the four Γ matrices (which come from the *AdS* space) used throughout the thesis is as follows:

$$\Gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

One can observe that for $k = 1, 2, 3$ we have $\Gamma_k = i\sigma_2 \otimes \sigma_k$. Besides, we introduce $\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$. Another matrix encountered in the text is

$$C_4 = i\Gamma_0\Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbb{C}P$ γ-matrices

The seven γ matrices (which come from the S^7 space) are as follows:

$$\gamma_1 = I_2 \otimes \Gamma_3,\ \gamma_2 = \sigma_3 \otimes \Gamma_1,\ \gamma_3 = i\sigma_2 \otimes i\Gamma_0\Gamma_1\Gamma_2$$

$$\gamma_4 = i\sigma_2 \otimes i\Gamma_2\Gamma_3,\ \gamma_5 = i\sigma_2 \otimes \Gamma_0\Gamma_3,\ \gamma_6 = -\sigma_1 \otimes \Gamma_1,\ \gamma_7 = I_2 \otimes \Gamma_3\Gamma_1,$$

The following matrices were also encountered in the text:

$$U = \mathrm{diag}(-1,-1,-1,3) \otimes i\sigma_2,$$
$$K_6 = \mathrm{diag}(1,1,1,0) \otimes i\sigma_2,$$
$$\epsilon = \mathrm{diag}(0,0,0,1) \otimes i\sigma_2$$

11D Γ-matrices

The 11D $\mathbf{\Gamma}$ matrices were used in formula (5.23) to construct the Wess-Zumino term of the M2 brane action. These matrices (which have dimensionality 32) can be built in the following way, as tensor products of the AdS and $\mathbb{C}P$ gamma-matrices defined above:

$$\mathbf{\Gamma}^A = \{\, i\Gamma^a\Gamma^5 \otimes I_8,\ a = 0...3;\ \Gamma^5 \otimes \gamma^b,\ b = 1...7\,\}$$

The 11D charge-conjugation matrix looks as follows:

$$C_{11} = C_4\Gamma_5 \otimes I_8 \tag{A.1}$$

and has the property

$$(\mathbf{\Gamma}^A)^{\mathrm{T}} = -C_{11}\mathbf{\Gamma}^A C_{11}^{-1} \tag{A.2}$$

The "v-fermions"

The $v_\lambda Q^\lambda$ matrix of v-fermions explicitly looks as follows:

$$v = v_\lambda Q^\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & v_{17} & v_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{27} & v_{28} \\ 0 & 0 & 0 & 0 & 0 & 0 & -v_{27}^* & -v_{28}^* \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{17}^* & v_{18}^* \end{bmatrix}, \tag{A.3}$$

where

$$v_{17} = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}(\psi_1 + i\,\psi_2),\ \ v_{18} = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}(\chi_1 + i\,\chi_2), \tag{A.4}$$

$$v_{27} =, \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}(\phi_1 + i\,\phi_2)\ \ v_{28} = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}(\xi_1 + i\,\xi_2). \tag{A.5}$$

On several occasions we used the complex fermion notation $\chi = \psi_1 + i\chi_1$, $\psi = \psi_2 + i\chi_2$, $\xi = \xi_1 - i\phi_1$, $\eta = \xi_2 - i\phi_2$. Besides, in Chapter 5 we made use of the gauge (5.37), which stands for $\phi_1 = \phi_2 = \xi_1 = \xi_2 = 0$.

1.1.1 The vector representation of $SO(6)$

One of the low-dimensional Lie group isomorphisms is as follows: $SU(4)/Z_2 \simeq SO(6)$. It is described by the diagram $1 \to Z_2 \to SU(4) \xrightarrow{f} SO(6) \to 1$, where the kernel is $Ker(f) = \{\pm 1\}$. In the representation theory it manifests itself in the following way: denote by V_F the fundamental representation of $SU(4)$, then $V_F^{\wedge 2}$ decomposes into 2 copies of the vector representation **6** of the group $SO(6)$. In other words, if the action of $SU(4)$ on a vector $v \in V_F$ has the form $v \to \omega v$, then the action of $SU(4)/Z_2$ on the skew-symmetric 4×4 matrix $m \in V_F^{\wedge 2}$ looks as follows: $m \to \omega m \omega^T$. It is clear that the action of the transformations $\pm\omega$ on m is the same, so indeed there is a Z_2-identification. Let us clarify, where the two copies come from. As a matter of fact, even though the matrix m is skew-symmetric and has dimension 4×4, its matrix elements are complex numbers, so the dimension of the representation $V_F^{\wedge 2}$ over the field $\mathbb{R}$ is equal to 12. Notice that ϵ_{ijkl} is an invariant tensor of the group $SU(4)$, since under the transformation $v_i \to \omega v_i$ $(i = 1, 2, 3, 4)$ the «volume element» $v_1 \wedge v_2 \wedge v_3 \wedge v_4$ is multiplied by $\det(\omega) = 1$ (a unimodular transformation preserves the volume). Hence we can decompose the obtained twelve-dimensional representation into two irreducible ones in the following way: $m = m_+ + m_-$, where $m_\pm$ is self-dual or anti-self-dual: $(m_\pm^{ij})^* = \pm\frac{1}{2}\epsilon_{ijkl}m_\pm^{kl}$

Appendix B

The magnon

2.1 The motion on γ-deformed S^3.

The metric of $AdS_5 \times S^5$, reduced to the $\mathbb{R} \times S^3$ takes the following form:

$$ds^2 = -dt^2 + \frac{d\chi^2}{4\chi(1-\chi)} + (1-\chi)d\phi_1^2 + \chi d\phi_2^2. \tag{B.1}$$

We will be looking for a solution of the equations of motion in the following form:

$$\phi_1(\sigma,\tau) = \omega\tau + \frac{p}{2r}(\sigma - v\tau) + \phi(\sigma - v\tau); \tag{B.2}$$

$$\phi_2(\sigma,\tau) = \nu\tau + \frac{\delta}{2r}(\sigma - v\tau) + \alpha(\sigma - v\tau); \tag{B.3}$$

$$\chi(\sigma,\tau) = \chi(\sigma - v\tau), \tag{B.4}$$

where $\delta = 2\pi(n_2 - \gamma J_1)$ and $\phi(\sigma - v\tau)$, $\alpha(\sigma - v\tau)$, $\chi(\sigma - v\tau)$ satisfy periodic boundary conditions.

Substituting the ansatz into the equations of motion, integrating the equations for ϕ and α once, and using the Virasoro constraints (2.15) , we get the following equations:

$$\phi' = -(\frac{v\omega}{1-v^2} + \frac{p}{2r}) - \frac{vA_1}{1-v^2}\frac{1}{1-\chi} \tag{B.5}$$

$$\alpha' = -(\frac{v\nu}{1-v^2} + \frac{\delta}{2r}) - \frac{vA_2}{1-v^2}\frac{1}{\chi} \quad \text{(B.6)}$$

$$\frac{(1-v^2)^2}{4}\chi'^2 = \kappa_0 + \kappa_1\chi + \kappa_2\chi^2 + \kappa_3\chi^3 \quad \text{(B.7)}$$

$$\omega A_1 + \nu A_2 + 1 = 0. \quad \text{(B.8)}$$

The constants κ_i are as follows:

$$\kappa_0 = -v^2 A_2^2 \quad \text{(B.9)}$$

$$\kappa_1 = 1 - \omega^2 + v^2(1 + A_2^2 - A_1^2) \quad \text{(B.10)}$$

$$\kappa_2 = -1 - \nu^2 + 2\omega^2 - v^2 \quad \text{(B.11)}$$

$$\kappa_3 = \nu^2 - \omega^2, \quad \text{(B.12)}$$

Thus, in the notation of section 3 one may write

$$f_0 = -(\frac{v\omega}{1-v^2} + \frac{p}{2r}); \quad f_1 = -\frac{vA_1}{1-v^2};$$
$$a_0 = -(\frac{v\nu}{1-v^2} + \frac{\delta}{2r}); \quad a_1 = -\frac{vA_2}{1-v^2};$$
$$\kappa = \frac{1-v^2}{2\sqrt{\omega^2 - \nu^2}}.$$

We also have the following expressions for the charges[1]:

$$\mathcal{J} = \frac{1}{1-v^2}(2rv^2 A_1 + \omega \int_{-r}^{r} d\sigma\,(1-\chi)) \quad \text{(B.13)}$$

$$\mathcal{J}_2 = \frac{1}{1-v^2}(2rv^2 A_2 + \nu \int_{-r}^{r} d\sigma\,\chi) = 0. \quad \text{(B.14)}$$

Our equations can be written in the following form:

$$\text{Periodicity of } \phi : \quad \frac{rv\omega}{1-v^2} + \frac{p}{2} = -\frac{v\,A_1}{1-v^2}\int_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{(1-\chi)|\chi'|}; \quad \text{(B.15)}$$

[1]From these expressions one can derive a linear relation between $E, \mathcal{J}, \mathcal{J}_2$:

$$\frac{1-v^2}{\mathcal{E}}\left(\frac{\mathcal{J}}{\omega} + \frac{\mathcal{J}_2}{\nu}\right) = 1 + v^2\left(\frac{A_1}{\omega} + \frac{A_2}{\nu}\right).$$

$$\text{Periodicity of } \alpha : \quad \frac{r v \nu}{1-v^2} + \pi\delta = -\frac{v\, A_2}{1-v^2} \int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{\chi|\chi'|}; \tag{B.16}$$

$$\text{Charge } \mathcal{J} \equiv \frac{J_1}{g} : \quad \mathcal{J} = \frac{2}{1-v^2}\left(r A_1 v^2 + \omega \int\limits_{\chi_{\min}}^{\chi_{\max}} d\chi\, \frac{(1-\chi)}{|\chi'|}) \right) \tag{B.17}$$

$$\text{Charge } \mathcal{J}_2 \equiv \frac{J_2}{g} = 0 : \quad 0 = r v^2 A_2 + \nu \int\limits_{\chi_{\min}}^{\chi_{\max}} d\chi\, \frac{\chi}{|\chi'|}, \tag{B.18}$$

and the periodicity condition for χ which in this case takes the form

$$\text{Length of string:} \quad \int_{-r}^{0} d\sigma = r = \int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{|\chi'|}\,. \tag{B.19}$$

