Modern aspects of topologically twisted gauge theories

Polynomial invariants and mock modular forms

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I would like to dedicate this thesis to my loving parents Yiannis and Zeta, my brother Alexis and my grandmother Alexandra.

Declaration

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work. I agree to deposit this thesis in the University's open access institutional repository or allow the library to do so on my behalf, subject to Irish Copyright Legislation and Trinity College Library conditions of use and acknowledgement.

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Abstract

In this dissertation we present new results in the field of topologically twisted gauge theories evaluated on compact four-manifolds without boundary. We focus on the Donaldson-Witten theory, that is the $\mathcal{N} = 2$ topologically twisted super Yang-Mills theory. We revisit and study the contribution on the Coulomb branch of the path integral of the low energy effective theory which is non-vanishing only for four-manifolds with $b_2^+ \leq 1$. We establish new ways to evaluate path integrals of this theory using mock modular forms. Furthermore we propose a new regularization and renormalization of the path integral required for conservation of the BRST symmetry of the topological theory. We conclude by generalizing these considerations to the Donaldson-Witten theory in the presence of supersymmetric surface defects.

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Chapter 1

Introduction

1.1 Motivation for this thesis

In the edge of the twentieth century, physics went through a revolution where a bunch of observed phenomena such as the photoelectric effect led to the discovery of two new theories. These theories were to shape not only the physics of the previous and future centuries but also they played a fundamental role to the enormous technological revolution that is still ongoing. These two newly established theories were no other than the theory of relativity and the theory of quantum mechanics. Both of them are two of the most successful theories of twentieth century physics and probably of equal importance and weight to Darwin's theory of the evolution of species. The theory of relativity was proposed by Einstein in his *miraculous years* between 1905 (special theory of relativity) and 1915 (general theory of relativity). The theory itself describes physical phenomena involving macroscopic objects, such as stars like our Sun, black holes, the precession of the perihelion of Mercury and also forms the base of our modern cosmological understanding since it explains the expansion of the universe among other phenomena. Relativity has been tested on our solar system and even to cosmological scales to a very high precision confirming beyond doubt its relevance to the universe. Quantum mechanics, on the other hand, is the theory that governs the physics of microscopic objects, such as molecules, atoms and elementary particles like the electrons, the neutrons and the protons. The validity of quantum mechanics is also beyond any doubt since its experimental verification and technological applications are omnipresent. For example, the spectrum of the hydrogen atom confirms that quantum mechanics indeed is absolutely necessary for describing the microscopic world while the transistors that every modern electronic device utilizes are a consequence of our understanding of quantum mechanics. Nevertheless, despite the fact that relativity and quantum mechanics have been studied by physicists for over a hundred years, to this day, they seem to be incompatible to each other.

This is something very puzzling since we know that there exist situations and phenomena that in order to be described both theories are needed, for example the physics near the singularities of black holes, or the singularity of the Big Bang. Although the special theory of relativity and quantum mechanics can be combined into the successful framework of **quantum field theory**, that we will describe in more detail below, this is not the case with the general theory of relativity. **String theory** was discovered in the seventies, originally as a theory to explain the short range interactions of the quarks in the atomic nuclei. Nevertheless, soon it was realized that string theory is a theory that can unify relativity and quantum mechanics, i.e. a theory of quantum gravity. String theory is to this day the only consistent theory of quantum gravity and the only promising theory to unify all fundamental forces of the nature, described by quantum field theory and the theory of general relativity, into a single mathematical framework.

In what follows, we will give a brief summary of the development of various fundamental aspects of theoretical physics, starting from quantum field theory and going to string theory. We call the attention of the reader on the very repeating pattern of the appearance of modern mathematics while we discuss these topics.

As we will explain below, the main theory to be studied in this thesis is a supersymmetric quantum field theory which has been shown to be of relevance within the context of string theory and D-branes. The reason for this approach is that advancing toward the end of the twentieth century, physicists and mathematicians managed to use the physical ideas of relativity, quantum mechanics and eventually string theory not to solve some physical problems but rather to solve various problems in modern mathematics. This was the first time in scientific history where mathematics was not a tool physicists use in order to tackle with their problems but instead physics provides the tools to solve mathematical problems and this thesis appears to have such a flavor to some extent. More generally it is evident by now that physics and mathematics are a dynamic duo, where one influences the other. This notion will become clear in the main body of this thesis. It would not be dishonest to state that the original motivation for this thesis came from this new point of view.

1.2 Quantum Field Theory

The study of the quantum mechanical theory of the electromagnetic fields and their interactions (for example the interactions between photons and electrons) led to the unification of quantum mechanics, special relativity and electrodynamics and eventually to the development of quantum electrodynamics, or as it is better known QED. Using this mathematical framework physicists started developing other similar theories that all together go under the umbrella name *quantum field theory* and have revolutionized physics by being the most experimentally accurate scientific theory we have developed. Quantum field theory interprets the particles we observe in the laboratories as excitations of fields that are omnipresent across the universe. A way to heuristically view what a field is the following: think of the sea as the field occupying the universe and the peaks of sea waves as the excitations of this field that we interpret as particles. In some places there are no sea waves, in other places there are sea waves propagating along one direction and in other places sea waves from different directions scatter and create new sea waves. Particle physics, which is governed by quantum field theory, can be interpreted as this sea picture. To return to QED, its success in explaining various phenomena such as the photoelectric one, the Lamb shift, and even the anomalous magnetic moment of the electron established the theory really quickly. Furthermore, a specific quantum field theory called *quantum chromodynamics*, or QCD, turned out to be the correct theory explaining the strong and weak nuclear interactions that are responsible for binding the atomic nuclei together and the radioactive decays respectively. As the Standard Model of particle physics was being developed it was realized that the electromagnetic and the weak nuclear forces fit in one framework, a quantum field theory known as the *electroweak theory*. What is very special about these quantum field theories we describe, is a characteristic feature they have. This is the possession of a local symmetry called *gauge symmetry* and it will also be apparent in the theories we will study later. For QCD the gauge symmetry is $SU(3)$ while for the electroweak theory it is $SU(2) \times U(1)$. When this symmetry is spontaneously broken the resulting symmetry for QED is the familiar $U(1)$.

Quantum field theory has been shown to be very successful and predicts numerous phenomena to a very high degree of accuracy already from its early days. It was realized though that the theory contained some mathematical inconsistencies that led to nonsensical answers and these inconsistencies had to be cured. In QED for example when one would try to compute some observable quantity (for example the cross section of some process) a possible answer could be infinity. This problem was cured by absorbing these infinite quantities into a finite number of parameters such as the coupling constants and the masses of the theory. This procedure is called *renormalization* and in the next chapter we will see how one can absorb infinite quantities and make integrals, that naively diverge, finite. Most quantum field theories fit into a general classification that splits them into two classes. Renormalizable and non-renormalizable. QED and QCD belong to the former class. The theory we will study in this thesis belongs to the latter class. In the early days it was thought that only the renormalizable theories made sense but our current understanding is that this is not the case. A very important example of a non-renormalizable theory is the theory of general relativity itself. The problem with non-renormalizable theories is that after specific energy scales they lose their predictive power and in some sense they do not describe the physics that their lower energy counterparts did. This behavior is an indicator that some new physics is needed above these energy scales and such problems led to string theory. Theories like general relativity take the name *effective field theories* in the sense that they do work pretty well in some range of energies but they are just dressed up low energy theories of some *mother theory*. On the other hand the Standard Model which is based on QCD and QED is a renormalizable theory. As a renormalizable theory it is expected to make accurate predictions at arbitrary large energy scales. Nevertheless for the Standard Moedel we also know that it should unify to a single theory at high enough energies. From this viewpoint even a renormalizable theory such as the Standard Model can be considered also as an effective field theory[1]. The notion of an effective field theory is very important since this thesis studies such non-renormalizable effective field theories.

1.3 Supersymmetry

As we explained, the Standard Model (the union of QCD and QED) is regarded as an effective field theory, a low energy approximation of some unified but unknown theory[2]. Eventually all theories must be unified to a single mother theory or "theory of everything" and we expect this to happen at energy scales of the order of anywhere between 10^{16} and 10^{18} GeV. Such energy scales are very large, actually enormously large, compared to the electroweak breaking scale which happens at 300 GeV and is one of the characteristic features of the Standard Model. The huge ratio of these two scales, which is called the *hierarchy problem*, is indeed extremely large and physicists believe that such a discrepancy requires justification. The hierarchy problem is indeed a problem because unlike the quarks the leptons have masses proportional to the electroweak symmetry breaking scale which itself is proportional to the mass of the scalar fields responsible for such a symmetry breaking. The scalar fields though, unlike the previously mentioned fields, are not protected by any gauge symmetry and it is not understood why their masses are not of the order 10^{16} or 10^{18} GeV. This unexplained phenomenon led to the development of supersymmetry.

Supersymmetry is a transformation property that fundamental particles are supposed to have and which transforms bosons to fermions and fermions to bosons. Therefore supersymmetry suggests that all fundamental particles have a supersymmetric partner with the opposite parity. Supersymmetry is not experimentally confirmed by 2019 but there are good

[1] Note that this point of view of an effective field theory is not universally agreed upon but many researchers take upon such an assumption.

[2] There exist various proposals for such theories, for example the $SU(5)$ unified theory or the $SO(10)$ unified theory which predict many unobserved to this day particles.

reasons to believe in its existence since it resolves various problems in quantum field theory including the hierarchy problem. This resolution is done by considering the Standard Model as a supersymmetric theory. The reason is very simple: if the scalar field (that is responsible for the electroweak symmetry breaking) and its fermionic partner transform in some chiral representation for an unknown gauge group G, then they both are required by supersymmetry to have vanishing bare masses and in such a case all masses of the standard model will then be related to the supersymmetry breaking scale.

Supersymmetry, furthermore, is an absolute requirement for the five standard string theories. In order to construct Standard Model like theories from string theory, space-time supersymmetry is a must, making supersymmetry tied to the problem of unifying quantum field theory with general relativity. Furthermore, supersymmetry is mathematically consistent and even mathematically interesting by itself.

1.4 String theory

As we explained earlier, string theory emerged in the 1960s as a proposal to decribe the strongly interactive hadrons in QCD as vibrating modes of a string. Nevertheless, this theory turned out to be very problematic for the following few reasons (which nowadays are seen as assets of the theory):

- String theory, in its simplest form, requires 26 dimensions of space-time in order to be consistent. Superstring theories require 10 dimensions of space-time (while the much later discovered M-theory requires 11 dimensions).
- The theory contains a massive particle with spin $s = 2$ and such a particle is not observed in hadronic physics.
- Strong interaction scattering amplitudes follow a power law fall off with respect to energy. Unlike, in string theory, the so-called Veneziano amplitudes fall off exponentially fast.

The above come at odds with string theory being a theory for hadronic physics and QCD was established as the only theory explaining the strong interactions. In the 1970s it was suggested that the massless spin two particle that appears in the spectrum of string theory could be the graviton, the quantum excitation of the gravitational field and this fact alone implied that the tension of the hypothetical vibrating string is related to the characteristic scale of gravity, the Planck scale which is of order 10^{19} GeV. Furthermore, it was realized that as the theory flows to lower energies it behaves like Einstein's general relativity. In the years to follow various string theory formulations were developed, including supersymmetry. By the 1980s the physicists had established five superstring theories known as Type I, Type IIA, Type IIB and $SO(32)$ and $E_8 \times E_8$ *heterotic* string theories. Some of those theories that were especially relevant for phenomenological models were subject to gravitational anomalies. A remarkable breakthrough of Green and Schwarz, the *anomaly cancellation*, established string theory as a viable scientific theory (although there were five of them[3]).

The five formulations of string theory are related to each other by the so-called *duality transformations*. Types IIA and IIB are related by T-duality as do the two heterotic theories. Supersting theories live on ten dimensional geometries and T-duality implies, in some cases as the above, that the physics of a theory in one geometry is equivalent to the physics of its T-dual theory in a different geometry. The simplest example is the one that relates IIA string theory on a circle of radius r to IIB theory on a circle of radius l_s^2/r, where l_s is the fundamental string length. Another type of duality is that of S-duality which relates the string coupling constant g_s to $1/g_s$. This type of duality is called weak-strong duality because it implies that a strongly coupled theory is equivalent to its weakly coupled S-dual theory. This is very important since if a theory is strongly coupled we do not have perturbative access to it but we can rely to its S-dual theory. The simplest example is that of type I string theory which is S-dual to the heterotic $SO(32)$. Type IIB string theory is self-dual which means that S-duality is a symmetry of this theory. T and S duality appear in quantum field theory as well and they will also play fundamental role in this thesis. It is worth mentioning that type IIA string theory is S-dual to M-theory.

Type IIA and IIB string theories have some other features, which are of great relevance to quantum field theory. When these theories are studied non-pertubratively, it can be realized that they contain other spatially extended objects except for strings. These objects are called Dp branes where p denotes the number of spatial dimensions (D stands for Dirichlet which is the nature of them as boundary conditions where fundamental strings are attached to) and they have tension that is proportional to $1/g_s$. Effectively D-branes (we skip the dimensionality for ease of notation) are hyper-surfaces embedded in the ambient ten dimensional space-time, that (super)string theory requires, where strings can end. Furthermore, D-branes are sources of p-form charge (RR charges). All of these objects except of the fundamental string ($p = 1$) become infinitely heavy as the string coupling goes to 0, and therefore they do not appear in perturbation theory. Type IIA theory has even valued branes while IIB has odd valued branes. Since fundamental strings can end on D-branes, this implies that quantum field theories live on the world-volume of the D-branes. For example, the Yang-Mills fields arise as the massless modes of open strings attached to them. Because of the ability to construct Yang-Mills like

[3]Today we know there is at least on more string theory, the Type 0 one.

theories using D-branes most phenomenological approaches to string theory with a view to construct Standard Model like theories rely on D-branes[4]. For example, it is speculated that the four-dimensions we interpret to live in are simply because we live on the world-volume of D3-branes which are themselves embedded in a higher (ten dimensional) space-time. The discovery of D-branes has given a new flavor to the physics of supersymmetric gauge theories which could be now seen as quantum field theories with string theoretic origin. In relation to the previous paragraph let us mention that S-duality has a gauge theoretic origin. Although physicists knew for a long time the notion of electric-magnetic duality in electrodynamics, Montonen and Olive showed this duality is preserved for gauge groups other than $U(1)$ while later Vafa and Witten presented strong evidence [84] that the $\mathcal{N} = 4$ super Yang-Mills gauge theory, that can be completely constructed from type IIB string reduction to four dimensions, exhibits S-duality or as it often referred to. Seiberg and Witten generalized the latter to the $\mathcal{N} = 2$ super Yang-Mills theory in four dimensions, the $\mathcal{N} = 2$ super QCD. Returning to the context of string theory, with the aid of S-duality we can study in depth type I, IIB and heterotic $SO(32)$ string theories at strong coupling. This is not the case for the remaining two string theories, type IIA and heterotic $E_8 \times E_8$ which at strong coupling they need an extra, eleventh, dimension to be described and approach at this limit the eleven dimensional M-theory. This extra dimension is of size $g_s l_s$ and geometrically is a circle (the "M-theory circle") for type IIA string theory and a line interval for the heterotic one. When this dimension is large, that is when $g_s l_s \gg 1$ we are residing outside perturbative string theory and new tools are needed to explain the corresponding physics. In 1995 it was observed that type IIA string theory is related to M-theory by dimensional reduction to ten dimensions. This can be understood in terms of an M2-brane which wraps the M-theory circle. When the radius of this circle goes to zero the M2-brane becomes the string of type IIA theory. At low energies M-theory can be approximated by an effective field theory, the eleven dimensional $\mathcal{N} = 1$ supergravity. Finally, M-theory itself contains extended objects as well, M5-branes and M2-branes which we mentioned above. The theory that lives on the world-volume of the former is the "mysterious" six dimensional $(2,0)$ super conformal field theory, the theory $\mathscr{X}$ as it is sometimes called. Using M5-branes and M2-branes various six dimensional quantum field theories can be constructed just like the theory $\mathscr{X}$. Although this thesis does not study any M-theoretic constructions many of the constructions that appear do have an M-theoretic origin and it would be of great interest to explore those aspects further.

[4]In the 1980s though, the heterotic theories were considered as better candidates for phenomenological models.

1.5 Supersymmetric gauge theories

The main body of this thesis studies a specific class of supersymmetric gauge theories many aspects of which have a string theoretic or M-theoretic construction. Supersymmetric gauge theories are interesting and useful for phenomenological reasons since they can be studied analytically and for topological theories solved analytically. Their mathematical attributes make them interesting for mathematical physicists and mathematicians as well though since they have been providing new insights in topology and geometry for the past thirty years.

Non-supersymmetric gauge theories, like electromagnetism but with non-abelian gauge group $SU(2)$ have been used by mathematicians to prove various results in algebraic and differential topology as we will explain in the next subsection. Some of these constructions the mathematicians invented will be of importance and use in later chapters of this thesis. A particular feature of any gauge theory, supersymmetric or not, is the set of classical minima of their actions, the instantons (and anti-instantons). Instantons have a moduli space that we will briefly describe later and for supersymmetric theories this moduli space can be understood using D-branes. The instanton moduli space plays a crucial role in many of these theories since the partition functions and correlation functions of many supersymmetric gauge theories localize on it. When we flow the supersymmetric theories of interest in this thesis to low energies the corresponding partition functions and correlation functions localize to a different moduli space, the *moduli space of vacua*. This space is the set of solutions of the vacuum equations (obtained from the scalar fields of the theory) modulo gauge transformations. What is special about this moduli space is that it is completely controlled by a single holomorphic function, the prepotential $\mathscr{F}$, which itself is related to instantons. For the theories of interest, which are studied on compact four-manifolds rather than flat space-time, the partition function or specific correlation functions have a geometric or rather topological interpretation since they hide indeed various topological data. This understanding led to a remarkable set of achievements and not only enhanced the faith of the usefulness of supersymmetric gauge theories, but also allowed physicists to enter into many mathematical disciplines in the fields of topology and geometry.

Currently, there exist many mathematical research directions whose main tool is gauge theory (and in some cases its connection to string theory and M-theory). This thesis will study some new aspects of towards that direction. The context of this thesis is a supersymmetric gauge theory that is topologically twisted (we will explain this below) and it is of interest since many of its correlation functions can be computed exactly and they do provide some topological invariants for the space the theory is studied on.

1.6 Gauge theory and mathematics

In this section we will briefly describe the impact that gauge theories have had in analysis, geometry and topology. Starting with the ideas of Simon Donaldson, there have been numerous developments in pure mathematics which have, to varying degrees, grown out of the ideas of gauge theories. These advancements are due to the realization that the gauge fields of particle physics can be viewed as connections over principal bundles in differential geometry, while matter fields are realized as sections of other kinds of vector bundles. This realization has led mathematicians and physicists to work on new kinds of questions, often shedding light on well-established problems. Less directly, various fundamental ideas and techniques, and especially the need to work with the infinite-dimensional gauge groups, have found a place in the general world-view of many mathematicians, influencing developments in other fields. The work in the area that lies between geometry and mathematical physics has been a prime example of the interaction between these fields which has been so fruitful over the past thirty years.

For the purposes of this thesis it is important to mention that Donaldson's work of gauge theories on smooth compact and oriented four-manifolds gave a partial answer to the classification problem of their smooth structures [17]. Donaldson, used gauge theory, instantons to be more precise, to define polynomials that are different for every inequivalent smooth structure one can equip a four-manifold with [15]. Nevertheless, skipping a lot of details, the computation of those polynomial invariants, the Donaldson invariants, are in most cases cumbersome[5]. A remarkable result came with the work of Witten [87] where he made a connection between Donaldson's theory and supersymmetric gauge theories. Witten was able to rediscover Donaldson's polynomial invariants using a $\mathcal{N}=2$ topologically twisted supersymmetric gauge theory on a four-manifold. This theory, that took the name Donaldson-Witten theory, was of particular importance since until that time the study supersymmetric gauge theories on arbitrary four-manifolds was not very well developed because such a task required to evaluate the theory on manifolds that admit a spin structure. With Witten's work the mathematical concept of Donaldson invariants and the problem of the smooth classification of four-manifolds started getting a more physical shape since Witten showed that the generating function of Donaldson invariants is equivalent to a specific correlation function in Donaldson-Witten theory

$$\Phi(p,\boldsymbol{x}) = Z_{\mathrm{DW}}(p,\boldsymbol{x}). \tag{1.1}$$

[5] We strictly mean the computation of Donaldson invariants and not their relation with the computationally more accesible Seiberg-Witten invariants that contain similar and in some cases equivalent information about the four-manifold.

The final breakthrough we need to mention in order to motivate this thesis is the fundamental paper of Moore and Witten [67]. In this remarkable work the authors brought Donaldson theory even closer to the interests of physicists since they studied the low energy effective action of Donaldson-Witten theory and wrote the famous formula

$$Z_{\mathrm{DW}}^{\mathrm{IR}} = Z_{u\text{-plane}} + Z_{\mathrm{SW}}. \tag{1.2}$$

Here, the left hand side denotes the path integral of the theory while the right hand side denotes the split into two contributions: the so-called u-plane (the space of quantum vacua in the Coulomb branch) and the Seiberg-Witten invariants contribution (coming from the *monopole equations*). This equation lies at the core of this thesis and we will study it to some detail in later chapters.

1.7 Contributions and structure of this thesis

This thesis contributes some new insights, results, and ideas regarding the u-plane integral or, more generally the Coulomb branch integral (for gauge groups of arbitrary rank). This thesis is consisted out of five chapters. Each one of the latter three ones are dedicated to the publication of a standalone paper. The order or presentation of these chapters will be different than the chronological order of publication of the papers that actually compose it. Nevertheless, we will briefly explain how these papers came to appear chronologically.

The first paper we published [44], together with my supervisor Jan Manschot, corresponds to Chapter 4. The contribution of this paper to the current literature is an alternative way to compute the u-plane integral via the theory of mock modular forms [93, 94]. Briefly, in this paper we find that by including to the path integral of the Donaldson-Witten theory a specific $\mathcal{Q}$-exact deformation (see section 2.3 for the definition of the $\mathcal{Q}$ differential) we are able to re-express the integrand of the u-plane integral as a specific kind of mock modular form called *indefinite theta function*. The computation of the u-plane integral then becomes a very simple residue integral. In this paper we computed the Donaldson invariants for $\mathbb{CP}^2$ and also for Hirzebruch surfaces $\mathbb{F}_\ell$. Our formulae are in agreement with the mathematics literature since we find the correct Donaldson invariants something that ultimately convinced us for the validity of our considerations.

The second paper [43] studied in detail the ramified Donaldson-Witten theory which is the Donaldson-Witten theory on a closed four-manifold with embedded surfaces[6] that support supersymmetric surface operators. This study was based on the previous technique of

[6]These surfaces sometimes are referred to as *surface defects*.

including a $\mathcal{Q}$-exact deformation to the path integral of the theory and derives a very simple formula for the computation of the ramified Donaldson invariants.

Such an inclusion of a $\mathcal{Q}$-exact operator was assumed to be safe and standard in the literature. Still, Greg Moore and Iurii Nidaiev, upon reading carefully our first paper, helped us realise that this might not be quite as trivial. This led to the paper [45], together with Jan Manschot, Greg Moore and Iurii Nidaiev, which studies carefully the insertion of $\mathcal{Q}$-exact operators to the path integral of the rank one Donaldson-Witten theory. By a careful analysis we realized that the insertion is well justified after regularizing the u-plane integral using techniques from the regularization of Petersson inner products of harmonic Maass forms [6]. This regularization technique is a new method for regularizing and renormalizing modular integrals and is a generalization (in an appropriate sense) of the standard method that was developed in [36].

All the necessary background will be described as required, in the appropriate order in the main four chapters of this thesis and also in the adjoined appendices with the exceptions of Seiberg-Witten and Donaldson-Witten theories which are required for all of the chapters and therefore we have dedicated Chapter 2 just for them. Whenever we think it is necessary we advise the reader to consult the literature by providing appropriate references.

Let us therefore summarize the structure of the present thesis which is composed by the main four chapters

- Chapter 2 - Review of Seiberg-Witten and Donaldson-Witten theories,
- Chapter 3 - Publication [45] and relevant background,
- Chapter 4 - Publication [44] and relevant background,
- Chapter 5 - Publication [43] and relevant background,

and complemented by Appendix A on modular forms, indefinite theta functions, etc., and Appendix B on surface operators.

Chapter 2

Seiberg-Witten theory and Donaldson-Witten theory

In this chapter we will lay down the basic background needed in order to understand the main contributions of this thesis. The technical tools required are the Seiberg-Witten technology of supersymmetric gauge theories and the topologically twisted $\mathcal{N}=2$ theory, the Donaldson-Witten theory. The goal is to use Seiberg-Witten theory and study the vacua of the Coulomb branch in the low energy effective action of the Donaldson-Witten theory.

2.1 Seiberg-Witten theory

In this section we give a brief overview of the very well-known Seiberg-Witten theory [74, 75]. A lot of this content is drawn from the excellent resources [48, 63, 64].

Seiberg-Witten theory is the low energy effective theory of the $\mathcal{N}=2$ SYM theory with gauge group $SU(2)$ or $SO(3)$ on flat $\mathbb{R}^4$. The global bosonic symmetry group is $SO(4)\times SU(2)_R \cong (SU(2)_l \times SU(2)_r)/\mathbb{Z}_2 \times SU(2)_R \times U(1)_R$ where the latter is an anomalous $U(1)$ symmetry. The Rotations around $\mathbb{R}^4$ are represented by $K = SO(4) \cong SU(2)_l \times SU(2)_r/\mathbb{Z}_2$ and the factor $SU(2)_R$ correspond to the R-symmetry. In terms of the corresponding Lie algebras this bosonic symmetry reads $\mathfrak{su}(2)_l \times \mathfrak{su}(2)_r \times \mathfrak{su}(2)_R \times \mathfrak{u}(1)_R$. For simplicity we focus on the pure gauge theory case with no additional matter representations so we will have no Higgs branches in the moduli space of vacua. The field content of the theory is a single $\mathcal{N}=2$ vector multiplet in the adjoint representation which is summarized in Table 2.1.

We want to study the vacua of the low energy effective theory. At this stage we will assume a generic gauge group G with Lie algebra $\mathfrak{g}$. Seiberg-Witten theory has a Lagrangian

Description	Field	Representation under K
gauge field	A	$(2,2,1)$
chiral spinor	ψ	$(2,1,2)$
anti-chiral spinor	$\bar{\psi}$	$(1,2,2)$
complex scalar (Higgs)	ϕ	$(1,1,1)$
auxiliary scalar	D	$(1,1,3)^0$

Table 2.1 The field content of pure $\mathcal{N}=2$ SYM theory and the corresponding representations.

description[1] with a potential term for the Higgs field and its classical vacua are given by minimizing this term. The condition for the vacua reads

$$[\phi,\bar{\phi}]=0. \tag{2.1}$$

The solution set $\mathcal{V}$ of the fields ϕ that satisfy the previous equation is a space with a natural action of the gauge group G [63, 74]. The quotient gives the *space of classical vacua of the theory*

$$\mathscr{B}^{\text{classical}}=\mathcal{V}/G. \tag{2.2}$$

Taking a closer look at Equation (2.1) we realize that it implies that ϕ is semi-simple and as a result it can be conjugated to a maximal torus $\mathfrak{t}\otimes\mathbb{C}$. Physically this means that we can perform a gauge transformation to the field ϕ and bring it to a diagonal form. When the Higgs field is (i) semi-simple[2] and (ii) regular[3], the stabilizer is the normalizer of the Cartan torus T, and the gauge group that remains unbroken is simply the maximal torus T with Lie algebra $\mathfrak{t}$ [63]. Therefore, the moduli space of classical vacua can be identified with

$$\mathscr{B}^{\text{classical}}=\frac{\mathfrak{t}\otimes\mathbb{C}\backslash\Delta}{W} \tag{2.3}$$

where W is the Weyl group and Δ is a subset of semi simple non-regular elements.

Returning to the simplest case $G=SU(2)$, the classical vacua are given by the solution set of

$$\text{Tr}([\phi,\bar{\phi}])^2=0. \tag{2.4}$$

[1] The specific Lagrangian is not of importance here so we do not include it but it can be easily read off from [48] or [83].

[2] This refers to the context of Lie algebras.

[3] A $n\times n$ diagonizable matrix is regular if and only if there are n distinct eigenvalues.

This term is minimized by normal matrices, therefore we can perform a gauge transformation and bring the ϕ to the form

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \tag{2.5}$$

uniquely up to a Weyl transformation which is just a $\mathbb{Z}_2$ automorphism given by the anti-podal map $a \to -a$. The classical vacua are pararemtrized by a single gauge invariant observable (that will be of fundamental importance in this thesis), the famous u parameter defined as

$$u^{\text{classical}} = \text{Tr}(\phi^2) = 2a^2. \tag{2.6}$$

This parameter can take any value in the complex plane - that we can simply refer to as the u-plane - and determines a classical vacuum of the theory on $\mathbb{R}^4$. At a generic point of the u-plane the gauge group $SU(2)$ is broken down to its maximal torus $U(1)$ and the $W^{\pm}$ bosons which are normally associated with the off-diagonal generators, have a mass proportional to $|a|$. Interestingly, there exists a specific value of u, the zero value, that these bosons become massless and the stabilizer of ϕ jumps back to $SU(2)$. This means that the non-abelian symmetry is restored at $u = 0$. We conclude therefore that $\mathscr{B}^{\text{classical}}$, the classical moduli space of vacua, is given by

$$\mathscr{B}^{\text{classical}} = \mathbb{C}\backslash\{0\}, \tag{2.7}$$

that is the punctured complex plane, or by adding the point at infinity we identify

$$\mathscr{B}^{\text{classical}} = \mathbb{CP}^1\backslash\{0\}. \tag{2.8}$$

The classical moduli space is subject to quantum effects and the description of low energy fluctuations around a chosen vacuum therefore gets corrections. Seiberg and Witten described in their seminal paper [74] the nature of these corrections and showed that although they do not change the topology of the moduli space they do change the singularity structure. Furthermore they showed that there is not any symmetry restoration at special values of the quantum moduli space unlike in the classical moduli space. In order to understand this point we need to study the low energy effective theory in detail. Unfortunately it is quite hard to integrate out massive degrees of freedom explicitly and as a result we have to understand the structure of the low energy theory by symmetry considerations and holomorphicity . By studying the most generic $\mathcal{N} = 2$ supersymmetric action involving a single vector multiplet Seiberg and Witten realized that the low energy $U(1)$ theory is completely determined by a single holomorphic function $\mathscr{F}(a)$ called the *prepotential*. Furthermore, by studying further the theory they were able derive the metric of the moduli space of vacua which reads

$$ds^2 = \mathrm{Im}(\tau) da d\bar{a}, \tag{2.9}$$

where

$$\tau(a) = \frac{\partial^2 \mathscr{F}(a)}{\partial a^2} \tag{2.10}$$

is the holomorphic gauge coupling of the theory. Using perturbation theory and the holomorphicity of τ it is possible to obtain the effective coupling at one-loop level [74]

$$\tau(a) = \frac{2\mathring{i}}{\pi} \log\left(\frac{a}{\Lambda}\right). \tag{2.11}$$

Here Λ is the dynamically generated scale of the low energy theory. Similarly we can find the prepotential at one-loop level [74]. Due to the constraints of $\mathscr{N} = 2$ supersymmetry this one-loop expression is in fact an exact perturbative answer. Since we begin with a non-abelian theory we expect instanton effects to give non-perturbative contributions to $\tau(a)$ and $\mathscr{F}(a)$ in addition of breaking some classical symmetries. The prepotential $\mathscr{F}(a)$ becomes

$$\mathscr{F}(a) = \frac{\mathring{i}}{2\pi} a^2 \log\left(\frac{a}{\Lambda}\right) + \sum_{k=1}^{\infty} \mathscr{F}_k \left(\frac{\Lambda}{a}\right)^{4k} a^2, \tag{2.12}$$

where $\mathscr{F}_k$ are all non-zero constant coefficients (since in a supersymmetric theory all instanton contributions come from zero modes). We see therefore that the problem of understanding the low energy effective action of the theory amounts into computing $\mathscr{F}_k$ for all k. This is the program of *instanton counting*[4] [69, 70].

Let us discuss further the structure of the Seiberg-Witten theory. The supersymmetry algebra of the theory contains a central charge function $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ where Γ is the lattice of electric and magnetic charges that, as we will explain shortly, is fibered over the quantum moduli space of vacua,

$$Z(n_e, n_m) = n_e a + n_m a_D. \tag{2.13}$$

Here, $(n_e, n_m) \in \Gamma$ and a_D is the dual of a defined as

$$a_D = \frac{\partial \mathscr{F}(a)}{\partial a}. \tag{2.14}$$

Together, (a, a_D) form a section of a flat line bundle, a local system actually, that undergoes monodromies along the moduli space of vacua. This hints that the quantum moduli space

[4]The mathematical definition of instantons is reviewed in Chapter 4.

might have some singularities that can affect the section (a, a_D) upon following the path of a closed loop around these singularities.

The problem at hand is to compute the holomorphic gauge coupling $\tau(a)$ for arbitrary a, a_D and u. In the quantum theory, the correct order parameter is the vacuum expectation value

$$u(a) = \frac{1}{16\pi^2}\langle \mathrm{Tr}(\phi^2)\rangle, \tag{2.15}$$

which only classically equals (2.6). Mathematically, the order parameter u from (2.6) is the generator of the $SU(2)$-equivariant cohomology of a point, $H^*_{SU(2)}(pt.) = A(\mathfrak{t}/W)$ where the right hand side denotes the coordinate ring of $\mathfrak{t}/W$. As we explained, in the quantum version of Equation (2.15) we have to take into consideration instanton effects. We can interpret u as being an element in the fractional field of $H^*_{SU(2)}(pt.)$ or in other words, we can interpret u as a rational function on $A(\mathfrak{t}/W)$ that can be roughly defined as[5] [68]

$$u := \frac{\sum_{n\geq 1} q^n \int_{\mathscr{M}_n} \mu(p)}{\sum_{n\geq 0} \int_{\mathscr{M}_n} \mathbf{1}}, \tag{2.16}$$

where q is a formal variable (for the moment), $\mathscr{M}_n$ is the moduli space of instantons, $p \in H_0(M, \mathbb{Q})$ and μ is a map called *slant product*. These objects will not be needed until chapter 4 so we will postpone their explicit definition and further discussion until then.

Furthermore, Seiberg and Witten showed that in the quantum theory the moduli space of vacua is the whole of the u-plane, every point of it is part of a $U(1)$ theory except for two special points, $u = \pm\Lambda^2$ where although the theory is still an abelian gauge theory, two new massless hypermultiplets appear, a monopole and a dyon and the theory is described by a $U(1)$ abelian gauge theory coupled to those hypermultiplets. It is precisely these two singular points that affect (a, a_D). In the classical limit, $\Lambda^2 \to 0$ these points come close together and we recover the classical moduli space of vacua $\mathscr{B}^{\text{classical}}$.

With these considerations on mind, Seiberg and Witten found that the exact solution of the theory is given by a family of elliptic curves fibered over the u-plane[6]. These curves Σ_u are called *Seiberg-Witten curves* and play a crucial role to this thesis. The curve that describes our moduli space of vacua can be written as

$$y^2 = 4x(x^2 - ux + \frac{\Lambda^4}{4}), \qquad x, y \in \mathbb{C}. \tag{2.17}$$

[5] Note that, by a slight abuse of notation, the u that appears in Equation (2.16) is not the classical one, rather it is the quantum corrected one.

[6] Elliptic curves appear for gauge theories of rank one. For higher rank gauge groups the corresponding Seiberg-Witten solution is described by hyper-elliptic curves.

Another presentation of this elliptic curve is its Weierstrass form which gives an isogenus curve

$$\tilde{y}^2 = 4\tilde{x}^3 - g_2\tilde{x} - g_3. \tag{2.18}$$

To transform (2.17) to (2.18) we need to make the substitutions $x = \alpha^{-2}\tilde{x} + \frac{1}{3}u$ and $y = \alpha^{-3}\tilde{y}$. The constants g_2 and g_3 are given in terms of Eisenstein series $E_k(\tau)$ as

$$g_2 = \frac{4\pi^4}{3}E_4, \tag{2.19}$$

$$g_3 = \frac{8\pi^6}{27}E_6. \tag{2.20}$$

See Equation (A.5) in Appendix A.1.2 for the definition of Eisenstein series. The parameter α that is involved in the transformation can be expressed as

$$\alpha = \frac{\sqrt{2}\pi}{\Lambda}\vartheta_2\vartheta_3, \tag{2.21}$$

where $\vartheta_j(\tau)$ are the classical Jacobi theta functions defined in Equation (A.10) in Appendix A.1.4. The determinant of the curve is given by

$$\Delta(u) = \frac{1}{4096}(u^2 - \Lambda^4), \tag{2.22}$$

and the solution set of $\Delta(u)$ corresponds to points in the u-plane that the curve degenerates. These are precisely the singular points $\pm\Lambda^2$ where the new hypermultiplet degrees of freedom appear.

This family of elliptic curves, which topologically correspond to genus one tori and are parametrized by u, comes equipped with a differential, the *Seiberg-Witten differential* λ_{SW} which satisfies

$$\frac{d\lambda_{\mathrm{SW}}}{du} = \frac{\sqrt{2}}{8\pi}\frac{dx}{y}. \tag{2.23}$$

By integrating this expression we easily find that (up to an exact differential) the Seiberg-Witten differential reads

$$\lambda_{\mathrm{SW}} = -\frac{\sqrt{2}}{4\pi}\frac{dx}{y}(2u - 4x). \tag{2.24}$$

The Seiberg-Witten differential completely determines a and a_D in terms of u by pairing it with the homology cycles $A, B \in H_1(\Sigma_u)$ as follows

$$a_D(u) = \oint_A \lambda_{\mathrm{SW}}, \tag{2.25}$$

$$a(u) = \oint_B \lambda_{\mathrm{SW}}. \tag{2.26}$$

This way the prepotential $\mathscr{F}(a)$ is implicitly determined through Equation (2.14). With the aid of Equation (2.10) the holomorphic gauge coupling τ can then be found using

$$\tau = \frac{da_D}{da}. \tag{2.27}$$

Using this relation we can also find the value of u as a function of τ [74]. These computations involve studying elliptic integrals of the first kind and other technical details that we will skip in this thesis.

Using (2.25) and (2.26) we can write the central charge of an arbitrary state of the theory with charge $\gamma \in \Gamma$ as

$$Z(\gamma) = \oint_\gamma \lambda_{\mathrm{SW}}. \tag{2.28}$$

We see therefore that γ is interpreted as a cycle in $H_1(\Sigma, \mathbb{Z})$. To be precise we have to substitute Σ with its closure $\overline{\Sigma}$ giving the identification

$$\Gamma \simeq H_1(\overline{\Sigma}, \mathbb{Z}). \tag{2.29}$$

In this way we have a natural interpretation of Equation (2.13) in terms of the homology of Σ_u. Recall that the mass of a BPS state of charge $\gamma \in \Gamma$ is determined by the absolute value of the central charge

$$M(\gamma) = |Z(\gamma)|. \tag{2.30}$$

The structure of the Coulomb branch is depicted in Figure 2.1. The weak coupling and the string coupling regions are separated by a co-dimension one wall, a *wall of marginal stability* W. This is defined by the condition

$$\arg a = \arg a_D, \tag{2.31}$$

or

$$\frac{a}{a_D} \in \mathbb{R}^+. \tag{2.32}$$

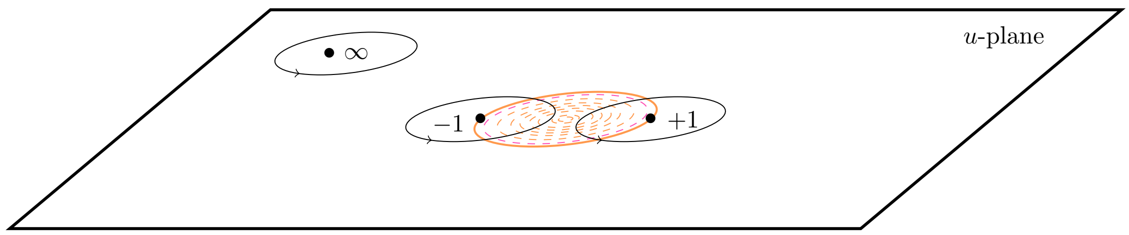

Figure 2.1 The Coulomb branch of the theory or the u-plane. We recognize three singular points around which monodromies have to be taken into account. The red region marks the strongly coupled region of the u-plane.

Because of the following property of the central charges of two BPS states

$$|Z(\gamma_1+\gamma_2)| \geq |Z(\gamma_1)|+|Z_{\gamma_2}|, \tag{2.33}$$

BPS states are allowed to combine or decay into other BPS states. This happens when the arguments of the central charges of electric and magnetic states align. For the case of pure $\mathscr{N}=2$ theory (as well as for $N_f=2,3$) the Coulomb branch splits into two regions. The strong coupling regions are located at the projection of the singular fibers and this is how one can understand the singular points of the elliptic curve Σ. Due to the non-local structure of the electric-magnetic charge lattice Γ, there is no unique description of the charges of the BPS states that they correspond to. We will see in the next paragraph how we deal with monodromies around the singular locus of $\mathscr{B}$. We would like to mention that in general there is a formula on determining the BPS spectrum in either side of the wall, it is Kontsevich's and Soibelman's wall-crossing formula [60] which connects the topic of BPS states with the topic of motivic Hall algebras and Donaldson-Thomas invariants. Note though, that this wall-crossing formula is different to the one we use in Chapters 4 and 5.

By now, one might have been convinced the low energy effective theory of the $\mathscr{N}=2$ SYM with gauge group $SU(2)$ can be understood by a family of elliptic curves Σ_u controlled by u and fibered over the u-plane and such that the curve degenerates at two points. Let us set $\Lambda=1$ such that the degeneration points are ± 1. By adding the point at infinity which corresponds to the semi-classical region of the theory, we see that topologically the moduli space of vacua that from now on we will call *Coulomb branch* is

$$\mathscr{B} = \mathbb{CP}^1 \backslash \{\infty, \pm 1\}. \tag{2.34}$$

This gives the information about the topology and the singularity structure of the Coulomb branch. But a closer observation of this hyperbolic Riemann surface yields a direct relation to modular $SL_2(\mathbb{Z})$ domains through Riemann's uniformization theorem. Actually the Coulomb branch is identified with

$$\mathscr{B} \cong \mathbb{H}/\Gamma^0(4), \tag{2.35}$$

where $\mathbb{H}$ is the Lobachevsky upper half-plane and $\Gamma^0(4)$ is a congruence subgroup of $SL_2(\mathbb{Z})$ (that will be defined shortly) that appears because the elliptic curve our theory is described by is a modular curve with respect to $\Gamma^0(4)$. Sometimes this group is called *monodromy group*. A way to get intuition is to realize that the u-plane in the quantum theory has three punctures that correspond to three singular points $\{\infty, \pm 1\}$. Let us study what happens say at $u = \infty$ by taking a closed loop,

$$u \to e^{2\pi i} u. \tag{2.36}$$

This action forces a and a_D to change. Since (a, a_D) is a section of a flat bundle, such a loop acts as a linear transformation on this section. For the point at infinity this is given locally by a *monodromy* matrix [74]

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}.$$

Near the singular point $u = +1$ the corresponding monodromy matrix is given by [74]

$$M_{+1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

while near the singular point $u = -1$ the monodromy matrix can be found by requiring that $M_{+1}M_{-1} = M_\infty$. Thus near $u = -1$ we have

$$M_{-1} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}.$$

These matrices generate the congruence subgroup $\Gamma^0(4)$ of $SL_2(\mathbb{Z})$ which is defined as

$$\Gamma^0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, b = 0 \mod 4 \right\}. \tag{2.37}$$

Let us discuss the structure of the Coulomb branch $\mathscr{B}$ as a modular domain. Had the monodromy group been simply $SL_2(\mathbb{Z})$ the moduli space of the elliptic curve Σ_u would had been simply the familiar key-hole shaped modular domain $\mathscr{F}_\infty$ (not to be confused with the prepotential) that we are familiar with from bosonic string theory and is defined as

$$\mathscr{F}_\infty = \left\{ \tau \in \mathbb{H} \mid -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2}, |\tau| \geq 1 \right\}. \tag{2.38}$$

Our curve though has as monodromy group the congruence subgroup $\Gamma^0(4)$ which is of index six. This means that it contains six images of $\mathscr{F}_\infty$ in its modular domain. That is, $SL_2(\mathbb{Z})$ can be written as

$$SL_2(\mathbb{Z}) = \bigcup_{i=1}^{6} \alpha_i^{-1}(\Gamma^0(4)), \tag{2.39}$$

where

$$\begin{aligned} \alpha_1 &= 1, \\ \alpha_2 &= T, \\ \alpha_3 &= T^2, \\ \alpha_4 &= T^3, \\ \alpha_5 &= S, \\ \alpha_6 &= T^2 S. \end{aligned} \tag{2.40}$$

Here T and S are the generators of $SL_2(\mathbb{Z})$ which act on τ as follows

$$T : \tau \to \tau + 1 \tag{2.41}$$

$$S : \tau \to -\frac{1}{\tau}. \tag{2.42}$$

Similarly, the $T' = T^4$ and $S' = ST^{-1}S$ generators of $\Gamma^0(4)$ act on τ following

$$T' : \tau \mapsto \tau + 4 \tag{2.43}$$

$$S' : \tau \mapsto \frac{\tau}{\tau + 1}. \tag{2.44}$$

Appendix A.1 contains more details on $SL_2(\mathbb{Z})$, its subgroup $\Gamma^0(4)$ and their generators. It is easy to see then that the modular domain $\mathbb{H}/\Gamma^0(4)$ which is isomorphic to the Coulomb branch $\mathscr{B}$, can be written as

$$\mathbb{H}/\Gamma^0(4) = \mathscr{F}_\infty \cup T\mathscr{F}_\infty \cup T^2\mathscr{F}_\infty \cup T^3\mathscr{F}_\infty \cup S\mathscr{F}_\infty \cup T^2 S\mathscr{F}_\infty. \tag{2.45}$$

The first four domains give the region near the singularity, or rather cusp, $\tau \to i\infty$ and correspond to the semi-classical region of the u-plane, $u \to \infty$. The region $S\mathscr{F}_\infty$ surrounds the cusp near $\tau = 0$ which corresponds to the monopole singular point at $u = +1$ and the region $T^2 S\mathscr{F}$ surrounds the cusp near $\tau = 2$ which corresponds to the dyon point at $u = -1$. These regions are portrayed in Figure 2.2 below.

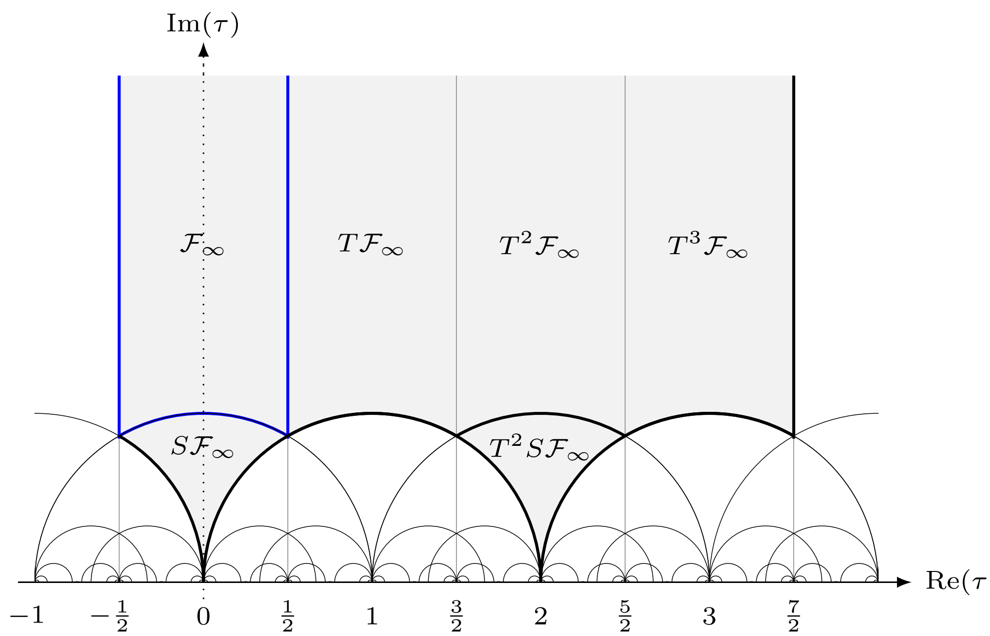

Figure 2.2 Upper-half plane $\mathbb{H}$ with the area bounded by blue ($\mathscr{F}_\infty$) a fundamental domain of $\mathbb{H}/SL_2(\mathbb{Z})$, and the shaded area a fundamental domain of $\mathbb{H}/\Gamma^0(4)$.

All quantities that appear in Seiberg-Witten theory can be written as combinations of standard modular forms. We close this section by mentioning some of them that will be needed in the rest of this thesis.

$$u(\tau) = \frac{1}{2}\frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{(\vartheta_2(\tau)\vartheta_3(\tau))^2}, \tag{2.46}$$

$$a(\tau) = \frac{1}{6}\left(\frac{2E_2(\tau) + \vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{\vartheta_2(\tau)\vartheta_3(\tau)}\right), \tag{2.47}$$

$$\frac{da(\tau)}{du(\tau)} = \frac{1}{2}\vartheta_2(\tau)\vartheta_3(\tau). \tag{2.48}$$

Remarks on Seiberg-Witten geometry and string theory

Seiberg-Witten geometry is sometimes referred to as *rigid special geometry* and it can emerge from completely string theoretic constructions through the moduli space of Calabi-Yau threefolds. Using the (Zamolodchikov) metric of the $\mathcal{N} = (2,2)$ superconformal field theories in $d = 2$, it can be shown that the (Kähler) prepotential of the moduli space of Calabi-Yau threefolds is related to the prepotential of the low energy effective theories of various string compactifications [77]. Let us describe this point a little further. In the context of type IIA string compactifications on a Calabi-Yau threefold X, the moduli of vector multiplets is identified with the deformations of the Kähler moduli of X. On the other hand, the moduli of the hypermultiplets is identified with deformation of the complex moduli of X. If we now switch to the type IIB picture these roles are reversed. This is precisely the idea behind *mirror symmetry*. Broadly speaking, the prepotential of X captures the classical geometry that can potentially admit quantum corrections in the full string theory picture. Nevertheless, there exist some *non-renormalization theorems* that ensure there is no quantum corrections to the moduli space of complex structures of X in type IIB string theory and therefore the exact coupling of vector multiplets can be computed by classical geometry. This fundamental idea has resulted in striking results where the exact solutions of various quantum field theories such as the superconformal field theory that corresponds to the quintinc threefold and its mirror [9].

Seiberg-Witten theory in specific is an example of a rigid supersymmetric Yang-Mills theory. Such theories are obtained by taking the $\alpha' \to 0$ limit of type II string theory on a certain Calabi-Yau background. In order to have theories with the desired gauge groups the Calabi-Yau background is usually a fibration of the compact $K3$ surface or the non-compact ALE space, both of which are hyper-Kähler, over a base $\mathbb{CP}^1$. Then, at the rigid limit, the Seiberg-Witten curves that we described previously can be roughly realized by looking close at the locus the singular fibers [40].

2.2 CohFT and BRST symmetry

As we will explain below, in order to study supersymmetric field theories on arbitrary compact four-manifolds we have to perform *topological twisting*. The resulting theories are some times called *cohomological field theories* (CohFTs), for reasons that will become clear later, and they have revolutionized the field of research in the interplay between algebraic and differential geometry and quantum field theory and string theory, the so-called *physical mathematics*. All kinds of different invariants of topological spaces are expressed nowadays through the language of CohFTs:

- **Donaldson invariants** [15] and Seiberg-Witten invariants [89] of smooth four-manifolds, which are some of the protagonists of this thesis, appear through the Donaldson-Witten theory. We will describe explicitly this connection in the present chapter.

- **Gromov-Witten invariants** of Calabi-Yau manifolds [88] are related to the topological string A-model which is nothing more than a specific kind of topological twist for the non-linear $\mathscr{N} = (2,2)$ sigma-model.

- **Seiberg-Witten invariants** of symplectic manifolds are related to a specific Gromov-Witten invariant of the same symplectic manifolds [82] and the former were originally understood by means of an abelian CohFT [89].

- **Donaldson-Thomas invariants** of Calabi-Yau manifolds [18] correspond to a six dimensional version of Donaldson-Witten theory that is related to the topological string B-model.

- **Vafa-Witten invariants** of four-manifolds are the subject of Vafa-Witten theory [84] which is one of the three twists of the $\mathscr{N} = 4$ SYM. Vafa-Witten invariants are currently a topic of intense research [81].

- **Geometric Langlands dualities** appear in physics by studying a reduction to two dimensions the Kapustin-Witten twisted $\mathscr{N} = 4$ SYM theory (this is the second twist of this theory out of the three available) [39]. In two dimensions this theory is related to a family of topological A-models and B-models.

All of the above theories have a very common ingredient, that is (at least) one BRST-like operator $\mathcal{Q}$ such that

$$\mathcal{Q}^2 = 0. \tag{2.49}$$

Let us make a remark at this point. This operator is the differential of some cohomology ring that appears in the theory we study. Some by now standard physical theories with their associated cohomology theories are shown in Table 2.2.

When this theory is a sigma model or a theory without gauge symmetries then Equation (2.49) is precise. If the theory we study involves some bundle with non-trivial gauge endomorphisms then this relationship must be taken modulo gauge transformations, in other words $\mathcal{Q}$ is to be the differential of some equivariant cohomology theory. The BRST operator $\mathcal{Q}$ can give a plethora of information about the theory since it divides the (gauge invariant) observables into three distinctive sets.

- Observables $\mathscr{O}$ for which $\{\mathcal{Q}, \mathscr{O}\} \neq 0$.

- Observables that are $\mathcal{Q}$-exact, i.e., observables that can be written as $\tilde{\mathcal{O}} = \{\mathcal{Q}, \mathcal{O}\}$ and such that $\{\mathcal{Q}, \tilde{\mathcal{O}}\} = 0$.
- Observables that are $\mathcal{Q}$-closed, i.e., observables $\mathcal{O}$ such that $\{\mathcal{Q}, \mathcal{O}\} = 0$. Of course, the $\mathcal{Q}$-exact observables are tautologically $\mathcal{Q}$-closed so we disregard them from this subset.

Cohomology theory	Physical theory
de Rham cohomology	supersymmetric QM on closed M_4
quantum de Rham cohomology	A-model on $\mathbb{R} \times S^1$
Dobeault cohomology	B-model on $\mathbb{R} \times S^1$
Khovanov homology of a knot K	CS theory on M_3
Lagrangian Floer symplectic homology	A-model on $\mathbb{R} \times I$ with boundary $L_{1,2}$
Floer instanton homology	4d gauge theory on M_4 with boundaries

Table 2.2 Various cohomology theories and some physical theories they are related to [31]. The first line of the table gives the Witten index. Quantum de Rham cohomology refers to the deformation quantization of the exterior algebra of the usual de Rham cohomology. Khovanov homology appears in the context of Chern-Simons theory which is a TQFT but not a CohFT. Floer homologies are related to dimensional reductions of gauge theories defined in four-dimensions. By M_d we denote a manifold of dimension d.

The following chapter will study in detail various $\mathcal{Q}$-exact operators so it is worthy to discuss them to some extent but before that we would like to discuss several important features that CohFTs have.

One of the most important examples of $\mathcal{Q}$-exact operator in CohFTs is the variation of the action functional $S[X]$ (assuming that the theory has a Lagrangian description) with respect to the Riemannian metric g we equip the underlying manifold M with. For a suitable operator W, we can express this variation as

$$\delta_g S[X] = \frac{1}{2} \int_M \sqrt{g}\, g^{mn} \{\mathcal{Q}, W_{mn}\}, \tag{2.50}$$

where by X we denote collectively all the fields of the theory and where m,n are Euclidean space indices. As explained in detail in [88], the path integral measure $[\mathscr{D}X]$ and the action functional $S[X]$ are both invariant under the global BRST symmetry generated by $\mathcal{Q}$. Then, the Ward-Takahashi identity for this symmetry suggests that the one-point function or the vacuum expectation value of a $\mathcal{Q}$-exact gauge invariant operator vanishes, that is

$$\langle \{\mathcal{Q}, \mathcal{O}\} \rangle = \int [\mathscr{D}X]\, e^{-S[X]} \{\mathcal{Q}, \mathcal{O}\} = 0. \tag{2.51}$$

In other words, the Ward-Takahashi identity says that the insertion of such a $\mathcal{Q}$-exact operator into a correlation function make it vanish,

$$\sum_i \langle \{\mathcal{Q}, \mathcal{O}_i\} \prod_{i\neq j} \mathcal{O}_j \rangle = 0. \tag{2.52}$$

In this thesis, and due to the nature of CohFTs, the distinction between $\mathcal{Q}$-exact and $\mathcal{Q}$-closed observables is of crucial importance. The former decouple from the latter since for a $\mathcal{Q}$-exact operator that can be written as $\{\mathcal{Q}, \mathcal{O}'\}$ we have

$$\langle \{\mathcal{Q}, \mathcal{O}'\} \prod_j \mathcal{O}_j \rangle = 0, \tag{2.53}$$

if

$$\{\mathcal{Q}, \mathcal{O}_j\} = 0, \qquad \forall j. \tag{2.54}$$

The fact that $\mathcal{Q}$-exact operators decouple from the $\mathcal{Q}$-closed ones is very important and has consequences for the nature of the topological observables of the theory. In a CohFT physical observables of the theory can only be the ones that are $\mathcal{Q}$-closed but not $\mathcal{Q}$-exact. We can see this by studying the variation with respect to the metric of the vacuum expectation value of an operator $\mathcal{O}$. This is given by

$$\delta_g \langle \mathcal{O} \rangle = \int [\mathcal{D}X]\, e^{-S[X]} (\delta_g \mathcal{O} - \mathcal{O} \delta_g \mathcal{L}), \tag{2.55}$$

where $\mathcal{L}$ is the Lagrangian of the theory. The right hand side of this equation vanishes if $\mathcal{O}$ is independent of the metric or if $\delta_g \mathcal{O}$ is at least $\mathcal{Q}$-exact.

Therefore, recalling Equation (2.50) as well as the previous paragraph, we arrive at the well-known and fundamental statement that the Hilbert space $\mathcal{H}$ of a CohFT is identified with a cohomology ring whose differential is $\mathcal{Q}$, that is

$$\mathcal{H} = \frac{\text{Ker } \mathcal{Q}}{\text{Im } \mathcal{Q}}. \tag{2.56}$$

In the case of Donaldson-Witten theory the underlying cohomology theory is either the de Rham cohomology or the Dolbeault cohomology (depending on the manifold under consideration) and specific correlation functions of observables in this complex give a physical construction of the famous Donaldson polynomial invariants [15, 17] whose definition will be recalled in Chapter 4.

2.3 Donaldson-Witten theory

Donaldson-Witten theory is the topologically twisted version of the $\mathcal{N}=2$ SYM. Let us describe why the twist is needed. Any $\mathcal{N}=2$ theory in four dimensions contains two supersymmetries that amount to a total of eight supercharges that are usually denoted as $Q_{\alpha I}$ and $\bar{Q}^{J}_{\dot{\alpha}}$. We want to be able to study supersymmetric theories on an arbitrary four-manifold M. The problem is that an arbitrary M will break all of those supersymmetries, the reason being that a generic M is not a *spin*-manifold. A manifold being *spin* or not determines wether it admits a spinor bundle whose sections are spinors just like ordinary fermions in flat space-time. Therefore, if we want to study the theory on curved compact manifolds we have to restrict to the few ones that are *spin* four-manifolds (as Pestun did with $\mathbb{S}^4$ [71]) or hyperKähler four-manifolds (such as the $K3$ surface). More physically, the supercharges of $\mathcal{N}=2$ supersymmetry anti-commute to translations and a generic four-manifold breaks translation symmetry. This seems very restrictive. Witten realized in his fundamental paper [89] that after performing the topological twist the supersymmetric theory is well defined on arbitrary M. The twisted theory is equivalent to the untwisted in flat space or on hyperKähler manifolds but on an arbitrary four-manifold it is only a subsector of the full (untwisted) theory.

We will now describe the twisting procedure. The global bosonic symmetry group of the theory is

$$H = SU(2)_l \times SU(2)_r \times SU(2)_R \times U(1)_R, \tag{2.57}$$

where $K = SU(2)_l \times SU(2)_r$ is the rotation group and $SU(2)_R \times U(1)_R$ is the internal R-symmetry group as we mentioned earlier. The flat space supersymmetry (spinorial) generators $Q_{\alpha I}$ and $\bar{Q}^{J}_{\dot{\alpha}}$ transform as $(2,1,2)^{-1}$ and $(2,1,2)^{-1}$ respectively. Twisting amounts on re-interpreting the rotation group by choosing an embedding

$$SU(2)_l \times SU(2)_R \to \mathrm{diag}(SU(2)_l \times SU(2)_R) = SU(2)'_l. \tag{2.58}$$

Under this redefinition the global symmetry group reads

$$H = SU(2)'_l \times SU(2)_r \times U(1)_R. \tag{2.59}$$

As a result of the twist the representations under which the fields transform are different and Table 2.3 shows what these representations change to and what the twisted fields are.

Effectively, under the twisting procedure all fields become differential forms. To be more precise, all twisted fields are sections of the deRham bundle, plus the gauge field which is a connection one-form. The gauge field $A \in \Omega^1(M, ad\,P)$ remains a connection over a principal

Field before twist	Rep. before twist	Rep. after twist	Field after twist
A	$(2,2,1)^0$	$(2,2)^0$	A
ψ	$(2,1,2)^{-1}$	$(2,2)^1$	ψ
$\bar{\psi}$	$(1,2,2)^1$	$(1,1)^{-1}\oplus(3,1)^{-1}$	η,χ
ϕ	$(1,1,1)^{-2}$	$(1,1)^{-2}$	ϕ
$\bar{\phi}$	$(1,1,1)^2$	$(1,1)^2$	$\bar{\phi}$
D	$(1,1,3)^0$	$(3,1)^0$	D

Table 2.3 Field content of the theory and representations before and after twisting.

$SU(2)$-bundle P (better understood as the connection on the associated adjoint bundle $ad\,P$). The scalar fields are sections of $ad\,P\otimes\mathbb{C}$ and D is a section of $\Omega^2_+(M,ad\,P)$. As for the Grassmann valued (non-spinorial) fermions we have the following characterizations. The scalar field η is a section of $\Omega^0(M,ad\,P)$, the one-form fermion ψ is a section of $\Omega^1(M,ad\,P)$ and the self-dual two-form fermion χ is a section of $\Omega^2_+(M,ad\,P)$. Notice that the previous three fields are Grassmann valued. Furthermore, the supercharges $Q_{\alpha I}$ and $\bar{Q}^J_{\dot{\alpha}}$ transform just as the fermions and in the twisted theory provide three operators one of which is the only supersymmetry that survives. We are referring of course to the scalar BRST operator

$$\mathcal{Q}=\varepsilon^{\dot{\alpha}\dot{\beta}}Q_{\alpha\beta}. \tag{2.60}$$

The other two operators we obtain from twisting the supersymmetries are a one-form $K=K_m dx^m$ and a self-dual two-form $L=L_{mn}dx^m\wedge dx^n$. The first operator, K, provides a canonical solution to the descent equations

$$\{\mathcal{Q},\mathcal{O}^{(i+1)}\}=d\mathcal{O}^{(i)}, \tag{2.61}$$

that involve observables $\mathcal{O}^{(i)}$ of form degree i, by setting $\mathcal{O}^{(i)}=K^i\mathcal{O}^{(i)}$. Integration of these operators over i-cycles in the homology of M give topological observables since $\{\mathcal{Q},K\}=d$, where d is the de Rham operator. The gauge invariant observables of the low energy abelian theory, that we are interested in, turn out to be the following ones:

$$Ku=\frac{1}{4\sqrt{2}}\frac{du}{da}\psi \tag{2.62}$$

$$K^2u=\frac{1}{32}\frac{d^2u}{da^2}\psi\wedge\psi-\frac{\sqrt{2}}{4}\frac{du}{da}(F_+-D) \tag{2.63}$$

$$K^3u=\frac{1}{2^7\sqrt{2}}\frac{d^3u}{da^3}\psi\wedge\psi\wedge\psi-\frac{3}{16}\frac{d^2u}{da^2}\psi\wedge(F_+-D)-\frac{3\sqrt{2}\,\mathring{\imath}}{8}(2d\chi-*d\eta). \tag{2.64}$$

Here, u is as in Equation (2.15),$F = dA$ is the two-form curvature of the connection and by $F_{\pm}$ we denote the self-dual and anti-self dual parts respectively (see Section 4.2 for more details). Also, $*$ is the Hodge star operator which for an oriented smooth four-manifold M, is a map that takes a p-form to a $(4-p)$-form,

$$*: \Omega^p(X) \to \Omega^{4-p}(X). \tag{2.65}$$

with $*^2 = 1$ for two-forms.

The last operator, L is a self-dual two-form that anti-commutes with the BRST operator $\mathcal{Q}$ to give

$$\{\mathcal{Q}, L\} = -(\bar{\sigma}_{mn})^{AB} \bar{Z}_{AB} dx^m \wedge dx^n, \tag{2.66}$$

where m,n are $SO(4)$ indices, A,B are $SU(2)_R$ indices, Z is the central charge and σ_{mn} is a combination of the Pauli matrices that is explained in Appendix B. In the same appendix we argue that for a Kähler surface S this commutator can be written as

$$\{\mathcal{Q}, L\} = \sqrt{2}\,\mathring{i}\,\bar{Z} J, \tag{2.67}$$

where $J \in \Omega^{1,1}(S)$ is the Kähler form associated with the Kähler metric of S.

Of specific interest, in the context of Donaldson invariants, is the operator

$$I(\boldsymbol{x}) = \frac{1}{4\pi^2} \int_{\boldsymbol{x}} \mathrm{Tr}\left(\frac{1}{8} \psi \wedge \psi - \frac{1}{\sqrt{2}} \phi F \right), \tag{2.68}$$

that arises from $O^{(2)} = K^2 u$ and where $\boldsymbol{x} \in H^2(M, \mathbb{Q})$ of course. In the low energy effective theory, and of relevance for Donaldson invariants, $\mathscr{O}^{(0)}$ is given by the Seiberg-Witten solution $2u$ (the multiplication by two is in order our results to agree with the mathematicians' conventions) and the surface operator $\mathscr{O}^{(2)}$ is identified with

$$\widetilde{I}_-(\boldsymbol{x}) = \frac{\mathring{i}}{\sqrt{2}\pi} \int_{\boldsymbol{x}} \left(\frac{1}{32} \frac{d^2u}{da^2} \psi \wedge \psi - \frac{\sqrt{2}}{4} \frac{du}{da} (F_- + D) \right), \tag{2.69}$$

where F is field strength of the remaining $U(1)$ gauge symmetry. To evaluate u-plane integrals using modern techniques of indefinite theta functions [94], we will add to this surface operator a $\mathcal{Q}$-exact operator

$$\widetilde{I}_+(\boldsymbol{x}) = -\frac{1}{4\pi} \int_{\boldsymbol{x}} \left\{ \mathcal{Q}, \frac{d\bar{u}}{d\bar{a}} \chi \right\}, \tag{2.70}$$

which can be written as

$$\widetilde{I}_+(\boldsymbol{x}) = -\frac{\mathring{\imath}}{\sqrt{2}\pi}\int_{\boldsymbol{x}}\left(\frac{1}{2}\frac{d^2\bar{u}}{d\bar{a}^2}\eta\,\chi + \frac{\sqrt{2}}{4}\frac{d\bar{u}}{d\bar{a}}(F_+ - D)\right). \tag{2.71}$$

This term couples to the self-dual part F_+ of F, whereas (2.69) involved only F_-.

Finally note that the renormalization group flow to the low energy theory gives rise to a contact term [52, 67], $\boldsymbol{x}^2 G(u)$, which is a consequence of the self-intersection of the cycle $\boldsymbol{x}$ appearing in the surface operators. Since the surface operators $\widetilde{I}_\pm$ are $\mathcal{Q}$-closed, the coefficient G of the contact term is necessarily holomorphic in u. It is conveniently expressed in terms of (quasi)-modular forms

$$G(u) = \frac{1}{24}\left(8u - E_2\left(\frac{du}{da}\right)^2\right), \tag{2.72}$$

where E_2 is the Eisenstein series of weight two defined by Equation (A.5) in Appendix A.1, or as a derivative to ϑ_4 [52]

$$G(u) = -\frac{1}{2\pi\mathring{\imath}}\left(\frac{du}{da}\right)^2 \partial_\tau \log(\vartheta_4). \tag{2.73}$$

If we want to emphasize the dependence of G on τ, we will write sometimes $G(\tau)$ instead of $G(u)$.