We have called the real roots of the equation $\chi_{\text{neg}}, \chi_{\min}, \chi_{\max}$ with the following ordering $\chi_{\text{neg}} \leq 0 \leq \chi_{\min} < \chi_{\max}$. Moreover, for the consistency of our approach we have to require that $\chi_{\min}, \chi_{\max} \in [0, 1)$, which will be justified by the solution. The fact that in the large J expansion one of the roots is negative can be easily proven. Indeed, in the strict $J \to \infty$ limit it follows from the work [19] that $\omega = 1,\ \nu = 0$, therefore the leading coefficient κ_3 of the polynomial in the r.h.s. of (B.7) is negative, and this should remain true for large J. The value of the r.h.s. of (B.7) at $\chi = 0$ is $\kappa_0 \leq 0$. These two facts together imply that there's a negative root χ_{neg}. Note also that the value of the r.h.s. of (B.7) at $\chi = 1$ is $-v^2 A_1^2 < 0$. This, together with the previous observation, implies that the two other roots of the polynomial either are both < 0 or both $\in [0, 1)$ or both > 1. We're interested in the case when they both lie in $[0, 1)$. We consider $(\chi_{\text{neg}}, \chi_{\min}, \chi_{\max})$ as independent variables that, together with all the previous variables (ν, ω, v, A_2), satisfy the following conditions which simply mean that $(\chi_{\text{neg}}, \chi_{\min}, \chi_{\max})$ are actually solutions of the cubic equation:

$$\chi_{\text{neg}} + \chi_{\min} + \chi_{\max} \quad = \quad -\frac{\kappa_2}{\kappa_3} \tag{B.20}$$

$$\chi_{\rm neg}\chi_{\rm min} + \chi_{\rm min}\chi_{\rm max} + \chi_{\rm neg}\chi_{\rm max} = \frac{\kappa_1}{\kappa_3} \tag{B.21}$$

$$\chi_{\rm neg}\chi_{\rm min}\chi_{\rm max} = -\frac{\kappa_0}{\kappa_3}. \tag{B.22}$$

We now switch to more convenient variables $(\widetilde{v}, \epsilon)$ instead of $\chi_{\rm min}, \chi_{\rm max}$ (leaving $\chi_{\rm neg}$ unaltered). These two sets are connected in the following way[2]:

$$\epsilon = \frac{\chi_{\rm min} - \chi_{\rm neg}}{\chi_{\rm max} - \chi_{\rm neg}}; \tag{B.23}$$

$$\widetilde{v}^2 = \frac{1 - \chi_{\rm max}}{1 - \chi_{\rm neg}}; \tag{B.24}$$

$$\chi_{\rm neg} = \chi_{\rm neg}. \tag{B.25}$$

Next we write the expressions for all integrals entering our equations:

$$\int\limits_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{d\chi}{\chi|\chi'|} = \frac{2\kappa}{(1-\widetilde{v}^2)^{3/2}(1-\chi_{\rm neg})^{1/2}(1+\chi_{\rm neg}\frac{\widetilde{v}^2}{1-\widetilde{v}^2})}\,\Pi\left(\frac{1-\chi_{\rm neg}}{1+\chi_{\rm neg}\frac{\widetilde{v}^2}{1-\widetilde{v}^2}}(1-\epsilon); 1-\epsilon\right);$$

$$\int\limits_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{d\chi}{(1-\chi)|\chi'|} = -\frac{2\kappa}{\widetilde{v}^2(1-\chi_{\rm neg})^{3/2}\sqrt{1-\widetilde{v}^2}}\,\Pi\left(\frac{\widetilde{v}^2-1}{\widetilde{v}^2}(1-\epsilon); 1-\epsilon\right); \tag{B.26}$$

$$\int\limits_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{d\chi}{|\chi'|} = \frac{2\kappa\, K(1-\epsilon)}{\sqrt{(1-\chi_{\rm neg})(1-\widetilde{v}^2)}};$$

$$\int\limits_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{d\chi\,\chi}{|\chi'|} = 2\kappa\frac{\chi_{\rm neg}\, K(1-\epsilon) + (1-\chi_{\rm neg})(1-\widetilde{v}^2)\, E(1-\epsilon)}{\sqrt{(1-\chi_{\rm neg})(1-\widetilde{v}^2)}};$$

$$\int\limits_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{d\chi\,(1-\chi)}{|\chi'|} = -2\kappa\frac{(\chi_{\rm neg}-1)\, K(1-\epsilon) + (1-\chi_{\rm neg})(1-\widetilde{v}^2)\, E(1-\epsilon)}{\sqrt{(1-\chi_{\rm neg})(1-\widetilde{v}^2)}}.$$

Thus, we have chosen the parameter ϵ rather than J as our expansion parameter. This means that we have to make an expansion of the system of equations (B.15)-(B.19) in ϵ and determine the corresponding coefficients in the expansion of various parameters, comparing powers of ϵ and/or $\log\epsilon$

[2]The purpose of introducing the variable ϵ should be clear — then the moduli of all tori in our expressions become $1-\epsilon$. The purpose of introducing $\widetilde{v}$ is the following: the first parameter of the Π-function in (B.26) becomes $\frac{\widetilde{v}^2-1}{\widetilde{v}^2}(1-\epsilon)$, so that it is in direct correspondence with an analogous parameter in the work [19].

which arise in this expansion. First of all, before solving the equations, we get rid of the variable r by plugging the expression for r from (B.19) into all other equations.

We make the following ansatz for our parameters:

$$\begin{aligned}
v(\epsilon) &= v_0(\epsilon) + v_1(\epsilon)\epsilon + O(\epsilon^2); \\
\widetilde{v}(\epsilon) &= \widetilde{v}_0(\epsilon) + \widetilde{v}_1(\epsilon)\epsilon + O(\epsilon^2); \\
\omega(\epsilon) &= \omega_0(\epsilon) + \omega_1(\epsilon)\epsilon + O(\epsilon^2); \\
\nu(\epsilon) &= \nu_1(\epsilon)\epsilon + O(\epsilon^2); \\
A_1(\epsilon) &= A_{1,0}(\epsilon) + A_{1,1}(\epsilon)\epsilon + O(\epsilon^2); \\
A_2(\epsilon) &= A_{2,1}(\epsilon)\epsilon + O(\epsilon^2); \\
\chi_{\text{neg}}(\epsilon) &= \chi_1(\epsilon)\epsilon + O(\epsilon^2); \\
\mathcal{J}(\epsilon) &= \mathcal{J}_0(\epsilon) + \mathcal{J}_1(\epsilon)\epsilon + O(\epsilon^2),
\end{aligned} \tag{B.27}$$

where we assume that all "coefficient" functions like $v_0(\epsilon), v_1(\epsilon), \widetilde{v}_0(\epsilon)$, etc. are terminating series in $\log\epsilon$ (this is the reason why expansions (B.27) are justified). This assumption will be proved aposteriori — by the solution that we will find.

We substitute (B.27) into our equations and expand these equations in ϵ^m, ignoring terms with logarithms (that is, treating any combination $\left(\sum\limits_{k=0}^{n} a_k(\log\epsilon)^k\right)\epsilon^m$ as just ϵ^m). Then we obtain a system of equations for our "coefficient" functions, which, when solved, exhibits the property of these functions mentioned above — that is, they're terminating series in powers of $\log\epsilon$.

In the course of expanding the above written equations we need an expansion for $\Pi(1-\alpha\,\epsilon, 1-\epsilon)$ as $\epsilon \to 0$ (α fixed and $0 < \alpha < 1$). To find such an expansion we make use of the following textbook identity for elliptic functions:

$$\Pi(1-\alpha\,\epsilon,1-\epsilon)=\frac{1}{\alpha(\alpha-1)\epsilon}\left[\alpha(1-\epsilon)K(1-\epsilon)-(1-\alpha\epsilon)\Pi(\frac{\alpha-1}{\alpha};1-\epsilon)\right]. \tag{B.28}$$

The meaning of using this identity is that it explicitly singles out the $\frac{1}{\epsilon}$ factor in the expansion. Once we have written $\Pi(1-\alpha\,\epsilon,1-\epsilon)$ in this form, we may use `Mathematica` to generate the expansions of functions in the r.h.s. of (B.28):

$$\Pi(1-\alpha\,\epsilon,1-\epsilon)=\frac{\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right)}{\sqrt{\frac{1}{\alpha}-1}\,\alpha\,\epsilon}+ \tag{B.29}$$
$$+\ \frac{\left(2\alpha\sqrt{\frac{1}{\alpha}-1}\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right)+(\alpha-1)(-\log(\epsilon/16)+1)\right)}{4(\alpha-1)}+$$
$$+\ \frac{\left(8\alpha^2\sqrt{\frac{1}{\alpha}-1}\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right)-(\alpha-1)(2\alpha+2(2\alpha+1)\log(\epsilon/16)+3)\right)\epsilon}{64(\alpha-1)}+O\left(\epsilon^2\right)$$

However, in our case α is not constant in ϵ but rather depends on ϵ in the following way:

$$\alpha(\epsilon)=\frac{\frac{\chi_{\rm neg}(\epsilon)}{\epsilon}+(1-\chi_{\rm neg}(\epsilon))(1-\widetilde{v}^2(\epsilon))}{1-\widetilde{v}^2(\epsilon)(1-\chi_{\rm neg}(\epsilon))}. \tag{B.30}$$

According to our ansatz (B.27) $\alpha(\epsilon)$ has a finite positive limit smaller than 1 as $\epsilon\to 0$ — this is the only thing, which is important for our expansions to be justified. That is, we plug the expansion of α in (powers and logarithms of) ϵ into the expansion for $\Pi(1-\alpha\,\epsilon,1-\epsilon)$ obtained at fixed α.

We also need to know the expansion of $\Pi\left(\frac{\widetilde{v}^2-1}{\widetilde{v}^2}(1-\epsilon);1-\epsilon\right)$ as $\epsilon\to 0$. It was constructed in the appendix of [19]. One has to use the identity

$$\Pi\left(\frac{v^2-1}{v^2}(1-\epsilon);1-\epsilon\right)= \tag{B.31}$$
$$=\ \frac{1}{(1-(1-v^2)\,\epsilon)\,K(\epsilon)}\left[\frac{1}{2}\pi v\sqrt{(1-v^2)\,(1-(1-v^2)\,\epsilon)}F\left(\arcsin\left(\sqrt{1-v^2}\right);\epsilon\right)+\right.$$
$$\left.+\ K(1-\epsilon)\left(\left(1-\left(1-v^2\right)\epsilon\right)K(\epsilon)-\left(1-v^2\right)(1-\epsilon)\Pi\left(\frac{v^2\epsilon}{1-(1-v^2)\,\epsilon};\epsilon\right)\right)\right]$$

In the r.h.s. there's only one function, which has an expansion that cannot be directly obtained by `Mathematica`, and its expansion looks as follows:

$$\Pi\left(\frac{v^2\epsilon}{1-(1-v^2)\epsilon};\epsilon\right)=\frac{\pi}{2}+\frac{1}{8}\left(2\pi v^2+\pi\right)\epsilon+\frac{1}{128}\pi\left(-8v^4+44v^2+9\right)\epsilon^2+$$
$$+\quad\frac{1}{512}\pi\left(16v^6-72v^4+206v^2+25\right)\epsilon^3+O\left(\epsilon^4\right). \tag{B.32}$$

Inverting the expansion

$$J(\epsilon)=J_0(\epsilon)+J_1(\epsilon)\epsilon+o(\epsilon), \tag{B.33}$$

we obtain ϵ as a function of J, that is we return to our original expansion in the limit $J\to\infty$:

$$\epsilon(J)=\frac{16}{e^2}e^{-\frac{\mathcal{J}}{\sin\frac{p}{2}}}\left[1-\frac{8}{e^2}e^{-\frac{\mathcal{J}}{\sin\frac{p}{2}}}\left(1-\mathcal{J}\,\frac{2-3\sin^2\frac{p}{2}}{2\sin\frac{p}{2}}\cos(\Phi)-\frac{1}{2}\mathcal{J}^2\cot^2\frac{p}{2}\cos\Phi\right)+\ldots\right]. \tag{B.34}$$