The BRST operator $\mathcal{Q}$ acts on the effective IR fields of the theory as follows

$$\begin{aligned}
&\{\mathcal{Q}, A\} = \psi, && \{\mathcal{Q}, a\} = 0 && \{\mathcal{Q}, \bar{a}\} = \sqrt{2}\,\mathring{\imath}\,\eta,\\
&\{\mathcal{Q}, D\} = (d\psi)_+, && \{\mathcal{Q}, \eta\} = 0, && \{\mathcal{Q}, \psi\} = 4\sqrt{2}\,da,\\
&\{\mathcal{Q}, \chi\} = \mathring{\imath}(F_+ - D).
\end{aligned} \tag{2.74}$$

Additionally, the BRST operator can be written as a derivative in the infinite dimensional field space as follows

$$\mathcal{Q} = \psi\frac{\partial}{\partial A} + (d\psi)_+\frac{\partial}{\partial D} + 4\sqrt{2}da\frac{\partial}{\partial\psi} + \sqrt{2}i\eta\frac{\partial}{\partial\bar{a}} + \mathring{\imath}(F_+ - D)\frac{\partial}{\partial\chi}. \tag{2.75}$$

The abelian low energy Donaldson-Witten theory is completely determined by the Seiberg-Witten solution from the previous section as it was understood in [67]. The Lagrangian of the theory reads

$$\begin{aligned}\mathscr{L} =& \frac{i}{16\pi}(\bar{\tau}F_+\wedge F_+ + \tau F_-\wedge F_-) + \frac{y}{8\pi}da\wedge * d\bar{a} - \frac{y}{8\pi}D\wedge *D \\ &- \frac{\tau}{16\pi}\psi\wedge *d\eta + \frac{\bar{\tau}}{16\pi}\eta\wedge d*\psi + \frac{\tau}{8\pi}\psi\wedge d\chi - \frac{\bar{\tau}}{8\pi}\chi\wedge d\psi \\ &+ \frac{\sqrt{2}i}{16\pi}\frac{d\bar{\tau}}{d\bar{a}}\eta\chi\wedge(F_+ + D) - \frac{\sqrt{2}i}{2^7\pi}\frac{d\tau}{da}\psi\wedge\psi\wedge(F_- + D) \\ &+ \frac{i}{3\pi\cdot 2^{11}}\frac{d^2\tau}{da^2}\psi\wedge\psi\wedge\psi\wedge\psi - \frac{\sqrt{2}i}{3\cdot 2^5\pi}\{\mathcal{Q}, \chi_{mn}\chi^{nr}\chi_r{}^m\}\sqrt{g}\,d^4x.\end{aligned} \tag{2.76}$$

We would like to close this section be adding a remark about the UV description of the theory that was studied in [88] and compare it with the IR description of Moore and Witten [67]. The partition function and correlation functions of the UV theory localize to the instanton moduli space (see Chapter 4) due to standard supersymmetric localization arguments. Witten showed that a specific correlation function reproduces physically the mathematically defined generating function of Donaldson invariants. In the IR such a localization arguments is not possible due to the fact that for each value of u there exists a different Lagrangian description (since the holomorphic gauge coupling τ and (a, a_D) vary as we move around the u-plane). This is why Seiberg-Witten analysis for the low energy theory is so crucial and it shows that the integrals over the Coulomb branch are modular integrals with respect to $\Gamma^0(4)$.

2.4 The u-plane integral

In this section we will introduce the u-plane integral. Simply put the u-plane integral is simply the path integral of the low energy Donaldson-Witten theory. Let us give first the topological data associated with the theory. We consider M to be a smooth, simply connected, closed (that is compact and without boundary) four-manifold and let $H^*(M)$ denote the cohomology ring of M with real coefficients. We denote by $b_i(M)$ the dimension of the vector space $H^i(M)$. Since the middle cohomology of M splits into a positive definite subspace $H^{2,+}(M)$ and a negative definite subspace $H^{2,-}(M)$, we denote the dimensions of each subspace by b_2^+ and b_2^-. Its basic topological numbers are its

- Euler character $\chi = 2 - 2b_1 + b_2$, and its
- signature $\sigma = b_2^+ - b_2^-$.

This thesis is dedicated to four-manifolds with $b_2^+ = 1$ only. The reason is that the u-plane integral vanishes for $b_2^+ > 1$ since it is not possible to soak up the fermionic zero modes in the path integral and only the singular points contribute to the theory and provide the

Seiberg-Witten invariants [67]. For $b_2^+ = 0$ the u-plane integral does contribute but the theory is not just a tree level theory. Rather, the theory receives contributions from one-loop determinants that may complicate the computations and we leave the study of this case for future considerations.

2.4.1 Remarks on four-manifolds

In this subsection we will make some clarifications on what four-manifolds we can evaluate Donaldson-Witten theory on. For a broader discussion on four-manifolds and their topology see Chapter 4.

Let us begin by stressing again that the u-plane integral only contributes for manifolds M with $b_2^+(M) = 1$. To be more precise, manifolds with $b_2^+ = 0$ also allow the study of the u-plane integral but for such manifolds the calculations are more involved due to the presence of one-loop determinants. We will be focusing on simply connected four-manifolds with $b_2^+(M) = 1$ then (which by a theorem of Wu are always at least almost complex manifolds since such four-manifolds always admit an almost complex structure, see [17]). Let us note that these four-manifolds are somewhat special in the extraordinary world of four-manifolds due to the wall-crossing phenomena that appear in their Donaldson invariants as observed by Göttsche and Zagier [27]. Some "easy to work with" examples of manifolds with $b_2^+ = 1$ are Kähler surfaces of Kodaira dimension $-\infty$, i.e. $\dim(H^0(M, K_M)) = 0$, where we denote by $K_M := c_1(\mathscr{K}_M)$, the first Chern class of the canonical line bundle $\mathscr{K}_M \in \mathrm{Pic}(X)$ [29, 86]. It is also useful to recall that K_M equals the second Stieffel-Whitney class $w_2(M)$ modulo elements in $H^2(M, 2\mathbb{Z})$.

These Kähler surfaces come in three families. Let M be a Kähler surface of such type.

1. If $K_M^2 > 0$ the surface M is rational or ruled,
2. if $K_M^2 = 0$ then the surface is a $\mathbb{CP}^1$ bundle over $\mathbb{T}^2$,
3. if $K_M^2 < 0$ then the surface is a $\mathbb{CP}^1$ bundle over a curve C_g of genus g greater than one.

For the first case, the rational and ruled surfaces, the Seiberg-Witten contributions vanish exactly because they admit a Kähler metric of positive scalar curvature. Specific examples of surfaces that have $b_2^+ = 1$ and positive scalar curvature are: the projective plane $\mathbb{CP}^2$, del Pezzo surfaces (blow-ups of the projective plane up to nice points), Hirzebruch surfaces $\mathbb{F}_l$ (they are defined as the projectivizations of the bundle $\mathscr{O}_{\mathbb{CP}^1} \oplus \mathscr{O}_{\mathbb{CP}^1}(-l)$), see [86]. As a matter of fact, it is a theorem that if a Kähler surface M admits a metric of positive scalar curvature then it is rational or ruled [49]. We want to stress the importance of such surfaces due to the fact that they allow us to probe the Coulomb branch of the theory (in both the usual version and ramified version to be discussed in Chapter 5) while most four-manifolds will not allow for this. These four-manifolds, therefore, provide an excellent lab to study quantitatively and qualitatively the low energy effective theories in their totality.

More generically, for any four-manifold the intersection form on the middle cohomology provides a natural bilinear form $B: H^2(M,\mathbb{R})\times H^2(M,\mathbb{R})\to\mathbb{R}$ that pairs degree two co-cycles,

$$B(\boldsymbol{k}_1,\boldsymbol{k}_2)=\int_M \boldsymbol{k}_1\wedge\boldsymbol{k}_2, \tag{2.77}$$

and whose restriction to $H^2(M,\mathbb{Z})\times H^2(M,\mathbb{Z})$ is an integral bilinear form with signature $(1,b_2-1)$. The bilinear form provides the quadratic form

$$Q(\boldsymbol{k}):=B(\boldsymbol{k},\boldsymbol{k})\equiv\boldsymbol{k}^2, \tag{2.78}$$

which can be brought to a simple standard form [17, Section 1.1.3]. We denote the period point by J, i.e., the harmonic two-form, satisfying

$$*J=J\in H^2(M,\mathbb{R}),\qquad J^2=1. \tag{2.79}$$

with $*$ the Hodge star operation. Using the period point, we can decompose elements $\boldsymbol{k}\in H^2(M)$ to its self-dual and anti-self-dual components: $\boldsymbol{k}_+=B(\boldsymbol{k},J)\,J$ and $\boldsymbol{k}_-=\boldsymbol{k}-\boldsymbol{k}_+$. the anti-self-dual part of $\boldsymbol{k}$. For later use, we mention that the canonical class is a characteristic vector of $H^2(M,\mathbb{Z})$ and satisfies

$$K_M^2=\sigma+8. \tag{2.80}$$

Finally, we equip M with a principal $U(1)$-bundle and connection one-form A with curvature $F=dA$ and we choose a fixed 't Hooft flux

$$\mathrm{Tr}\left(\frac{[F]}{4\pi}\right)=2\boldsymbol{\mu}\in H^2(M,\mathbb{Z}), \tag{2.81}$$

that corresponds to the first Chern class of the bundle. For more details see Chapter 4.

2.4.2 Definition of the u-plane integral

The u-plane integral or Coulomb branch integral $\Phi_{\boldsymbol{\mu}}^J$ [67] of Donaldson-Witten theory, without any operator insertions, is defined as the usual path integral over the infinite dimensional field space,

$$\Phi_{\boldsymbol{\mu}}^J=\int[\mathscr{D}a\,\mathscr{D}\bar{a}\,\mathscr{D}A\,\mathscr{D}\eta\,\mathscr{D}\psi\,\mathscr{D}\chi\,\mathscr{D}D]\,e^{-\mathscr{S}}\equiv\langle 1\rangle. \tag{2.82}$$

The notation above suggests that the u-plane integral depends on two very important quantities:

- the period point J for reasons that become clear in Chapter 4,
- the conjugacy class $\boldsymbol{\mu} \in \mathbb{Z}_2$ that effectively conveys the information of the gauge group ($SU(2)$ or $SO(3)$) via the fluxes of the gauge field (see Equation (2.81)).

We will review in this section that the path integral is well-defined and reduces to a modular integral over the domain $\mathbb{H}/\Gamma^0(4)$. Furthermore, for the chosen class of four-manifolds, $\Phi^J_{\boldsymbol{\mu}}$ reduces to a finite dimensional integral over the zero modes [67], while since we restrict to simply connected four-manifolds there exist no ψ zero modes[7]. The path integral of the effective theory on the Coulomb branch becomes then

$$\Phi^J_{\boldsymbol{\mu}} = \sum_{\boldsymbol{k}\in H^2(M,\mathbb{Z})} \int da\wedge d\bar{a}\wedge d\eta\wedge d\chi\wedge dD\; A(u)^{\chi(M)}B(u)^{\sigma(M)}\; e^{-\int_M \mathscr{L}_0}, \tag{2.83}$$

where the sum over the $\boldsymbol{k}$ magnetic fluxes is really a sum over all topological line bundles (which are classified by their first Chern class). Also, by abuse of notation, a, $\bar{a}$, η, χ and D denote their zero modes, (constant functions on M. The Lagrangian $\mathscr{L}_0$ is the Lagrangian $\mathscr{L}$ (2.76) restricted to the zero modes including the ones of the gauge field. The functions $A(u)$ and $B(u)$ are curvature couplings; they are holomorphic functions of u, given by [67, 90]

$$\begin{aligned} A(u) &= \alpha\left(\frac{du}{da}\right)^{\frac{1}{2}}, \\ B(u) &= \beta\,(u^2-1)^{\frac{1}{8}}, \end{aligned} \tag{2.84}$$

where α and β are numerical factors. In more general theories including matter, such as the $N_f = 4$ theory, they may depend on parameters such as masses and coupling constants.

We will first integrate over the auxiliary field and to do so we introduce the Lagrangian $\mathscr{L}_{0,D}$, which consists of the terms in the zero-mode Lagrangian $\mathscr{L}_0$ involving D,

$$\mathscr{L}_{0,D} = -\frac{y}{8\pi}D\wedge D + \frac{\sqrt{2}\,i}{16\pi}\frac{d\bar{\tau}}{d\bar{a}}\eta\chi\wedge D. \tag{2.85}$$

[7] Due to the fact that $\pi(X) = 0$.

After a Wick rotation $D \to \mathring{\imath} D$ in the field space, the Gaussian integration over the D-zero modes yields the following result

$$\int dD\, e^{-\int_M \mathscr{L}_{0,D}} = 2\pi \mathring{\imath} \sqrt{\frac{2}{y}}. \tag{2.86}$$

Since we have integrated out D we can proceed with the integration of the fermionic degrees of freedom. The only remaining term involving fermion zero modes in $\mathscr{L}_0$ is

$$\mathscr{L}_{0,f} = \frac{\sqrt{2}\mathring{\imath}}{16\pi}\frac{d\bar{\tau}}{d\bar{a}}\eta\chi \wedge F_+. \tag{2.87}$$

Integrating over the η and χ zero modes we obtain the following contribution to the path integral

$$\int d\eta \wedge d\chi\, e^{-\int_M \mathscr{L}_{0,f}} = \frac{\sqrt{2}\mathring{\imath}}{4}\frac{d\bar{\tau}}{d\bar{a}} B(\boldsymbol{k}, J), \tag{2.88}$$

where the vector $\boldsymbol{k}$ equals the U(1) flux $[F]/4\pi \in H^2(M,\mathbb{Z}) + \boldsymbol{\mu}$.

The sum over these magnetic fluxes in Equation (2.83) takes the form of a Siegel-Narain theta function [67] (which we can think of as a generalization of the classical Jacobi theta functions)

$$\Psi^J_{\boldsymbol{\mu}}\left[\mathscr{K}_{\mathrm{p}}\right](\tau,\bar{\tau}) = \sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}} \mathscr{K}_{\mathrm{p}}(\boldsymbol{k})\,(-1)^{B(\boldsymbol{k},K_M)}\, q^{-\frac{\boldsymbol{k}_-^2}{2}}\bar{q}^{\frac{\boldsymbol{k}_+^2}{2}}, \tag{2.89}$$

where Λ is an integer lattice (for a complex surface S it can be viewed as the lattice of connected components of the Picard group of S) that can be identified with $H^2(M,\mathbb{Z})$ and q is a complex number defined as

$$q := e^{2\pi \mathring{\imath}\tau}. \tag{2.90}$$

Furthermore, the function $\mathscr{K}_{\mathrm{p}}$ is called *kernel* of the Siegel-Narain theta function and for the kernel of the partition function of the theory it reads

$$\mathscr{K}_{\mathrm{p}}(\boldsymbol{k}) = -\frac{\pi}{\sqrt{y}} B(\boldsymbol{k}, J), \tag{2.91}$$

which follows from multiplying (2.86) and (2.88), and dividing by the factor $\frac{d\bar{\tau}}{d\bar{a}}$ since this provides the change of variables from the Coulomb branch parameter a to $\tau \in \mathbb{H}/\Gamma^0(4)$. When we consider correlation functions in the next section, instead of just the partition function, we will find different expressions for the kernel depending on the inserted fields. In order to be clear which *u*-plane integral we are interested in we set

$$\Phi_{\boldsymbol{\mu}}^{J} = \Phi_{\boldsymbol{\mu}}^{J}[\mathscr{K}_{\mathrm{p}}]. \tag{2.92}$$

We can express the integrand of the partition function (2.83) more compactly, using Matone's formula [59]

$$\frac{du}{d\tau} = 4\pi\mathring{\imath}(1-u^2)\left(\frac{da}{du}\right)^2, \tag{2.93}$$

and the identities (2.46) and (2.48). This maps the Coulomb branch integral to the following integral over the modular domain $\mathbb{H}/\Gamma^0(4)$

$$\Phi_{\boldsymbol{\mu}}^{J} = \int_{\mathbb{H}/\Gamma^0(4)} d\tau\wedge d\bar{\tau}\;\tilde{\nu}(\tau)\,\Psi_{\boldsymbol{\mu}}^{J}[\mathscr{K}_{\mathrm{p}}](\tau,\bar{\tau}), \tag{2.94}$$

with the *measure factor* defined as

$$\tilde{\nu}(\tau) := 8\mathring{\imath}(1-u^2)\frac{da}{du}\vartheta_4(\tau)^{\sigma}. \tag{2.95}$$

The whose modular transformations of $\tilde{\nu}$ for the two generators S' and T' of the $\Gamma^0(4)$ subgroup are

$$\begin{aligned} \tilde{\nu}\left(\frac{\tau}{\tau+1}\right) &= (\tau+1)^{2-b_2/2}e^{-\frac{\pi\mathring{\imath}\sigma}{4}}\tilde{\nu}(\tau), \\ \tilde{\nu}(\tau+4) &= -\tilde{\nu}(\tau). \end{aligned} \tag{2.96}$$

The measure $\tilde{\nu}(\tau)$ behaves near the weak coupling cusp $\tau\to\mathring{\imath}\infty$ as $\tilde{\nu}\sim q^{-\frac{3}{8}}$, and near the monopole cusp, $\tau_M = -1/\tau\to\mathring{\imath}\infty$ as $\tilde{\nu}\sim q_M^{\frac{1+\sigma}{8}}$. Note that using the relation

$$\tilde{\nu}(\tau) = \nu(\tau)\frac{da}{d\tau}, \tag{2.97}$$

we can recast the integral in terms of the integration measure $\int_{\mathscr{B}} da\wedge d\bar{a}$ so that the dependence on the Coulomb branch parameter is obvious. Furthermore, for theories of higher rank the integral seems to be easier analyzed as an integral over the higher dimensional Coulomb branch as we will explain in Chapter 4.

An important requirement for the partition function (2.94) is the modular invariance of the integrand under $\Gamma^0(4)$ transformations. In other words, the partition function must be invariant under this generalized electric-magnetic duality. We can easily determine the modular transformations of $\Psi_{\boldsymbol{\mu}}^{J}[\mathscr{K}_{\mathrm{p}}]$ from those of $\Psi_{\boldsymbol{\mu}}^{J}[1]$ (listed in Appendix A.1.5). The effect of inserting the kernel $\mathcal{K}_{\rm p}$ in $\Psi^J_{\boldsymbol{\mu}}[1]$ is to increase the weight by $(\frac{1}{2},\frac{3}{2})$ (the factor $1/\sqrt{y}$ contributes $(\frac{1}{2},\frac{1}{2})$ and $B(\boldsymbol{k},J)$ contributes $(0,1)$ to the total weight). We then arrive at

$$\begin{aligned}\Psi^J_{\boldsymbol{\mu}}[\mathcal{K}_{\rm p}]\left(\frac{\tau}{\tau+1},\frac{\bar\tau}{\bar\tau+1}\right) &= (\tau+1)^{\frac{b_2}{2}}(\bar\tau+1)^2 e^{\frac{\pi i}{4}\sigma}\,\Psi^J_{\boldsymbol{\mu}}[\mathcal{K}_{\rm p}](\tau,\tau),\\ \Psi^J_{\boldsymbol{\mu}}[\mathcal{K}_{\rm p}](\tau+4,\bar\tau+4) &= e^{2\pi i B(\boldsymbol{\mu},K)}\,\Psi^J_{\boldsymbol{\mu}}[\mathcal{K}_{\rm p}](\tau,\bar\tau),\end{aligned} \tag{2.98}$$

where we used the fact that $K_M^2 = 8+\sigma$. We see that the integrand of the partition function (2.94) is invariant under the $\tau\mapsto\frac{\tau}{\tau+1}$ transformation. However, if $B(\boldsymbol{\mu},K)=0 \mod \mathbb{Z}$, the $\tau\mapsto\tau+4$ does multiply the integrand by -1, but one can show that $\Psi^J_{\boldsymbol{\mu}}[\mathcal{K}_{\rm p}]$ vanishes in this case, such that there is no violation of the duality. We conclude therefore that the Coulomb branch integral of Equation (2.94) is well-defined since the measure $d\tau\wedge d\bar\tau$ transforms as a mixed modular form of weight $(-2,-2)$ while the product $\tilde\nu\,\Psi^J_{\boldsymbol{\mu}}[\mathcal{K}_{\rm p}]$ is a mixed modular form of weight $(2,2)$ for the congruence subgroup $\Gamma^0(4)$.

In Chapters 3 and 4 we proceed with the evaluation of $\Phi^J_{\boldsymbol{\mu}}[\mathcal{K}_{\rm p}]$ for various $\mathcal{K}_p$. In the next section we will continue by considering correlation functions of BRST exact observables, which need to satisfy the same requirements on modular invariance of the integrand as above. We finish this chapter by summarizing in Table 2.4 the modular weights of the various ingredients that appear in u-plane integral for future use.

Ingredient	Mixed weight
$d\tau\wedge d\bar\tau$	$(-2,-2)$
y	$(-1,-1)$
$\partial_{\bar\tau} f$	$(k,2)$ if f has weight $(k,0)$
$\tilde\nu(\tau)$	$(2-b_2/2,0)$
$\Psi^J_{\boldsymbol{\mu}}[1]$	$\frac{1}{2}((b_2-1),1)$

Table 2.4 Modular weights of functions relevant for the u-plane integral. Transformations are in $SL_2(\mathbb{Z})$ for the first three rows, while in $\Gamma^0(4)$ for the last two rows.

Chapter 3

BRST symmetry and evaluation of u-plane integrals

The present chapter of this thesis deals with a careful and thorough analysis of Equations (2.51) and (2.55) which are of fundamental importance for CohFTs. In Donaldson-Witten theory (and generalizations of it with matter representations and higher ranks gauge groups) the operator $\mathcal{Q}$ can be written as a derivative in the field space, as we have seen in Equation (2.75). As a result a vacuum expectation value such as $\langle\{\mathcal{Q},\mathcal{O}\}\rangle$ can receive contributions from boundaries or non-compact regions of the field space. This might cause trouble even for the simplest gauge invariant observables such as polynomials in the fields of the theory. In specific, the vacuum expectation values of some of these observables seems to diverge precisely near the singularities of the Coulomb branch of the low energy Donaldson-Witten theory. It is further known that for asymptotically conformal twisted theories such as the $\mathcal{N}=2$ SYM with gauge group $SU(2)$ and number of flavors $N_f=4$ as well as the Argyres-Douglas (A_1,A_2) theory, boundary contributions lead to a, normally unexpected, continuous metric dependence for some correlation functions.

This chapter will mainly focus on $\mathcal{Q}$-exact observables on the Coulomb branch of the Donaldson-Witten theory. We will identify which vacuum expectation values (or more generic correlation functions) seem to diverge and instead of disregarding these observables as "non-physical" we will introduce a new regularization and renormalization procedure for modular integrals inspired by the work of mathematicians [6]. This regularization and renormalization procedure will ensure the decoupling of the $\mathcal{Q}$-exact observables from the theory while being, at the same time, consistent with previous results [27, 52, 67].

Let us briefly explain the new regularization and renormalization procedure of Section 3.2 that will follow. To do that we have to describe the contribution of the Coulomb branch to the path integral in some more detail. As we have already stressed, the contribution of the Coulomb branch is non-vanishing for four-manifolds with $b_2^+ \leq 1$. This fact makes such four-manifolds a powerful arena for the analysis of this phase of the theory. We will concentrate on four-manifolds with $b_2^+ = 1$, for which the path integral reduces to an integral over the Coulomb branch parameter $u = \frac{1}{16\pi^2}\left\langle \mathrm{Tr}(\phi^2)\right\rangle$ [52, 67] that we described in Section 2.1. Recall that ϕ is the Higgs field of the theory valued in the adjoint representation, and $\langle \mathscr{O} \rangle$ denotes the vacuum expectation value (or one-point function) of a gauge invariant operator $\mathscr{O}$ in a normalized vacuum state of the theory on $\mathbb{R}^4$. Recall that the order parameter u determines the effective coupling constant $\tau \in \mathbb{H}$. Changing variables from u to τ maps the u-plane to six $SL_2(\mathbb{Z})$ images of the fundamental domain $\mathscr{F}_\infty = \mathbb{H}/SL_2(\mathbb{Z})$ in the upper-half plane as we explained around Equation (2.45). As a result, the path integral can generically be written as a sum of integrals of the form

$$L_{m,n,s} = \int_{\mathscr{F}_\infty} d\tau \wedge d\bar{\tau}\; q^m \bar{q}^n y^{-s}, \tag{3.1}$$

where $\tau = x + \mathring{\imath}y$ is the effective holomorphic coupling of the theory as we explained in the previous chapter. Such type of modular integrals have also appeared in (bosonic) string theory in the context of one-loop amplitudes [14, 36, 50], and much earlier in the mathematics literature as the Petersson product for cusp forms [72].

The integral (3.1) is finite for $m+n > 0$ and $s \in \mathbb{R}$, and also for $m+n = 0$ with $s > 1$. The integrand however diverges exponentially for $y = \mathrm{Im}(\tau) \to \infty$ if $m+n < 0$. For a large class of such (m,n), namely when one of the two numbers is non-negative, the integral can be evaluated using a, by now standard, prescription [5, 14, 36]. Simply put, this prescription is to carry out first the integral over $x = \mathrm{Re}(\tau)$ and then the integral over y, such that

$$L_{m,n,s} \sim \delta_{m,n} \int^{\infty} dy\, y^{-s} e^{-4\pi y n}, \tag{3.2}$$

where we have just highlighted the potentially divergent part. The (m,n,s) encountered for the famous Donaldson-Witten observables in the formulation of [67] are all such that this regularization applies.

On the other hand, the condition that one element of the pair (m,n) is non-negative, may appear artificial. Actually, as suggested above, we will present observables within Donaldson-Witten theory which lead to integrals as (3.1) that have both m and n negative. The integrand in (3.2) diverges in such cases, and the standard prescription does not cure the infinity. The examples we present are in fact $\mathcal{Q}$-exact, such that the divergence leads to some tension with the expectation that vacuum expectation values of $\mathcal{Q}$-exact operators vanish in topological field theory as described in [88]. Nevertheless, instead of excluding these operators on the basis of their boundary behavior, we will demonstrate that they do vanish once appropriately regularized and renormalized.

One observable we will study in this context (that will be of particular importance in Chapter 4) is the $\mathcal{Q}$-exact insertion we described in the previous chapter

$$\int_{\boldsymbol{x}} \{\mathcal{Q}, \mathrm{Tr}[\bar{\phi}\,\chi]\} = \int_{\boldsymbol{x}} \frac{d\bar{u}}{d\bar{a}} F_+ + \dots, \tag{3.3}$$

where $\bar{\phi}$ is the complex conjugate of the Higgs field ϕ, χ is the self-dual Grassmann valued two-form field, F_+ is the self-dual part of the curvature F of the gauge connection and $\boldsymbol{x}$ is a two-cycle in the rational homology ring of M. The dots in (3.3) represent terms involving fermions and the auxiliary field. This operator has appeared previously in the context of the CohFT interpretation of Witten-like indices [65], and more recently for the evaluation of Coulomb branch integrals using indefinite theta functions in [44] and also in [43].

This chapter therefore, based on [45], proposes a new renormalization prescription for the u-plane integral[1], which is based on the analytic continuation of the incomplete Gamma function. This renormalization was recently developed by the authors of [6] in the context of modular integrals. See also [8] and [20]. For all $\mathcal{Q}$-exact operators which are regular in the interior of the u-plane, that is away from the strong and weak coupling cusps, we show that this prescription ensures the decoupling of $\mathcal{Q}$-exact states from $\mathcal{Q}$-closed states. It reduces to the standard prescription described below equation (3.1) where applicable, while it also could in principle be applied to evaluate correlation functions for non-$\mathcal{Q}$-closed observables. We hope that the new regularization makes the evaluation of new observables possible, and that this will lead to further useful results concerning topologically twisted theories and four-manifold topology.

3.1 BRST exact observables

In what follows we will present and manipulate a number of $\mathcal{Q}$-exact operators that we will bring into a suitable form that will facilitate the computation of their vacuum expectation values using the regularization and renormalization procedure we described above.

[1] With "u-plane integral", we refer to correlation functions on the Coulomb branch of rank one Donaldson-Witten theory, while "Coulomb branch integral" is used for arbitrary rank. Nevertheless, in many occasions in the literarure these terms are interchanged.

3.1.1 An anti-holomorphic $\mathcal{Q}$-exact observable

In this section we will analyze the u-plane integral with the insertion of a specific anti-holomorphic $\mathcal{Q}$-exact surface observable that will be of crucial importance in Chapter 4. Our analysis will demonstrate that this is an observable whose vacuum expectation value appears to diverge rather than vanish as suggested by the Ward-Takahashi identity and this observable will motivate the new regularization in the next section.

The observable we are interested in is

$$\begin{aligned} I_+(\boldsymbol{x}) &= -\frac{1}{4\pi}\int_{\boldsymbol{x}} \{\mathcal{Q},\{L,\mathrm{Tr}[\bar\phi^2]\}\} \\ &= -\frac{1}{4\pi}\int_{\boldsymbol{x}} \{\mathcal{Q},\mathrm{Tr}[\bar\phi\,\chi]\}\,, \end{aligned} \tag{3.4}$$

where $\boldsymbol{x}\in H_2(M,\mathbb{Q})$ is a two-cycle, and L is the twisted supersymmetry generator discussed in Section 2.3. The subscript $+$ is to indicate that it involves a self-dual two-form field, and is in a sense a self-dual counterpart of the holomorphic, anti-self dual Donaldson observable $I_-(\boldsymbol{x})$ as it was explained in [44]. Using the action of the L operator, we can determine the image of $I_+(\boldsymbol{x})$ in the IR theory, denoted by $\widetilde{I}_+(\boldsymbol{x})$, in terms of the fields of the effective low energy theory,

$$\begin{aligned} \widetilde{I}_+(\boldsymbol{x}) &= -\frac{1}{4\pi}\int_{\boldsymbol{x}} \left\{\mathscr{Q},\frac{d\bar u}{d\bar a}\chi\right\} \\ &= -\frac{\mathring{\imath}}{\sqrt{2}\pi}\int_{\boldsymbol{x}} \left(\frac{1}{2}\frac{d^2\bar u}{d\bar a^2}\eta\,\chi+\frac{\sqrt{2}}{4}\frac{d\bar u}{d\bar a}(F_+-D)\right). \end{aligned} \tag{3.5}$$

Note that in Equation (2.71) we quoted this result without using the L operator. Just as we did with the partition function (2.94), we first integrate over the D zero mode using (2.86) and we obtain

$$\int dD\,\left[\int_{\boldsymbol{x}} D\right] e^{-\int_M \mathscr{L}_{0,D}} = 2\pi\,\mathring{\imath}\sqrt{\frac{2}{y}}\left(\frac{\sqrt{2}\,\mathring{\imath}}{4y}\frac{d\bar\tau}{d\bar a}\int_{\boldsymbol{x}}\eta\chi\right). \tag{3.6}$$

Next, we will integrate over the fermionic zero modes, that is, we will consider the integral

$$\int d\eta\wedge d\chi\wedge dD\,\widetilde{I}_+(\boldsymbol{x})\,e^{-\int_M \mathscr{L}_{0,f}+\mathscr{L}_{0,D}} \tag{3.7}$$

After some work, this integral evaluates to $\frac{d\bar{\tau}}{d\bar{a}}\mathscr{K}_+(\boldsymbol{k})$, with the kernel $\mathscr{K}_+(\boldsymbol{k})$ given by

$$\mathscr{K}_+(\boldsymbol{k}) := \frac{2B(\boldsymbol{x},J)}{\sqrt{y}}\left(-\frac{1}{2}\frac{d^2\bar{u}}{d\bar{a}\,d\bar{\tau}} + \frac{\mathring{\imath}}{8y}\frac{d\bar{u}}{d\bar{a}} + \frac{\pi\mathring{\imath}}{2}\frac{d\bar{u}}{d\bar{a}}\boldsymbol{k}_+^2\right). \tag{3.8}$$

Once combined with the sum over the $U(1)$ fluxes, Equation (3.5) can be written in a compact form. One arrives at a total derivative with respect to $\bar{\tau}$,

$$\Psi_{\boldsymbol{\mu}}^J[\mathscr{K}_+](\tau,\bar{\tau}) = -\partial_{\bar{\tau}}\left(\frac{B(\boldsymbol{x},J)}{\sqrt{y}}\frac{d\bar{u}}{d\bar{a}}\Psi_{\boldsymbol{\mu}}^J[1](\tau,\bar{\tau})\right). \tag{3.9}$$

This expression demonstrates that $\Psi_{\boldsymbol{\mu}}^J[\mathscr{K}_+]$ vanishes for $B(\boldsymbol{\mu},K_M) = \frac{1}{2} \mod \mathbb{Z}$, since $\Psi_{\boldsymbol{\mu}}^J[1]$ vanishes in this case (it can be seen by explicitly summing the first few positive and negative terms). If non-vanishing, $\Psi_{\boldsymbol{\mu}}^J[\mathscr{K}_+]$ has the required modular properties. This means that it transforms with modular weight $(\frac{b_2}{2},2)$, and changes by a sign under $\tau \mapsto \tau+4$.

Since we obtained the Siegel-Narain theta function for the $\mathcal{Q}$-exact operator $\widetilde{I}_+$ we can proceed and compute the vacuum expectation value of $\widetilde{I}_+$. We arrive at the integral

$$\langle\widetilde{I}_+(\boldsymbol{x})\rangle = -\int_{\mathbb{H}/\Gamma^0(4)} d\tau\wedge d\bar{\tau}\;\partial_{\bar{\tau}}\left(\widetilde{\nu}(\tau)\frac{B(\boldsymbol{x},J)}{\sqrt{y}}\frac{d\bar{u}}{d\bar{a}}\Psi_{\boldsymbol{\mu}}^J[1]\right), \tag{3.10}$$

for some kernel $\mathscr{K}$. We can easily evaluate this integral using Stokes' theorem (see Section 4.7). This reduces to arcs close to the three cusps of $\mathbb{H}/\Gamma^0(4)$, $\tau \to \mathring{\imath}\infty$, 0 and 2. In the Coulomb branch picture this is a contour integral that localizes at the singularities of $\mathscr{B}$. But something unexpected is to be discovered here. Since $\frac{d\bar{u}}{d\bar{a}}$ diverges as $\bar{q}^{-\frac{1}{8}}$ for $\tau \to \mathring{\imath}\infty$ and $\tilde{\nu}(\tau)$ diverges as $q^{-\frac{3}{8}}$ for the same limit, we find integrals $L_{m,n,s}$ (3.1) with both m and $n<0$ for the cusp at $\mathring{\imath}\infty$. The standard prescription to resolve this issue, mentioned above Equation (3.2), therefore cannot cure this divergence if $\Psi_{\boldsymbol{\mu}}^J[\mathscr{K}_+] \sim q^{\frac{1}{4}}$ for $\tau \to \mathring{\imath}\infty$. Nevertheless, we will explain in Section 3.3 that the integral can be properly regularized and renormalized. This will result in the expected $\langle\widetilde{I}_+(\boldsymbol{x})\rangle = 0$, in agreement with the global BRST symmetry and the rules of CohFT.

3.1.2 General $\mathcal{Q}$-exact observables

Motivated by the example $\widetilde{I}_+(\boldsymbol{x})$, we would like to make an analysis of general $\mathcal{Q}$-exact observables in this subsection. We assume that these observables satisfy the constraints of single-valuedness on the u-plane (eventually this translates to modular invariance under $\Gamma^0(4)$), and that these would be automatically satisfied if we derive them from $\mathcal{Q}$-exact UV operators. We will find that the integrands of the u-plane integrals for such observables always take the form of a total $\bar\tau$-derivative, $\int d\tau\wedge d\bar\tau\,\partial_{\bar\tau}(\text{modular form of weight two})$. This fact makes their evaluation much easier as we will see in Section 3.3.

The $\mathcal{Q}$-closed and exact observables admit a natural grading by their form degree. For the restriction we have imposed on M we only have three generic cases therefore and we present them below.

0-form operators

The most generic zero-form operator $\mathcal{O}_0$ can be written as

$$\mathcal{O}_0 = V_0(a,\bar a) + V_1(a,\bar a)\,\eta, \tag{3.11}$$

where we require that V_j are real-analytic functions in the real and imaginary part of a on the interior of the u-plane, in other words, the V_j do not have singularities away from the weak and strong coupling cusps in the u-plane. In other words, $V_j \in \mathcal{O}_{\mathbb{CP}^1}(U)$, with $U = \mathbb{CP}^1\backslash\{S\}$ and S the finite dimensional set of singularities, in our case $S = \{\pm 1, \infty\}$. After acting with $\mathcal{Q}$ on this expression using Equation (2.74), we find that the most general $\mathcal{Q}$-exact zero-form operator is

$$G_0 = \{\mathcal{Q},\mathcal{O}\} = \sqrt{2}\,\mathring{\imath}\,\partial_{\bar a}V_0(a,\bar a)\,\eta. \tag{3.12}$$

The vacuum expectation value $\langle G_0\rangle$ vanishes after integration over the fermionic modes, since G_0 is Grassmann odd and the action only contains Grassmann even terms. We thus find that any zero-form operator satisfies the Ward identity (2.51). Moreover, any product operator of the form

$$\prod_j \mathcal{O}_{0,j}, \tag{3.13}$$

with $\mathcal{O}_{0,j}$ being $\mathcal{Q}$-exact zero-form operators is also of the form (3.11).

Let us next consider $\mathcal{Q}$-closed zero-form observables. We deduce from (3.12) that for any $\mathcal{Q}$-closed observable the η^0 term is necessarily holomorphic, thus

$$C_0 = W_0(a) + W_1(a,\bar a)\,\eta, \tag{3.14}$$

where W_j are again real-analytic functions on the interior of the u-plane. For single valuedness of the u-plane integrand, W_0 must be invariant under $\Gamma^0(4)$ transformations. For example the famous point operator is u. Comparing (3.12) and (3.14), we deduce that there exists $\mathcal{Q}$-closed forms, linear in η, which are not $\mathcal{Q}$-exact. They do however not contribute to correlation functions since they are Grassmann odd. For the same reason, the Ward-Takahashi identity

(2.53) is satisfied for 0-form observables. Specifically, for any 0-form observable $\mathcal{O}_0$ we have

$$\langle \{\mathcal{Q}, \mathcal{O}_0\} \prod_j \mathcal{O}_{0,j} \rangle = 0, \tag{3.15}$$

if all $\mathcal{O}_{0,j}$ are $\mathcal{Q}$-closed 0-form observables.

2-form operators

We continue with $\mathcal{Q}$-exact two-form operators $G_2 = \{\mathcal{Q}, \mathcal{O}_2\}$. We let $\mathcal{O}_2$ be the most generic two-form operator, expressed as

$$\mathcal{O} = \sum_{X \in \{\chi, F_\pm, D, \psi\wedge\psi\}} V_{X,j}(a,\bar{a})\, \eta^j X. \tag{3.16}$$

where $V_{X,j}(a,\bar{a})$ are again real-analytic functions without singularities away from the strong and weak coupling singularities. Comparing with Equation (3.5), we find that for $\widetilde{I}_+$ the function $V_{\chi,0}(a,\bar{a})$ reads

$$V_{\chi,0}(a,\bar{a}) = -\frac{1}{4\pi}\frac{d\bar{u}}{d\bar{a}}, \tag{3.17}$$

with all other $V_{X,j}$ equal to 0. Acting with $\mathcal{Q}$ on $\mathcal{O}$ we obtain the following expression for the two-form operator

$$\begin{aligned} G_2 = {} & \sqrt{2}\,\mathring{\imath}\, \partial_{\bar{a}} V_{\chi,0}\eta\,\chi + \mathring{\imath}\, V_{\chi,0}\,(F_+ - D) \\ & + \sum_{i=\pm} \Big(\sqrt{2}\,\mathring{\imath}\, \partial_{\bar{a}} V_{F_i,0}\eta\, F_i + V_{F_i,0}\,(d\psi)_i \Big) \\ & + \sqrt{2}\,\mathring{\imath}\, \partial_{\bar{a}} V_{D,0}\eta\, D + V_{D,0}\,(d\psi)_+ \\ & + \sqrt{2}\,\mathring{\imath}\, \partial_{\bar{a}} V_{\psi\wedge\psi,0}\eta\; \psi\wedge\psi - 8\sqrt{2}\, V_{\psi\wedge\psi,0}\, \psi\wedge da \\ & + \sum_{i=\pm} \Big(-\mathring{\imath} V_{\chi,1}\eta\; (F_i + -D) - V_{F_i,1}\eta\,(d\psi)_i \Big) \\ & - V_{D,1}\eta\,(d\psi)_+ - 4\sqrt{2}\, V_{\psi\wedge\psi,1}\eta\; \psi\wedge da. \end{aligned} \tag{3.18}$$

In correlation functions we integrate G_2 over a two-cycle $\boldsymbol{x} \in H_2(M,\mathbb{Z})$. For simplicity of notation, we set

$$G_2(\boldsymbol{x}) \equiv \int_{\boldsymbol{x}} G_2. \tag{3.19}$$

To evaluate $\langle G_2(\boldsymbol{x})\rangle$ for the class of four-manifolds relevant to this paper, we reduce to zero modes and integrate over the η and χ zero modes. This ensures that all terms on the right hand side of Equation (3.20) have a vanishing contribution to $\langle G_2(\boldsymbol{x})\rangle$, *except* the two terms with $V_{\chi,0}$. We will proceed with only these two terms, which is similar to the analysis in

Subsection 3.1.1. Integrating over the D zero modes gives

$$\begin{aligned}&\int dD\, G_2(\boldsymbol{x})\, e^{-\int_M \mathscr{L}_{0,D}} = \\ &2\pi\mathring{\imath}\sqrt{\frac{2}{y}}\left[\int_{\boldsymbol{x}}\left(\sqrt{2}\,\mathring{\imath}\,\partial_{\bar{a}}V_{\chi,0}+\frac{\sqrt{2}}{4y}\frac{d\bar{\tau}}{d\bar{a}}V_{\chi,0}\right)\eta\chi+4\pi\mathring{\imath}V_{\chi,0}B(\boldsymbol{k}_+,\boldsymbol{x})\right].\end{aligned} \tag{3.20}$$

Integrating subsequently over the η and χ zero modes gives the sum over fluxes $\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_2]$, with kernel

$$\mathscr{K}_2 = -\frac{4\pi}{\sqrt{y}}B(\boldsymbol{x},J)\left(\partial_{\bar{\tau}}V_{\chi,0}-\frac{\mathring{\imath}}{4y}V_{\chi,0}-\pi\mathring{\imath}V_{\chi,0}\boldsymbol{k}_+^2\right), \tag{3.21}$$

which can be simplified to

$$\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_2](\tau,\bar{\tau}) = \partial_{\bar{\tau}}\left(\frac{4\pi B(\boldsymbol{x},J)}{\sqrt{y}}V_{\chi,0}\Psi^J_{\boldsymbol{\mu}}[1](\tau,\bar{\tau})\right). \tag{3.22}$$

We can easily deduce the modular properties of $V_{\chi,0}$ necessary for single-valuedness of the integrand: $V_{\chi,0}$ must have weight $(0,-1)$, and transform with the same multiplier system[2] as $d\bar{u}/d\bar{a}$.

Our next aim is to consider correlation functions of a $\mathcal{Q}$-exact operator with a $\mathcal{Q}$-closed operator. To this end, let us analyze the form of the most general $\mathcal{Q}$-closed two-form operator which can be written as

$$C_2 = \sum_X W_{X,j}X\,\eta^j. \tag{3.23}$$

Since we integrate over a closed two-cycle, $\int_{\boldsymbol{x}}\mathscr{O}$, the right hand side of (3.20) can vanish up to a total derivative. The relations this imposes on the functions $W_{*,j}$ are easily read off from (3.20). We obtain

$$\begin{aligned}&W_{\psi\wedge\psi,0} = \frac{1}{8\sqrt{2}}\partial_a W_{F_-,0}, \qquad W_{\chi,1} = \sqrt{2}\partial_{\bar{a}}W_{F_+,0},\\ &W_{F_-,0} = W_{F_+,0}+W_{D,0} = \text{holomorphic},\\ &W_{D,1}+W_{F_+,1} = W_{F_-,1} = W_{\psi\wedge\psi,1} = W_{\chi,0} = 0.\end{aligned} \tag{3.24}$$

[2] A multiplier system for a subgroup of $SL_2(\mathbb{R})$ can be thought of as an analogue of the group character, albeit slightly more complicated. We will not need the full definition here but the interested reader can consult standard references in modular forms.

Therefore, we see that C_2 can be expressed as

$$\begin{aligned} C_2 = W_{F_-,0}\,(F_- + D) + \frac{1}{8\sqrt{2}}\partial_a W_{F_-,0}\,\psi\wedge\psi \\ + \sqrt{2}\partial_{\bar{a}} W_{F_+,0}\eta\chi + W_{F_+,0}\,(F_+ - D) \\ + W_{F_+,1}(a,\bar{a})\,\eta\,(F_+ - D), \end{aligned} \tag{3.25}$$

and the action of the BRST operator on C_2 gives

$$\{\mathcal{Q}, C_2\} = d\left(W_{F_-,0}(a)\,\psi\right). \tag{3.26}$$

The first two terms in (3.25) are holomorphic and do match the terms in the standard Donaldson surface observable as derived using the descent formalism, $K^2 u$ [67]. Comparing with Equation (2.17) in [67], we find that for the surface observable[3]

$$W_{F_-,0}(a) = -\frac{\mathring{\imath}}{4\pi}\frac{du}{da}. \tag{3.27}$$

The last three terms on the right hand side of (3.25) are $\mathcal{Q}$-exact, and the first two are of form which do not automatically vanish.

We consider next the correlation function for the product of a $\mathcal{Q}$-exact and a $\mathcal{Q}$-closed operator, $G_2(\boldsymbol{x})\,C_2(\boldsymbol{x}')$. As before, the path integral restricts to zero modes and Grassmann even terms. Integration over the zero modes of D, η and χ gives

$$\begin{aligned} &\int d\eta\wedge d\chi\wedge dD\, G_2(\boldsymbol{x})\,C_2(\boldsymbol{x}')\,e^{-\int_M \mathscr{L}_{0,D}+\mathscr{L}_{0,f}} = \\ &16\pi^2\,\frac{d\bar{\tau}}{d\bar{a}} B(\boldsymbol{x},J)\,B(\boldsymbol{x}',J)\,B(\boldsymbol{k},J)\left[\left(\partial_{\bar{\tau}}\frac{V_{\chi,0}W_{F_+,0}}{\sqrt{y}}\right) - \frac{V_{\chi,0}W_{F_+,0}}{\sqrt{y}}\,\pi\,\mathring{\imath}\,\boldsymbol{k}_+^2\right] \\ &+16\pi^2\,\frac{d\bar{\tau}}{d\bar{a}} W_{F_-}\,B(\boldsymbol{x},J)\,B(\boldsymbol{x}',\boldsymbol{k}_-)\,\left[\left(\partial_{\bar{\tau}}\frac{V_{\chi,0}}{\sqrt{y}}\right) - \frac{V_{\chi,0}}{\sqrt{y}}\,\pi\,\mathring{\imath}\,\boldsymbol{k}_+^2\right], \end{aligned} \tag{3.28}$$

where the first line is due to the product of the non-holomorphic $\mathcal{Q}$-exact part of C_2 with G_2, and the second line is the contribution of the product from the holomorphic part of C_2 with G_2. Note that the term in brackets on the second line is very similar to (3.21), since the holomorphic part commutes with $\partial_{\bar{\tau}}$. Using (3.28), we may write the u-plane integrand for

[3] Note that we find a different sign for the $\psi\wedge\psi$ term compared to [44, 67].

$\langle G_2(\boldsymbol{x})\, C_2(\boldsymbol{x}')\rangle$ as[4]

$$\begin{aligned} &B(\boldsymbol{x},J)\, B(\boldsymbol{x}',J)\, \partial_{\bar\tau}\Big(\widetilde{\nu}\, V_{\chi,0}\, W_{F_+,0}\, \Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_2^{(2)}/\,\mathring{\imath}]\Big) \\ &+ B(\boldsymbol{x},J)\, \partial_{\bar\tau}\Big(\widetilde{\nu}\, W_{F_-,0}\, V_{\chi,0}\, \Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_-]\Big)\,. \end{aligned} \tag{3.29}$$

where the kernels read

$$\begin{aligned} \mathscr{K}_2^{(2)} &= \frac{16\pi^2\,\mathring{\imath}}{\sqrt{y}}\, B(\boldsymbol{k},J), \\ \mathscr{K}_- &= \frac{16\pi^2}{\sqrt{y}} B(\boldsymbol{x}',\boldsymbol{k}_-). \end{aligned} \tag{3.30}$$

We have thus demonstrated that the u-plane integral takes for this correlation function also the form of a total derivative.

For an arbitrary product of $\mathcal{Q}$-exact two-form operators,

$$\left\langle \prod_{j=1}^{\ell} G_2^{(j)}(\boldsymbol{x}_j) \right\rangle, \tag{3.31}$$

we will prove in Subsection 3.1.2 that the integrand takes a similar form. Namely, the sum over fluxes can written as

$$\partial_{\bar\tau}\left(\left(\prod_{j=1}^{\ell} B(\boldsymbol{x}_j,J)\, V^{(j)} \right) \Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_2^{(\ell)}] \right), \tag{3.32}$$

where the kernel $\mathscr{K}_2^{(\ell)}$ can be expressed in terms of the Hermite polynomial $H_{\ell-1}$. See Equation (3.50) for the precise expression. Using the results of Vignéras [85], one can show that $\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_2^{(\ell)}]$ has the required modular properties. We have thus demonstrated that any product of $\mathcal{Q}$-exact two-form observables can be expressed as a total $\bar\tau$-derivative. The form of the kernel ensures moreover that the integrand is well-defined, as long as the functions $V^{(\ell)}$ transform with the same weight and multiplier system as $\frac{d\bar u}{d\bar a}$ under $\Gamma^0(4)$ transformations.

4-form operators

Since the four-manifolds we consider are simply connected there are no three-form operators. We can therefore, jump to the most generic $\mathcal{Q}$-exact four-form operators that we will treat similarly to the most generic two-form operators from the previous section.

[4] We have divided by $\mathring{\imath}$ in the first kernel of $\Psi^J_{\boldsymbol{\mu}}$ for consistency with Equation (3.50) at the end of this subsection.

If we leave aside terms which have an odd number of fermionic fields and terms involving derivatives, the most generic four-form operator takes the form

$$\begin{aligned} G_4 &= \{\mathcal{Q}, V_{\chi,F_+}\chi\wedge F_+ + V_{\chi,D}\chi\wedge D\} \\ &= \sqrt{2}\,\mathring{\imath}\,\partial_{\bar a}V_{\chi,F_+}\,\eta\chi\wedge F_+ + \mathring{\imath}\,V_{\chi,F_+}\,(F_+ - D)\wedge F_+ \\ &\quad + \mathring{\imath}\,V_{\chi,D}(F_+ - D)\wedge D. \end{aligned} \tag{3.33}$$

We aim to evaluate $\langle\int_M G_4\rangle$and to accomplish that we will use Equation (2.86), (3.6) and

$$\int dD\left[\int_M D\wedge D\right]e^{-\int_M \mathscr{L}_{0,D}} = -\frac{8\pi^2\,\mathring{\imath}}{y}\sqrt{\frac{2}{y}}. \tag{3.34}$$

Then we find

$$\begin{aligned} \int dD\int_M G_4\,e^{-\int_M \mathscr{L}_{0,D}} &= 2\pi\,\mathring{\imath}\sqrt{\frac{2}{y}}\Big[\sqrt{2}\,\mathring{\imath}\,\partial_{\bar a}V_{\chi,F_+}\int_M \eta\chi\wedge F_+ \\ &+ \mathring{\imath}\,V_{\chi,F_+}\int_M (F_+ - \frac{\sqrt{2}\,\mathring{\imath}}{4y}\frac{d\bar\tau}{d\bar a}\eta\chi)\wedge F_+ + \mathring{\imath}V_{\chi,D}\int_M (F_+\wedge\frac{\sqrt{2}\,\mathring{\imath}}{4y}\frac{d\bar\tau}{d\bar a}\eta\chi) + \frac{4\pi\,\mathring{\imath}}{y}V_{\chi,D}\Big] \end{aligned} \tag{3.35}$$

Integrating over the fermionic zero modes gives the kernel $\mathscr{K}_4$ for $\Psi^J_{\boldsymbol{\mu}}$,

$$\mathscr{K}_4(\boldsymbol{k}) = \frac{8\sqrt{2}\pi^2 B(\boldsymbol{k},J)}{\sqrt{y}}\left(\partial_{\bar\tau}V_{\chi,F_+} - \frac{\mathring{\imath}}{4y}V_{\chi,F_+} - \pi\,\mathring{\imath}\,V_{\chi,F_+}\boldsymbol{k}_+^2\right). \tag{3.36}$$

Also $\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_4]$ can be expressed as an anti-holomorphic derivative in accordance to the results of the previous subsection

$$\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_4] = 4\sqrt{2}\,\mathring{\imath}\,\partial_{\bar\tau}\left(V_{\chi,F_+}\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_{\mathrm{p}}]\right), \tag{3.37}$$

with the kernel $\mathscr{K}_{\mathrm{p}}$ as in Equation (2.91).

General $\mathcal{Q}$-exact operator

We have already seen examples of classes of $\mathcal{Q}$-exact operators with the property that their u-plane integrand can be expressed as a total derivative with respect to $\bar\tau$. We will demonstrate here that this is not a mere coincidence but a generic feature of $\mathcal{Q}$-exact operators in Donaldson-Theory. To this end, we reduce to the zero mode sector from the beginning and include the $\mathcal{Q}$-exact part of the Lagrangian in the observable. Recall that the zero mode

Lagrangian can be expressed as

$$\mathscr{L}_0 = \frac{i\tau}{8\pi} F \wedge F + \{\mathcal{Q}, W\}, \tag{3.38}$$

with $W = -\frac{iy}{8\pi}\,\chi\,(F_+ + D)$. We can rewrite

$$\{\mathcal{Q}, \mathscr{O}\}\, e^{-\mathscr{L}_0} = \left\{\mathcal{Q}, \widetilde{\mathscr{O}}\right\} q^{-\boldsymbol{k}^2/2}, \tag{3.39}$$

with $\widetilde{\mathscr{O}} = \mathscr{O}\, e^{-\int_M \{\mathcal{Q}, W\}}$. This will simplify the integrations over the fermion and auxiliary field zero modes.

To this end, let us expand $\widetilde{\mathscr{O}}$ in terms of η and χ, and integrate χ over a two-cycle $\boldsymbol{x} \in H_2(M, \mathbb{Q})$, such that the operator belongs to $H_0(M)$. The expansion then reads

$$\widetilde{\mathscr{O}}(\boldsymbol{x}) = \sum_{m=0,1} \widetilde{\mathscr{O}}_{m,0}\, \eta^m + \sum_{m=0,1} \widetilde{\mathscr{O}}_{m,1}\, \eta^m \int_{\boldsymbol{x}} \chi, \tag{3.40}$$

where $\widetilde{\mathscr{O}}_{m,n}$ are functions of a, $\bar{a}$, $\int_{\boldsymbol{x}} F$ and $\int_{\boldsymbol{x}} D$. With the aid of the supersymmetry algebra of Equation (2.74) restricted to zero modes, we find

$$\left\{\mathcal{Q}, \widetilde{\mathscr{O}}(\boldsymbol{x})\right\} = \sqrt{2}\, i\, \partial_{\bar{a}} \widetilde{\mathscr{O}}_{0,0}\, \eta + \sqrt{2}\, i\, \partial_{\bar{a}} \widetilde{\mathscr{O}}_{0,1}\, \eta \int_{\boldsymbol{x}} \chi - i \int_{\boldsymbol{x}} (F_+ - D) \sum_{m=0,1} \widetilde{\mathscr{O}}_{m,1}\, \eta^m. \tag{3.41}$$

Only the term with $\widetilde{\mathscr{O}}_{0,1}$ survives the integration over fermion zero modes,

$$\int d\eta\, d\chi \left\{\mathcal{Q}, \widetilde{\mathscr{O}}(\boldsymbol{x})\right\} = -\sqrt{2}\, i\, B(\boldsymbol{x}, J)\, \partial_{\bar{a}} \widetilde{\mathscr{O}}_{0,1}, \tag{3.42}$$

where

$$\widetilde{\mathscr{O}}_{0,1} = \mathscr{O}_{0,1}\, q^{-\boldsymbol{k}_-^2/2} \bar{q}^{\boldsymbol{k}_+^2/2} \exp\left(\frac{y}{8\pi} \int_M D^2\right). \tag{3.43}$$

We thus find that the u-plane integrand can be expressed as a total $\bar{\tau}$-derivative for any $\mathcal{Q}$-exact observable. Moreover, the only term of $\widetilde{\mathscr{O}}$ which contributes to the integrand is linear in χ and independent of η.

Finally, we will now consider a product $\prod_{j=1}^{\ell} G_2(\boldsymbol{x}_j)$ of form degree two $\mathcal{Q}$-exact operators. We can express this product as $\{\mathcal{Q}, \mathscr{O}^{(\ell)}\}$, with

$$\mathscr{O}^{(\ell)} = V^{(1)} \int_{\boldsymbol{x}_1} \chi \times \prod_{j=2}^{\ell} G_2^{(j)}(\boldsymbol{x}_j), \tag{3.44}$$

and

$$G_2^{(j)} = \sqrt{2}\,\mathring{\imath}\,\partial_{\bar{a}} V^{(j)}\,\eta\,\chi + \mathring{\imath}\,V^{(j)}\,(F_+ - D), \qquad j = 1,\ldots,\ell. \tag{3.45}$$

The coefficient of the term $\eta^0 \int_{\boldsymbol{x}_1} \chi$, that is $\mathscr{O}_{0,1}^{(\ell)}$, is given by

$$\mathscr{O}_{0,1}^{(\ell)} = \mathring{\imath}^{\ell-1}\, B(F_+ - D, J)^{\ell-1}\, V^{(1)} \prod_{j=2}^{\ell} V^{(j)}\, B(\boldsymbol{x}_j, J). \tag{3.46}$$

Integrating out the auxiliary field leads to expressions in terms of the Hermite polynomials as suggested below Equation (3.32). Recall the following integral formula for the Hermite polynomials,

$$H_\ell(s) = \frac{2^\ell}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt\,(s - it)^\ell\, e^{-t^2}. \tag{3.47}$$

The first few polynomials H_ℓ read

$$\begin{aligned} H_0(s) &= 1, \\ H_1(s) &= 2s, \\ H_2(s) &= 4s^2 - 2. \end{aligned} \tag{3.48}$$

Using the identity (3.47), we find

$$\begin{aligned} &\int dD\, \mathscr{O}_{0,1}^{(\ell)} \exp\!\left(\frac{y}{8\pi}\int_M D^2\right) = \\ &2\sqrt{\pi}\left(\mathring{\imath}\sqrt{\frac{2\pi}{y}}\right)^{\ell} H_{\ell-1}(\sqrt{2\pi y}\,B(\boldsymbol{k},J)) \prod_{j=1}^{\ell} V^{(j)}\, B(\boldsymbol{x}_j, J). \end{aligned} \tag{3.49}$$

We now arrive at Equation (3.32) with kernel

$$\mathscr{K}_2^{(\ell)} = -2\,\mathring{\imath}\,\sqrt{2\pi}\left(\mathring{\imath}\sqrt{\frac{2\pi}{y}}\right)^{\ell} H_{\ell-1}(\sqrt{2\pi y}\,B(\boldsymbol{k},J)). \tag{3.50}$$

3.1.3 A holomorphic self-dual operator

In the previous subsections we saw examples of a holomorphic operator combined with the anti-self-dual part of the field strength, and an anti-holomorphic operator combined with the self-dual part of the field strength. The low energy expressions of these operators illustrate that they neatly satisfy the constraints of the duality group $\Gamma^0(4)$. In this section, we consider a $\mathcal{Q}$-exact operator which is holomorphic in a and involves the self-dual part of the field strength. We denote the UV operator by $I_\diamond$, and it reads explicitly

$$I_\diamond(\boldsymbol{x}) = \int_{\boldsymbol{x}} \{\mathcal{Q}, \mathrm{Tr}(\phi\,\chi)\}. \tag{3.51}$$

What is curious about this operator is that the integrand is not a descendant of the descent operators K or L. Therefore the low energy IR operator that it flows to, $\widetilde{I}_\diamond(\boldsymbol{x})$, does not follow in a straight forward manner from the UV expression. The discussion in this subsection is therefore of a more speculative nature.

We take the following ansatz for the IR observable

$$I_\diamond(\boldsymbol{x}) \xrightarrow{\text{RG flow}} \widetilde{I}_\diamond(\boldsymbol{x}) = \int_{\boldsymbol{x}} \left(\frac{du}{da}\,(F_+ - D) + g(\tau)\,\frac{du}{da}\,(F_+ + D) \right), \tag{3.52}$$

where $g(\tau)$ is an unknown function, which we aim to fix below using modular invariance. From the UV definition, one would expect that g vanishes, but that would spoil the modular invariance of the inegrand. We will require that g is a non-perturbative (instantonic) correction, and vanishes exponentially fast in the weak-coupling limit $\tau \to i\infty$. Furthermore, we will then demonstrate that g is uniquely determined by modularity.

The procedure is similar as is the previous sections. Integration over the fermion zero modes after insertion of $\widetilde{I}_\diamond(\boldsymbol{x})$, leads to the kernel

$$\mathscr{K}_\diamond(\boldsymbol{k}) = \frac{i B(\boldsymbol{x}, J)}{2\sqrt{2y}}\,\frac{du}{da} \left\{ \left(4\pi \boldsymbol{k}_+^2 + \frac{1}{y}\right) - g(\tau)\left(4\pi \boldsymbol{k}_+^2 - \frac{1}{y}\right) \right\}. \tag{3.53}$$

Recall that the integrand has to be modular invariant under $\Gamma^0(4)$. To satisfy this requirement and also the fact that g is non-perturbative, we set

$$g(\tau) = \frac{\pi}{6}\left(2E_2(\tau) - \vartheta_3(\tau)^4 - \vartheta_4(\tau)^4\right) = -16\pi q + O(q^2), \tag{3.54}$$

where E_2 is the second Eisenstein series, and the ϑ_j are the classical Jacobi theta series (A.10). We set furthermore

$$\widehat{g}(\tau) = \frac{\pi}{6}\left(2\widehat{E}_2(\tau) - \vartheta_3(\tau)^4 - \vartheta_4(\tau)^4\right), \tag{3.55}$$

with $\widehat{E}_2(\tau)$ the non-holomorphic Eisenstein series $\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi y}$ (a quasi-modular form). We see that $\widehat{g}$ transforms as a weight two modular form of $\Gamma^0(4)$, and that for $\tau \to i\infty$, the function $\widehat{g}(\tau)$ behaves as $-\frac{1}{y} + O(q)$. We can now express the Siegel-Narain theta function $\Psi^J_{\boldsymbol{\mu}}$ of this operator as a total derivative with respect to $\bar{\tau}$ in the following way

$$\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_\diamond](\tau,\bar{\tau}) = \frac{i}{2} B(\boldsymbol{x},J)\frac{du}{da}\frac{d}{d\bar{\tau}}\left(\sqrt{y}\,\widehat{g}(\tau)\,\Psi^J_{\boldsymbol{\mu}}[1](\tau,\bar{\tau})\right). \tag{3.56}$$

where as before we have not included the term $\frac{d\bar{\tau}}{d\bar{a}}$, which is the Jacobian for the change of variable to $\bar{\tau}$. One may verify that $\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_\diamond]$ has the same transformation properties as $\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_+]$.

3.1.4 Summary

Let us give a summary of the results of this section. We have found that vacuum expectation values of $\mathcal{Q}$-exact operators can be expressed as integrals whose integrands can be written as a total $\bar{\tau}$-derivative after integration over the auxiliary field D and the fermionic zero modes. The vacuum expectation value of a $\mathcal{Q}$-exact operator takes therefore the form

$$\langle\{\mathcal{Q},\mathscr{O}\}\rangle = \int_{\mathbb{H}/\Gamma^0(4)} d\tau\wedge d\bar{\tau}\;\partial_{\bar{\tau}}\left(\widetilde{\nu}\;W_{\mathscr{O}}\,\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_{\mathscr{O}}]\right), \tag{3.57}$$

for some non-holomorphic function $W_{\mathscr{O}}$ and kernel $\mathscr{K}_{\mathscr{O}}$, which both depend on $\mathscr{O}$. Given the total derivative, we can easily evaluate the integral using Stokes' theorem, which reduces the integral to three arcs around the cusps of $\mathbb{H}/\Gamma^0(4)$.

For a more standard treatment, we map the integral over $\mathbb{H}/\Gamma^0(4)$ to an integral over $\mathscr{F}_\infty$, using Equation (2.39), by mapping the six $SL_2(\mathbb{Z})/\Gamma^0(4)$ images of $\mathscr{F}_\infty$ in $\mathbb{H}/\Gamma^0(4)$ back to $\mathscr{F}_\infty$. Equation (3.57) can then be expressed as

$$\langle\{\mathcal{Q},\mathscr{O}\}\rangle = \int_{\mathscr{F}_\infty} d\tau\wedge d\bar{\tau}\;\partial_{\bar{\tau}}F_{\mathscr{O}}, \tag{3.58}$$

where $F_{\mathscr{O}}$ is the sum of the six transformations of $\widetilde{\nu}(\tau)\,W_{\mathscr{O}}\,\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_{\mathscr{O}}]$ by the elements of $SL_2(\mathbb{Z})/\Gamma^0(4)$. It has a q-expansion of the form

$$F_{\mathscr{O}} = y^{-s}\sum_{m,n} c(m,n)\,q^m\bar{q}^n, \tag{3.59}$$

or a finite sum of such terms with different s. For the $\mathcal{Q}$-exact operator $\widetilde{I}_+(\boldsymbol{x})$, we have seen that m and n can be both negative leading to a divergence for $\tau\to i\infty$. Moreover, it is possible that $m=n<0$, for which the standard renormalization does not apply. We introduce a new regularization and renormalization prescription in the next section which also treats terms with $m=n<0$ and shows that (3.58) is convergent.

3.2 Renormalization of modular integrals

The previous section discussed the importance of integrals of the form

$$\mathscr{I}_f = \int_{\mathscr{F}_\infty} d\tau \wedge d\bar{\tau}\, y^{-s} f(\tau,\bar{\tau}), \tag{3.60}$$

for supersymmetric field theories, where f is a non-holomorphic modular form of weight $(2-s, 2-s)$, and $\mathscr{F}_\infty$ a fundamental domain for the modular group (see below for further explanation). We will discuss in this section the evaluation and regularization of integrals of this form, which has been developed in the mathematical literature in the context of inner products for weakly holomorphic modular forms [6].[5]

3.2.1 Renormalization of integrals over $\mathscr{F}_\infty$

We start by considering the integral over a single term $q^m \bar{q}^n$ in the Fourier expansion of f.[6] To this end, consider the set $\mathscr{T}$ of triples (m,n,s), defined by

$$\mathscr{T} = \{m,n \in \mathbb{R}, s \in \mathbb{Z}/2 \,|\, m-n \in \mathbb{Z}\}, \tag{3.61}$$

For $(m,n,s) \in \mathscr{T}$, we consider the integral

$$L_{m,n,s} = \int_{\mathscr{F}_\infty} d\tau \wedge d\bar{\tau}\, y^{-s} q^m \bar{q}^n, \tag{3.62}$$

where $\mathscr{F}_\infty$ is the standard keyhole fundamental domain $\mathscr{F}_\infty = \mathbb{H}/SL_2(\mathbb{Z})$ pictured in Figure 2.2. Since $\mathscr{F}_\infty$ is non-compact and the integrand may diverge at the limit $y \to \infty$, this is an improper integral. It should be understood as the limiting value of integrals over compact domains, which approach $\mathscr{F}_\infty$. To this end, we introduce the compact domain $\mathscr{F}_Y$ by restricting $\mathrm{Im}(\tau) \leq Y$ for some $Y \geq 1$.[7] The boundaries of $\mathscr{F}_Y$ are given by the following

[5] A weakly holomorphic modular form $f(\tau)$ is a modular form which is holomorphic on the interior of $\mathbb{H}$ but may diverge for $\tau \to i\infty \cup \mathbb{Q}$.

[6] We will justify in section 3.2.2 that the Fourier series and the integral can be exchanged.

[7] One may consider a more general upper bound with Y being a function of $\mathrm{Re}(\tau) = x$. This choice will not affect the final result.

arcs

$$\begin{aligned} 1: &\quad \tau = \tfrac{1}{2} + \mathring{\imath} y, &\quad y \in [\tfrac{1}{2}\sqrt{3}, Y], \\ 2: &\quad \tau = x + \mathring{\imath} Y, &\quad x \in [-\tfrac{1}{2}, \tfrac{1}{2}], \\ 3: &\quad \tau = -\tfrac{1}{2} + \mathring{\imath} y, &\quad y \in [\tfrac{1}{2}\sqrt{3}, Y], \\ 4: &\quad \tau = \mathring{\imath}\, e^{\mathring{\imath}\varphi}, &\quad \varphi \in [-\tfrac{\pi}{6}, \tfrac{\pi}{6}]. \end{aligned} \tag{3.63}$$

In the limit, $\lim_{Y\to\infty} \mathscr{F}_Y$ we recover $\mathscr{F}_\infty$. We then regularize $L_{m,n,s}$ as

$$L_{m,n,s}(Y) = \int_{\mathscr{F}_Y} d\tau \wedge d\bar{\tau}\, y^{-s}\, q^m \bar{q}^n, \tag{3.64}$$

for $(m,n,s) \in \mathscr{T}$, and define

$$L_{m,n,s} = \lim_{Y\to\infty} L_{m,n,s}(Y), \tag{3.65}$$

provided the limit exists. To study the dependence on Y, we split the compact domain $\mathscr{F}_Y$ into $\mathscr{F}_1$ plus a rectangle $[-\frac{1}{2}, \frac{1}{2}] \times [1, Y]$ as shown below in Figure 3.1. The split of $\mathscr{F}_Y$, gives for $L_{m,n,s}(Y)$ the following expression

$$L_{m,n,s}(Y) = \int_{\mathscr{F}_1} d\tau \wedge d\bar{\tau}\; y^{-s}\, q^m \bar{q}^n - 2\,\mathring{\imath} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_1^Y dx \wedge dy\;\, y^{-s}\, q^m \bar{q}^n. \tag{3.66}$$

The first term on the right hand side is finite and independent of Y. In the second term, we integrate over x, which gives zero unless $m = n$,

$$-2\,\mathring{\imath}\, \delta_{m,n} \int_1^Y dy\, y^{-s}\, e^{-4\pi m y}. \tag{3.67}$$

We thus find that $\lim_{Y\to\infty} L_{m,n,s}(Y)$ converges, except for the cases

- $m = n < 0$, or
- if $m = n = 0$ with $s \le 1$.