We now write out explicitly the expansions of the parameters entering the equations of motion:

$$\begin{aligned}
\chi_{\rm neg}(\mathcal{J}) &= -\frac{16}{e^2}\sin^2\frac{p}{2}\,\sin^2\frac{\Phi}{2}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots, && \text{(B.35)}\\
\chi_{\max}(\mathcal{J}) &= \sin^2\frac{p}{2}+\frac{8}{e^2}\sin\frac{p}{2}\,\cos^2\frac{p}{2}\,\cos\Phi\,(3\,\sin\frac{p}{2}+\mathcal{J})\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,\\
\chi_{\min}(\mathcal{J}) &= \frac{16}{e^2}\sin^2\frac{p}{2}\,\cos^2\frac{\Phi}{2}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,\\
v(\mathcal{J}) &= \cos\frac{p}{2}-\frac{4}{e^2}\sin\frac{p}{2}\,\cos\frac{p}{2}\,\cos\Phi\,(\sin\frac{p}{2}+\mathcal{J})\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,\\
\omega(\mathcal{J}) &= 1+\frac{8}{e^2}\sin^2\frac{p}{2}\,\cos\Phi\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,\\
\nu(\mathcal{J}) &= \frac{4}{e^2}\cos\frac{p}{2}\,\sin\Phi\,(2\sin\frac{p}{2}+\mathcal{J})\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,\\
f_0(\mathcal{J}) &= -\frac{p}{\mathcal{E}}-\frac{\cos\frac{p}{2}}{\sin^2\frac{p}{2}}+\frac{\cos\Phi\,\sin p\left(2\mathcal{J}\cos p+6\mathcal{J}-\sin\frac{p}{2}+3\sin\frac{3p}{2}\right)}{2e^2\,\sin^4\frac{p}{2}}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,\\
f_1(\mathcal{J}) &= \frac{\cos\frac{p}{2}}{\sin^2\frac{p}{2}}+\frac{\cos\Phi\,\sin p\left(\sin\frac{3p}{2}-2\mathcal{J}(\cos p+3)-11\sin\frac{p}{2}\right)}{2e^2\,\sin^4\frac{p}{2}}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,\\
a_0(\mathcal{J}) &= -\frac{\delta}{\mathcal{E}}-\frac{4}{e^2}\left(\mathcal{J}+2\sin\frac{p}{2}\right)\sin\Phi\,\cot^2\frac{p}{2}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,\\
a_1(\mathcal{J}) &= \frac{8}{e^2}\sin\frac{p}{2}\,\sin\Phi\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\ldots,
\end{aligned}$$

where

$$\Phi=\frac{\delta}{2^{3/2}\cos^3(\frac{p}{4})}, \tag{B.36}$$

and the solution exists for all $p\in[-\pi;\pi]$ (if and) only if

$$|\delta|=|2\pi(n_2-\gamma J)|\le\pi. \tag{B.37}$$

This means that for the undeformed $AdS_5 \times S^5$, that is $\gamma = 0$, the only possible choice is $n_2 = 0$, or $\delta = 0$. In this case all formulas reduce to what was found in [19].

To obtain the dispersion relation one should expand (B.19) with respect to ϵ and then substitute the expansion (B.34) of ϵ in terms of J. The dispersion relation with the first correction has the following form:

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin\frac{p}{2} \left(1 - \frac{4}{e^2} \sin^2\frac{p}{2} \cos\Phi \, e^{-\frac{\mathcal{J}}{\sin p/2}} + \ldots\right); \qquad \text{(B.38)}$$

$$\Phi = \frac{\delta}{2^{3/2}\cos^3\frac{p}{4}}; \quad |\delta| = |2\pi(n_2 - \gamma J)| \leq \pi.$$

Appendix C

Symmetry algebra

The J-matrices – generators of $SO(3)$ – used in section 2, are defined as follows:

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{C.1}$$

They satisfy the usual condition $[J_k,\, J_l] = -\epsilon_{klm} J_m$.

The matrix W defined in (3.5) looks as follows:

$$W = \begin{pmatrix} 0 & 0 & \frac{\omega_1}{2} & -\frac{i\omega_1}{2} & \frac{\omega_2}{2} & -\frac{i\omega_2}{2} \\ 0 & 0 & -\frac{i\omega_1}{2} & -\frac{\omega_1}{2} & -\frac{i\omega_2}{2} & -\frac{\omega_2}{2} \\ -\frac{\omega_1}{2} & \frac{i\omega_1}{2} & 0 & 0 & 0 & 0 \\ \frac{i\omega_1}{2} & \frac{\omega_1}{2} & 0 & 0 & 0 & 0 \\ -\frac{\omega_2}{2} & \frac{i\omega_2}{2} & 0 & 0 & 0 & 0 \\ \frac{i\omega_2}{2} & \frac{\omega_2}{2} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.2}$$

In the main text we have used a separate notation for the first row of this matrix:

$$\widehat{w} = (\frac{\omega_1}{2},\, -\frac{i\omega_1}{2},\, \frac{\omega_2}{2},\, -\frac{i\omega_2}{2}) = (\omega_1,\, \omega_2) \otimes \frac{1}{2}(1, -i). \tag{C.3}$$

The fermionic matrix θ looks as follows:

$$\theta = \left(\begin{array}{llllll} \kappa_1^{+1} - \kappa_2^{-1} & -i\left(\kappa_1^{+1} + \kappa_2^{-1}\right) & \frac{1}{2}\left(\chi_{1,1}^{+} - \chi_{1,2}^{-}\right) & -\frac{1}{2}i\left(\chi_{1,1}^{+} + \chi_{1,2}^{-}\right) & \frac{1}{2}\left(\chi_{1,2}^{+} + \chi_{1,1}^{-}\right) & -\frac{1}{2}i\left(\chi_{1,2}^{+} - \chi_{1,1}^{-}\right) \\ \kappa_1^{-1} + \kappa_2^{+1} & i\left(\kappa_1^{-1} - \kappa_2^{+1}\right) & \frac{1}{2}\left(\chi_{2,1}^{+} - \chi_{2,2}^{-}\right) & -\frac{1}{2}i\left(\chi_{2,1}^{+} + \chi_{2,2}^{-}\right) & \frac{1}{2}\left(\chi_{2,2}^{+} + \chi_{2,1}^{-}\right) & -\frac{1}{2}i\left(\chi_{2,2}^{+} - \chi_{2,1}^{-}\right) \\ -\bar\kappa_1^{-1} - \bar\kappa_2^{+1} & i\left(\bar\kappa_1^{-1} - \bar\kappa_2^{+1}\right) & \frac{1}{2}\left(\bar\chi_{2,2}^{-} - \bar\chi_{2,1}^{+}\right) & -\frac{1}{2}i\left(\bar\chi_{2,1}^{+} + \bar\chi_{2,2}^{-}\right) & \frac{1}{2}\left(-\bar\chi_{2,2}^{+} - \bar\chi_{2,1}^{-}\right) & -\frac{1}{2}i\left(\bar\chi_{2,2}^{+} - \bar\chi_{2,1}^{-}\right) \\ \bar\kappa_1^{+1} - \bar\kappa_2^{-1} & i\left(\bar\kappa_1^{+1} + \bar\kappa_2^{-1}\right) & \frac{1}{2}\left(\bar\chi_{1,1}^{+} - \bar\chi_{1,2}^{-}\right) & \frac{1}{2}i\left(\bar\chi_{1,1}^{+} + \bar\chi_{1,2}^{-}\right) & \frac{1}{2}\left(\bar\chi_{1,2}^{+} + \bar\chi_{1,1}^{-}\right) & \frac{1}{2}i\left(\bar\chi_{1,2}^{+} - \bar\chi_{1,1}^{-}\right) \end{array} \right)$$

At this point we would like to remind the reader that the kappa-gauge corresponds to setting $\chi^+ = \chi^- \equiv \chi$ in the matrix written above.

We can equally express the fields $\kappa^{\pm 1}$, $\chi^{\pm 0}$ in terms of the n_{ij}, i.e. elements of the matrix θ. Namely,

$$\kappa_1^{+1} = \tfrac{1}{2}(n_{11} + i\, n_{12}),\ \kappa_2^{+1} = \tfrac{1}{2}(n_{21} + i\, n_{22}) \tag{C.4}$$
$$\chi_{1\mathrm{i}}^{+0} = n_{13} + i\, n_{14},\ \chi_{1\dot{2}}^{+0} = n_{15} + i\, n_{16}$$
$$\chi_{2\mathrm{i}}^{+0} = n_{23} + i\, n_{24},\ \chi_{2\dot{2}}^{+0} = n_{25} + i\, n_{26}$$

The 'conjugate' combinations are

$$\kappa_1^{-1} = \tfrac{1}{2}(n_{21} - i\, n_{22}),\ \kappa_2^{-1} = -\tfrac{1}{2}(n_{11} - i\, n_{12}) \tag{C.5}$$
$$\chi_{1\mathrm{i}}^{-0} = n_{15} - i\, n_{16},\ \chi_{1\dot{2}}^{-0} = -(n_{13} - i\, n_{14})$$
$$\chi_{2\mathrm{i}}^{-0} = n_{25} - i\, n_{26},\ \chi_{2\dot{2}}^{-0} = -(n_{23} - i\, n_{24})$$

First of all, we describe explicitly the matrix generators of the $su(2) \oplus su(2) \oplus u(1)$ algebra. We introduce the following matrices:

$$t_k = -\tfrac{i}{2}\Delta(\sigma_k),\ u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2,$$
$$s_1 = \tfrac{1}{2}\sigma_1 \otimes i\sigma_2,\ s_2 = -\tfrac{1}{2} i\sigma_2 \otimes I_2,\ s_3 = \tfrac{1}{2}\sigma_3 \otimes i\sigma_2, \tag{C.6}$$

I_2 being the 2×2 identity matrix and Δ the diagonal embedding: $\Delta(a) = I_2 \otimes a$. In these notations t_k describe the $su(2)|_{4\times4}^{\mathrm{AdS}}$, u is the $u(1)|_{2\times2}$ U(1)-charge from $\mathbb{CP}^3$ and s_k describe the $su(2)|_{4\times4}^{\mathbb{CP}}$. These matrices (after corresponding embeddings into 10×10 matrices) satisfy the necessary reality conditions (for example, the s_i are real) and the following commutation relations:

$$[t_i,\, t_j] = \epsilon_{ijk}\, t_k,\ [s_i,\, s_j] = \epsilon_{ijk}\, s_k,\ [t_i,\, s_j] = [t_i,\, u] = [s_i,\, u] = 0. \tag{C.7}$$

3.1 The charges

In this appendix we explain the notations for the worldsheet fields which enter the expressions for the charges appearing in the symmetry algebra (3.26).