Let us denote by this $\mathscr{D}$ set, in other words

$$\mathscr{D} = \{(m,n,s) \in \mathscr{T} \,|\, m = n < 0\} \cup \{(0,0,s) \in \mathscr{T} \,|\, s \le 1\}. \tag{3.68}$$

The correlation functions discussed in Section 3.1 give rise to $(m,n,s) \in \mathscr{D}$, suggesting that $\mathcal{Q}$-exact observables might diverge rather than vanish. To resolve the tension of this apparent divergence with the structure of topologically twisted theories, we would like to regularize and renormalize such integrals. The cases with $m = n = 0$ are renormalized in the

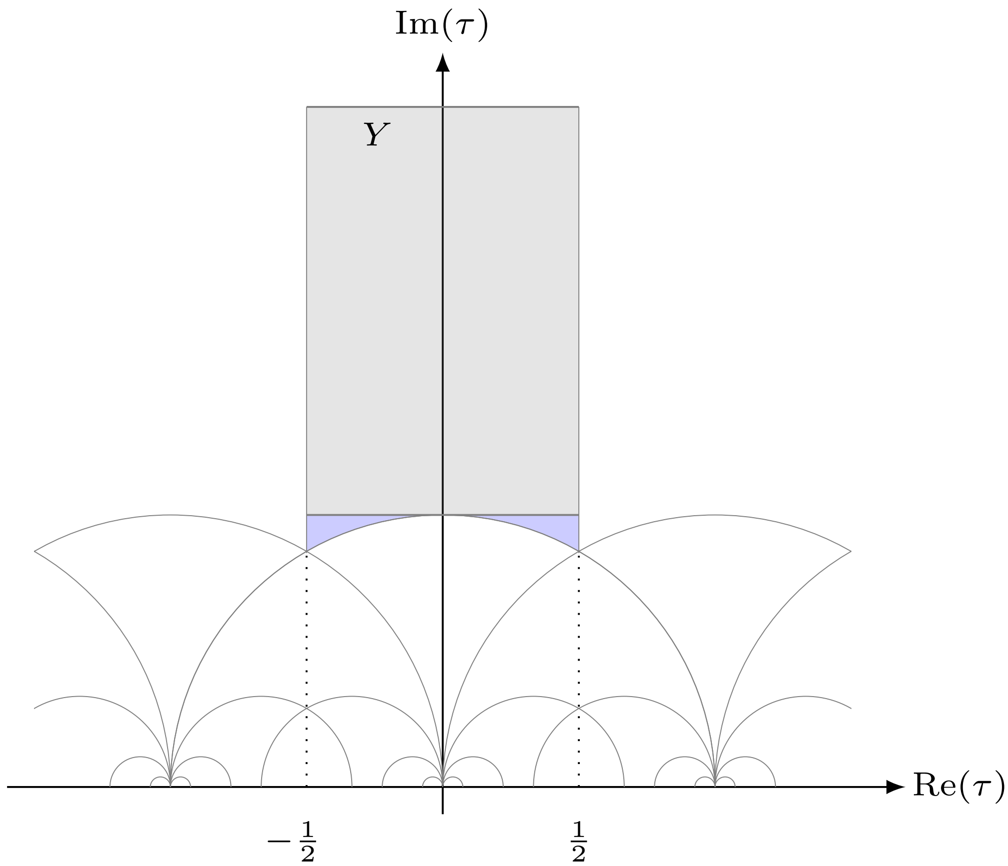

Figure 3.1 Splitting of $\mathscr{F}_Y$ into $\mathscr{F}_1$ (the blue region) and the rectangle $\mathscr{R}_Y$ (gray region).

standard way[8] [5, 14, 36] as is the constant term of the integral for sufficiently large s, which gives 0 for (3.67) if $s = 1$ and otherwise $2\mathring{\imath}/(s-1)$. The trouble is in treating the cases with $m = n < 0$, and to achieve this we put forward in this section a regularized and renormalized version $L^{\mathrm{r}}_{m,n,s}$, of $L_{m,n,s}$ for all $(m,n,s) \in \mathscr{T}$.

Before introducing $L^{\mathrm{r}}_{m,n,s}$, let us note that the limit of the sum

$$\lim_{Y\to\infty}\left[L_{m,n,s}(Y) + 2\mathring{\imath}\,\delta_{m,n}\int_1^Y dy\, y^{-s}\, e^{-4\pi m y}\right] = L_{m,n,s}(1) \tag{3.69}$$

is finite. In the definition for $L^{\mathrm{r}}_{m,n,s}$, we will subtract from the two terms in the brackets, an appropriately regularized counter part of the second term. To this end, let us introduce the

[8] This was described in the beginning of Chapter 3.

generalized exponential integral $E_\ell(z)$. For $\mathrm{Re}(z) > 0$, $E_\ell(z)$ is defined by

$$E_\ell(z) = \int_1^\infty e^{-zt} t^{-\ell} dt. \tag{3.70}$$

Integral shifts of the parameter ℓ are related by partial integration

$$e^{-z} = z E_\ell(z) + \ell E_{\ell+1}(z). \tag{3.71}$$

We can also express $E_\ell(z)$ in terms of the incomplete Gamma function $\Gamma(k,z)$ defined as

$$\Gamma(k,z) = \int_z^\infty e^{-t} t^{k-1} dt = z^k E_{1-k}(z). \tag{3.72}$$

With the analytic continuation of $\Gamma(k,z)$, we can extend the domain of $E_\ell(z)$ to the full complex plane. We define

$$E_\ell(z) = \begin{cases} z^{\ell-1} \displaystyle\int_z^\infty e^{-t} t^{-\ell} \, dt, & \text{for } z \in \mathbb{C}^*, \\ \dfrac{1}{\ell - 1}, & \text{for } z = 0,\ \ell \neq 1, \\ 0, & \text{for } z = 0,\ \ell = 1, \end{cases} \tag{3.73}$$

where for non-integral ℓ, we fix the branch of $t^{-\ell}$ by specifying that the argument of any complex number $\rho \in \mathbb{C}^*$ is in the domain $(-\pi, \pi]$. For $s \in \mathbb{R}^+$, we have

$$\mathrm{Im}(E_\ell(-s)) = -\frac{\pi s^{\ell-1}}{\Gamma(\ell)}. \tag{3.74}$$

In terms of this function $E_\ell(z)$, we can finally define $L^{\mathrm{r}}_{m,n,s}$ for all $(m,n,s) \in \mathscr{T}$:

$$L^{\mathrm{r}}_{m,n,s} = \int_{\mathscr{F}_1} d\tau \wedge d\bar{\tau}\, y^{-s} q^m \bar{q}^n - 2\hat{\imath}\, \delta_{m,n} E_s(4\pi m), \tag{3.75}$$

which regularizes and renormalizes the ill-defined $L_{m,n,s}$ from Equation (3.62) .

3.2.2 Modular invariant integrands

We provide in this subsection the prescription to renormalize $\mathscr{I}_f$. Let us start with the integral of a modular form over the fundamental domain,

$$\mathscr{I}_f = \int_{\mathscr{F}_\infty} d\tau \wedge d\bar{\tau}\, y^{-s} f(\tau,\bar{\tau}), \tag{3.76}$$

where $f(\tau,\bar{\tau})$ is a non-holomorphic modular form for $SL_2(\mathbb{Z})$ of weight $(2-s,2-s)$, with Fourier expansion

$$f(\tau,\bar{\tau}) = \sum_{m,n\gg-\infty} c(m,n)\, q^m \bar{q}^n, \tag{3.77}$$

where the $c(m,n)$ are only non-zero if $m-n\in\mathbb{Z}$ by the requirement that f is a modular form. We assume that f is in fact a function on $\mathbb{H}\times\bar{\mathbb{H}}$, which satisfies

$$f\left(\frac{a\tau+b}{c\tau+d},\frac{a\sigma+b}{c\sigma+d}\right) = (c\tau+d)^{2-s}(c\sigma+d)^{2-s} f(\tau,\sigma), \tag{3.78}$$

where for $s\in\mathbb{Z}+\frac{1}{2}$, we specify the branch of the square root by requiring that the argument of $c\tau+d$ is in $(-\pi,\pi]$. For a single factor $(c\tau+d)^{2-s}$, consistency of the square root and $SL_2(\mathbb{Z})$ requires a non-trivial multiplier system. For $f(\tau,\sigma)$, the multiplier systems for τ and σ are complex conjugate and multiply to one on the right hand side of (3.78).

For the physical correlation functions of Section 3.1, we have to allow f with a finite number of polar terms, i.e., there is an $M\in\mathbb{Z}$ such that $c(m,n)=0$ if $m<M$ or $n<M$, such that the number of terms with $m+n<0$ is finite. For sufficiently large m and n, where the integral behaves well, double application of the well-known saddle point argument shows that the coefficients $c(m,n)$ are bounded by

$$c(m,n) < e^{\sqrt{Km}+\sqrt{Kn}}, \tag{3.79}$$

for some constant $K>0$. The sum over m and n is therefore absolutely convergent for $\mathrm{Im}(\tau)<\infty$.

Due to the terms with $m+n\leq 0$, the integrand in (3.76) diverges for $y\to\infty$, such that the integral is ill-defined. If there are no terms with $m=n<0$, the integral is defined using the well-known regularization [5, 14, 36], but we have seen in Section 3.1 also terms with $m=n<0$ may appear in correlation functions on the Coulomb branch. To regularize these integrals, we introduce a cut-off Y for $\mathrm{Im}(\tau)$ as in Subsection 3.2.1, and define the integral

$\mathscr{I}_f(Y)$ of f over this domain $\mathscr{F}_Y$ (3.63),

$$\mathscr{I}_f(Y) = \int_{\mathscr{F}_Y} d\tau \wedge d\bar{\tau}\, y^{-s} f(\tau,\bar{\tau}). \tag{3.80}$$

We regularize the divergence of $\mathscr{I}_f(Y)$ by subtracting terms involving the generalized exponential function $E_s(z)$ defined in (3.73). More precisely, we replace $\mathscr{I}_f$ by its regularized and renormalized version $\mathscr{I}_f^{\mathrm{r}}$, defined as

$$\mathscr{I}_f^{\mathrm{r}} = \lim_{Y\to\infty} \left[\mathscr{I}_f(Y) - 2\mathring{\imath} \sum_{m \gg -\infty} c(m,m)\, Y^{1-s} E_s(4\pi m Y) \right]. \tag{3.81}$$

Let us verify that the limit is well-defined. Since the domain $\mathscr{F}_Y$ is compact and the sum over m and n is absolutely convergent on $\mathscr{F}_Y$, we can exchange the double integral and the sum. Thus, we have

$$\mathscr{I}_f(Y) = \sum_{m,n \gg -\infty} c(m,n)\, L_{m,n,s}(Y), \tag{3.82}$$

with $L_{m,n,s}(Y)$ as in (3.66). We substitute this expression in (3.81). Using

$$\int_1^Y dy\, y^{-s}\, e^{-4\pi m y} = E_s(4\pi m) - Y^{1-s} E_s(4\pi m Y),$$

we arrive at

$$\mathscr{I}_f^{\mathrm{r}} = \sum_{m,n \gg -\infty} c(m,n)\, L^{\mathrm{r}}_{m,n,s}\,, \tag{3.83}$$

with $L^{\mathrm{r}}_{m,n,s}$ as in (3.75). This is finite since there are at most a finite number of terms with $m = n < 0$, and the sum over the other m and n is absolutely convergent.

3.2.3 Evaluation using Stokes' theorem

If we assume that the integrand can be expressed as a total derivative with respect to $\bar{\tau}$, we can evaluate the integral using Stokes' theorem, and we will find that $\mathscr{I}_f^{\mathrm{r}}$ takes an elegant form in this case. To this end, let us write $y^{-s} f(\tau,\bar{\tau})$ as

$$\partial_{\bar{\tau}} \widehat{h}(\tau,\bar{\tau}) = y^{-s} f(\tau,\bar{\tau}), \tag{3.84}$$

such that the integrand of (3.76) is in fact exact and equal to $-d(d\tau\, \widehat{h})$. Note that this does not imply that $d\bar{\tau}\, \partial_{\bar{\tau}} \widehat{h}$ is exact, since $d\widehat{h} = d\tau\, \partial_\tau \widehat{h} + d\bar{\tau}\, \partial_{\bar{\tau}} \widehat{h}$. For our application to modular integrals, $\widehat{h}(\tau,\bar{\tau})$ transforms as a modular form of weight two. Equation (3.84) can be integrated using $E_\ell(z)$. For $s\neq 1$,[9]

$$\widehat{h}(\tau,\bar\tau)=h(\tau)+2\,\mathring{\imath}\,y^{1-s}\sum_{m,n\gg-\infty}c(m,n)\,q^{m-n}E_s(4\pi ny), \tag{3.85}$$

while for $s=1$, the terms with $n=0$ in the sum should be replaced by

$$-2\,\mathring{\imath}\log(y)\sum_{m\gg-\infty}c(m,0)\,q^m.$$

The $c(m,n)$ in (3.85) are the Fourier coefficients of f (3.77), and h is a (weakly) holomorphic function with Fourier expansion

$$h(\tau)=\sum_{\substack{m\gg-\infty\\ m\in\mathbb{Z}}}d(m)\,q^m. \tag{3.86}$$

Since there are no holomorphic modular forms of weight two for $SL_2(\mathbb{Z})$, the function $h(\tau)$ is uniquely determined by the coefficients $d(m)$ with $m<0$. However, since the $d(m)$, $m<0$, are not determined by the $c(m,n)$, there the space of weakly holomorphic modular forms of weight two gives an ambiguity in $h(\tau)$. Below Equation (3.90) we will discuss that the integral $\mathscr{I}_f^{\mathrm{r}}$ is independent of this ambiguity.

Note that if $f=f(\bar\tau)$ is a (weakly) anti-holomorphic, $\widehat{h}(\tau,\bar\tau)$ is a Maass form annihilated by the weight s hyperbolic Laplacian, and in this case almost satisfies the requirements for a harmonic Maass form [7].[10] Moreover, if f is anti-holomorphic, $h(\tau)$ is a mock modular form with shadow $\overline{f}$ [93, 94] (also see Appendix A.2).

The modular properties of $\widehat{h}(\tau,\bar\tau)$ imply interesting transformations for $h(\tau)$. Let us consider this for the case that f depends on both τ and $\bar\tau$, but is such that the $c(m,n)$ in (3.85) are only non-vanishing for $n>0$ (or $n\geq 0$ and $s>1$). We can then express $\widehat{h}$ as

$$\widehat{h}(\tau,\bar\tau)=h(\tau)+2^s\int_{-\bar\tau}^{i\infty}\frac{f(\tau,-v)}{(-\mathring{\imath}(v+\tau))^s}dv, \tag{3.87}$$

[9] We follow here the convention for Maass forms as in [8]. In other literature on Maass forms such as [6], $E_\ell(s)$ is sometimes replaced by the function $s^{\ell-1}W_\ell(-s/2)=\mathrm{Re}(E_\ell(s))$, $(s\neq 0)$. This has no effect for $s>0$, but terms with $s<0$ lead to additional contributions involving $\mathrm{Im}(E_\ell(s))$ in the final result for $\mathscr{I}_f^{\mathrm{r}}$ (3.90). Reference [6, Definition 3.1] corrected for this in the definition of their inner-product.

[10] A harmonic Maass form of weight k is annihilated by the weight k hyperbolic Laplacian, whereas the weight of $\widehat{h}(\tau,\bar\tau)$ is 2 independently of s.

Note that the two terms on the right hand side are separately invariant under $\tau \to \tau+1$, while the transformation of the integral under $\tau \to -1/\tau$ implies for $h(\tau)$,

$$h(-1/\tau) = \tau^2 \left(h(\tau) + 2^s \int_0^{i\infty} \frac{f(\tau,-v)}{(-i(v+\tau))^s}\, dv \right). \tag{3.88}$$

Let us return now to the generic case with $f(\tau,\bar\tau)$ of the form (3.77) and evaluate $\mathscr{I}_f^{\mathrm{r}}$. The integral over $\mathscr{F}_Y$ can then be carried out using Stokes' theorem, which reduces to a contribution from the interval $[-\frac{1}{2}+iY, \frac{1}{2}+iY]$. We thus find that the integral over $\mathscr{F}_Y$ in (3.81) equals for $s \neq 1$,

$$d(0) + 2i \lim_{Y\to\infty} \sum_{m \gg -\infty} Y^{1-s}\, c(m,m)\, E_s(4\pi m Y), \tag{3.89}$$

using expression (3.85) for $\widehat{h}$. For $s=1$, we apply the renormalization by analytic continuation in s mentioned below (3.68), which gives the same result.

The last step remaining is to combine (3.89) with the other term in equation (3.81), which gives

$$\mathscr{I}_f^{\mathrm{r}} = d(0). \tag{3.90}$$

As a result the only contribution to the integral arises from the constant term of the Fourier seried of $h(\tau)$. This obviously reduces to the regularization for $\mathscr{I}_f$ if either m or n is non-negative [36, 67]. We mentioned below equation (3.86), that there is an ambiguity in h due to the possibility to add a weakly holomorphic modular form of weight two. Since the constant terms of such modular forms vanish, the result (3.90) does not depend on this ambiguity.

To see that this statement is true, let $C(\tau)$ be a weakly holomorphic modular form of weight two. Recall that the first cohomology of $\mathscr{F}_\infty$ is trivial, $H^1(\mathscr{F}_\infty) = 1$, and the one-form $C(\tau)\, d\tau$ is necessarily exact. The period $\int_Y^{Y+1} C(\tau)\, d\tau$ therefore vanishes, which implies that its constant term vanishes. Indeed, a basis of weakly holomorphic modular forms of weight two is given by derivatives of powers of the modular invariant J-function, $\partial_\tau\big(J(\tau)^\ell\big)$, $\ell \in \mathbb{N}$, which have all vanishing constant terms.

3.3 Evaluation of correlation functions of $\mathcal{Q}$-exact observables

We return to the u-plane integrals for correlation functions of $\mathcal{Q}$-exact observables $\langle\{\mathcal{Q},\mathcal{O}\}\rangle$, where $\{\mathcal{Q},\mathcal{O}\}$ may be a product of $\mathcal{Q}$-exact and $\mathcal{Q}$-closed operators as discussed in Section 3.1. As discussed in Subsection 3.1.2, the corresponding u-plane integrals take the form of a total $\bar\tau$-derivative for $\mathcal{Q}$-exact observables. This is the key property for their evaluation, and we can therefore treat all such correlation function simultaneously as indicated in Section 3.1.4.

Using the regularization and renormalization scheme of Section 3.2, we will show that the correlation functions of the form $\langle\{\mathcal{Q},\mathcal{O}\}\rangle$ vanish, confirming the Ward-Takahashi identities of the BRST symmetry. Recall from Subsection 3.1.4, that $\langle\{\mathcal{Q},\mathcal{O}\}\rangle$ can be expressed as

$$\langle\{\mathcal{Q},\mathcal{O}\}\rangle = \int_{\mathcal{F}_\infty} d\tau\wedge d\bar\tau\, \partial_{\bar\tau}F_{\mathcal{O}}, \tag{3.91}$$

with

$$F_{\mathcal{O}}(\tau,\bar\tau) = y^{-s}\sum_{m,n} c(m,n)\, q^m\bar q^n, \tag{3.92}$$

where only a finite number of $c(m,n)\neq 0$ for $m+n<0$. Let us first evaluate (3.91) using Section 3.2.3. Since

$$\partial_{\bar\tau}F_{\mathcal{O}} = -\mathring{\imath}\, y^{-s}\sum_{m,n} c(m,n)\,(2\pi n+\tfrac{1}{2}s y^{-1})\, q^m\bar q^n, \tag{3.93}$$

we can identify $F_{\mathcal{O}}$ with $\widehat{h}_1+\widehat{h}_2$ following (3.84). Here $\widehat{h}_1$ is of the form (3.85) and $\widehat{h}_2$ as well, but with s replaced by $s+1$. $F_{\mathcal{O}}$ is a (non-holomorphic) modular form of weight two, and the discussion in Section 3.1 did not include a holomorphic function h_1+h_2. Indeed, since $F_{\mathcal{O}}$ is a modular form of weight two, vanishing of h_1+h_2 is consistent with the modular properties. The sum of constant terms $d_1(0)+d_2(0)$ thus vanishes, which demonstrates that $\langle\{\mathcal{Q},\mathcal{O}\}\rangle$ vanishes.

Alternatively, one may start from (3.81) with $f=\partial_{\bar\tau}F_{\mathcal{O}}$, such that $\langle\{\mathcal{Q},\mathcal{O}\}\rangle$ reads

$$\begin{aligned}\langle\{\mathcal{Q},\mathcal{O}\}\rangle = \lim_{Y\to\infty}\Bigg[&\int_{\mathcal{F}_Y} d\tau\wedge d\bar\tau\,\partial_{\bar\tau}F_{\mathcal{O}}\\ &-2Y^{-s}\sum_{m\gg-\infty} c(m,m)(2\pi m Y\, E_s(4\pi mY)+\frac{s}{2}E_{s+1}(4\pi mY))\Bigg].\end{aligned} \tag{3.94}$$

To evaluate the integral over $\mathscr{F}_Y$, we use Stokes' theorem. Modular invariance of the integrand implies that only the arc at $\mathrm{Im}(\tau) = Y$ contributes. Using (3.71) for the second line, we arrive again at the desired result

$$\begin{aligned}\langle\{\mathcal{Q},\mathscr{O}\}\rangle &= \sum_m c(m,m) \lim_{Y\to\infty} \left[Y^{-s}\, e^{-4\pi Y m} - Y^{-s}\, e^{-4\pi Y m}\right] \\ &= 0.\end{aligned} \tag{3.95}$$

We have thus demonstrated that the correlation function of a generic $\mathcal{Q}$-exact observable vanishes with the current prescription.

Given that the vacuum expectation value of any $\mathcal{Q}$-exact observable vanishes, power series of $\mathcal{Q}$-exact observables vanish as well. We have in particular

$$\langle (1 - e^{\alpha\{\mathcal{Q},\mathscr{O}\}})\, \mathscr{O}' \rangle = 0, \tag{3.96}$$

for arbitrary $\alpha \in \mathbb{C}$ and assuming that $\mathscr{O}'$ is $\mathcal{Q}$-closed. We can therefore safely add $\mathcal{Q}$-exact terms to the action. This will justify the inclusion of $e^{\widetilde{I}_+(\boldsymbol{x})}$ in the u-plane integrand in Chapter 4 (also in references [43, 44]). It was, in fact, precisely this question which motivated the article that this chapter is based upon.

3.4 Discussion and Summary

We have revisited the evaluation of correlation functions on the Coulomb branch of Donaldson-Witten theory. While vanishing of correlation functions of BRST-exact observables is important for the topological nature of the theory, we have seen families of BRST-exact observables whose correlation functions appear to diverge due to contributions from the boundary of field space. The divergences become most manifest after a change of variables from u to the complexified coupling constant $\tau \in \mathbb{H}/\Gamma^0(4)$. Depending on the observable, the integrand may contain terms $q^m \bar{q}^n$ with m, n both negative, which diverge for $\tau \to i\infty$.

We have demonstrated that such divergences can be cured using a new prescription to regularize and renormalize the integrals over modular fundamental domains. This prescription employs the analytic continuation of the incomplete Gamma function, and was recently developed for for the definition of regularized inner products of weakly holomorphic modular forms [6]. Strikingly, this results in a vanishing expectation value for the correlation functions of BRST-exact observables in Donaldson-Witten theory, confirming its BRST symmetry. With the new regularization we have demonstrated that all valid $\mathcal{Q}$-exact observables decouple from the $\mathcal{Q}$-closed operators. A central aspect of our analysis was that $\mathcal{Q}$-exact observables lead to a u-plane integrand which is a total derivative with respect to $\bar\tau$. We will further elaborate on this aspect in the next two chapters.

As we have restricted our analysis to Donaldson-Witten theory and four-manifolds with $b_2^+=1$, there are immediate directions for future work. We plan to analyze in future work the BRST symmetry of other twisted theories including those with matter and with superconformal symmetry. We would like to extend our discussion also to four-manifolds with $b_2^+=0$, where one-loop determinants contribute in addition to the zero modes.

Besides the $\mathcal{Q}$-closed observables, the new prescription also renormalizes correlation functions of observables outside the $\mathcal{Q}$-cohomology, which are "unphysical" from the point of view of the physical theory. An example is $\langle \mathrm{Tr}(\bar\phi^2)\rangle_{\mathbb{R}^4}$. We leave it for future work to see whether such correlation functions may contain interesting information.

Another potential area of applications are (bosonic) string amplitudes where previous regularizations were initially developed [14, 36]. In particular, it is a standard result that the one-loop contribution $\mathscr{A}_{1-\mathrm{loop}}$ to the vacuum energy in the bosonic string is divergent due to the presence of a tachyon. What is curious about the new regularization and renormalization prescription presented above is that it gives a definite finite value for this amplitude. Recall that

$$\mathscr{A}_{1-\mathrm{loop}} = i\mathscr{I}_f, \tag{3.97}$$

with $f(\tau,\bar\tau)=|\eta(\tau)|^{-48}$. After performing the regularization and renormalization prescription we find for $\mathscr{A}^{\mathrm{r}}_{1-\mathrm{loop}}$ the following result

$$\begin{aligned}\mathscr{A}^{\mathrm{r}}_{1-\mathrm{loop}} &= -\mathring{\imath}\,\frac{2^{27}\pi^{14}}{\Gamma(14)} + \sum_{m,n\geq 1} c(m,n)\mathrm{Re}(L^{\mathrm{r}}_{m,n,14}) \\ &= -\mathring{\imath}\,196620.04\ldots + 64021.15\ldots,\end{aligned} \tag{3.98}$$

where we used $\mathrm{Im}(E_\ell)(-s) = -\frac{\pi s^{\ell-1}}{\Gamma(\ell)}$ for $s>0$. Note here that the contribution of the tachyon is the negative part of the amplitude. It is an open question what the physical consequences of this result are, if any. We hope to return to this question in the future.

Chapter 4

Donaldson-Witten theory and indefinite theta functions

The fundamental papers by Witten [88] and by Moore and Witten [67] provided an incredible amount of physical insight in the field of differential topology related to smooth four-manifolds. The generating function of Donaldson invariants, that we referred to in Subsection 1.6 and will analytically define briefly, could be viewed after these publications as a correlation function of a specific operator of the UV and IR Donaldson-Witten theory respectively.

In this chapter, continuing in the path of these fundamental papers, we will study in detail how the inclusion of the $\mathcal{Q}$-exact surface observable $\widetilde{I}_+$ that was introduced in Section 3.1.1 to the path integral of Donaldson-Witten theory affects the correlation function that gives the Donaldson invariants. In Donaldson-Witten theory, if the metric of the compact four-manifold M has positive scalar curvature, the partition function is completely determined by the integral over the Coulomb branch parameter a, while more generally the Coulomb branch integral captures the wall-crossing behavior of both Donaldson polynomials and Seiberg-Witten invariants. This chapter will show that after the addition of this $\mathcal{Q}$-exact observable to the u-plane integrand, the integrand can be written as a total derivative to the anti-holomorphic Coulomb branch coordinate $\bar{a}$ using Zwegers' indefinite theta functions. In this way, we will reproduce Göttsche's expressions for Donaldson invariants of rational surfaces in terms of indefinite theta functions for any choice of Riemannian metric g.

We begin in the first few sections by recalling some standard facts about four-manifolds as well as the geometric construction of Yang-Mills theories and then we continue by recalling the definition of Donaldson invariants in Section 4.5 . Then we proceed to Section 4.6 where we incorporate the $\widetilde{I}_+$ operator in the u-plane integral that corresponds to the generating function of Donaldson invariants. In Section 4.7 we evaluate the integral and in Section 4.8 we discuss generalizations to higher rank gauge groups.

This chapter is dedicated to the memory of Sir Michael Atiyah, who pioneered the field of geometry and Yang-Mills theory, and happened to pass away during the writing of this chapter. I was fortunate to meet him and speak to him once...

4.1 Some facts about four-manifolds

In this section we will briefly summarize some results known about four-manifolds in the era before Donaldson. The reader who is familiar with the classical topological invariants of four-manifolds or not interested in them can safely jump to the next section.

The integer homology and cohomology structure of closed, smooth, oriented four-manifolds is as follows:

$$\begin{aligned}
H_0(M) = \mathbb{Z} = \mathbb{Z}\cdot[pt.] \qquad & H^4(M)\\
H_1(M) = \pi_1(M)^{\text{abel.}} \qquad & H^3(M)\\
H_2(M) \qquad & H^2(M) = \mathrm{Hom}(H_2(M),\mathbb{Z})\oplus H_1(M)_{\text{torsion}}\\
H_3(M) \qquad & H^1(M) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})\\
H_4(M) = \mathbb{Z} = \mathbb{Z}\cdot[M] \qquad & H^0(M) = \mathbb{Z} = \mathbb{Z}\cdot[1]
\end{aligned}$$

For simply connected closed four-manifolds, $\pi_1 = 0$, this looks much simpler. With the aid of the Hurewicz theorem the corresponding homology and cohomology reads:

$$\begin{aligned}
H_0(M) = \mathbb{Z} = \mathbb{Z}\cdot[pt.] \qquad & H^4(M) = \mathbb{Z}\\
H_1(M) = 0 \qquad & H^3(M) = 0\\
H_2(M) = \mathbb{Z}^d \qquad & H^2(M) = \mathbb{Z}^d\\
H_3(M) = 0 \qquad & H^1(M) = 0\\
H_4(M) = \mathbb{Z} = \mathbb{Z}\cdot[M] \qquad & H^0(M) = \mathbb{Z} = \mathbb{Z}\cdot[1]
\end{aligned}$$

We see that the only relevant topological invariant is the dimension of the second homology or cohomology group, b_2. Cohomology groups are slightly more interesting, for our purposes, compared to the homology groups because they encode some extra information. This is because cohomology classes can be paired via the wedge product as we have seen in the previous chapter. For simply connected four-manifolds this pairing is given by the bilinear form (2.77) which we also recall below, in Equation (4.12). Of course, if we use singular cohomology with integer coefficients instead of differential forms over M, the bilinear form yields an integer, and in that language, the wedge product is what is usually called the cup product.

Using Poincaré duality, we can interpret the bilinear form in terms of homology instead of cohomology. Then, an element of $H_2(M)$ can be viewed as a surface embedded in M, and if x_1 and x_2 are two such surfaces, they will generically intersect in a finite set of points. If these are counted with appropriate signs, the number of points in the intersection will be the integer that corresponds exactly to the integer that is yielded by the bilinear form which we can call in this case *intersection form*. We see therefore that there is a natural way to associate a bilinear form on $H_2(M)$, or equivalently on $H^2(M)$, which is furthermore symmetric, integer valued, and non-degenerate. By choosing some basis for $H_2(M)$, this intersection form, B can be viewed as a symmetric unimodular matrix $A \in \mathbb{Z}^{b_2 \times b_2}$.

The intersection form B is a topological invariant of the four-manifold M. To be more precise, every simply connected compact four-dimensional manifold without boundary gives rise to an integer valued unimodular matrix A as above. This is nice since matrices are very convenient and easy to understand but the problem is that the intersection form requires choosing a basis on $H_2(M)$. Changing this basis with an invertible integer-valued matrix S results to $A \to S^T AS$, and both A and $S^T AS$ represent the same bilinear form but in different bases.

The classification of symmetric integer valued unimodular bilinear forms, up to a change of basis by such a matrix S, is a difficult subject in general. If we relax this requirement slightly and allow any change of basis with respect to a real valued S, the classification of these bilinear forms amounts to counting the number of positive eigenvalues and the number of negative eigenvalues (since the determinant is ± 1, there are no zero eigenvalues). Let b_2^+ be the number of positive eigenvalues and b_2^- the number of negative eigenvalues. Since we are only allowed integer change of bases, the problem is more difficult than this. We still do have b_2^+ and b_2^-, but several matrices may have the same values for b_2^+ and b_2^- but not be equivalent. Actually, the classification of definite symmetric bilinear forms (that means that either $b_2^+ = 0$ or $b_2^- = 0$) is not known at all, but it is known that the number of these forms, of even moderate size, is quite large. Therefore we cannot hope to classify four-manifolds with this technique.

On the other hand the classification of the indefinite case (neither $b_2^+ = 0$ or $b_2^- = 0$) is actually much better and we know the classification of these completely. Let us define some important terminology at this point. An integral intersection form Q is called even if $Q(a) \equiv 0 \mod 2$. Otherwise it is called odd. For even quadratic forms it is possible to change the basis so that the matrix is diagonal with eigenvalues ± 1 on the diagonal. This basis can be chosen so that the matrix breaks up into 2×2 blocks and 8×8 blocks, where the 2×2 blocks are the matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the 8×8 blocks are each the Cartan matrix for the exceptional Lie group E_8

$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}.$$

Note that the matrix E_8 is definite, so there must be at least one H, else we would be considering the definite case that we described previously.

Parity	Indefinite	Definite
Odd	$m\langle 1\rangle \oplus n\langle -1\rangle$	$\pm\mathbb{1}$, $E_8 \oplus \langle 1\rangle$, many unknown
Even	$mH \oplus nE_8$	nE_8, $SO(32)$, Leech lattice, many unknown

Table 4.1 Some facts about the classification of intersection forms for generic four-manifolds.

In table 4.2 we summarize some standard facts about some of the most studied four-manifolds. Two of these four-manifolds, the Hirzebruch surfaces $\mathbb{F}_\ell$ and the projective plane $\mathbb{CP}^2$ will be studied later in Sections 4.7.2 and 4.7.3 respectively. As for the rest, by $\overline{\mathbb{CP}}^2$ we denote the standard $\mathbb{CP}^2$ but with reverse orientation, and # denotes the connected sum.