More precisely, we need to explain how indices of various fields are lowered and raised, since without this understanding it is impossible to check the covariance of the expressions that we obtain, even if lower indices are always contracted with upper ones. First of all, $Z \equiv z_i\sigma_i$ and $P_z \equiv P_{z_i}\sigma_i$. For matrix elements of these matrices we use the notation $Z^a{}_b$ and $(P_z)^a{}_b$ respectively. We need to use this shifted notation for the indices, since otherwise it would not be clear, what Z_1^2 and Z_2^1 stand for. The indices in our notation should be (as usual) read from left to right, that is, for instance $Z^1{}_2$ is the element in the first row and second column, etc. As for the fermions, we use notation χ^a_α, $\kappa^{a,+1}$, κ_b^{-1}. The $\kappa^{\pm 1}$ have different positions of the index, since they're in conjugate representations[1]. Obviously, the conjugate of a field transforms in a representation, conjugate to the one of this field. Thus, conjugation changes the position of the index. For example, $\bar{\chi}^\alpha_a \equiv (\chi^a_\alpha)^*$, $\bar{\kappa}^{+1}_a \equiv (\kappa^{a,+1})^*$, $(Z^*)_a{}^b \equiv (Z^a{}_b)^*$, etc. Starting from this point, one can raise or lower indices, using ϵ^{ab} and $\epsilon^{\alpha\beta}$. For instance, $v^a \equiv \epsilon^{ab}v_b$ and $v_a = -\epsilon_{ab}v^b$. Once the minus sign in the previous formula has been written out explicitly, $\epsilon^{ab} = \epsilon_{ab}$.

We remind the reader that ϵ_{ab} is the Clebsch-Gordan coefficient for coupling two spins $\frac{1}{2}$ to obtain spin 0, whereas $(\epsilon\sigma_i)_{ab}$ are the Clebsch-Gordan coefficients for coupling two spins $\frac{1}{2}$ to obtain spin 1. This means, for instance, that $\epsilon_{ab}v_a w_b$ is a scalar, whereas $(\epsilon\sigma_i)_{ab}v^a w^b$ is a vector.

[1]These representations are equivalent, as we discussed in the text. However, we prefer to define the fermions precisely this way to get rid of some extra ϵ-symbols. We should just bear in mind that the indices in this case should be contracted as $\kappa^{a,+1}\kappa_a^{-1}$ or $(\kappa_a^{\pm})^*\kappa_a^{\pm}$, etc.

3.2 Geodesics

As is well-known,the Penrose limit is an expansion in the vicinity of a geodesic. We call geodesics γ_1 and γ_2 equivalent, if γ_2 can be obtained from γ_1 by action of the isometry group. Since a geodesic is determined as a solution of a second order differential equation, it is determined by the initial point $\gamma(0)$ and velocity $\dot{\gamma}(0)$. Obviously, velocities $s\dot{\gamma}(0)$ define the same geodesic for any nonzero s (the only difference comes from the dilation of an affine parameter on the geodesic). Thus, if G acts transitively on $\mathcal{M}$ and H acts transitively on $\mathrm{P}(V_\perp)$ (P denoting projectivization), then all geodesics are equivalent. In our case $\mathrm{P}(V_\perp) = \mathbb{R}\mathrm{P}^5$. A stronger condition is that, instead of the action on $\mathbb{R}P^5$, H should act transitively on S^5, which might be more convenient and is probably satisfied in many cases. Another wording is that the representation of H on V should be irreducible over $\mathbb{R}$. For instance, this is the case for the manifold under consideration, since V decomposes as $V_\perp = 3 \oplus \bar{3}$ over $\mathbb{C}$, but is irreducible over $\mathbb{R}$ under the action of $H = U(3)$. From the former viewpoint, $U(3)$ also acts transitively on S^5, which, among other things, gives rise to a coset $U(3)/U(2) = S^5$ (and even, cancelling the $U(1)$ factors, $SU(3)/SU(2) = S^5$).

There's an important exception, however, which we have omitted in the argumentation presented above. It is the case, when two geodesics 'touch' at some point $p \in \mathcal{M}$. Definition of touching is obvious and means that they both pass through the point p and have the same velocity direction (once again, up to $\pm$, that is 'backward' and 'forward' are not distinguished), i.e. $\dot{\gamma}_1(p) \propto \dot{\gamma}_2(p)$. In this case, the solution of the differential equation is not specified by the point p and the velocity at this point. This may well happen, since for the uniqueness of a solution a differential equation should have a regular r.h.s. (we assume that we are dealing with a system of first-order

ODEs, written in the form $\dot{y}_i = f_i(\{y_j\})$).[2]

For the moment we consider the question with geodesics as not totally settled, at least for us it is unclear at the moment whether any of the geodesics can touch in $\mathbb{CP}^3$. Of course, it should be possible to check this by a direct calculation, namely, solution of the geodesic equation.

[2]For instance, consider a very simple example of equation (reminiscent of the equation of the classical giant magnon) $z' = \sqrt{z}$ with initial data $z(0) = 0$. It has two solutions: $z \equiv 0$ and $z = \frac{t^2}{4}$. This is due to the fact that $\frac{d}{dz}(r.h.s.) = \frac{1}{2\sqrt{z}}$ is not bounded in the vicinity of $z = 0$.

Appendix D

The rotating string

4.1 The coset space $\mathrm{AdS}_4 \times \mathbb{CP}^3$

To make the thesis self-contained, in this appendix we recapitulate the basic facts about the description of the coset space $\mathrm{AdS}_4 \times \mathbb{CP}^3 = \mathrm{OSP}(2,2|6)/\mathrm{SO}(3,1) \times \mathrm{U}(3)$.

Consider 10×10 supermatrices

$$A = \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix}, \tag{D.1}$$

where X and Y are even (bosonic) 4×4 and 6×6 matrices, respectively. The 4×6 matrix θ and the 6×4 matrix η are odd, i.e. linear in fermionic variables. As the matrix superalgebra, the Lie superalgebra $\mathfrak{osp}(6|4)$, is spanned by supermatrices A satisfying two conditions

$$A^{st} = -CAC^{-1}, \qquad A^{\dagger} = -\Gamma A \Gamma^{-1}, \tag{D.2}$$

where $C = \mathrm{diag}(C_4, \mathbb{I}_6)$ and $\Gamma = \mathrm{diag}(\Gamma_0, -\mathbb{I}_6)$. Here A^{st} stands for the supertranspose of A:

$$A^{st} = \begin{pmatrix} X^t & -\eta^t \\ \theta^t & Y^t \end{pmatrix}. \tag{D.3}$$

The bosonic subalgebra of $\mathfrak{osp}(6|4)$ is $\mathfrak{usp}(2,2) \oplus \mathfrak{so}(6)$. Explicitly, the fermionic matrices obey

$$\eta = -\theta^t C_4\,, \qquad \theta^* = i\Gamma_2\theta\,, \tag{D.4}$$

i.e. fermions are symplectic Majorana with the total number of real fermionic components equal to 24.

We further introduce a 6×6 matrix K_6 and a 10×10 matrix $K = \mathrm{diag}(K_4, K_6)$:

$$K_6 = \mathbb{I}_3 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K = \mathbb{I}_5 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{D.5}$$

This matrices can be used to define an automorphism Ω of order four of the complexified algebra $\mathfrak{osp}(4|6)$

$$\Omega(A) = \begin{pmatrix} K_4 X^t K_4 & K_4\eta^t K_6 \\ -K_6\theta^t K_4 & K_6 Y^t K_6 \end{pmatrix} = -\Sigma K A^{st} K^{-1}\Sigma^{-1}\,. \tag{D.6}$$

Here $\Sigma = \mathrm{diag}(\mathbb{I}_4, -\mathbb{I}_6)$ is the grading matrix. The orthosymplectic condition for A implies

$$\Omega(A) = (\Sigma K C)A(\Sigma K C)^{-1} \equiv \Upsilon A \Upsilon^{-1}\,, \tag{D.7}$$

i.e. Ω is an inner automorphism. Explicitly,

$$\Upsilon = \begin{pmatrix} \Gamma_5 & 0 \\ 0 & -K_6 \end{pmatrix}. \tag{D.8}$$

As the vector space, $\mathcal{A} = \mathfrak{osp}(6|4)$ can be decomposed under Ω into the direct sum of homogeneous components: $\mathcal{A} = \sum_{k=0}^{3} \mathcal{A}^{(k)}$, where the projection $A^{(k)}$ of a generic element $A \in \mathfrak{osp}(6|4)$ on the subspace $\mathcal{A}^{(k)}$ is define as

$$A^{(k)} = \frac{1}{4}\Big(A + i^{3k}\Omega(A) + i^{2k}\Omega^2(A) + i^k\Omega^3(A)\Big)\,. \tag{D.9}$$

In particular, $\mathcal{A}^{(0)} = \mathfrak{so}(3,1) \oplus \mathrm{u}(3)$.

Throughout Chapter 4 we use the generators T_{ij} of $\mathfrak{so}(6)$ defined as $T_{ij} = E_{ij} - E_{ji}$, where E_{ij} are the standard matrix unities. We also introduce the following six matrices T_6 which are Lie algebra generators of $\mathfrak{so}(6)$ along the $\mathbb{CP}^3$ directions:

$$\begin{aligned} T_1 &= E_{13} - E_{31} - E_{24} + E_{42}\,, & T_2 &= E_{14} - E_{41} + E_{23} - E_{32}\,, \\ T_3 &= E_{15} - E_{51} - E_{26} + E_{62}\,, & T_4 &= E_{16} - E_{61} + E_{25} - E_{52}\,, \\ T_5 &= E_{35} - E_{53} - E_{46} + E_{64}\,, & T_6 &= E_{36} - E_{63} + E_{45} - E_{54}\,. \end{aligned} \tag{D.10}$$

These generators are normalized as $\mathrm{tr}(T_i T_j) = -4\delta_{ij}$.

According to [11], a generic SO(6) element parametrizing the coset space $\mathbb{CP}^3 = \mathrm{SO}(6)/\mathrm{U}(3)$ can be written as

$$g_{\mathbb{CP}} = e^{y_i T_i}\,. \tag{D.11}$$

We parametrize $\mathbb{CP}^3$ by means of the spherical coordinates $(r, \varphi, \theta, \alpha_1, \alpha_2, \alpha_3)$, or, alternatively, by means of three complex inhomogeneous coordinate w_i,

$$\begin{aligned} y_1 + i y_2 &= r\,\sin\theta\cos\frac{\alpha_1}{2}\,e^{\frac{i}{2}(\alpha_2+\alpha_3)+\frac{i}{2}\varphi} = \frac{r}{|w|}w_1\,, \\ y_3 + i y_4 &= r\,\sin\theta\sin\frac{\alpha_1}{2}\,e^{-\frac{i}{2}(\alpha_2-\alpha_3)+\frac{i}{2}\varphi} = \frac{r}{|w|}w_2\,, \\ y_5 + i y_6 &= r\,\cos\theta\,e^{i\varphi} = \frac{r}{|w|}w_3\,, \end{aligned} \tag{D.12}$$

where $|w|^2 = \bar{w}_k w_k$ and $\sin r = \frac{|w|}{\sqrt{1+|w|^2}}$. The geodesic circle described by the angle φ corresponds to taking $\theta = 0$ and $r = \frac{\pi}{4}$, or, equivalently, $w_3 = e^{i\varphi}$ and $w_1 = 0 = w_2$. If we further extract a geodesic angle φ by introduce one real field x and two complex fields v_1 and v_2:

$$w_3 = (1-x)e^{i\varphi}\,, \quad w_1 = \frac{1}{\sqrt{2}}v_1 e^{i\varphi/2}\,, \qquad w_2 = \frac{1}{\sqrt{2}}v_2 e^{i\varphi/2}\,, \tag{D.13}$$

then the corresponding quadratic action for the $\mathbb{CP}^3$ fluctuation modes around the (S, J)-string solution coincides with the plane-wave action obtained in [11].

4.2 Kappa-symmetry

Here we present an independent analysis of κ-symmetry transformations in the background of the (S, J)-string. As was explained in [11], κ-symmetry acts on the coset element by multiplication from the right:

$$g \to g e^{\epsilon} = g' g_c \,, \tag{D.14}$$

where g_c is a compensating group element from the denominator of the coset. We see that at linear order in χ and ϵ we get

$$g \to g_{\text{O}} g_{\chi} g_{\text{B}} e^{\epsilon} = g_{\text{O}} e^{\chi} e^{g_{\text{B}} \epsilon g_{\text{B}}^{-1}} g_{\text{B}} \approx g_{\text{O}} e^{\chi + g_{\text{B}} \epsilon g_{\text{B}}^{-1}} g_{\text{B}} \tag{D.15}$$

Thus, at the linearized level the fermion matrix χ changes under the κ-symmetry variation as

$$\chi \to \chi + g_{\text{B}} \epsilon g_{\text{B}}^{-1} \,. \tag{D.16}$$

Note also that the compensation matrix g_c which depends on the even number of fermions does not arise for the linearized transformations. The parameter $\epsilon = \epsilon^{(1)} + \epsilon^{(2)}$ in the above formula is the one found in [11], e.g.,

$$\begin{aligned} \epsilon^{(1)} &= A^{(2)}_{\alpha,-} A^{(2)}_{\beta,-} \kappa^{\alpha\beta}_{++} + \\ &+ \kappa^{\alpha\beta}_{++} A^{(2)}_{\alpha,-} A^{(2)}_{\beta,-} + A^{(2)}_{\alpha,-} \kappa^{\alpha\beta}_{++} A^{(2)}_{\beta,-} - \frac{1}{8} \text{str}(\Upsilon^2 A^{(2)}_{\alpha,-} A^{(2)}_{\beta,-}) \kappa^{\alpha\beta}_{++} \,, \end{aligned} \tag{D.17}$$

where $\kappa^{\alpha\beta}_{++}$ is the κ-symmetry parameter[1]. It is easy to find

$$A^{(2)}_{\tau} = -\frac{1}{2} g_{\text{B}}^{-1} \Big(g_{\text{O}}^{-1} \partial_\tau g_{\text{O}} + G (g_{\text{O}}^{-1} \partial_\tau g_{\text{O}})^t G^{-1} \Big) g_{\text{B}} \,, \tag{D.18}$$

$$A^{(2)}_{\sigma} = -\frac{1}{2} g_{\text{B}}^{-1} (\partial_\sigma G G^{-1}) g_{\text{B}} \,. \tag{D.19}$$

[1]We present the complete analysis for $\epsilon^{(1)}$ only, the computation of $\epsilon^{(2)}$ goes along the same lines.

Hence, in the conformal gauge

$$A^{(2)}_{\tau,-} = \frac{1}{2}(A^{(2)}_{\tau} - A^{(2)}_{\sigma}) \equiv g_{\rm B}^{-1}\hat{A}g_{\rm B}\,, \tag{D.20}$$

where

$$\hat{A} = -\frac{1}{4}\Big(g_{\rm O}^{-1}\partial_\tau g_{\rm O} + G(g_{\rm O}^{-1}\partial_\tau g_{\rm O})^t G^{-1} - \partial_\sigma G G^{-1}\Big)\,. \tag{D.21}$$

An element G entering the last formula is determined from eq.(4.5) to be

$$G = \left(\begin{array}{cc} e^{-\frac{i}{2}\rho\Gamma_2}K_4 e^{-\frac{i}{2}\rho\Gamma_2} & 0 \\ 0 & e^{\frac{\pi}{4}T_5}K_6 e^{-\frac{\pi}{4}T_5} \end{array}\right) = \left(\begin{array}{cc} e^{-i\rho\Gamma_2}K_4 & 0 \\ 0 & e^{\frac{\pi}{2}T_5}K_6 \end{array}\right) \tag{D.22}$$

We also note that since we pulled out the factor $g_{\rm B}$ out of $A^{(2)}$, the matrix $\hat{A}$ is not element of the space $\mathcal{A}^{(2)}$.

Thus, under κ-symmetry transformation the fermionic matrix χ is shifted by

$$g_{\rm B}\epsilon^{(1)}g_{\rm B}^{-1} = \hat{A}\hat{A}\kappa + \kappa\hat{A}\hat{A} + \hat{A}\kappa\hat{A} - \frac{1}{8}{\rm str}(\Upsilon^2\hat{A}\hat{A})\kappa\,.$$

In order find out implementation of this formula for χ, we have to understand the structure of the matrix $\hat{A}$. Calculations reveal the following remarkably simple formula

$$\hat{A} = \frac{\omega_2}{4}\left(\begin{array}{cc} -\frac{i}{\omega_2}e^{-\frac{i}{2}\rho\Gamma_2}\big(\varkappa\cosh\rho\Gamma_0 - \omega_1\sinh\rho\Gamma_1 + \rho'\Gamma_2\big)e^{\frac{i}{2}\rho\Gamma_2} & 0 \\ 0 & T_{34} + T_{56} \end{array}\right).$$

The non-trivial Virasoro constraint written in terms of $\hat{A}$ implies

$$0 = 4\,{\rm str}(\hat{A}\hat{A}) = \rho'^2 + \omega_1^2\sinh^2\rho - \varkappa^2\cosh^2\rho + \omega_2^2\,, \tag{D.23}$$

which is an equation for the function ρ. An important about the matrix $\hat{A}$ is that it is not constant on the world-sheet, quite in opposite to the point-particle case. On the other hand, since an expression

$$\varkappa\cosh\rho\Gamma_0 - \omega_1\sinh\rho\Gamma_1 + \rho'\Gamma_2 \tag{D.24}$$

multiplied with i takes values in $\mathcal{A}^{(2)}$, one can always find a similarity transformation with an element V from SO(3,1), which brings $\hat{A}$ to a constant matrix, e.g, to Γ_0, namely,

$$\varkappa \cosh \rho \Gamma_0 - \omega_1 \sinh \rho \Gamma_1 + \rho' \Gamma_2 = \omega_2 V \Gamma_0 V^{-1}\,, \tag{D.25}$$

where we have taken into account that on solutions of the Virasoro constraint (D.23), the eigenvalues of matrix (D.24) are $\pm\omega_2$. For instance, one can take

$$V = v \left(\begin{array}{cccc} \omega_1 \sinh \rho - i\rho' & 0 & 0 & \omega_2 - \kappa \cosh \rho \\ 0 & \omega_1 \sinh \rho + i\rho' & \omega_2 - \kappa \cosh \rho & 0 \\ 0 & \omega_2 - \kappa \cosh \rho & \omega_1 \sinh \rho - i\rho' & 0 \\ \omega_2 - \kappa \cosh \rho & 0 & 0 & \omega_1 \sinh \rho + i\rho' \end{array} \right), \tag{D.26}$$

where the unessential normalization constant is fixed by requiring $\det V = 1$. Indeed, one can check that on solutions of the Virasoro constraint the relation (D.25) is satisfied. Thus, the matrix $\hat{A}$ exhibits the following factorizable structure

$$\hat{A} = \frac{\omega_2}{4} \mathscr{V} \mathscr{A} \mathscr{V}^{-1}\,, \tag{D.27}$$

where we have introduced two matrices:

$$\mathscr{A} = \left(\begin{array}{cc} -i\Gamma_0 & 0 \\ 0 & T_{34} + T_{56} \end{array} \right), \qquad \mathscr{V} = \left(\begin{array}{cc} e^{-\frac{i}{2}\rho\Gamma_2} V & 0 \\ 0 & \mathbb{I} \end{array} \right), \tag{D.28}$$

where, in particular, matrix $\mathscr{A}$ does not depend on the world-sheet variables. We thus see that under a linearized κ-symmetry transformation the combination $\mathscr{V}^{-1}\chi\mathscr{V}$ undergoes a shift by an element

$$\frac{\omega^2}{16}\Big[\mathscr{A}^2(\mathscr{V}^{-1}\kappa\mathscr{V}) + \mathscr{A}(\mathscr{V}^{-1}\kappa\mathscr{V})\mathscr{A} + (\mathscr{V}^{-1}\kappa\mathscr{V})\mathscr{A}^2 - \frac{1}{8}\mathrm{str}(\Upsilon^2\mathscr{A}^2)(\mathscr{V}^{-1}\kappa\mathscr{V})\Big]\,.$$

An easy calculation shows that the matrix above has a structure

$$\frac{\omega_2^2}{16} \left(\begin{array}{cc} 0 & \varepsilon \\ -\varepsilon^t C_4 & 0 \end{array} \right), \tag{D.29}$$

where ε the matrix ε depends on 8 fermions only, i.e. the rank of the on-shell κ-symmetry transformations is equal to eight, confirming thereby the conclusions of [11]. Thus, our analysis shows that transformation (D.29) suffices to gauge away from the general element

$$\mathscr{V}^{-1}\chi\mathscr{V} = \begin{pmatrix} 0 & V^{-1}e^{-\frac{i}{2}\rho\Gamma_2}\theta \\ -(V^{-1}e^{-\frac{i}{2}\rho\Gamma_2}\theta)^t C_4 & 0 \end{pmatrix} \tag{D.30}$$

precisely eight fermions.

Finally, we note that in section (4.4.2) we made an additional rotation of χ with the matrix W given by eq.(4.25). To find how the new fermionic matrix transforms under κ-symmetry, we have to rotate the parameter κ in the same way $\kappa \to W\kappa W^{-1}$. In the $\mathbb{CP}^3$ sector this rotation effectively leads to modifying the matrix $\mathscr{A}$ in the following way

$$\mathscr{A} \ \to \ \mathscr{A} = \begin{pmatrix} -i\Gamma_0 & 0 \\ 0 & T_6 \end{pmatrix},$$

which is the consequence of $e^{\frac{\pi}{4}T_5}(T_{34}+T_{56})e^{-\frac{\pi}{4}T_5} = T_6$. This new matrix $\mathscr{A}$ coincides with the one used in the paper [11], where it was concluded that the corresponding κ-symmetry transformations allow one to make the gauge choice

$$\theta T_{56} = 0\,,$$

which puts to zero the fifth and the sixth column of θ.

Appendix E

Low-energy dynamics

5.1 The particle spectrum in different gauges: an example

The problem we described in Section 5.3 is related to a singular gauge choice for a part of the kappa-symmetry.