Therefore we have seen a few examples that show a variety of intersection forms. A natural question to ask is *which intersection forms are possible? Does any intersection form we can cook up belong to some four-manifold?* Furthermore, not much was known about whether it was possible for two manifolds to have the same intersection form (of course they would have the same b_2, and in particular, the same homology and cohomology).

In the 1980s two breakthroughs suddenly added remarkable clarity to the status of the world of simply connected four-manifolds, and they happened at roughly the same time. On

Four-manifold	B	b_2	b_2^+	b_2^-
$\mathbb{S}^4$	0	0	0	0
$\mathbb{F}_0$	H	2	1	1
$\mathbb{CP}^2$	$\langle 1 \rangle$	1	1	0
$\overline{\mathbb{CP}}^2$	$\langle -1 \rangle$	1	0	1
$K3$	$3H \oplus 2E_8$	22	19	3
$(\mathbb{CP}^2)^{\# m} \# (\overline{\mathbb{CP}}^2)^{\# n}$	$m\langle 1 \rangle \oplus n\langle -1 \rangle$	$m+n$	m	n
$K3^{\# m} \# \mathbb{F}_0^{\# n}$	$(3m+n)H \oplus 2mE_8$	$22m+2n$	$19m+n$	$3m+n$

Table 4.2 Topological information for various four-manifolds.

the one hand Michael Freedman published work that was completely of topological nature and on the other hand Simon Donaldson published differential geometric work that used gauge theory, in particular instantons. These two breakthroughs were complementary in the sense that they addressed two disjoint sides of the questions we posed previously. The work of Freedman classified topological manifolds[1] (where the coordinate charts need not patch together smoothly while for smooth manifolds the coordinate charts patch together smoothly, in other words C^∞-differentiably) up to homeomorphism as opposed to Donaldson's work which gave insight to the classification of four-manifolds up to diffeomorphism. In other words, Donaldson's work looked at the problem of the smooth structures of four-manifolds. As it turned out the topological and smooth classification are very different.

The work of Freedman, published in 1982, showed that for simply connected compact four-manifolds without boundary, all intersection forms depicted in Table 4.3 are possible, and with an additional $\mathbb{Z}_2$-valued invariant, the Kirby–Siebenmann invariant[2], these data completely determine the four-manifold up to homeomorphism. Therefore, the question of classifying simply connected compact and without boundary topological four-manifolds up to homeomorphism was finally solved[3]. The idea behind Freedman's work is to show that a more sophisticated version of the way classification for dimensions five and higher is achieved actually works for dimension four as well but since this work did not involve any physics, we will not discuss Friedman's idea here but rather refer to the book [35].

On the other hand, Donaldson's revolutionary work, that started with his seminal paper [16], dealt with smooth four-manifolds using differential-geometric techniques (and soon

[1] For two topological manifolds to be homeomorphic, all that is necessary is the existence of a continuous map from one to the other with a continuous inverse

[2] The Kirby-Siebenmann invariant is an element of $H^4(M,\mathbb{Z}_2)$ that conveys information about the piecewise linear structure of the four-manifold M.

[3] Unfortunately this does not include the fact that intersection forms are not classified.

after algebro-geometric techniques). This work did not result in as complete an answer, to the smooth classification, as in the topological case, but what Donaldson discovered resulted in a simplification along a completely different direction. Donaldson was able to show that the intersection form must be either indefinite (in which case we know how to classify such intersection forms) or plus or minus the identity. We realize therefore that the situation where we did not know how to classify intersection forms, the case where it was definite, is the situation where this classification is unnecessary, since smooth manifolds cannot have them as intersection forms anyway[4].

Parity	Definite	Indefinite
Odd	$m\langle 1\rangle \oplus n\langle -1\rangle$	$\pm\mathbb{1}$
Even	$mH \oplus nE_8$	none

Table 4.3 Classification of intersection forms for simply connected closed four-manifolds.

Furthermore, using pure $SU(2)$ Yang-Mills theory, Donaldson managed to find a systematic, albeit computationally difficult, way to classify smooth structures of compact and without boundary smooth four-manifolds. An introduction to the mathematical aspects of this theory can be found in [17]. In 1988 Witten [88] found a topological quantum field theory, the Donaldson-Witten theory of course, and gave a physical formulation of Donaldson's smooth four-manifold invariants. Later, Moore and Witten studied the low energy dynamics of the Donaldson-Witten theory, that we already analytically described in the previous chapter. In Section 4.5 we will explain how this thesis further contributes not only to the low energy dynamics of the Donaldson-Witten theory but also to the computational complexity of Donaldson invariants. Below we will review Donaldson's construction of his polynomial invariants and see explicitly how gauge theory plays such an important role. Before that we will briefly review the necessary background in Subsections 4.2, 4.3 and 4.4.

4.2 Bundles and connections over four-manifolds

Let us consider a principal G-bundle $P \to M$ over a smooth four-manifold M for a Lie group G. To any such bundle we can associate a vector bundle $E \to X$ with structure group G. We equip this bundle with a connection [5] A which is defined by the covariant derivative. Let

[4] With the exception of the identity and minus the identity.

[5] It is customary in the physics literature to define a connection through its local connection one-form A, that is the *gauge field*. In the mathematical literature the connection is defined as the covariant derivative. Here we will use these two related notions interchangeably unless a confusion arises.

$\Omega^p(M) = \bigwedge^p TX$ denote the space of p-forms over M. Locally $A \in \Omega^1(X, \mathrm{End}(E))$. The corresponding curvature of the connection is $F_A \in \Omega^2(X, \mathrm{End}(E))$. Under a local trivilization of E, with coordinates $\{x^i\}$ on X, the connection can be written as

$$A = A_m dx^m. \tag{4.1}$$

The curvature of the connection, defined as $F_A = d_A A = dA + [A, A]$, can be written in the same local patch as

$$F_A = F_{mn} dx^m \wedge dx^n, \tag{4.2}$$

where in both previous equations Einstein summation convention is implied. The matrix F_{mn} reads

$$F_{mn} = \frac{\partial A_n}{\partial x^n} - \frac{\partial A_n}{\partial x^m} + [A_m, A_n], \tag{4.3}$$

with the commutator taken over the Lie algebra $\mathfrak{g}$.

Let us give a couple of remarks about the curvature of the connection. First of all, the curvature F_A satisfies the Bianchi identity

$$d_A F_A = 0, \tag{4.4}$$

and this can be easily shown by the definition of the curvature. Furthermore, under a bundle automorphism that maps $A \to u(A)$ the curvature transforms as

$$F_{u(A)} = u F_A u^{-1}. \tag{4.5}$$

4.3 Geometry of Yang-Mills Theories

Let us denote by $(\bullet, \bullet)$ the natural metric on $\Omega^p(M)$. Then, for $a, b \in \Omega^p(M)$, we have for the wedge product,

$$a \wedge *b = (a, b) d\mu, \tag{4.6}$$

where by $d\mu$ we denote the Riemannian volume element and by $*$ the Hodge star operator. The space of two-forms $\Omega^2(M)$ can be split as

$$\Omega^2(M) = \Omega^+(M) \oplus \Omega^-(M), \tag{4.7}$$

where $\Omega^+(M) \subset \Omega^2(M)$ is the subspace of self-dual two-forms and $\Omega^-(M) \subset \Omega^2(M)$ is the subspace of anti-self-dual two-forms, in other words, they are the ± 1 eigenspaces of the

Hodge star operator. Therefore, for a $a \in \Omega^{\pm}(M)$ we have

$$a \wedge a = \pm |a|^2 d\mu. \tag{4.8}$$

To the exterior derivative $d : \Omega^p \to \Omega^{p+1}$ we associate its adjoint operator $d^* : \Omega^p \to \Omega^{p-1}$. Then, for $a \in \Omega^{p-1}(M)$ and $b \in \Omega^p(M)$ we have that

$$da \wedge b = a \wedge d^* b. \tag{4.9}$$

For M as above, we can express the adjoint operator as $d^* = \pm * d^*$.

Let us assume that M is an oriented smooth and compact four-manifold. Then, the Hodge theorem tells us that the de Rham cohomology of M can be represented by the space of harmonic forms $\mathscr{H}(M)$. For each class $a \in H^2(M,\mathbb{R})$ there exists a unique harmonic representative (that by abuse of notation we denote with the same letter) $a \in \mathscr{H}(M)$,

$$da = d^* a = 0. \tag{4.10}$$

We see, therefore, that the Hodge star operator preserves the harmonic forms and as a result the second cohomology group $H^2(M,\mathbb{R})$ can be decomposed as

$$H^2(M,\mathbb{R}) = \mathscr{H}^+(M) \oplus \mathscr{H}^-(M). \tag{4.11}$$

Here, we denote by $\mathscr{H}^{\pm}(M)$ the space of the self-dual and anti-self-dual harmonic two-forms. To get a better understanding, recall that to every four-manifold we associate a bilinear form

$$B : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) \to \mathbb{R} \tag{4.12}$$

that pairs two-cycles with two-cocycles on the de Rham cohomology. For two classes $\boldsymbol{a},\boldsymbol{b} \in H^2(M,\mathbb{R})$, we have

$$B(\boldsymbol{a},\boldsymbol{b}) = \int_M \boldsymbol{a} \wedge \boldsymbol{b}, \tag{4.13}$$

and B has signature (b_2^+, b_2^-). The quadratic form associated to M is a map

$$Q : H^2(M,\mathbb{R}) \to \mathbb{R}, \tag{4.14}$$

and is defined through the bilinear form B (these objects were also defined in (2.77) and (4.12) but we recall them here for convenience). For a class $a \in H^2(M,\mathbb{R})$ we have

$$Q(\boldsymbol{a}) = B(\boldsymbol{a},\boldsymbol{a}) = \int_X a \wedge a. \tag{4.15}$$

We see therefore that $\mathscr{H}^{\pm}$ can be thought of as the maximal positive and negative definite subspaces of the bilinear form B. Their dimensions are denoted as

$$b_2^{\pm}(M) = \dim(\mathscr{H}^{\pm}(M)), \tag{4.16}$$

with the second Betti number of M being $b_2(M) = b_2^+(X) + b_2^-(X)$. In this thesis we will focus on smooth compact four-manifolds with $b_2^+ = 1$ for reasons that we explained in detail in Subsection 2.4.1. It is important to note that the spaces $\mathscr{H}^{\pm}$ are orthogonal to each other with respect to the bilinear form B. This means that for $a \in \mathscr{H}^+(X)$ and $b \in \mathscr{H}^-(X)$ we have

$$a \wedge b = a \wedge *b = 0. \tag{4.17}$$

This decomposition of $\mathscr{H}^{\pm}(M)$ can be extended to the space of two-forms with values in $\mathrm{End}(E)$ as

$$\Omega^2(M,\mathrm{End}(E)) = \Omega^+(M,\mathrm{End}(E)) \oplus \Omega^-(M,\mathrm{End}(E)). \tag{4.18}$$

This will play a central role in this thesis since we will extensively use this decomposition in order to write the class of the curvature F_A as $[F_A]/4\pi = \boldsymbol{k}$ with

$$\boldsymbol{k} = \boldsymbol{k}_+ + \boldsymbol{k}_-. \tag{4.19}$$

Let us remark that if the spaces $\Omega^{\pm}(M,\mathrm{End}(E))$ are of dimension greater than one we will be using bold faced symbols.

Having set all this notation we are ready to define the notion of an ant-self dual connection which physicists call instantons. Because of (4.18) the curvature of a connection A splits as

$$F_A = F_A^+ + F_A^-. \tag{4.20}$$

Definition 4.3.1 *A connection A whose curvature satisfies* $F_A^+ = 0$ *corresponds to an* instanton *and is called anti-self-dual (ASD) connection. If the curvature satisfies* $F_A^- = 0$ *it is called anti-instanton or self-dual (SD) connection. Finally, if* $F_A = 0$ *the connection is called flat.*

In this section, we will study the equation $F_A^+ = 0$. For example, if $M = \mathbb{R}^4$ with local coordinates $\{x_1, x_2, x_3, x_4\}$, an ASD connection satisfies

$$\begin{aligned} F_{12} + F_{34} &= 0, \\ F_{14} + F_{23} &= 0, \\ F_{13} + F_{42} &= 0. \end{aligned} \tag{4.21}$$

Locally these equations are valid on any smooth four-manifold.

We can now study instantons, in a more physical context by considering a classical Yang-Mills theory. Let M be a compact smooth four-manifold. The geometric setting of Yang-Mills theory contains a principal G-bundle $P \to M$ with associated vector bundle $E \to X$, as before. For simplicity let us restrict the structure group to $G = SU(n)$. The Lie algebra $Lie(G) = \mathfrak{su}(n)$ consists of traceless skew-hermitian matrices

$$X^\dagger = -X, \tag{4.22}$$

such that $\mathrm{Tr}(M^2) = -|M|^2$. Then, for the curvature[6] F of a connection A on E, using Equation (4.20) we have

$$\mathrm{Tr}(F \wedge F) = -(|F^+|^2 - |F^-|^2) d\mu, \tag{4.23}$$

and

$$\mathrm{Tr}(F \wedge *F) = -|F|^2 d\mu. \tag{4.24}$$

At this point let us recall that the second Chern class of such bundle E is defined as

$$c_2(E) = -\frac{1}{8\pi^2} \int_M \mathrm{Tr}(F \wedge F), \tag{4.25}$$

and it is a topological invariant of E and independent of the choice of connection A. Sometimes we write $c_2(E) = k$ and call it instanton number or instanton charge and it is a positive

[6] We will skip the subscript A for the curvature F_A when it is clear to which connection it corresponds.

integer[7]. The Yang-Mills functional is defined as

$$\begin{aligned} S[A] &= ||F||^2 \\ &= \int_M |F|^2 d\mu \\ &= -\int_M \mathrm{Tr}(F \wedge *F). \end{aligned} \tag{4.26}$$

It is easy to see that if the connection A is ASD, then it corresponds to a true minimum of $S[A]$. We say that instantons minimize the action. The Euler-Lagrange equation for $S[A]$ is given by

$$d_A^* F = 0, \tag{4.27}$$

and together with the Bianchi identity (4.4) they give Maxwell's equations. Instantons and anti-instantons are trivially solutions of the Euler-Lagrange equation.

4.4 Moduli space of instantons

We can understand the functional $S[A]$ as a function on the affine infinite dimensional space of all connections that we denote by $\mathscr{A}$. Due to the automorphisms $\mathrm{Aut}(E)$ of the bundle E the action functional is invariant under gauge transformations, that is vertical automorphisms $u : E \to E$ that are sections of $\mathrm{Aut}(E)$. Gauge transformations form an infinite-dimensional Lie group $\mathscr{G} = \Gamma(\mathrm{Aut}(E))$ where the group structure is given by pointwise multiplication. This group is called *the group of gauge transformations*. Two connections that differ by a gauge transformation are called gauge equivalent. Therefore we see that there is a gauge redudancy in the space of all connections $\mathscr{A}$. We can consider the orbit space $\mathscr{B}$ (not to be confused with the Coulomb branch from Chapter 3) which is defined as the quotient space

$$\mathscr{B} = \mathscr{A} \Big/ \mathscr{G}, \tag{4.28}$$

and a point $[A]$ represents the set of gauge equivalent classes of a connection $A \in \mathscr{A}$. The orbit space $\mathscr{B}$ generically has nontrivial topology and just as $\mathscr{A}$ it also is infinite dimensional. One can study it geometrically by equipping it with a structure of a Banach manifold though. Since we are interested in ASD connections that live in $\mathscr{A}$, or to be more precise in $\mathscr{B}$, and not just arbitrary connections we can consider the space

$$\mathscr{M}(M, g) = \{A \in \mathscr{B} | F^+ = 0\}. \tag{4.29}$$

[7] Equivalently, for anti-instantons or SD connections k is a negative integer.

This is called the *moduli space of instantons* and for a generic Riemannian metric g on M it can be given the structure of a smooth, finite dimensional manifold. When it is clear we will denote $\mathscr{M}(M,g)$ simply as $\mathscr{M}$. Let us remark the the smoothness condition on $\mathscr{M}$ is valid as long as we consider irreducible connections on E. Although reducible connections have to be disregarded because they produce singularities in the moduli space we will soon see that this actually can be used to our advantage. For a good metric on M the dimension of $\mathscr{M}$ is given by

$$\dim(\mathscr{M}) = 4c_2(E) - \dim(G)(1 - b_1(M) + b_2^-(M)). \tag{4.30}$$

From now on we will assume that $G = SU(2)$ or $G = SO(3)$ since for higher rank structure groups some of the following statements are not quite accurate. Let us give a few properties of $\mathscr{M}$ that will be useful for later considerations.

- $\mathscr{M}$ is orientable. This can be seen by calculating its determinant line bundle $\bigwedge^n TM$, where n is the top exterior power, as the determinant of the Atiyah-Singer index bundle on $\mathscr{M}$ (see [63]) and making sure that it is a trivial bundle.

- $\mathscr{M}$ carries a universal instanton bundle $\mathbb{E}$ with a universal connection $\mathbb{A}$. This is a principal G-bundle on $M \times \mathscr{M} := \mathfrak{M}$ with the following properties: for every $[A] \in \mathscr{M}$, the restriction $\mathbb{E}|_{M\times\{[A]\}}$ is isomorphic to E and similarly the restriction of the universal connection $\mathbb{A}|_{M\times\{[A]\}}$ lies in the equivalence class $[A] \in \mathscr{M}$ as a connection on E. The universal instanton bundle will be used for the definition of Donaldson invariants.

- $\mathscr{M}$ is non-compact. In order to define integration over $\mathscr{M}$ we need to compactify it. This is also quite important for the definition of Donaldson invariants.

The natural compactification of $\mathscr{M}$ is the Uhlenbeck compactification $\overline{\mathscr{M}}$. Intuitively, $\mathscr{M}$ is compactified by including to it the ideal instantons that correspond to reducible connections. Physically, ideal instantons are field configurations such that the square norm of the curvature $|F|^2$ is a Dirac delta function. This means that these configurations have spike-like energy density at a specific point over M being flat anywhere else. The compactified moduli space $\overline{\mathscr{M}}$ has a natural stratification to topological sub-sectors and is defined as

$$\overline{\mathscr{M}} = \bigcup_{0\leq m\leq k} \mathscr{M}_{k-m} \times \mathrm{Sym}^m(M), \tag{4.31}$$

where $k = c_2(E)$ and $\mathrm{Sym}^m(M) = M/S_m$ with S_m the permutation group of m points. This space is highly singular but compact. As a matter of fact its boundary contains a component isomorphic to M as can be easily seen from the definition.

Moduli space of stable vector bundles

Although the theory of Donaldson invariants that will be explained later is defined through the universal instanton bundle $\mathbb{E}$ and involves integrals over $\mathscr{M}$, something special happens if the underlying four-manifold is a projective surface over $\mathbb{C}$.

Let S be such a surface with a very ample line bundle $\mathscr{L}$ over it and let g be the Hodge metric on S which comes from the projective embedding defined by the sections of $\mathscr{L}$. We will discuss below that exists an identification between the space $\mathscr{M}(S,\mathscr{L})$ of isomorphism classes of $\mathscr{L}$-slope stable rank-two holomorphic vector bundles $\mathscr{E}$ on X with $c_1(\mathscr{E}) = 0$ and the moduli space $\mathscr{M}(S,g)$ of ASD connections on a principal $SU(2)$-bundle P over S.

Recall that there exists a very deep connection between objects in differential geometry and complex geometry, the Hitchin-Kobayahsi correspondence. To state the correspondence we need the notion of μ-stability. A holomorphic vector bundle is called μ-stable if and only if for every subbundle $\mathscr{F} \subset \mathscr{E}$ we have

$$\frac{\deg(\mathscr{F})}{\mathrm{rnk}(\mathscr{F})} < \frac{\deg(\mathscr{E})}{\mathrm{rnk}(\mathscr{E})}, \tag{4.32}$$

where the degree of a bundle is defined as $\deg(\mathscr{E}) = \int_S B(c_1(\mathscr{E}), J)$ and $\mathrm{rnk}(\mathscr{E})$ is the rank of $\mathscr{E}$. If we replace $<$ with $\leq$ we have semi-stability instead of stability. For fixed $c_1(E)$, J is sometimes called *stability condition* and represents $c_1(\mathscr{L})$.

The statement of the Hitching-Kobayashi correspondence is for a principal $SU(2)$-bundle P over X the induced connection on the associated bunle $\mathscr{E}$ is ASD if and only if it the metric on $\mathscr{E}$ is Hermitian-Einstein. On the other hand such a vector bundle is μ-stable if and only if it is indecomposable and has a Hermitian-Einstein metric. Therefore the Hitchin-Kobayashi correspondence implies that there exists a bijection between $\mathscr{M}(X,g)$ and $\mathscr{M}(S,\mathscr{L})$. The Hitchin-Kobayashi correspondence applies for various types of complex manifolds. For Kähler surfaces it is also reffered to as Donaldson-Uhlenbeck-Yau theorem.

Furthermore just as $\mathscr{M}(S,g)$ has a natural compactification $\overline{\mathscr{M}}(S,g)$, so does $\mathscr{M}(S,\mathscr{L})$ whose natural compactification is the Gieseker-Mayurama compactification $\overline{\mathscr{M}}(S,\mathscr{L})$ that we will not explicitly describe but rather mention that there exists a one-to-one map between $\overline{\mathscr{M}}(S,\mathscr{L})$ and the Uhlenbeck compactification [17]. Intuitively the Gieseker-Mayurama compactification includes in the moduli space the so-called *point-like instantons* or non-commutative instantons. These correspond to sheaves over S, so the compactification yields a moduli space of sheaves which has better properties than the moduli space of vector bundles.

4.5 Donaldson invariants

Donaldson invariants [15, 17] were defined by Donaldson as *smooth structure invariants* of the underlying differentiable four-manifold. Let us restrict to gauge groups $SU(2)$ and $SO(3)$ for simplicity (as they were originally defined) and the reader interested in higher rank groups is advised to consult [58]. To define Donaldson invariants we need the notion of Donaldson's μ map which is defined through the universal instanton bundle $\mathbb{E} \to \mathfrak{M}$ as follows

$$\mu : H_i(M) \to H^{4-i}(\mathscr{M}), \tag{4.33}$$

such that

$$\mu(\boldsymbol{x}) = c_2(\mathbb{E})/\boldsymbol{x}. \tag{4.34}$$

Here, by $/$ we denote the *slant product* which is defined as

$$/ : H^p(\mathfrak{M}, \mathbb{Q}) \times H_q(M, \mathbb{Q}) \to H^{p-q}(\mathscr{M}, \mathbb{Q}). \tag{4.35}$$

Another way to understand the μ map is as follows. If we denote the projection of $\mathfrak{M}$ to M by p_1 and to $\mathscr{M}$ by p_2 respectively, we can write

$$\mu(\boldsymbol{y}) = p_{2*}[p_1^*(\boldsymbol{y}) \cup c_2(\mathbb{E})], \tag{4.36}$$

where p_{2*} is the push forward in cohomology (integration along the fibers of p_2), and the argument of p_1^* is the Poincaré dual of $\boldsymbol{y} \in H_i(M, \mathbb{Q})$ that we denote by the same symbol. We may now proceed to the definition Donaldson invariants as polynomials on $H_0(M, \mathbb{Q}) \oplus H_2(M, \mathbb{Q})$. For classes $p \in H_0(M, \mathbb{Q})$ and $\boldsymbol{x} \in H_2(M, \mathbb{Q})$ the degree d Donaldson invariant for a fixed $c_1(E)$ is defined as

$$\mathscr{D}^{\gamma,J}_{t,m}(p, \boldsymbol{x}) := \int_{\mathscr{M}_\gamma} \mu(\boldsymbol{x})^m \cup \mu(p)^t, \tag{4.37}$$

where the integration is non-vanishing only for $d = 2m + 4t$, $\mu(p) \in H^4(\mathscr{M}, \mathbb{Q})$, $\mu(\boldsymbol{x}) \in H^2(\mathscr{M}, \mathbb{Q})$ and $\mathscr{M}_k$ is the k-instanton moduli space. The information of Donaldson invariants can be repackaged conveniently in a generating function,

$$\Phi^J_{\boldsymbol{\mu}}(p, \boldsymbol{x}) = \sum_{k \geq 0} \sum_{\substack{m,t > 0 \\ 2m+4t=d}} \frac{\mathscr{D}^{\gamma,J}_{t,m}(p, \boldsymbol{x})}{t!m!} \alpha^t \beta^m, \tag{4.38}$$

where α and β are formal variables. Although in principle Donaldson invariants do not depend on the Riemannian metric g we equip the four-manifold M with, when $b_2^+ = 1$ a piece-wise dependence on g is developed for the following reason. For $b_2^+ > 1$ two generic metrics in the space of Riemannian metrics are path connected. On the other hand, for $b_2^+ = 1$ non-generic metrics form a codimension one subset in the space of Riemannian metrics, a collection of *walls*, and two generic metrics might not be necessarily path connected. As a consequence, Donaldson invariants for this case are only piece-wise constant with respect to a given g. To be more precise, we have a *chamber structure* on the period domain, that is the connected component C of the positive cone in the middle cohomology $H_2(M,\mathbb{R})$ and Donaldson invariants will remain constant only when the period point $J(g)$ (this is the cohomology class of of the self-dual harmonic two-forms modulo scalar components and very often denoted as $\omega(g)$) stays within a chamber. But, one might ask what happens as we "cross the wall". Then the differences of Donaldson invariants between the two sides of the wall are given by the *wall-crossing formula* that we will also derive, from a physical picture, shortly. Such a physical derivation of the wall-crossing formula was first presented in [67], together with the correspondence between Donaldson invariants and Donaldson-Witten theory from an IR point of view) and in this chapter we will present how some of these computations are simplified, and in some sense generalized slightly, by considering the $\mathcal{Q}$-exact operator $\widetilde{I}_+$.

The rest of this chapter

Before we continue with the analysis of $\widetilde{I}_+$ in the low energy effective Donaldson-Witten theory, we would like to make some further comments. The work of Göttsche [26] and Göttsche and Zagier [27] connected Donaldson invariants to the subject of modular forms, which is at first sight rather distant from the above. Göttsche and Zagier realized that Donaldson invariants of rational surfaces, as determined earlier for example in [22, 46, 51], could be written as a residue of a combination of modular forms and so-called indefinite theta functions. The latter enjoy much scientific interest in recent years, in part due to their connection to Ramanujan's mock theta functions [93, 94]. An indefinite theta function $\Theta : \mathbb{H} \to \mathbb{C}$ is a holomorphic q-series defined as a sum over an indefinite lattice Λ with signature $(1, n-1)$. The sum is convergent, since the sum is restricted to lattice points with negative definite norm (for the convention taken in this paper). However, Θ does not transform as a modular form under $SL_2(\mathbb{Z})$ transformations. The latter can however be cured thanks to the seminal work of Zwegers'. In specific one may add a specific non-holomorphic function R to Θ such that the sum $\widehat{\Theta} = \Theta + R$ transforms as a modular form. Interestingly, the $\bar{\tau}$-derivative $\Psi = \partial_{\bar{\tau}}\widehat{\Theta}$, turns out to be a Siegel-Narain theta function associated to Λ, whose modular properties are more easily determined using the standard Poisson resummation technique.

Let us now return to the physical u-plane integral to explain the main result of this chapter. As we already saw in Chapter 3 the integral can be expressed as an integral over the fundamental domain $\mathbb{H}/\Gamma^0(4)$, where $\Gamma^0(4) \in SL_2(\mathbb{Z})$ is the electric-magnetic duality group, after a change of variables from $u \in \mathscr{B}$ to the effective coupling constant $\tau \in \mathbb{H}$. Subsequently the technique of "lattice reduction" can be applied to evaluate the integral when $b_2(M) > 1$ [52, 67]. This technique was originally developed in the context of one loop amplitudes in string theory [14, 36] and also has major mathematical applications [5]. For the manifold $\mathbb{CP}^2$, with $b_2(\mathbb{CP}^2) = 1$, the integrand was realized as a total derivative to $\bar{\tau}$ using Zagier's modular completion of the class number generating function [91].

While the u-plane integral does explain the occurrence of modular forms via Seiberg-Witten theory in the context of Donaldson invariants, and is verified for low instanton number with the mathematical results of [26, 27], the final result of [67] has quite a different form from [26, 27]. Later work by Griffin, Malmendier and Ono proved agreement for all instanton numbers in [28, 53, 54] in specific cases of rational surfaces.

The present chapter, based on [44], demonstrates that the indefinite theta functions of Göttsche can be derived more directly from the u-plane integral. To this end, we add the $\mathcal{Q}$-exact term $\widetilde{I}_+$ that we introduced in Equation (2.71) to the effective action of the Donaldson-Witten theory used in [67], which does not modify the value of the integral by the usual rules of topological field theory as proven in Chapter 3. Using techniques of indefinite theta functions developed by Zwegers [94], we show that the modified integrand is a total $\bar{\tau}$-derivative for an arbitrary four-manifold with $b_2^+ = 1$. The integrand of the u-plane integral equals what is known as the "shadow" of the indefinite theta series (up to an overall multiplicative function). As a result, the u-plane integral can be immediately evaluated for a generic choice of metric, and reproduces precisely Göttsche's results for complex algebraic surfaces. The same technique can be applied when matter is included, and we hope similar techniques can be developed for gauge groups with rank larger than one, theories of class S [23, 25] or more general non-Lagrangian theories. We moreover expect that it may be applied more widely as an alternative for "lattice reduction" to evaluate modular integrals.

More generically, it is quite interesting to note that the Coulomb branch integral provides both the holomorphic and non-holomorphic terms of the indefinite theta function and this is far from unique in Donaldson-Witten theory. In other cases where such mock modular forms appear in physics, such as in Vafa-Witten theory [84], AdS_3 gravity [57], black holes [2, 13, 55], or the moonshine phenomenon [10], the holomorphic part has usually the clearest physical interpretation, whereas the non-holomorphic term is typically less well understood.

We are now ready to proceed to the next section which discusses the $\mathcal{Q}$-exact surface operator, and how it modifies the integrand of the u-plane integral (in analogy to our considerations in Chapters 2 and 3).

4.6 The u-plane integral and the $\mathcal{Q}$-exact operator

As we have already discussed in great detail, the u-plane integral is the path integral over the Coulomb branch of topologically twisted $\mathcal{N}=2$ supersymmetric gauge theory with gauge group $SU(2)$ or $SO(3)$. As we have further explained we restrict in the following to four-manifolds with $b_2^+=1$. The corresponding lattices Λ for such four-manifolds are completely classified as we just described, see Table 4.3.

4.6.1 The topologically twisted path integral

The path integral of the effective theory on the Coulomb branch is given by

$$\Phi^J_{\boldsymbol{\mu}}(p,\boldsymbol{x})=\int[\mathscr{D}X]\,\nu(\tau)\,e^{-\int_M \mathscr{L}+2pu+\widetilde{I}_-(\boldsymbol{x})+\widetilde{I}_+(\boldsymbol{x})+\boldsymbol{x}^2G(u)}, \tag{4.39}$$

where $[\mathscr{D}X]$ stands for the path integral measure of the fields $[\mathscr{D}A\mathscr{D}a\mathscr{D}\bar{a}\mathscr{D}\eta\mathscr{D}\psi\mathscr{D}\chi\mathscr{D}D)]$ and $G(u)$ is the contact term from equation (2.72). As discussed in the previous section $\Phi^J_{\boldsymbol{\mu}}$ depends discontinuously on the metric g, and may jump across walls of marginal stability. The metric dependence of $\Phi^J_{\boldsymbol{\mu}}$ is only through the period point $J=J(g)$ [46].

Our goal is evaluate the path integral (4.39). In [44] we did so by substituting for D the solution to its equation of motion. This yields a missing numerical factor that we have to take into account. The correct result has already been presented in Section 3.1.1. In order to obtain this result we had to perform an integration over the fermionic zero modes. We woud like to elaborate slightly on the point of how the fermion zero modes contribute here. To understand the contribution to the u-plane integral from these fields it is useful to discuss their scaling behavior under a Weyl transformation (overall rescaling) of the metric $\lim_{t\to\infty}=t^2g_0$ for a fixed metric g_0.[8] The scaling dimensions of the zero modes naturally equals their form degree. These equal the scaling dimensions of the quantum fluctuations of the fields, except for η, whose quantum fluctuation has dimension two instead of zero [67, Section 2.3]. Thus we see that the terms of the Lagrangian involving η and χ have scaling dimension larger than four, except when we replace η by its zero mode η_0. Similarly, the term involving η in the surface operator (2.71) has dimension two if we replace η by its zero mode. One can

[8] This is a one-parameter family of metrics and belongs to a single chamber in the positive cone.

show that the corrections due to the quantum fluctuations of χ do not survive in the limit $t \to \infty$, assuming that $b_1(M) = 0$ [67].

Next we can combine the result of Section 3.1.1 with the the insertion to the path integral of the operator that will generate for us the Donaldson invariants. That is we want to compute the vacuum expectation value

$$\left\langle e^{2pu+\boldsymbol{x}^2 G(u)} \right\rangle := \Phi^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}). \tag{4.40}$$

We find therefore,

$$\Phi^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = \int_{\mathbb{H}/\Gamma^0(4)} d\tau \wedge d\bar{\tau}\ \tilde{\nu}(\tau)\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})[\mathscr{K}_+]\, e^{2pu+\boldsymbol{x}^2 G(u)}. \tag{4.41}$$

Here $\mathscr{K}_+$ is the kernel (3.8), $\tilde{\nu}(\tau)$ is the measure factor (2.97) and $\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}_+]$ is given by equation (3.9). Note that although we only denote its holomorphic arguments but it is a non-holomorphic function that is defined as

$$\Psi^J_{\boldsymbol{\mu}}(\tau,\bar{\tau},\boldsymbol{\rho},\bar{\boldsymbol{\rho}}) = \exp\Big(-2\pi y \boldsymbol{b}_+^2\Big) \sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}} \partial_{\bar{\tau}} \left(\sqrt{2y}B(\boldsymbol{k}+\boldsymbol{b},\underline{J})\right)$$
$$(-1)^{B(\boldsymbol{k},K_M)} \exp\Big(-\pi \mathring{\imath}\, \bar{\tau}\boldsymbol{k}_+^2 - \pi \mathring{\imath}\, \tau \boldsymbol{k}_-^2 - 2\pi \mathring{\imath} B(\boldsymbol{k}_+,\bar{\boldsymbol{\rho}}) - 2\pi \mathring{\imath} B(\boldsymbol{k}_-,\boldsymbol{\rho})\Big), \tag{4.42}$$

where we have defined

$$\boldsymbol{\rho} = \frac{\boldsymbol{x}}{2\pi}\frac{du}{da}, \tag{4.43}$$

and

$$\boldsymbol{b} = \frac{\mathrm{Im}(\boldsymbol{\rho})}{y} \tag{4.44}$$

Note that in the equation above and for the rest of this chapter we will not explicitly show the dependence of $\Psi^J_{\boldsymbol{\mu}}$ on its kernel $\mathscr{K}_+$. Also, we identify the lattice Λ with $H^2(M,\mathbb{Z})$ as below equation (2.89).