To clarify the situation we present an example from the hard core of gauge theory, where a similar phenomenon occurs. Namely, we will consider the Abelian $U(1)$ Higgs model with the standard Lagrangian:

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4}F_{\mu\nu}^2 + \overline{D}^{\mu}\phi^*\, D_{\mu}\phi - \frac{\widehat{g}}{4}(\phi^*\phi - v^2)^2 \qquad \text{(E.1)}$$

In the above clearly ϕ is the Higgs field, the covariant derivative is $D_\mu \phi = (\partial_\mu + i\, g\, A_\mu)\,\phi$ and the gauge transformations are

$$\phi \to e^{i\, g\, \alpha}\,\phi,\ A_\mu \to A_\mu - \partial_\mu \alpha$$

It is possible to choose the so-called "unitary" gauge, which corresponds to setting ϕ to be real. In this gauge the Lagrangian (E.1) obtains the following form:

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4}F_{\mu\nu}^2 + (\partial_\mu\phi)^2 + g^2\, A^2\phi^2 - \frac{\widehat{g}}{4}(\phi^2 - v^2)^2 \qquad \text{(E.2)}$$

Now, as long as $v \neq 0$ in order to stabilize the potential $V = \frac{\widehat{g}}{4}(\phi^2 - v^2)^2$ one usually makes a shift $\phi = v + \varphi$, which, among other things, produces the following quadratic form:

$$\mathcal{L}^{(2)}_{\text{Higgs}} = -\frac{1}{4}F^2_{\mu\nu} + (\partial_\mu \varphi)^2 + g^2 v^2 A^2 - \widehat{g}\, v^2 \varphi^2 \tag{E.3}$$

In particular, for $v \neq 0$ the quadratic form above is nondegenerate, its zeros describing the spectrum of the theory: 3 particles of mass $m_1 = gv$ (which come from the gauge field A_μ) and 1 particle of mass $m_2 = \widehat{g}^{1/2} v$ (which comes from the scalar φ). One should thus expect that in the limit $v \to 0$ we would obtain 4 massless particles. However, in practice this limit is rather subtle, and this is due to the fact that the quadratic Lagrangian (E.3) becomes gauge-invariant in the limit $v \to 0$, despite the fact that a gauge has already been chosen. This is of course an artifact of the combination of gauge choice and the perturbation expansion, since the gauge invariance is broken, as it should be, by the interaction terms that we have dropped in (E.3). The same statement can be reformulated, if one looks at the propagator $D_{\mu\nu}$ of the gauge field

$$D_{\mu\nu}(k) = \frac{1}{k^2 - g^2 v^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{g^2 v^2}\right), \tag{E.4}$$

which clearly is singular of order $\sim 1/v^2$ when $v \to 0$. Thus, the situation is similar to the case we are considering in this thesis, since, as explained in Section ??, the propagators of some of the fermions behave as $1/u^2$ in the $u^2 \to 0$ limit (equivalently, their quadratic Lagrangian is proportional to u^2).

What is the solution to this problem? According to the general logic explained above, we need to find a more suitable gauge, so that the quadratic part of the Lagrangian is nondegenerate even in the $v \to 0$ limit. There are many gauges at our disposal, for instance the Feynman gauge, in which the quadratic Lagrangian looks as follows (when $v = 0$):

$$\mathcal{L}^{(2)}_{\text{Higgs}} = -\frac{1}{2}(\partial_\mu A_\nu)^2 + |\partial_\mu \phi|^2 + \bar{c}\,\Box\, c, \tag{E.5}$$

c, $\bar{c}$ being the Faddeev-Popov ghosts. In particular the cohomology of the BRST Q operator in the A_μ sector consists of two states, which describe the two polarizations of the massless vector field. Apart from them, we also have the two massless scalar fields ϕ, $\bar{\phi}$, thus the spectrum of the model indeed consists of 4 massless particles. Notice, however, that in this case two of these particles come from A_μ and two from ϕ, whereas in the case of the unitary gauge we had three particles from A_μ and one from ϕ. The different ways of splitting the sectrum is of course a natural consequence of gauge invariance of the model. The important point is that both approaches give the same spectrum in the limit $v \to 0$, however only one of them is applicable for perturbative calculations.

In using the quite involved construction described in Chapter 5 we have in mind the simple idea of choosing a proper gauge for our model.

5.2 The $osp(8|4)$ superalgebra

This section of the Appendix provides a matrix realization of the $osp(8|4)$ superalgebra. The discussion here is in many ways parallel to the one of [11], since the $osp(6|4)$ algebra described there is very similar to the one of our interest.

Generators of the $osp(8|4)$ can be thought of as $4|8 \times 4|8$ supermatrices:

$$A = \begin{pmatrix} X & \vartheta \\ \eta & Y \end{pmatrix}, \tag{E.6}$$

where X and Y are bosonic matrices of dimensions 4×4 and 8×8 respectively. The matrix Y belongs to $so(8)$ and, as such, is real and antisymmetric:

$$Y^* = Y, \quad Y^{\mathrm{T}} = -Y \tag{E.7}$$

The matrix X belongs to $sp(4)$ and can be characterized by the following properties:

$$X^* = i\Gamma_2 X (i\Gamma_2)^{-1}, \quad X^{\mathrm{T}} = -C_4 X C_4^{-1} \tag{E.8}$$

As for the fermions, η is related to ϑ via

$$\eta = -\vartheta^{\mathrm{T}} C_4 \tag{E.9}$$

and the reality property reads

$$\vartheta^* = i\Gamma_2 \vartheta \tag{E.10}$$

5.3 Embeddings $SO(6) \hookrightarrow SO(8)$

The purpose of this section of the Appendix is to prove that the standard diagonal embedding of $OSP(6|4) \subset OSP(8|4)$ is the one relevant for our purposes.

A Lie superalgebra can be decomposed into its bosonic and fermionic components in a standard way: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. Then, $\mathcal{L}_0$ is represented on $\mathcal{L}_1$, since $[\mathcal{L}_0, \mathcal{L}_1] \subset \mathcal{L}_1$. Thus, a natural question arises which representations arise in this way. The answer to this question (among others) was given by Kac [77]. For the case of the $osp(8|4)$ superalgebra the corresponding fermionic module is $sp_4 \otimes so_8$, where sp_4 and so_8 are the standard (defining) representations of the corresponding algebras. In other words, for practical purposes one can consider the representation in terms of $8|4 \times 8|4$ supermatrices:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{E.11}$$

where A is a standard representation of $sp(4)$ and D is the standard representation of $so(8)$ [1], whereas B and C are the fermionic algebra

[1] It is well-known that $Spin(8)$ has 3 different irreps of dimension 8, related by the so-called triality, so in this case the theorem of Kac rules out two of them, which are the chiral and anti-chiral spinorial ones.

elements, subject to natural reality properties (besides, C is conjugate to B, in the sense that it is uniquely determined by the latter). In fact, representations of both of these bosonic algebras can be conveniently described in terms of gamma-matrices. Let us denote by $\Gamma_\mu, \mu = 0,1,2,3$ the $D = 4$ gamma-matrices, and by $\gamma_\alpha, \alpha = 1...6$ (and also γ_7 on slightly separate grounds) the $D = 6$ (respectively $D = 7$) gamma-matrices. They satisfy the following defining conditions:

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta^{(4)}_{\mu\nu}, \quad \eta^{(4)} = \mathrm{diag}(+,-,-,-) \tag{E.12}$$

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta^{(6)}_{\alpha\beta}, \quad \eta^{(6)} = \mathrm{diag}(-,-,-,-,-,-) \tag{E.13}$$

In these signatures the Γ matrices may be chosen to be imaginary, and the γ matrices may be chosen to be real (this is the choice of Majorana bases for both Clifford algebras[2]). The $sp(4)$ algebra is then generated by [11] $\frac{1}{2}[\Gamma_\alpha, \Gamma_\beta]$, $i\Gamma_\alpha$, so all the generators are real. The $so(8)$ algebra is generated by standard E_{ij} matrices (with 1 on the ij-place and -1 on the ji place), so (not surprisingly) it is real too. In this setup the fermions should be chosen real as well.

There are different ways to represent $SO(6)$ on the 8-dimensional vector space of the $SO(8)$ vector representation. One of them is the standard diagonal embedding, which can be continued to the embedding $OSP(6|4) \subset OSP(8|4)$ in a simple way:

$$G_{OSP(6|4)} = \left(\begin{array}{cc|c} A & B & \\ C & D & \\ \hline & & 1_2 \end{array}\right), \tag{E.14}$$

The other representation is a faithful[3] $Spin(6)$ representation (and, as such,

[2]Note however that throughout the thesis we used a different representation of the Γ matrices, see Appendix A.

[3]We remind the reader that a representation r of a group $\mathcal{G}$ on a vector space V, that is a homomorphism $r : \mathcal{G} \to \mathrm{GL}(V)$, is called faithful, if r is injective.

not a representation of $SO(6)$). It may be constructed in the following way. Let $x_1 \dots x_8$ be the 8 coordinates in the vector space, on which $SO(8)$ is represented. We can form complex combinations $X_1^{\pm} = x_1 \pm ix_2$, etc. Then, those $SO(8)$ transformations which correspond to analytic (linear) maps of $X^+_{1,2,3,4}$ form an $SU(4)$. In this way $SU(4) = Spin(6)$ is represented irreducibly on the 8-dimensional real vector space (if one considered the vector space over $\mathbb{C}$, the representation of $SU(4)$ would split as $4 + \bar{4}$). The above definition is equivalent to the following: one needs to choose those matrices from $SO(8)$ which commute with a given complex structure in $\mathbb{R}^8$. For definitiveness we choose the simplest complex structure, which in our conventions is given by γ_7. Clearly, such matrices are $\frac{1}{2}[\gamma_\alpha, \gamma_\beta]$, $\alpha, \beta = 1 \dots 6$ (as well as γ_7 itself, which thus extends $SU(4)$ to $U(4)$). The latter can be split in two groups: the ones, which lie in $SU(3) \subset SU(4)$ (we call them L_i's), and the ones which lie in the compliment (we call them $\mathbb{T}_i$ and U). These are the following:

$$L_1 = \gamma_{26} + \gamma_{15}, \quad L_2 = \gamma_{35} + \gamma_{46} \quad L_3 = \gamma_{14} - \gamma_{23} \tag{E.15}$$

$$L_4 = \gamma_{16} - \gamma_{25}, \quad L_5 = \gamma_{36} - \gamma_{45}, \quad L_6 = \gamma_{13} + \gamma_{24} \tag{E.16}$$

$$L_7 = \gamma_{12} - \gamma_{34}, \quad L_8 = \gamma_{12} + \gamma_{34} - 2\gamma_{56}, \quad U = \gamma_{12} + \gamma_{34} + \gamma_{56}. \tag{E.17}$$

$$\mathbb{T}_1 = 1/2(\gamma_{26} - \gamma_{15}), \quad \mathbb{T}_2 = 1/2(\gamma_{35} - \gamma_{46}), \quad \mathbb{T}_3 = 1/2(\gamma_{14} + \gamma_{23}) \tag{E.18}$$

$$\mathbb{T}_4 = 1/2(\gamma_{16} + \gamma_{25}), \quad \mathbb{T}_5 = 1/2(\gamma_{36} + \gamma_{45}), \quad \mathbb{T}_6 = 1/2(\gamma_{13} - \gamma_{24}) \tag{E.19}$$

In the above U is an element of $SU(4)$ which commutes with $SU(3)$.

In fact, the $SU(3)$ group, generated by L's is the same, as the $SU(3)$ subgroup of the diagonal $SO(6)$ embedding. In the notations of the paper [57], the projector $P_6 = \mathrm{diag}(1,1,1,1,1,1,0,0)$ leaves L_i invariant $P_6 L_i P_6 = L_i$, and annihilates $\mathbb{T}_i$: $P_6 \mathbb{T}_i P_6 = 0$.