4.6.2 Modular invariance of the integrand

For completeness, we discuss in this subsection invariance of the integrand under the $\Gamma^0(4)$ duality group of Seiberg-Witten theory, which is an important consistency requirement for the integrand. Since the integration $d\tau \wedge d\bar{\tau}$ transforms with weight $(-2,-2)$, the integrand in (4.41) must have modular weight $(2,2)$. Let us start with the Siegel-Narain theta function $\Psi^J_{\boldsymbol{\mu}}$ (4.42). A general form of such theta functions which suits our purposes is given in Appendix

A, equation (A.13). To compare (4.41) with that equation, we set $\boldsymbol{z} = \boldsymbol{\rho}$ and $\boldsymbol{b} = \mathrm{Im}(\boldsymbol{\rho})/y$, with $\boldsymbol{\rho}$ as defined in (4.43).

We see that $\boldsymbol{\rho}$ appears in $\Psi^J_{\boldsymbol{\mu}}$ as an elliptic variable. Indeed, since $\frac{da}{du}$ is a modular form of weight one under $\Gamma^0(4)$, $\boldsymbol{\rho}$ transforms as an elliptic variable (can be thought of as a fugacity factor). More precisely, one verifies using the properties of the Jacobi theta functions (A.11), that $\boldsymbol{\rho}$ transforms under the two generators of $\Gamma^0(4)$ as

$$\begin{aligned} \boldsymbol{\rho}(\tau+4) &= -\boldsymbol{\rho}(\tau), \\ \boldsymbol{\rho}\left(\frac{\tau}{\tau+1}\right) &= \frac{\boldsymbol{\rho}(\tau)}{\tau+1}. \end{aligned} \tag{4.45}$$

Note that this differs by a minus sign from the usual transformation of an elliptic variable under $\tau \to \tau+4$. Using these transformations, identifying K in (A.13) with the canonical class K_M and also the fact that $\boldsymbol{k} = \boldsymbol{l} + \boldsymbol{\mu} + \frac{K_M}{2}$ (we can perform the shift by $\frac{K_X}{2}$ since K_X is a characteristic vector of the lattice Λ as follows from the Hirzebruch-Riemann-Roch theorem[9]) we can find the modular transformations of $\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})$. The transformation properties of $\Psi^J_{\boldsymbol{\mu}}$ under $SL_2(\mathbb{Z})$ and $\Gamma^0(4)$ are given in Appendix A.1.5.

With these transformations at hand, we can determine the action of $\Gamma^0(4)$ generators on $\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})$. Recall that $\boldsymbol{\mu} \in H^2(M,\mathbb{Z}/2)$ in the path integral. We then find for the generator $\tau \to \tau+4$ of $\Gamma^0(4)$

$$\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})\big|_{\tau\mapsto\tau+4} = -\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}). \tag{4.46}$$

The action of the second generator gives

$$\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})\big|_{\tau\mapsto\frac{\tau}{\tau+1}} = (\bar\tau+1)^2(\tau+1)^{\frac{b_2}{2}}\exp\left(-\frac{\pi i\boldsymbol{\rho}^2}{\tau+1} + \frac{\pi i}{4}\sigma(M)\right)\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}), \tag{4.47}$$

where we used $K_M^2 = 8+\sigma(M)$ for simply connected four-manifolds with $b_2^+ = 1$.

[9] A characteristic vector $K \in \Lambda$ is defined as follows: for $\boldsymbol{v} \in \Lambda$ we have $\boldsymbol{v}^2 = B(\boldsymbol{v},K) \mod 2$. It is a fact that a characteristic vector always exists. The Hirzebruch-Riemann-Roch theorem, or Riemann-Roch theorem for surfaces, states that for a complex surface X and for a line bundle $L = \mathcal{O}(D)$, where D is an effective divisor, we have

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}B(D,D-K_X) + \chi(\mathcal{O}_X).$$

Since the Euler character of any line bundle is an integer, by taking their difference and multiplying by two we get that $B(D,D-K_X) \in 2\mathbb{Z}$.

More generically the modular properties of $\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}]$ depend on $\mathscr{K}$. The modular transformations under the $SL_2(\mathbb{Z})$ generators for $\Psi^J_{\boldsymbol{\mu}}[1]$ are

$$\begin{aligned}\Psi^J_{\boldsymbol{\mu}+K/2}[1](\tau+1,\bar\tau+1,\boldsymbol{z},\bar{\boldsymbol{z}}) &= e^{\pi \mathring{\imath}(\boldsymbol{\mu}^2-K^2/4)}\Psi_{\boldsymbol{\mu}+K/2}[1](\tau,\bar\tau,\boldsymbol{z}+\boldsymbol{\mu},\bar{\boldsymbol{z}}+\boldsymbol{\mu}),\\ \Psi^J_{\boldsymbol{\mu}+K/2}[1]\left(-\frac{1}{\tau},-\frac{1}{\bar\tau},\frac{\boldsymbol{z}}{\tau},\frac{\bar{\boldsymbol{z}}}{\bar\tau}\right) &= (-\mathring{\imath}\tau)^{\frac{n-1}{2}}(\mathring{\imath}\bar\tau)^{\frac{1}{2}}\exp(-\pi\mathring{\imath}\boldsymbol{z}^2/\tau+\pi\mathring{\imath}K^2/2)\\ &\quad\times(-1)^{B(\boldsymbol{\mu},K)}\Psi^J_{K/2}[1](\tau,\bar\tau,\boldsymbol{z}-\boldsymbol{\mu},\bar{\boldsymbol{z}}-\boldsymbol{\mu})\end{aligned} \tag{4.48}$$

and the ones for arbitrary kernel $\mathscr{K}$ can easily be derived from those.

Next we discuss the contact term $e^{\boldsymbol{x}^2G(\tau)}$ with $G(u)$ given in (2.72). Due to the special transformations of the weight two Eisenstein series E_2 given in (A.7), the contact term transforms as follows

$$\begin{aligned}e^{\boldsymbol{x}^2G(\tau+4)} &= e^{\boldsymbol{x}^2G(\tau)},\\ e^{\boldsymbol{x}^2G\left(\frac{\tau}{\tau+1}\right)} &= e^{\boldsymbol{x}^2G(\tau)+\frac{\pi\mathring{\imath}}{\tau+1}\boldsymbol{\rho}^2}.\end{aligned} \tag{4.49}$$

The remaining term in the integrand is the measure factor $\tilde\nu(\tau)$. Using the identity

$$\left(\frac{\left(\frac{2\mathring{\imath}}{\pi}\frac{du}{d\tau}\right)^2}{u^2-1}\right)^{\frac{1}{8}} = \vartheta_4(\tau), \tag{4.50}$$

we can write $\tilde\nu(\tau)$ as in (2.95) that we recall here for convenience

$$\tilde\nu(\tau) = -8\mathring{\imath}(u^2-1)\frac{da}{du}\,\vartheta_4(\tau)^{\sigma(M)}. \tag{4.51}$$

If we express further u and da/du in terms of Jacobi theta functions and use the transformation properties (A.11) under $\Gamma^0(4)$, one finds Equations (2.96) that we also recall here for convenience

$$\begin{aligned}\tilde\nu(\tau+4) &= -\tilde\nu(\tau),\\ \tilde\nu\left(\frac{\tau}{\tau+1}\right) &= (\tau+1)^{2-\frac{b_2(M)}{2}}e^{-\frac{\pi\mathring{\imath}\sigma(M)}{4}}\tilde\nu(\tau).\end{aligned} \tag{4.52}$$

Combining now (4.46), (4.47), (4.49) and (4.52), we conclude that the integrand of (4.41) has indeed weight (2,2) under $\Gamma^0(4)$ as required.

In the next section we proceed with the evaluation of the integral and soon after we provide explicit results for a few complex rational surfaces.

4.7 Evaluation of *u*-plane integrals

In this section we will discuss how to explicitly evaluate the *u*-plane integral of Equation (4.41) that is the physical recast of the generating function of Donaldson invariants. We show how modern techniques from analytic number theory aid towards the computation of Donaldson invariants while physically they appear in the *u*-plane integral.

4.7.1 General strategy

Recall that in Section 3.1 we analyzed the vacuum expectation values of four operators in Donaldson-Witten theory, which all lead to integrants of the form $\tilde{\nu}(\tau)\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}](\tau,\bar{\tau})$, with different kernels $\mathscr{K}$ (depending on the insertion at hand). Therefore the *u*-plane integral can be written as (omitting for now the τ and $\boldsymbol{\rho}$ dependence)

$$\Phi^J_{\boldsymbol{\mu}}[\mathscr{K}] = \int_{\mathbb{H}/\Gamma^0(4)} d\tau\wedge d\bar{\tau}\;\tilde{\nu}(\tau)\,\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}]. \tag{4.53}$$

Note that here we have chosen to exclude the generators of Donaldon invariants, $e^{2pu+\boldsymbol{x}^2G(u)}$ for the moment since they are necessarily holomorphic and will not affect what follows.

An efficient technique to evaluate these integrals is to express their integrand as a total derivative with respect to $\bar{\tau}$ of a function $\widehat{\mathscr{H}}$[10],

$$\frac{d}{d\bar{\tau}}\widehat{\mathscr{H}}^J_{\boldsymbol{\mu}}[\mathscr{K}](\tau,\bar{\tau}) = \tilde{\nu}(\tau)\,\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}](\tau,\bar{\tau}), \tag{4.54}$$

and which by a simple change of variables is equivalent to an anti-holomorphic derivative with respect to $\bar{u}$. In order to integrate over the *u*-plane, we remove three disks of radius $r \ll 1$ around each of the singularities $\{\infty,-1,+1\}$, giving three boundaries that we denote by $\partial_j\mathscr{B}$ for $j=1,2,3$. See figure 4.1. Then, a simple application of Stokes' theorem gives then the following "localization formula"

$$\Phi^J_{\boldsymbol{\mu}}[\mathscr{K}] = \sum_j \oint_{\partial_j\mathscr{B}} du\,\left(\frac{d\tau}{du}\right)\widehat{\mathscr{H}}^J_{\boldsymbol{\mu}}[\mathscr{K}]. \tag{4.55}$$

The inverse map $u^{-1}:\mathscr{B}\to\mathbb{H}/\Gamma^0(4)$, as discussed schematically around Equation (2.39), maps each of the boundaries $\partial_j\mathscr{B}$ to arcs in $\mathbb{H}/\Gamma^0(4)$ near the cusps $\{i\infty,0,2\}$ as displayed in figure 2.2. The equivalent picture in the Coulomb branch or the *u*-plane is displayed in Figure 4.1 below.

[10]The reasons for this notation is that $\widehat{\mathscr{H}}$ is the non-holomorphic modular completion of some non-modular but holomorphic indefinite theta function [94].

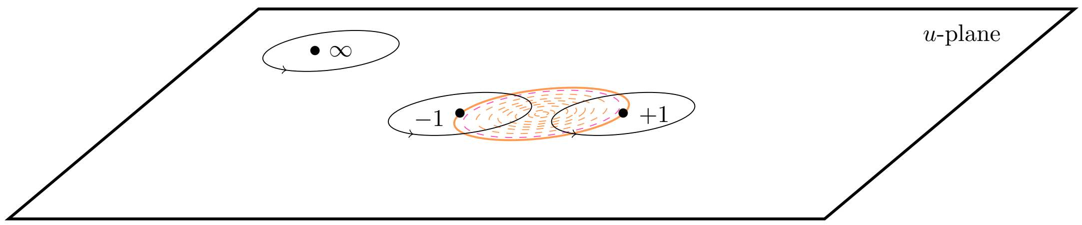

Figure 4.1 Boundaries of the u-plane after removing neighborhoods of the singularies $\{\infty,-1,+1\}$. The region enclosed by the dashed circle denotes the strong coupling region of the u-plane.

In order modular invariance of the integrand to be ensured, the function $\widehat{\mathscr{H}_{\boldsymbol{\mu}}^J}[\mathscr{K}]$ is required to transform as modular form of weight $(2,0)$ with trivial multiplier system, which one may hope to determine explicitly using methods from analytic number theory, especially the theory of mock modular forms as developed in [94] (see also [93] and [7, 13] for reviews). Assuming the existence of such a suitable $\widehat{\mathscr{H}_{\boldsymbol{\mu}}^J}[\mathscr{K}]$, it is quite straightforward to apply the discussion of Section 3.2. To relate the integral over $\mathbb{H}/\Gamma^0(4)$ to an integral over $\mathscr{F}_\infty$, we map the six different images of $\mathscr{F}_\infty$ of $\mathbb{H}/\Gamma^0(4)$ as displayed in Figure 2.2, back to $\mathscr{F}_\infty$ using $SL_2(\mathbb{Z})/\Gamma^0(4)$ transformations of τ. After this inverse mapping procedure, we can use the modular properties of the integrand in order to express each of the six integrands as a series in q and $\bar{q}$, after which the techniques of Section 3.2 can be applied. To this end, one can use the relations (A.14) for $\Psi_{\boldsymbol{\mu}}^J$, while the q-series for $\tilde{v}(\tau)$ follow from the standard relations for Jacobi theta function. Since the maps $\tau\mapsto\tau-n$, $n=1,2,3$ do not change the constant part of the integrand, we find that $\Phi_{\boldsymbol{\mu}}^J[\mathscr{K}]$ evaluates to

$$\Phi_{\boldsymbol{\mu}}^J[\mathscr{K}]=4\left[\widehat{\mathscr{H}_{\boldsymbol{\mu}}^J}[\mathscr{K}](\tau,\bar{\tau})\right]_{q^0}+\left[\tau\mapsto-\tfrac{1}{\tau}\right]_{q^0}+\left[\tau\mapsto\tfrac{2\tau-1}{\tau}\right]_{q^0}, \qquad (4.56)$$

where the subscript denotes that we only pick up the q^0 coefficient of the q-series expansion of $\widehat{\mathscr{H}_{\boldsymbol{\mu}}^J}[\mathscr{K}]$ as a result of the residue integral (4.55). The factor of four that multiply the first bracket is due to the fact that there are four copies of $\mathscr{F}_\infty$ in $\mathbb{H}/\Gamma^0(4)$. The last two brackets are obtained using modular transformations of the integrand $\widehat{\mathscr{H}}[\mathscr{K}]$, under the S and a T^2S transformation respectively.

Let us make an important remark at this point. There exists the possibility to add to $\widehat{\mathscr{H}_{\boldsymbol{\mu}}^J}[\mathscr{K}]$ a holomorphic "integration constant" $s_{\mathscr{K}}$, which necessarily is a weight two modular form for $\Gamma^0(4)$. This function will be mapped to a weight two modular form for $SL_2(\mathbb{Z})$ by the inverse mapping. As discussed in Section 3.2.2, there are no holomorphic $SL_2(\mathbb{Z})$ modular forms with weight two, and the weakly holomorphic ones have a vanishing constant term (this can be seen by expanding E_2). Therefore, we conclude that there is no ambiguity due to the integration constant, just as with elementary integration.

On the other hand, the integration constant $s_{\mathscr{K}}$ can modify the contribution from each cusp, since a holomorphic modular form of weight two for $\Gamma^0(4)$ exists. It is explicitly given by $\vartheta_2(\tau)^4+\vartheta_3(\tau)^4$, and while it does indeed contribute with a factor of four at the cusp at infinity, the contributions of the two cusps together add up to zero. We can make a natural choice of the integration constant using the asymptotic behaviour of our theta functions; we require that the exponential behavior of $\mathscr{H}_{\boldsymbol{\mu}}^J$ for $\tau\to \mathring{\imath}\infty$ matches the behavior of $\tilde{\nu}\,\Psi_{\boldsymbol{\mu}}^J$ in this limit. For example, a consequence of this is that the wall-crossing of the Seiberg-Witten invariants is naturally cancelled by the wall-crossing at the strong-coupling cusps of the u-plane integral.

In the remainder of this section, we determine such an $\widehat{\mathscr{H}}$ such that it allows us to re-derive the Donaldson invariants for Hirzebruch surfaces $\mathbb{F}_\ell$ and the projective plane $\mathbb{CP}^2$, as well as the wall-crossing formula for $\Phi_{\boldsymbol{\mu}}^J$ for an arbitrary simply connected four-manifold M with $b_2^+=1$.

The main technique to express $\Psi_{\boldsymbol{\mu}}^J[\mathscr{K}](\tau,\bar\tau)$ as a total anti-holomorphic derivative, is therefore by making use of indefinite theta series $\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}[\widehat{\mathscr{K}}]$ with kernel $\widehat{\mathscr{K}}$ [94],

$$\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}[\widehat{\mathscr{K}}]=\sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}}\widehat{\mathscr{K}}(\boldsymbol{k})\,(-1)^{B(\boldsymbol{k},K_M)}q^{-\boldsymbol{k}^2/2}w^{-B(\boldsymbol{k},\boldsymbol{\rho})}. \tag{4.57}$$

where $w=e^{2\pi\mathring{\imath}}$. For the case of the BRST exact operator $\widetilde{I}_+$ that is of interest for this chapter, the indefinite theta function can be explicitly written as

$$\begin{aligned}\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{\rho})=\sum_{\boldsymbol{k}\in\Lambda+\mu}\frac{1}{2}\Big(E(\sqrt{2y}B(\boldsymbol{k}+\boldsymbol{b},\underline{J}))-\operatorname{sgn}(\sqrt{2y}B(\boldsymbol{k}+\underline{J}'))\Big)\\ \times(-1)^{B(\boldsymbol{k},K_M)}q^{-\boldsymbol{k}^2/2}w^{-B(\boldsymbol{k},\boldsymbol{\rho})},\end{aligned} \tag{4.58}$$

with $E(t):\mathbb{R}\to[-1,1]$ a reparametrization of the error function,

$$E(t)=2\int_0^t e^{-\pi u^2}du=\operatorname{Erf}(\sqrt{\pi}t), \tag{4.59}$$

and $\underline{J} = J/\sqrt{Q(J)}$ is the normalization of J as before.

The kernel $\widehat{\mathscr{K}}$ depends on J and $J' \in \Lambda$ and J' is a null vector, this means that[11] $Q(J') = 0$. Actually this is what allows us to derive a formula for the Donaldson invariants for *any* suitable Riemannian metric g that we equip M with. Furthermore, the kernel satisfies

$$\frac{d}{d\bar{\tau}} \widehat{\mathscr{K}}(\boldsymbol{k}) = \mathscr{K}(\boldsymbol{k})\, e^{-2\pi y \boldsymbol{k}_+^2}. \tag{4.60}$$

See appendix A for more details on this function. The function $\widehat{\mathscr{H}_{\boldsymbol{\mu}}}[\mathscr{K}]$ in (4.54) and (4.55) takes the form

$$\widehat{\mathscr{H}_{\boldsymbol{\mu}}^J}[\mathscr{K}] = \tilde{\nu}(\tau)\, \widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}[\widehat{\mathscr{K}}]. \tag{4.61}$$

and it is straightforward to show that indeed Equation (4.54) is satisfied. If we include the generators of the Donaldson invariants the function we seek reads

$$\widehat{\mathscr{H}_{\boldsymbol{\mu}}^J}[\mathscr{K}] = \tilde{\nu}(\tau)\, e^{2pu+\boldsymbol{x}^2 G(\tau)}\, \widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}[\widehat{\mathscr{K}}]. \tag{4.62}$$

Finally, we close this subsection by briefly mentioning the *wall-crossing formula* which was earlier derived from the u-plane integral in [67]. We include a more complete discussion in Section 5.3.6 of Chapter 5. This formula gives the discontinuous change of $\Phi_{\boldsymbol{\mu}}^J$ under the variation of a metric with period point J_0 to one with period point $J_1 \in H^2(M)$. It is clear from the discussion above that the difference $\Delta\Phi_{\boldsymbol{\mu}}^{J_1 J_0} = \Phi_{\boldsymbol{\mu}}^{J_1} - \Phi_{\boldsymbol{\mu}}^{J_0}$ is given by

$$\Delta\Phi_{\boldsymbol{\mu}}^{J_1 J_0}(p,\boldsymbol{x}) = \int_{\mathbb{H}/\Gamma^0(4)} d\tau \wedge d\bar{\tau}\; \tilde{\nu}(\tau) \left(\Psi_{\boldsymbol{\mu}}^{J_1} - \Psi_{\boldsymbol{\mu}}^{J_0}\right) e^{2pu+\boldsymbol{x}^2 G(\tau)}. \tag{4.63}$$

Therefore, the contribution from the cusp at $i\infty$ simply reads

$$\Delta\Phi_{\boldsymbol{\mu}}^{J_1 J_0}(p,\boldsymbol{x}) = 4 \left[\tilde{\nu}(\tau)\, \Theta_{\boldsymbol{\mu}}^{J_1 J_0}(\tau,\boldsymbol{\rho})\, e^{2pu+\boldsymbol{x}^2 G}\right]_{q^0}, \tag{4.64}$$

while the contributions of other cusps are canceled by the wall-crossing of the Seiberg-Witten invariants [67]. This nicely reproduces Göttsche's wall-crossing formula [26, Theorem 3.3] and the expression of Göttsche-Zagier in terms of an indefinite theta series a là Zwegers [27, Corollary 4.3].

[11] Note that such a vector can always be found and since it does not have to represent a period point in the positive cone the resulting expression only depends on the metric represented by J.

4.7.2 Application to the Hirzebruch surfaces $\mathbb{F}_\ell$

In this subsection, we specialize the four-manifold M to one of the Hirzebruch surfaces $\mathbb{F}_\ell$. A Hirzebruch surface is a fibration $\pi : \mathbb{F}_\ell \to \boldsymbol{C}$ with fiber $\boldsymbol{f} \cong \mathbb{CP}^1$ over a base $\boldsymbol{C} \cong \mathbb{CP}^1$. The base and the fiber form a basis for $H^2(\mathbb{F}_\ell, \mathbb{Z})$, in terms of which the canonical class K_ℓ is expressed as $K_\ell = -2\boldsymbol{C} - (2+\ell)\boldsymbol{f}$. The intersection matrix for $(\boldsymbol{C}, \boldsymbol{f})$ is

$$Q_{\mathbb{F}_\ell} = \begin{pmatrix} -\ell & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.65}$$

Note in particular that $\boldsymbol{f}$ is an element of $H^2(\mathbb{F}_\ell)$ with vanishing norm, $\boldsymbol{f}^2 = 0$. Two Hirzebruch surfaces $\mathbb{F}_{\ell_1}$ and $\mathbb{F}_{\ell_2}$ are (real) diffeomorphic if $\ell_1 = \ell_2 \mod 2$, while they are complex diffeomorphic only for $\ell_1 = \ell_2$. For a complete description of Hirzebruch surfaces we advise the reader to consult [86].

Our goal is to evaluate $\Phi^J_{\boldsymbol{\mu}}$ for $\mathbb{F}_\ell$ using (4.62). To do this we consider the indefinite theta function (4.58) with the quadratic form (4.65) above. We set $J' = \boldsymbol{f}$, which is fixed by the fact that no stable bundles, in the sense of Equation 4.32, exist for metrics with this period point. Indeed for $J = \boldsymbol{f}$, we get that $\Theta^{J\boldsymbol{f}}_{\boldsymbol{\mu}}$ vanishes. One may show that only the cusp at ∞ contributes to the integral for $\mathbb{F}_\ell$, and we arrive thus for $\Phi^J_{\boldsymbol{\mu}}$ at the following expression

$$\Phi^J_{\boldsymbol{\mu}}(p, \boldsymbol{x}) = 32\,\mathring{\imath} \left[(u^2-1)\, \frac{da}{du}\, \Theta^{J\boldsymbol{f}}_{\boldsymbol{\mu}}(\tau, \boldsymbol{\rho})\, e^{2pu + \boldsymbol{x}^2 G(\tau)} \right]_{q^0}. \tag{4.66}$$

We can simplify the expression for $\Theta^{J\boldsymbol{f}}_{\boldsymbol{\mu}}$ and express it as a (generalized) Appell sum. To this end, we write $\boldsymbol{k}$ as $\boldsymbol{k} = \boldsymbol{m} + n\boldsymbol{f}$, with $\boldsymbol{m}$ such that

$$\frac{B(\boldsymbol{m}+\boldsymbol{b}, J)}{B(\boldsymbol{f}, J)} \in [0,1). \tag{4.67}$$

Then $\Theta^{J\boldsymbol{f}}_{\boldsymbol{\mu}}$ takes the form

$$\begin{aligned} \Theta^{J\boldsymbol{f}}_{\boldsymbol{\mu}} = & \sum_{\substack{\boldsymbol{m}\in\Lambda \\ B(\boldsymbol{m}+\boldsymbol{b},J)/B(\boldsymbol{f},J)\in[0,1)}} \sum_{n\in\mathbb{Z}} (-1)^{B(\boldsymbol{m},K_\ell)} q^{-\frac{\boldsymbol{m}^2}{2}} e^{2\pi \mathring{\imath} B(\boldsymbol{\rho},\boldsymbol{m})} \\ & \times \frac{1}{2}\Big(\operatorname{sgn}(B(\boldsymbol{m}+\boldsymbol{b},J) + nB(\boldsymbol{f},J)) - \operatorname{sgn}(B(\boldsymbol{m}+\boldsymbol{b},\boldsymbol{f})) \Big) \\ & \times (-1)^{nB(\boldsymbol{f},K_\ell)} q^{-nB(\boldsymbol{f},\boldsymbol{m})} e^{-2\pi \mathring{\imath} n B(\boldsymbol{\rho},\boldsymbol{f})}. \end{aligned} \tag{4.68}$$

We can carry out the sum over n as a geometric series to find

$$\Theta_{\boldsymbol{\mu}}^{J\boldsymbol{f}}(\tau,\boldsymbol{\rho}) = \sum_{\substack{\boldsymbol{m}\in\Lambda+\boldsymbol{\mu} \\ B(\boldsymbol{m}+\boldsymbol{b},J)/B(\boldsymbol{f},J)\in[0,1)}} \frac{(-1)^{B(\boldsymbol{m},K_\ell)}q^{-\boldsymbol{m}^2/2}e^{-2\pi i B(\boldsymbol{\rho},\boldsymbol{m})}}{1-q^{-B(\boldsymbol{f},\boldsymbol{m})}e^{-2\pi i B(\boldsymbol{\rho},\boldsymbol{f})}}, \tag{4.69}$$

where we used that $B(\boldsymbol{f},K_\ell) = -2$.

Of particular interest in the literature is the suitable polarization

$$J_\varepsilon = \frac{\varepsilon(\boldsymbol{C}+\ell\boldsymbol{f})+\boldsymbol{f}}{\sqrt{\ell\varepsilon^2+\varepsilon}}, \tag{4.70}$$

with ε sufficiently small such that no walls are crossed between $\boldsymbol{f}$ and J_ε for the rank two vector bundles. If $B(\boldsymbol{\mu},\boldsymbol{f}) \in \mathbb{Z}+\frac{1}{2}$ the condition on $\boldsymbol{m}$ has no solutions in agreement with the fact there are no stable bundles for such metrics. If $B(\boldsymbol{\mu},\boldsymbol{f}) \in \mathbb{Z}$ we have the solutions $\boldsymbol{m} = 0$ and $\boldsymbol{m} = \frac{1}{2}f$, due to strictly semi-stable bundles. We find therefore

$$\begin{aligned} \Theta_0^{J_\varepsilon \boldsymbol{f}}(\tau,\boldsymbol{\rho}) &= \frac{1}{1-e^{-2\pi i B(\boldsymbol{\rho},\boldsymbol{f})}}, \\ \Theta_{\boldsymbol{f}}^{J_\varepsilon J'}(\tau,\boldsymbol{\rho}) &= \frac{-e^{-\pi i B(\boldsymbol{\rho},\boldsymbol{f})}}{1-e^{-2\pi i B(\boldsymbol{\rho},\boldsymbol{f})}} = \frac{i}{2\sin(\pi B(\boldsymbol{\rho},\boldsymbol{f}))}. \end{aligned} \tag{4.71}$$

Using the formula for the measure term (2.95) and letting $\boldsymbol{x} = x_C\boldsymbol{C}+x_f\boldsymbol{f} \in H^2(\mathbb{F}_\ell,\mathbb{R})$, we arrive at the following non-vanishing generating function for Donaldson invariants for this suitable polarization J_ε,

$$\begin{aligned} \Phi_0^{J_\varepsilon}(p,\boldsymbol{x}) &= -16\left[(u^2-1)\frac{da}{du}e^{2pu+\boldsymbol{x}^2G(u)}\cot(\tfrac{1}{2}x_C\,du/da)\right]_{q^0}, \\ \Phi_{\frac{1}{2}\boldsymbol{f}}^{J_\varepsilon}(p,\boldsymbol{x}) &= 16\left[(u^2-1)\frac{da}{du}e^{2pu+\boldsymbol{x}^2G(u)}\frac{1}{\sin(\frac{1}{2}x_C\,du/da)}\right]_{q^0}, \end{aligned} \tag{4.72}$$

where we expressed $\Phi_0^{J_\varepsilon}$ in terms of $\cot(x)$ using the fact that only odd powers of x_f contribute to the expansion of the right hand side. This is in agreement with [27, Theorem 5.3] and [67, Section 8.2].

4.7.3 Application to the projective plane $\mathbb{CP}^2$

As a second and final example for this chapter, we consider the complex projective plane $\mathbb{CP}^2$ which will allow us to see another application of indefinite theta functions to the u-plane integral. Since $b_2(\mathbb{CP}^2) = 1$, in this case the period point of the metric is proportional to the hyperplane class $H \in \mathrm{Pic}(\mathbb{CP}^2)$. Since there is thus no chamber dependence, we omit it from the notation. The sum over $U(1)$ fluxes Ψ_μ is given by[12]

$$\begin{aligned}\Psi_\mu(\tau,\rho) = &\exp\left(-2\pi y b^2\right) \sum_{k\in\mathbb{Z}+\mu} \partial_{\bar\tau}\left(\sqrt{2y}(k+b)\right) \\ &\times (-1)^{3k}\,\bar q^{\frac{k^2}{2}}\, e^{-2\pi i\bar\rho k},\end{aligned} \tag{4.73}$$

where we have used that the canonical class $K_{\mathbb{P}^2}$ equals $3H$.

Since the lattice $H^2(\mathbb{CP}^2,\mathbb{Z})$ is one-dimensional, we can not directly apply the indefinite theta function to integrate over the Coulomb branch. However, we can extend the one-dimensional lattice to a two-dimensional lattice by dividing and multiplying by the Jacobi theta function ϑ_4 defined in (A.10). We write therefore,

$$\Psi_\mu = \frac{\vartheta_4(\tau)}{\vartheta_4(\tau)}\Psi_\mu. \tag{4.74}$$

Although this seems like a tautology, geometrically we can interpret these manipulations in terms of the blow-up $\widehat{\mathbb{CP}}^2$ of $\mathbb{CP}^2$. Note that the measure (4.51) differs by a factor ϑ_4^{-1} for $\mathbb{CP}^2$ and $\widehat{\mathbb{CP}}^2$. Including the summation over $\mathbb{Z}$ in θ_4 in the lattice sum, $\Psi_\mu(\tau,\rho)$ reads

$$\begin{aligned}\Psi_\mu(\tau,\rho) = &\frac{\exp\left(-2\pi\tau_2 b^2\right)}{\theta_4(\tau)} \\ &\times \sum_{(k_1,k_2)\in\mathbb{Z}^2+(\mu,0)} \partial_{\bar\tau}\left(\sqrt{2\tau_2}(k_1+b)\right)(-1)^{3k_1+k_2}\,\bar q^{\frac{k_1^2}{2}} q^{\frac{k_2^2}{2}} e^{-2\pi i\bar\rho k_1}.\end{aligned} \tag{4.75}$$

Our earlier discussion shows that $\Psi_\mu(\tau,\rho)$ can be expressed as an anti-holomorphic derivative,

$$\frac{1}{\vartheta_4(\tau)}\partial_{\bar\tau}\widehat\Theta_\mu^{JJ'}(\tau,\rho), \tag{4.76}$$

where $\widehat\Theta_{\boldsymbol\mu}^{JJ'}$ is the completion of the indefinite theta function $\Theta_{\boldsymbol\mu}^{JJ'}$ whose associated lattice Λ is the two-dimensional lattice with diagonal quadratic form $\mathrm{diag}(1,-1)$. The two-dimensional parameters $\boldsymbol\mu$ and $\boldsymbol\rho$ are given by $(\mu,0)$ and $(\rho,0)$ respectively, whereas the two parameters $J,J' \in \Lambda\otimes\mathbb{R}$ are given by $J=(1,0)$ and $J'=(1,1)$ respectively.

The lattice sum in $\Theta_{\boldsymbol\mu}^{JJ'}$ can be partially carried out using a geometric series, leading to the expression

$$\Theta_\mu^{JJ'}(\tau,\rho) = w^\mu(-1)^{2\mu}\sum_{\ell\in\mathbb{Z}+\mu}\frac{(-1)^\ell q^{\frac12\ell^2+\mu\ell}}{1-wq^\ell}, \tag{4.77}$$

[12] We omit the boldface font here for k, b and ρ, since they are elements of one-dimensional spaces.

with $w = e^{2\pi i\rho}$. This is, up to a prefactor, a specialization of the Appell sum [94]

$$A(u,v,\tau) = e^{\pi i u} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}. \tag{4.78}$$

Treating first the case $\mu = \frac{1}{2}$, we arrive at the following expression for the generating function $\Phi_{\frac{1}{2}}(p,x)$

$$\Phi_{\frac{1}{2}}(p,x) = -32\,i \left[(u^2-1)\, \frac{da}{du}\, e^{2pu + x^2 G(u)}\, \Theta^{JJ'}_{(\frac{1}{2},0)}(\tau,\rho) \right]_{q^0}, \tag{4.79}$$

which gives for the first few terms

$$\Phi_{\frac{1}{2}}(0,x) = 1 + \frac{3}{16}\frac{x^4}{4!} + \frac{29}{32}\frac{x^8}{8!} + \frac{69525}{4096}\frac{x^{12}}{12!} + O(x^{16}). \tag{4.80}$$

These terms are in agreement with [67] and [22, Theorem 4.4], while the full series matches the expression of Göttsche [26, Theorem 3.5].