In order to get the diagonal embedding in this way, one needs to project the γ-matrices (exactly as described in the appendix to paper [57]): $T_i =$

$P_6\gamma_i P_6$, $i = 1...6$. We have chosen the Majorana gamma-matrices γ in such a way, that these projections give precisely the T-matrices, defined in the paper [11] (this is a confirmation that the embedding has been chosen correctly).

Once we know what the correct embedding is, we can proceed to define the Hopf fiber bundle. As described in [57], in order to do this we need to represent the sphere S^7 as a coset $SU(4) \times U(1)/SU(3) \times U'(1)$, where the 'gauge group' $U'(1)$ is generated by the element U, whereas translation along the fiber S^1 is generated by γ_7. Note that one may write

$$\gamma_7 = K_6 + \epsilon, \tag{E.20}$$

where ϵ is defined in Appendix A and

$$U = K_6 - 3\epsilon, \tag{E.21}$$

so the new space is indeed produced by a twisting of the original $U(1)$ gauge group generated by K_6 with the new direction ϕ, that appears as the angle of $SO(2)$ (generated by ϵ) in $SO(6) \times SO(2) \subset SO(8)$, both subgroups embedded diagonally.

As explained in Appendix 5.4, dimensional reduction corresponds to dropping the einbein $e^7 = d\phi - \mathcal{A}$, which describes the fiber. In terms of the coset, it implies an additional gauging of a $U(1)$ subgroup, generated by the fiber translations, that is by γ_7. On the other hand, as it follows from the formulas (E.20, E.21) above, gauging both U and γ_7 is the same as gauging K_6 and ϵ, and thus we return to the $\mathbb{C}P^3$ space, as we should. $\triangleright$

Thus, the $OSP(8|4)/SO(7) \times SO(1,3)$ coset element can be chosen as follows:

$$g = g_{OSP(6|4)}\, e^{\varphi\epsilon}\, e^{\upsilon_\lambda Q^\lambda}. \tag{E.22}$$

Here $g_{OSP(6|4)}$ is the $OSP(6|4)$ coset element, which can be taken, for instance, from [11], and schematically it looks as follows: $g_{OSP(6|4)} =$

$g_{bosons} h_{24fermions}$. v_λ are the additional 8 fermions absent in the $OSP(6|4)$ coset. The matrix $v_\lambda Q^\lambda$ can be found in the Appendix A.

5.4 Dimensional reduction in detail

This section of the Appendix is dedicated to the explanation of how the Kaluza-Klein reduction is performed in our setup. In part 5.4.3 we prove that the reduction preserves the $OSP(6|4)$ subgroup of the $OSP(8|4)$ isometry group of the $AdS_4 \times S^7$ background.

5.4.1 Metric term

In this section we will follow the line of reasoning adopted in [45].

Suppose we have an 11D metric which can depend on the fermions as well as on the bosons. This metric is subject to an important qualification – it has a linearly realized U(1) isometry, which we will take to be the shift $z \to z + a$, a being an arbitrary constant. Then the metric can be written in the following way:

$$ds^2 = g_{ab}dx^a dx^b + B_a dx^a dz + C dz^2 \tag{E.23}$$

Assume that the three membrane coordinates are σ, τ, y. Let us set $z = y$, then the pullback of the metric written above to the membrane worldvolume can be written as follows:

$$\widehat{G}_{\alpha\beta} dx^\alpha dx^\beta = \left(g_{ab}\partial_\alpha x^a \partial_\beta x^b + B_a \partial_\alpha x^a \partial_\beta z + C\partial_\alpha z \partial_\beta z\right) dx^\alpha dx^\beta \tag{E.24}$$

In the above formula the indices α, β run from 1 to 3. Now, when $\alpha, \beta = 1, 2$ we have

$$\widehat{G}_{\alpha\beta} = g_{ab}\partial_\alpha x^a \partial_\beta x^b \tag{E.25}$$

When $\beta = 3$,

$$\widehat{G}_{\alpha 3} = \frac{1}{2} B_a \partial_\alpha x^a \tag{E.26}$$

Finally, if $\alpha = \beta = 3$, we get

$$\widehat{G}_{33} = C \tag{E.27}$$

If we calculate the determinant of $\widehat{G}_{\alpha\beta}$, it will of course be some function of g_{ab}, B_a, C. Since $\det(\widehat{G}) \neq \det(g)$, g cannot be regarded as the pullback of the correct 10D string metric. However, the following Kaluza-Klein construction cures this drawback. Indeed, the correct pullback h_{ab} is intoduced by the following decomposition of $\widehat{G}$ (here $i = 1, 2$):

$$\widehat{G} = \Phi^{-2/3} \begin{pmatrix} h_{ij} + \Phi^2 A_i A_j & \Phi^2 A_i \\ \Phi^2 A_j & \Phi^2 \end{pmatrix} \tag{E.28}$$

The point of this decomposition is that now for *any* Φ, A_i the following holds true:

$$\det(\widehat{G}) = \det(h) \tag{E.29}$$

Thus, the Nambu-Goto actions of the membrane and the string coincide up to a factor of the radius of the fiber, which we denote by r:

$$\int d\sigma d\tau dy \sqrt{\det \widehat{G}} = r \int d\sigma d\tau \sqrt{\det h}. \tag{E.30}$$

One can read off the following from (E.28):

$$\Phi^{-2/3}\Phi^2 = \Phi^{4/3} = \widehat{G}_{33} \tag{E.31}$$

$$\Phi^{-2/3}\Phi^2 A_i = \Phi^{4/3} A_i = \widehat{G}_{i3} \tag{E.32}$$

$$\Phi^{-2/3}(h_{ij} + \Phi^2 A_i A_j) = \widehat{G}_{ij} \tag{E.33}$$

We need to express $\Phi^{4/3}$, A_i and, ultimately, h_{ij} from these expressions:

$$\Phi^{4/3} = \widehat{G}_{33} \tag{E.34}$$

$$A_i = \frac{\widehat{G}_{3i}}{\widehat{G}_{33}} \tag{E.35}$$

$$h_{ij} = \sqrt{\widehat{G}_{33}}\,(\widehat{G}_{ij} - \frac{\widehat{G}_{3i}\widehat{G}_{3j}}{\widehat{G}_{33}}) \tag{E.36}$$

The last line of this equation gives us the sought for pullback (to the 2D string worldsheet) of the 10D metric. Its determinant is equal to the determinant of $\widehat{G}$, as explained above. It is a general answer, independent of the representation of the vielbeins (in this argumentation the vielbeins are irrelevant, since from the very beginning we're dealing with the metric, and of course one can always choose the vielbeins in a pretty form to satisfy the given metric).

The procedure we have just explained is equivalent to writing the original metric (E.23) in the form

$$ds^2 = \widetilde{g}_{ab} dx^a dx^b + C(dz - \widehat{B}_a dx^a)^2 \tag{E.37}$$

then dropping the last term and multiplying the first one by an appropriate factor of C, such that the determinant is unchanged.

5.4.2 Wess-Zumino term

Here we explain the technical side of the Wess-Zumino term construction. It was announced in (5.23) that the Wess-Zumino term for the M2 brane action looks as follows:

$$\mathcal{F} = \frac{1}{8}\epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} E^{\bar{a}} \wedge E^{\bar{b}} \wedge E^{\bar{c}} \wedge E^{\bar{d}} + \widehat{\lambda}\, E^{\alpha} \wedge [\mathbf{\Gamma}_A, \mathbf{\Gamma}_B]_{\alpha}^{\beta}\, E_{\beta} \wedge E^A \wedge E^B. \tag{E.38}$$

The bosonic vielbeins in the above formula are normalized in a canonical way, namely so that the $AdS_4 \times S^7$ metric is written in the form $ds^2 = \eta_{11}^{AB} E_A E_B$.

The fermionic vielbeins E^{α} which enter this formula are the fermionic components of the supercurrent

$$J = -g^{-1}\, dg = E^a T_a + E^{\alpha} Q_{\alpha} + A^{ab} \Omega_{ab}, \tag{E.39}$$

We will not write out the matrix form of the supergenerators Q_{α}, since this is to a large extent irrelevant for our problem, but will rather deal with components of the matrix $\mathcal{Q} = E^\alpha Q_\alpha$. It is a supermatrix with zero bosonic components, so it has "off-diagonal" form. We will call its top right block Θ (its bottom left block $\widehat{\Theta}$ is related to it via $\widehat{\Theta} = -\Theta^{\mathrm{T}} C_4$) and write its matrix components as $\Theta_{\bar{a}a}$, where $\bar{a} = 0...3$ is the row number and $a = 1...8$ is the column number.

After this preparational work we may write the term $E^\alpha \wedge [\mathbf{\Gamma}_A, \mathbf{\Gamma}_B]_\alpha^\beta E_\beta$ as follows:

$$E^\alpha \wedge [\mathbf{\Gamma}_A, \mathbf{\Gamma}_B]_\alpha^\beta E_\beta = \Theta^{\bar{a}a} \wedge [\mathbf{\Gamma}_A, \mathbf{\Gamma}_B]_{\bar{a}a}^{\bar{b}b} \Theta_{\bar{b}b} = \Theta_{\bar{a}a} (C_{11})^{\bar{a}a,\bar{b}b} \wedge [\mathbf{\Gamma}_A, \mathbf{\Gamma}_B]_{\bar{b}b}^{\bar{c}c} \Theta_{\bar{c}c}$$

The double-index notation is very convenient for our choice of the $\mathbf{\Gamma}$-matrices (see Appendix A), since all of them have the form $A_4 \otimes B_8$, and clearly $(A_4 \otimes B_8)_{\bar{a}a}^{\bar{b}b} \equiv (A_4)_{\bar{a}}^{\bar{b}} (B_8)_a^b$. The C_{11} matrix raises/lowers the indices, but the interpretation of its own indices is still the same: $(C_{11})^{\bar{a}a,\bar{b}b} = -(C_{11})_{\bar{a}a,\bar{b}b} = (C_4 \Gamma_5)^{\bar{a}\bar{b}} \delta^{ab}$ (the minus sign is due to the fact that $C_{11}^2 = -1$).

The requirement of the closedness of the four-form in (E.38) fixes the value of $\widehat{\lambda}$:

$$\widehat{\lambda} = \frac{i}{4} \tag{E.40}$$

.

5.4.3 $OSP(6 \mid 4)$ invariance of the 10D theory

We mentioned above that the form (E.22) of the coset provides for the $OSP(6|4)$ invariance of the ten-dimensional theory. That is what we're going to prove here. For simplicity we will consider just the $\mathbb{C}P^3$ part of the problem, or the $SO(6)$ symmetry group. Let $\Omega \in SO(6) \subset SO(8)$ act on $g(w, \theta, v)$ (w and θ, v are the bosonic and fermionic fields of the coset respectively) from the left, that is $g' = \Omega g(w, \theta)$. From the properties of

$g_{OSP(6|4)}$ it follows that we can write

$$g' = g_{OSP(6|4)}(w', \theta')\, \omega(w, \theta)\, e^{\varphi \epsilon}\, e^{v_\lambda Q^\lambda}, \tag{E.41}$$

where ω is the compensating element from the stabilizer $U(3)$ of the $SO(6)/U(3)$ coset. Let us write $\omega = \omega_{SU(3)}\omega_{U(1)}$, where $\omega_{SU(3)}$ belongs to $SU(3) \subset U(3)$ and therefore also to $SO(7)$, and $\omega_{U(1)} = e^{\nu K_6} \in U(1) \subset U(3)$ does not belong to $SO(7)$. Then $\omega_{U(1)} e^{\varphi\epsilon}\, e^{v_\lambda Q^\lambda} = e^{\varphi\epsilon}\, e^{v_\lambda Q^\lambda}\omega_{U(1)} = e^{(\varphi+3\nu)\epsilon}(e^{-3\nu\epsilon} e^{v_\lambda Q^\lambda} e^{3\nu\epsilon}) e^{\nu U}$, where, as before, $U = K_6 - 3\epsilon \in SO(7)$ belongs to the stabilizer of the $SO(8)/SO(7)$ coset. We denote

$$\varphi' = \varphi + 3\nu, \tag{E.42}$$

$$v'_\lambda Q^\lambda = e^{-3\nu\epsilon} v_\lambda Q^\lambda e^{3\nu\epsilon}, \tag{E.43}$$

then we may write

$$g' = g_{OSP(6|4)}(w', \theta')\, e^{\varphi'\epsilon}\, e^{v'_\lambda Q^\lambda}\, e^{\nu U}\, \omega_{SU(3)}. \tag{E.44}$$

The key property to observe is that, since ν is a function of $w, \bar{w}, \theta$, the variations of all fields involve only the fields $w, \bar{w}$ and θ, v (we have included v here, since the variation of v is $\delta v \sim \nu\, v$), but not φ. In order to see why this is important we write down the 11D metric in the Kaluza-Klein form (in the presence of fermions):

$$ds^2 = ds^2_{10D} + \widehat{\mathcal{D}}\, (d\varphi - \mathcal{A}^{(1)})^2, \tag{E.45}$$

where $\widehat{\mathcal{D}}$ is the dilaton. Since the variations of all the fields do not depend on φ, the only way for (E.45) to be invariant is for both $d\varphi - \mathcal{A}^{(1)}$ and $\widehat{\mathcal{D}}$ to be invariant.

5.5 Low-energy limit: integrating out the massive fields

Usually the low-energy limit implies that we need to get rid in one or another way of all the massive fields in the theory. As explained in Section 5.7, in the present model finding the low-energy limit is a rather subtle endeavour. This is due to the fact, that the Lagrangian contains terms of various dimensions. Here by dimension we always mean the simple canonical dimension, which in turn is determined by the behaviour of the propagator of a given field: all scalar fields have dimension 0 and fermions have dimension 1/2, the derivative being clearly of dimension one. Let us make it clear that so far we have only fixed the conformal gauge $\gamma_{+-} = \gamma_{-+} = 1$ and set to zero the massive θ fermions. Thus, we should keep in mind that we have the v-fermions, some of which are redundant and are subject to the additional kappa-symmetry transformations and, last but not least, some of the bosonic fields in the Lagrangian (there are 10 bosonic fields) are unphysical and are subject to the Virasoro constraints. Ultimately we want to express the two gauge artifacts $z_\pm$ in terms of the physical fields. Of course, this is best done using the light-cone gauge [103], but for our purposes it will be enough to use a shortcut which we will now describe. The Virasoro constraints look as follows:

$$0 = V_1 = \partial_+ z_+ - z_1 + \frac{i}{2}(\xi_1\xi_2 + \phi_1\phi_2 - \chi_1\chi_2 - \psi_1\psi_2) + ... \tag{E.46}$$

$$0 = V_2 = \partial_- z_- + z_1 + \frac{i}{2}(\xi_1\xi_2 + \phi_1\phi_2 - \chi_1\chi_2 - \psi_1\psi_2) + ..., \tag{E.47}$$

The part of the Lagrangian, which contains the z fields, is:

$$\begin{gathered}\mathcal{L}_z = 2z_2^2 + 4z_1\partial_+ z_+ - 4z_1\partial_- z_- + \\ +4\partial_- z_- \partial_+ z_+ + 2i(\partial_- z_- + \partial_+ z_+)(\xi_1\xi_2 + \phi_1\phi_2 - \psi_1\psi_2 - \chi_1\chi_2) + \\ +4iz_1(\xi_1\psi_2 + \xi_2\psi_1 - \phi_1\chi_2 - \phi_2\chi_1) + 2iz_2(\chi_2\psi_1 + \chi_1\psi_2 - \xi_1\phi_2 - \xi_2\phi_1)\end{gathered} \tag{E.48}$$

We remind the reader that the physical fields z_1, z_2 are massive, so we may set them to zero everywhere, except for the terms written above (since some of these terms have a 'subcritical' dimension, that is dimension smaller than 2). For this reason we did not write out the kinetic terms of the z_1, z_2 fields above (we're going to integrate out the z_1, z_2 fields, in the same fashion as the W and Z bosons can be integrated out in the low-energy limit of the Standard Model).

Next we plug $\partial_+ z_+, \partial_- z_-$ from (E.46, E.47) into (E.48) to obtain:

$$\begin{gathered}\mathcal{L}_z = 4z_1^2 + 2z_2^2 + \\ +4iz_1(\xi_1\psi_2 + \xi_2\psi_1 - \phi_1\chi_2 - \phi_2\chi_1) + 2iz_2(\chi_2\psi_1 + \chi_1\psi_2 - \xi_1\phi_2 - \xi_2\phi_1) + \\ +2(\xi_1\xi_2\phi_1\phi_2 - \xi_1\xi_2\chi_1\chi_2 - \xi_1\xi_2\psi_1\psi_2 - \phi_1\phi_2\chi_1\chi_2 - \phi_1\phi_2\psi_1\psi_2 + \chi_1\chi_2\psi_1\psi_2)\end{gathered} \tag{E.49}$$

As an intermediate result, we get the correct masses (2 and 4) for the AdS physical fields (this should be compared with the spectrum obtained for the first time in [55]). We can now easily integrate out the fields z_1 and z_2:

$$\begin{aligned}\mathcal{L}_z = 3\xi_1\xi_2\phi_1\phi_2 - 2\xi_1\xi_2\chi_1\chi_2 - 2\xi_1\phi_1\chi_2\psi_2 - 3\xi_1\phi_2\chi_1\psi_2 - \\ -\xi_1\phi_2\chi_2\psi_1 - \xi_2\phi_1\chi_1\psi_2 - 3\xi_2\phi_1\chi_2\psi_1 - 2\xi_2\phi_2\chi_1\psi_1 - \\ -2\phi_1\phi_2\psi_1\psi_2 + 3\chi_1\chi_2\psi_1\psi_2\end{aligned} \tag{E.50}$$

Recall that (E.50) is only the part of the Lagrangian which initially depended on the z fields. There's another part, which we would have obtained, had we simply set all the z fields to zero and dropped the vertices with higher derivatives. This part looks as follows:

$$\begin{gathered}\mathcal{L}_0 = 2\,(\tfrac{\delta_{ij}}{1+|w_k|^2} - \tfrac{\bar{w}_i\, w_j}{(1+|w_k|^2)^2})\,(\partial_+ w_i \partial_- \bar{w}_j + \partial_- w_i \partial_+ \bar{w}_j) + \\ +i(\xi_1 - \psi_1)\partial_+(\xi_1 - \psi_1) + i(\phi_1 + \chi_1)\partial_+(\phi_1 + \chi_1) + \\ -i(\xi_2 - \psi_2)\partial_-(\xi_2 - \psi_2) - i(\phi_2 + \chi_2)\partial_-(\phi_2 + \chi_2) + \\ +2\,\tfrac{\bar{w}_i\partial_+ w_i - w_i \partial_+ \bar{w}_i}{1+|w_k|^2}\,(\xi_1 - \psi_1)(\phi_1 + \chi_1) - 2\,\tfrac{\bar{w}_i\partial_- w_i - w_i \partial_- \bar{w}_i}{1+|w_k|^2}\,(\xi_2 - \psi_2)(\phi_2 + \chi_2) +\end{gathered} \tag{E.51}$$

$$+3(\xi_1\xi_2\phi_1\chi_2 - \xi_1\xi_2\phi_2\chi_1 - \xi_1\phi_1\phi_2\psi_2 - \xi_1\chi_1\chi_2\psi_2 +$$
$$+\xi_2\phi_1\phi_2\psi_1 + \xi_2\chi_1\chi_2\psi_1 + \phi_1\chi_2\psi_1\psi_2 - \phi_2\chi_1\psi_1\psi_2)+$$
$$+5(\xi_1\xi_2\chi_1\chi_2 + \phi_1\phi_2\psi_1\psi_2) + 6(\xi_1\phi_2\chi_1\psi_2 + \xi_2\phi_1\chi_2\psi_1)-$$
$$-\xi_1\phi_1\chi_2\psi_2 + \xi_1\phi_2\chi_2\psi_1 + \xi_2\phi_1\chi_1\psi_2 - \psi_2\phi_2\chi_1\psi_1$$

The indices i, j, k in this formula run over the values $1, 2, 3$. As explained in Chapter 5, the second and third lines of (E.51) determine the kappa-symmetry transformations, and it is clear, that the quartic terms in this formula are not invariant under this transformation (indeed, the only invariant combination is $(\xi_1 - \psi_1)(\xi_2 - \psi_2)(\phi_1 + \chi_1)(\phi_2 + \chi_2)$). However, according to the general logic that we have explained, the complete low-energy Lagrangian is the sum of (E.50) and (E.51):

$$\mathcal{L} = 2\left(\tfrac{\delta_{ij}}{1+|w_k|^2} - \tfrac{\bar{w}_i w_j}{(1+|w_k|^2)^2}\right)(\partial_+ w_i \partial_- \bar{w}_j + \partial_- w_i \partial_+ \bar{w}_j)+ \tag{E.52}$$
$$+i(\xi_1 - \psi_1)\partial_+(\xi_1 - \psi_1) + i(\phi_1 + \chi_1)\partial_+(\phi_1 + \chi_1)+$$
$$-i(\xi_2 - \psi_2)\partial_-(\xi_2 - \psi_2) - i(\phi_2 + \chi_2)\partial_-(\phi_2 + \chi_2)+$$
$$+2\tfrac{\bar{w}_i\partial_+ w_i - w_i\partial_+\bar{w}_i}{1+w_i^2}(\xi_1 - \psi_1)(\phi_1 + \chi_1) - 2\tfrac{\bar{w}_i\partial_- w_i - w_i\partial_-\bar{w}_i}{1+w_i^2}(\xi_2 - \psi_2)(\phi_2 + \chi_2)+$$
$$+3(\xi_1 - \psi_1)(\xi_2 - \psi_2)(\phi_1 + \chi_1)(\phi_2 + \chi_2)$$

Thus, the result is invariant under the residual kappa transformations, as it should be. Now we can safely set to zero 4 of the 8 fermions to obtain the Lagrangian announced in Chapter 5. Note that in order to obtain (5.43) or (5.46), one needs to rescale the fermions by a factor of 2 ($v \to 2\,v$) and divide the Lagrangian by 4.

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