Next we consider the case $\mu = 0$. The series $\Phi_0(p,x)$ can be determined similarly using multiplication and division by ϑ_4. However, we notice from (4.77) that $\widehat{\Theta}_0(\tau,\rho)$ is then divergent for small ρ, which is at odds with the Donaldson invariants being polynomials in $\boldsymbol{x}$. The resolution is that the holomorphic integration constant mentioned below (4.56) is non-vanishing in this case. Using the blow-up formula, one finds that the constant equals

$$C(\tau,\rho) = \frac{\vartheta_4(\tau,0)}{\vartheta_4(\tau,\rho)}\, \partial_\rho \ln\left(\frac{\vartheta_1(\tau,\rho)}{\vartheta_4(\tau,\rho)} \right), \tag{4.81}$$

leading to the following expression for $\Phi_0(p,x)$

$$\Phi_0(p,x) = -32\,i \left[(u^2-1)\, \frac{da}{du}\, e^{2pu + x^2 G(u)} \left(C(\tau,\rho) + e^{-\pi i \rho} A(\rho, -\tfrac{1}{2}\tau, \tau) \right) \right]_{q^0}. \tag{4.82}$$

We would like to relate this to the expression of [26, Theorem 3.5]. We recall the periodicity property of the Appell function (4.78) from [94, Chapter 1],

$$\frac{A(u,v,\tau)}{\vartheta_1(v,\tau)} - \frac{A(u+z,v+z,\tau)}{\vartheta_1(v+z,\tau)} = \frac{\eta^3\, \vartheta_1(u+v+z,\tau)\, \vartheta_1(z,\tau)}{\vartheta_1(u,\tau)\, \vartheta_1(v,\tau)\, \vartheta_1(u+z,\tau)\, \vartheta_1(v+z,\tau)}. \tag{4.83}$$

Letting $v=-\frac{1}{2}\tau$ and taking the limit $z\to\frac{1}{2}\tau$, we find

$$C(\tau,\rho)+e^{-\pi i\rho}A(\rho,-\tfrac{1}{2}\tau,\tau)=\frac{\vartheta_4(\tau)}{\eta(\tau)^3}\sum_{\substack{k_1\in\mathbb{Z}\\ k_2\in\mathbb{Z}+\frac{1}{2}}}(\mathrm{sgn}(k_1+a)-\mathrm{sgn}(k_1+k_2+a))\\ \times k_2\,(-1)^{k_1+k_2}e^{2\pi i\rho k_1}q^{-\frac{k_1^2}{2}+\frac{k_2^2}{2}}. \tag{4.84}$$

Substitution of this expression in (4.82) reproduces the expression in [26, Theorem 3.5]. For completeness, we list the first few terms in the expansion

$$\Phi_0(p,x)=-\frac{3}{2}x+\frac{x^5}{5!}+3\frac{x^9}{9!}+54\frac{x^{13}}{13!}+O(x^{17}), \tag{4.85}$$

which are in agreement with [22, Theorem 4.2]. We could also arrive at the right hand side of (4.84) by multiplying and dividing in (4.75) by $\vartheta_1(z,\tau)$ instead of $\vartheta_4(\tau)$, and then taking the limit $z\to 0$ in $\Theta_0^{JJ'}(\tau,\rho,z)/\vartheta_1(z,\tau)$. A similar procedure was used in the context of D3-instanton corrections [3, Section 4].

4.8 On the extension to gauge groups of arbitrary rank

Donaldson-Witten theory can be generalized to theories with a gauge group G with rank $r>1$ [52, 58] using the corresponding Seiberg-Witten geometries [4, 19, 41, 78]. Mochiziku [62] developed an algebraic-geometric framework to discuss higher rank Donaldson invariants but this is beyond the scopes of this thesis. This section generalizes the $\mathcal{Q}$-exact surface operator (2.70) to theories with arbitrary rank gauge group G, and discusses the sum over $U(1)^r$ fluxes of the Coulomb branch integrand for a four-manifold M with $b_1=0$ and $b_2^+=1$. We keep this section relatively short and refer the reader for the details to [58].

Let us consider the Coulomb branch of a $\mathcal{N}=2$ supersymmetric Yang-Mills theory whose gauge group G has rank r. We denote the Cartan elements of the Lie algebra by H_K, $K=1,\dots,r$. Then, the vacuum expectation value of the scalar component of the $\mathcal{N}=1$ chiral superfield can classicaly be brought to the form

$$\phi=\sum_{K=1}^{r}a^K H_K. \tag{4.86}$$

The a^K provide local special coordinates on the Coulomb branch moduli space. Alternatively, one can consider the r Weyl invariant Casimirs u_K, $K=1,\dots,r$, as coordinates on the Coulomb branch. At a generic point on the Coulomb branch, the field content consists of

r copies of the effective $U(1)$ theory described in Section 2, which are distinguished by a superscript: A^K, ψ^K, ..., for $K = 1, \dots, r$. The effective coupling $\tau_{KL} = \tau_{KL}(a^M)$ is now an $r \times r$ matrix. The effective Coulomb branch theory breaks down at the locus where gauge bosons become massless, or more geometrically, the corresponding Seiberg-Witten curve becomes singular.

Most aspects of the rank one Donaldson-Witten theory generalize to rank r without much effort. For example after topological twisting, the action of the $\mathcal{Q}$ operator on the low energy IR fields is given by

$$\begin{aligned} &\{\mathcal{Q}, A^K\} = \psi^K, && \{\mathcal{Q}, a^K\} = 0 && \{\mathcal{Q}, \bar{a}^K\} = \sqrt{2}\,\mathring{\imath}\,\eta^K, \\ &\{\mathcal{Q}, D^K\} = (d\psi)_+, && \{\mathcal{Q}, \eta^K\} = 0, && \{\mathcal{Q}, \psi^K\} = 4\sqrt{2}\,da^K, \\ &\{\mathcal{Q}, \chi^K\} = \mathring{\imath}(F_+ - D)^K. \end{aligned} \tag{4.87}$$

The effective Lagrangian on the Coulomb branch is similarly a straightforward generalization of the rank one case [58].

There is a larger freedom for the construction of surface operators in the higher rank theories. Starting from any invariant function $\mathscr{U} = \mathscr{U}(a^K)$ of the coordinates a^K, one may construct a suitable surface operator. The operator $\widetilde{I}_-$ (2.69) generalized to general r takes the form [58]

$$\widetilde{I}_-(\boldsymbol{x}) = \frac{\mathring{\imath}}{\sqrt{2}\pi} \int_{\boldsymbol{x}} \left(\frac{1}{32\sqrt{2}\pi} \mathscr{U}_{KL} \psi^K \wedge \psi^L - \frac{\sqrt{2}}{4} \mathscr{U}_K (F_- + D)^K \right), \tag{4.88}$$

where the subscripts indicate differentiation to a^K:

$$\mathscr{U}_K = \frac{d\,\mathscr{U}}{da^K}, \qquad \mathscr{U}_{KL} = \frac{d^2\,\mathscr{U}}{da^K da^L}. \tag{4.89}$$

The generalization of the $\mathcal{Q}$-exact surface operator $\widetilde{I}_+$ (2.71) is similarly given by

$$\begin{aligned} \widetilde{I}_+(\boldsymbol{x}) &= -\frac{1}{4\pi} \int_{\boldsymbol{x}} \{\mathcal{Q}, \{L, \mathrm{Tr}[\bar{\phi}_K \bar{\phi}^K]\}\} \\ &= -\frac{1}{4\pi} \int_{\boldsymbol{x}} \{\mathcal{Q}, \bar{\mathscr{U}}_K \chi^K\}, \end{aligned} \tag{4.90}$$

where the trace is taken over the Lie algebra. Then, using the supersymmetry algebra (4.87) this operator can be written as

$$\widetilde{I}_+(\boldsymbol{x}) = -\frac{\mathring{\imath}}{2\sqrt{2}\pi}\int_{\boldsymbol{x}}\left(\bar{\mathscr{U}}_{KL}\eta^K\chi^L + \frac{1}{\sqrt{2}}\bar{\mathscr{U}}_K(F_+ - D)^K\right). \tag{4.91}$$

Our next aim is derive the sum over the $U(1)^r$ fluxes $\boldsymbol{k}^K$, the function $\Psi^J_{r,\boldsymbol{\mu}}$, when both $\widetilde{I}_+$ and $\widetilde{I}_-$ are inserted in the path integral. After integrating out the auxiliary fields D^K, we find that $\Psi^J_{r,\boldsymbol{\mu}}$ is given by

$$\begin{aligned}\Psi^J_{r,\boldsymbol{\mu}}(\tau_{KL},\boldsymbol{\rho}_K) &= \frac{1}{\sqrt{\det v}}e^{-2\pi v_{KL}\boldsymbol{b}^K_+\boldsymbol{b}^L_+}\sum_{\boldsymbol{k}\in\Gamma}(-1)^{B(\boldsymbol{k}^K W_K, K_M)}\mathscr{K}(\boldsymbol{k},\boldsymbol{\rho},\omega)\\ &\times\exp\Big(-\pi\mathring{\imath}\,\bar{\tau}_{KL}B(\boldsymbol{k}^K_+,\boldsymbol{k}^L_+) - \pi\mathring{\imath}\,\tau_{KL}B(\boldsymbol{k}^K_-,\boldsymbol{k}^L_-) - 2\pi\mathring{\imath}\,B(\boldsymbol{k}^K_+,\bar{\boldsymbol{\rho}}_K) - 2\pi\mathring{\imath}\,B(\boldsymbol{k}^K_-,\boldsymbol{\rho}_K)\Big)\end{aligned} \tag{4.92}$$

where $v_{KL} = \mathrm{Im}(\tau_{KL})$, W_K are the components of the Weyl vector of G, and we introduced

$$\begin{aligned}\boldsymbol{\rho}_K &= \frac{\boldsymbol{x}}{2\pi}\mathscr{U}_K \in H^2(M,\mathbb{C})\\ \boldsymbol{b}^K &= v^{KL}\mathrm{Im}(\boldsymbol{\rho}_L) \in H^2(M,\mathbb{R}),\end{aligned} \tag{4.93}$$

in analogy to the rank one case. The kernel $\mathscr{K}$ in (4.92) is given by the integral over the fermion zero modes

$$\begin{aligned}\mathscr{K}(\boldsymbol{k},\boldsymbol{\rho},\omega) &= \int\left[\prod_{K,L=1}^{r} d\eta_0^K d\chi_0^L\right]\exp\Big(-\frac{\sqrt{2}\mathring{\imath}}{4}\int_M \bar{\mathscr{F}}_{KLM}\eta_0^K\chi_0^L\wedge(\boldsymbol{k}_+ - \boldsymbol{b}_+)^M\\ &-\frac{\mathring{\imath}}{\sqrt{2}}\bar{\boldsymbol{\rho}}_{KL}\eta_0^K\chi_0^L + \frac{1}{64\pi}v^{KP}\bar{\mathscr{F}}_{KLM}\bar{\mathscr{F}}_{PQR}\eta_0^L\chi_0^M\wedge\eta_0^Q\chi_0^R\Big),\end{aligned} \tag{4.94}$$

where we have defined

$$\mathscr{F}_{KLM} = \frac{d\tau_{KL}}{da^M}. \tag{4.95}$$

Carrying out this integral for $G = SU(3)$ (which is a gauge group of rank two), we arrive at

$$\begin{aligned}\mathscr{K}(\boldsymbol{k},\boldsymbol{\rho},\omega) &= \tfrac{1}{8}\Big(\bar{\mathscr{F}}_{11K}B(\boldsymbol{k}^K - \boldsymbol{b}^K, J) + 2B(\bar{\boldsymbol{\rho}}_{11}, J)\Big)\Big(\bar{\mathscr{F}}_{22L}B(\boldsymbol{k}^L - \boldsymbol{b}^L, J) + 2B(\bar{\boldsymbol{\rho}}_{22}, J)\Big)\\ &-\tfrac{1}{8}\Big(\bar{\mathscr{F}}_{12K}B(\boldsymbol{k}^K - \boldsymbol{b}^K, J) + 2B(\bar{\boldsymbol{\rho}}_{12}, J)\Big)\Big(\bar{\mathscr{F}}_{12L}B(\boldsymbol{k}^L - \boldsymbol{b}^L, J) + 2B(\bar{\boldsymbol{\rho}}_{12}, J)\Big)\\ &+\frac{1}{32\pi}(\bar{\mathscr{F}}_{11K}\bar{\mathscr{F}}_{22L} - \bar{\mathscr{F}}_{12K}\bar{\mathscr{F}}_{12L})v^{KL}.\end{aligned}$$

We leave it for future work to express $\Psi^J_{r,\boldsymbol{\mu}}$ as a total derivative of $\bar{a}^K$ (note that for simple linear quiver gauge theories with gauge group $\prod_i SU(2)_i$, $i = 1,\dots,n$ this is trivial and therefore not so interesting from a physical point of view).

4.9 Discussion and Summary

In this chapter we have discussed in detail the insertion of the $\mathcal{Q}$-exact operator to the path integral of Donaldson-Witten theory or equivalently to the generating function of Donaldson invariants for a four-manifold M with $b_2^+ = 1$. We have shown that for gauge group $SU(2)$ and $SO(3)$ the integrand may be expressed as a total $\bar{\tau}$-derivative of an indefinite theta function precisely due to the insertion of the $\mathcal{Q}$-exact surface operator $\widetilde{I}_+$ which couples to the self-dual part of the field strength F. This allows to readily evaluate the integral, and to express it as a sum of contributions over the cusps of the integration domain $\mathbb{H}/\Gamma^0(4)$. In this way, we reproduce the result of Göttsche, who expressed generating series of Donaldson invariants in terms of a residue of an indefinite theta function. Furthermore, we discussed in detail how to apply this in two examples: the Hirzebruch surfaces $\mathbb{F}_\ell$ and the projective plane $\mathbb{CP}^2$. We concluded with the case of higher rank gauge groups which still require careful study. There are other possible directions to consider in correlation to the previous discussion such as theories with matter representations.

Chapter 5

Ramified Donaldson-Witten theory and mock modular forms

In the final part of this thesis, we continue by investigating the ramified[1] u-plane integral based on [43] and inspired by the works of Tan [79, 80]. We study the low energy dynamics of Donaldson-Witten theory in the presence of surface defects that support non-local supersymmetric surface operators. These defects correspond to real codimension two surfaces[2] embedded in a four-manifold M. In the presence of such defects it is possible to define the so-called *ramified Donaldson invariants* associated to M [47]. A familiar (albeit slightly different) example of one-dimensional defects (codimension three with respect to the dimension of M) are line operators like the electrically charged Wilson lines and the magnetically charged 't Hooft (disorder) operators. The study of such operators in supersymmetric quantum field theories was initiated around ten years ago with the works [30, 33, 34] for $\mathcal{N}=4$ theories, [1, 30, 79, 80] for $\mathcal{N}=2$ theories, [42] in the context of Klebanov-Witten theory and very interestingly [12] in the context of higher CohFTs and Donaldson-Thomas theory, aspects of which we hope to return to in the future. A general and complete treatment of surface operators (mainly for $\mathcal{N}=2$ theories) is found in [32] along with many references within. The mathematical interest in these defects lies in the fact that in their presence it is possible to define the *ramified Donaldson invariants* associated to M [47].

In this chapter we take a fresh look at the ramified u-plane integral by adding to the Lagrangian of the ramified Donaldson-Witten theory the $\mathcal{Q}$-exact surface operator $\tilde{I}_+$ that couples to the self-dual part of the curvature of the (in an appropriate sense *extended*) gauge bundle, in the presence of surface defects. In accordance to Chapter 4, this $\mathcal{Q}$-exact insertion

[1] This terminology is due to [34]

[2] In an algebraic setting they correspond to genus g smooth algebraic curves.

allows us to write the ramified u-plane integral also as an integral of a total anti-holomorphic derivative of an indefinite theta function à la Zwegers [94].

We will see in this chapter that just like in the case of ordinary Donaldson-Witten theory, mock modular forms appear to play a similar role in the ramified theory generalizing the former elegantly. In Section 5.1 we review the notion of ramified Donaldson invariants and recall some of the definitions that appeared earlier for the ease of the reader. In Section 5.2 we recall how supersymmetric surface operators appear in $\mathcal{N}=4$ and mainly in $\mathcal{N}=2$ theories in four dimensions. In Section 5.3.1 we give a quick overview of the u-plane integral as it appears in the usual Donaldson-Witten theory. In Section 5.3 we describe in some detail how the u-plane is modified in the presence of the embedded surfaces, we include the $\mathcal{Q}$-exact deformation and show that the integral localizes at the cusp at infinity for specific manifolds of Kodaira dimension $-\infty$. In Section 5.4 we conclude with some remarks and some discussion. In appendix B.2 we discuss how surface operators correspond to lifts of the maximal torus to the Cartan subalgebra.

5.1 Ramified Donaldson invariants

In Chapter 4 we gave a brief review of the definition of the usual Donaldson invariants. In this section we will briefly review the notion of ramified Donaldson invariants following closely [47, 79]. Let M be a smooth, closed (compact without boundary) and simply connected four-manifold equipped with a Riemannian metric. Let $\mathscr{E}$ be a $G=SO(3)$ principal bundle over X that can be lifted to a $G=SU(2)$ principal bundle for $w_2(\mathscr{E})=0$.

$$\begin{array}{ccc} G & \longrightarrow & \mathscr{E} \\ & & \downarrow \\ & & M \end{array}$$

As in Chapter 4 we denote by $\mathfrak{g}=\mathrm{Lie}(G)$ and by $\mathfrak{t}=\mathrm{Lie}(\mathbb{T})$ the Cartan subalgebra, where $\mathbb{T}$ is the maximal torus of G. Let us recall that the middle integral cohomology is isomorphic to a lattice $\Lambda=\mathbb{Z}^{b_2}$ and splits into two orthogonal components $\mathbb{Z}^{b_2^+,0}\perp\mathbb{Z}^{0,b_2^-}$ where we have exactly $b_2=b_2^++b_2^-$. The lattice comes equipped with a uni-modular quadratic form $Q:H^2(M,\mathbb{R})\to\mathbb{R}$ and a bilinear form $B:H^2(M,\mathbb{R})\times H^2(M,\mathbb{R})\to\mathbb{R}$ (see definitions (2.77) and (4.12) in Chapter 3). By restricting only to integral classes in $H^2(M,\mathbb{Z})$ both the quadratic and bilinear forms are $\mathbb{Z}$ valued.

An embedded closed surface $\boldsymbol{S}\hookrightarrow M$ is a genus g complex curve[3] embedded into M. Therefore by $Q(\boldsymbol{S})$ we denote the self-intersection number of $\boldsymbol{S}$ and also by $\bar{M}=M\backslash\boldsymbol{S}$ we

[3]The genus of this curve can be determined by the adjunction formula $g(\boldsymbol{S})=1+\frac{1}{2}(\boldsymbol{S}^2+B(\boldsymbol{S},K_M))$.

denote the complement of $\boldsymbol{S}$. Near an open neighborhood of $\boldsymbol{S}$ we can split the vector bundle to a sum of complex line bundles over M as $\mathscr{E} = \mathscr{L} \oplus \mathscr{L}^{-1}$. We denote the curvature of a connection on $\mathscr{L}$ by $F_{\mathscr{L}}$. Let $A \in \Omega^1(\bar{M}, \mathrm{g})$ denote the local G-connection one-form which becomes singular as it approaches $\boldsymbol{S}$. Then locally this connection takes the form

$$A = \mathring{\imath}\, \alpha d\theta + \text{regular} \tag{5.1}$$

with $\alpha \in \sigma_3\mathbb{R}$ and $\sigma_3 = \mathrm{diag}(1,-1) \in \mathrm{t}$. The angular coordinate comes from $z = re^{\mathring{\imath}\theta}$ where z is a holomorphic coordinate normal to $\boldsymbol{S}$ and the connection is singular exactly as $z \to 0$. Due to the coordinate singularity we just discussed there exists a non-trivial gauge-invariant holonomy $\mathrm{Hol}_\gamma(A) = e^{-2\pi\alpha}$ of the connection contouring some small loop γ around $\boldsymbol{S}$. Note though that if $\mathrm{Hol}_\gamma(A)$ is trivial, such that γ is contractible, then the connection is an ordinary connection on $\mathscr{E}$ defined over M. These embedded surfaces are of interest since they will support supersymmetric surface operators.

Using the above we can define the so-called *ramified Donaldson invariants* for a four-manifold M with the presence of an embedded complex curve $\boldsymbol{S}$. The ramified Donaldson invariants $\widetilde{\mathscr{D}}$ are defined very similarly to the ordinary ones, that is they are polynomials on the homology of $\bar{M}$ with rational coefficients

$$\widetilde{\mathscr{D}} : \mathrm{Sym}[H_0(\bar{M},\mathbb{Q}) \oplus H_2(\bar{M},\mathbb{Q})] \to \mathbb{Q}. \tag{5.2}$$

Let $\mathscr{M}_{\tilde{\gamma}}$ denote the moduli space of ramified G-instantons with instanton number $\tilde{k} = \int_{\bar{M}} c_2(\mathscr{E})$ and $\tilde{\gamma} := \gamma(\mathscr{E})$. Also recall that the Euler characteristic of M is $\chi = \sum_i (-1)^i b_i$, the signature of M is $\sigma = b_2^+ - b_2^-$ while by $l = -\int_{\boldsymbol{S}} c_1(\mathscr{L})$ we denote the magnetic flux number.

The dimension of the moduli space of ramified G-instantons, that for brevity we denote by s, is [79]

$$s = 8k - \frac{3}{2}(\chi + \sigma) + 4l - 2(g-1) \tag{5.3}$$

where $k = -\int_M c_2(\mathscr{E})$ and we assume there are no reducible connections. Then, the corresponding degree s *ramified Donaldson invariant* is defined as

$$\widetilde{\mathscr{D}}^{\gamma,J}_{t,m}(p,\boldsymbol{x}) := \int_{\mathscr{M}_{\tilde{\gamma}}} \bar{\mu}(\boldsymbol{x})^m \cup \bar{\mu}(p)^t, \tag{5.4}$$

where J denotes a choice of polarization in $H^2(M,\mathbb{R})$, a period point. The ramified Donaldson invariants are the numbers $\widetilde{\mathscr{D}}_{t,m}$ and are given, like in the case of ordinary Donaldson invariants, through the intersection theory of $\mathscr{M}_{\tilde{\gamma}}$. In analogy to the ordinary case, there exists a universal ramified instanton bundle and a ramified slant product that defines Donaldson's map $\bar{\mu}_D$ such that

$$\bar{\mu}_D : H_i(\bar{M},\mathbb{Q}) \to H^{4-i}(\mathscr{M}_{\tilde{\gamma}},\mathbb{Q}). \tag{5.5}$$

The generating function of ramified Donaldson invariants is given by summing over all vector bundles $\mathscr{E} \to M$ for a fixed $\boldsymbol{\mu}$ and varying $\tilde{k}$:

$$\tilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = \sum_{\tilde{k}\geq 0} \sum_{\substack{m,t>0\\ 2m+4t=s}} \frac{\widetilde{\mathscr{D}}_{t,m}(p,\boldsymbol{x})}{t!m!}\alpha^t\beta^m, \tag{5.6}$$

where α and β are formal variables as in the ordinary case. This generating function corresponds physically to a specific correlation function of the ramified Donaldson-Witten theory.

The ramified Donaldson invariants also develop a metric dependence and wall-crossing phenomena: $\tilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x})$ jumps discontinuously when crossing a wall of marginal stability in the positive cone[4] of X and in general we have $\widetilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) \neq \widetilde{\Phi}^{J'}_{\boldsymbol{\mu}}(p,\boldsymbol{x})$ for two different period points $J,J' \in H^2(X,\mathbb{R})$. Therefore, strictly speaking, these functions are not quite smooth structure invariants but rather piecewise invariants. The difference of the ramified Donaldson polynomials between two period points belonging to different chambers is given by the wall-crossing formula that will be discussed in Section 5.3.

5.2 Review of surface operators in four dimensions

In this section we will briefly recall some well known facts about surface operators in $\mathscr{N}=2$ supersymmetric Yang-Mills theories. Surface operators in topological field theories first appeared in [33] where the authors consider the GL-twist of the $\mathscr{N}=4$ super Yang-Mills theory, the Kapustin-Witten topological field theory [39]. In order to set the stage and get some intuition about surface operators we begin by considering the non-compact four-manifold $\mathbb{C}^2 \cong \mathbb{C}_a \times \mathbb{C}_b$ spanning x_0,x_1,x_2,x_3 and consider a co-dimension two defect supported at $\mathbb{C}_a$ with coordinates x_0 and x_1 which is localized at $x_2=x_3=0$.

The bosonic fields on $\mathbb{C}_a$ are the two components of the original gauge field spanning $\mathbb{C}_a$, that is A_0,A_1 and four of the six scalars of the $\mathscr{N}=4$ multiplet. There is a two dimensional supersymmetric theory living on $\mathbb{C}_a$. Along $\mathbb{C}_b$ it is required that the normal components A_2,A_3 for the gauge field and ϕ_2,ϕ_3 for the scalars, have a suitable singular behavior as they approach $\mathbb{C}_a$. Supersymmetry then requires that $A=A_2dx^2+A_3dx^3$, $F=dA$ and

[4] See Subsection 5.3.6 for definition.

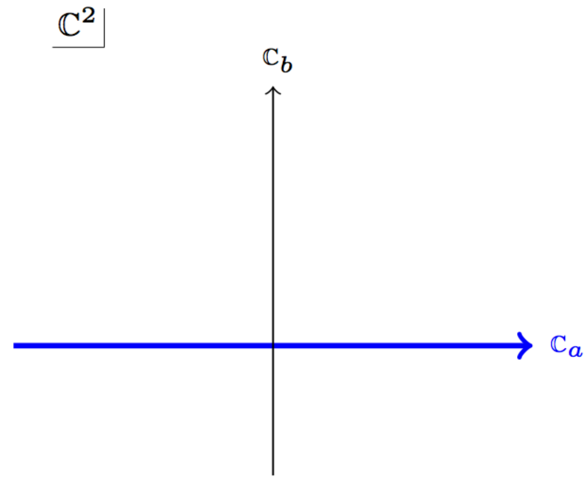

Figure 5.1 A surface defect supported on $\mathbb{C}_a \subset \mathbb{C}^2$.

$\phi = \phi_2 dx^2 + \phi_3 dx^3$ satisfy Hitchin's equations

$$\begin{aligned} F + \phi \wedge \phi &= 0, \\ d_A \phi &= 0, \\ d_A * \phi &= 0. \end{aligned} \tag{5.7}$$

Next we want to set $x_2 + \mathring{\imath} x_3 = z = re^{\mathring{\imath}\theta}$ and move to $\mathscr{N} = 2$ supersymmetry where we do not have all of the six scalar fields. Theories with $\mathscr{N} = 2$ supersymmetry in four dimensions admit half-BPS surface operators and the corresponding two-dimensional theory on $\mathbb{C}_a$ preserves $\mathscr{N} = (2,2)$ supersymmetry [24]. The BPS equations for the (untwisted) $\mathscr{N} = 2$ theory are $F = 0$ and $d_A \phi = 0$, where ϕ is the adjoint valued Higgs field. Let us consider the simple case where $\phi = 0$. Then the BPS configurations correspond to irreducible flat connections on X. Therefore any such surface operator is in a one-to-one correspondance with an irreducible flat connection on the vector bundle $\mathscr{E} \to M$ restricted on $\bar{M}$ which is singular along $\boldsymbol{S}$ or, in other words, a surface operator supported along $\boldsymbol{S} \hookrightarrow M$ corresponds to an irreducible flat connection on $\mathscr{E} \to \bar{M}$. The flatness condition is obvious by taking Hitchin's equations and letting ϕ to be trivial. The curvature of the connection should be vanishing on $\bar{M}$ but not on $\boldsymbol{S}$. Instead, we can consider an *extended* vector bundle $\mathscr{E}' \to M$ with connection one-form $\mathscr{A}$ such that the curvature two-form is given by $\mathscr{F} = F - 2\pi\alpha\delta_S$ with α interpreted as the electric charge of the surface operator. This bundle extension was first implemented in the context of the ramified geometric Langland's program in [33] where connection to parabolic Higgs bundles was described. The BPS condition[5] gives the flatness

[5]Note that for the twisted theory the BPS equation is $\mathscr{F}^+ = 0$ [48, Chapter 5].

condition

$$\mathscr{F} = 0 \tag{5.8}$$

or equivalently $F = 2\pi\alpha\delta_{\boldsymbol{S}}$. Since F is the two-form curvature we can split it in self-dual $F^+ = 2\pi\alpha\delta_{\boldsymbol{S}}^+$ and anti-self-dual $F^- = 2\pi\alpha\delta_{\boldsymbol{S}}^-$ parts where $\delta_{\boldsymbol{S}}$ denotes the Poincaré two-form dual to the homology class $\boldsymbol{S}$ supported on. Let us make a clarification of the notation we use. Note that $\boldsymbol{S}$ belongs to fundamental class of its own homology $H_2(\boldsymbol{S})$ while we denote by $\delta_{\boldsymbol{S}}$ the two-form Poincaré dual of the Dirac delta function supported on $\boldsymbol{S}$. As explained in detail in [33, 79, 80] in principle $\alpha \in \mathbb{T} = \mathfrak{t}/\Lambda_{\text{cochar.}}$ where $\Lambda_{\text{cochar.}} = \text{Hom}(\text{U}(1), \mathbb{T})$ is the co-character lattice [33]. This is not quite the case though because we have extended the bundle $\mathscr{E}$ to $\mathscr{E}'$ which amounts in lifting α to the Cartan subalgebra $\mathfrak{t}$ and there are many inequivalent such lifts yielding the same holonomy for A. Each possible lift corresponds to a different surface operator therefore. We include some discussion on the lifts of surface operators in Appendix B.2. Note that in this extended bundle $\mathscr{E}'$ we can include a theta-like angle term with a contribution to the path integral as

$$e^{i\eta\int_{\boldsymbol{S}} F}, \tag{5.9}$$

where the exponent measures the magnetic flux of $\mathscr{E}'$ through $\boldsymbol{S}$. Note that this integral is proportional to the monopole number, with $\eta \in \mathfrak{t}$ as well. In that sense the pair (α, η) corresponds to electric and magnetic charges of a dyon-like surface operator that we can interpret to be supported on $\boldsymbol{S}$. Mathematically such type of extensions of vector bundles can be described in the context of parabolic bundles [12, 61] which are very interesting objects on their own with connections to the Riemann-Hilbert problems or the Painvelé equations among others (for example see [37] for parabolic bundles over $\mathbb{CP}^1$). We hope to expand in this context in a future work on ramified Vafa-Witten theory (that is the theory in the presence of embedded divisors).

5.3 The ramified u-plane integral

5.3.1 Some facts about the u-plane integral

In this short subsection we recall some facts about the u-plane integral for the convenience of the reader. Most of the contents of this subsection have been discussed in detail in Chapter 2.

We begin with the classical moduli space of vacua of $\mathscr{N} = 2$ theories as well as their quantum version which are discussed in 2.1. We recall here that for rank one gauge algebras the coordinate ring of $\mathfrak{t}/W$ is identified with the equivariant cohomology of a point $H_G^*(pt)$

which has a single generator $u := \frac{1}{16\pi^2}\mathrm{Tr}(\phi^2)$ and the Coulomb branch $\mathscr{B}$ is a quantum lift of the classical moduli space of vacua defined through the moduli space of instantons [67, 68]. The natural coordinate in $\mathscr{B}$ is u that can now be seen as an element of the field of fractions of $H^*_G(pt)$. We described earlier that in Donaldson-Witten theory the partition function Z_{DW}, receives contributions from two different terms:

$$Z_{\mathrm{DW}} = Z_u + Z_{\mathrm{SW}}. \tag{5.10}$$

The left hand side of the equality above corresponds precisely to the generating function of Donaldson invariants. The right hand side is composed by Z_u, the u-plane integral, that we have already explained in detail, and Z_{SW} given by

$$Z_{\mathrm{SW}} = \sum_s Z(u_s), \tag{5.11}$$

where we sum contributions to Z_{DW} from all $s \in \mathscr{B}$ such that the discriminant of the Seiberg-Witten curve is zero, i.e., the Seiberg-Witten contributions. That is, the union of s is a divisor along which the elliptic fiber over $\mathscr{B}$, which is the Seiberg-Witten curve (2.17), becomes singular. Recall also that for four-manifolds with $b_2^+ = 1$ that admit a metric of positive scalar curvature these contributions Z_{SW} vanish [67] (see Section 2.4.1) but our result is independent of this fact.

We will shortly see that in the presence of surface defects the u-plane integral will be altered slightly, mainly due to the fact that we consider a different (extended) vector bundle than the one considered in Chapters 3 and 4 (also in [44, 45, 67, 79]). A detailed derivation of the ramified u-plane integral can be found in [79].

5.3.2 Analysis of the ramified u-plane integral

Naturally, the ramified u-plane integral is the path integral over the Coulomb branch $\mathscr{B}$ of the Donaldson-Witten gauge theory in the presence of surface defects with gauge group $SU(2)$ or $SO(3)$. Let us first, write down the Lagrangian of the low energy Donaldson-Witten theory in the presence of surface defects.

$$\begin{aligned}\mathscr{L} =&\frac{\mathring{i}}{16\pi}(\bar{\tau}F_+\wedge F_+ + \tau F_-\wedge F_-) + \frac{y}{8\pi}da\wedge *d\bar{a} - \frac{y}{8\pi}D\wedge *D\\ &-\frac{\tau}{16\pi}\psi\wedge *d\eta + \frac{\bar{\tau}}{16\pi}\eta\wedge d*\psi + \frac{\tau}{8\pi}\psi\wedge d\chi - \frac{\bar{\tau}}{8\pi}\chi\wedge d\psi\\ &+\frac{\sqrt{2}\mathring{i}}{16\pi}\frac{d\bar{\tau}}{d\bar{a}}\eta\chi\wedge(F_+ + D) - \frac{\sqrt{2}\mathring{i}}{2^7\pi}\frac{d\tau}{da}\psi\wedge\psi\wedge(F_- + D)\\ &+\frac{\mathring{i}\eta_{\mathrm{m}}}{4}\mathscr{F}\wedge\delta_{\boldsymbol{S}} + \frac{\mathring{i}}{3\pi\cdot 2^{11}}\frac{d^2\tau}{da^2}\psi\wedge\psi\wedge\psi\wedge\psi - \frac{\sqrt{2}\mathring{i}}{3\cdot 2^5\pi}\{\mathcal{Q},\chi_{mn}\chi^{nr}\chi_r{}^m\}\sqrt{g}\,d^4x.\end{aligned} \tag{5.12}$$

This Lagrangian is identical to the one of the usual Donaldson-Witten theory except for the term that is proportional to η_{m} which appears[6] due to the presence of the surface defect and the fact that we consider the field strengths $\mathscr{F}$ for connections on the extended vector bundle $\mathscr{E}$. Just like in the ordinary Donaldson-Witten theory, the ramified one also contains a BRST-like nilpotent scalar supercharge $\mathcal{Q}$ whose cohomology provides the physical observables of the theory. The supersymmetric algebra in the presence of the surface defect reads exactly as in (2.74) but with the substitutions

$$\begin{aligned} A &\to \mathscr{A},\\ F &\to \mathscr{F},\end{aligned} \tag{5.13}$$

with $\mathscr{A}$ the connection of the low energy $U(1)$ line bundle. The path integral of the theory is

$$Z_{\mathrm{rDW}} = \int[\mathscr{D}X]\, e^{-\int_M \mathscr{L}[\varphi]},$$

where by X we collectively denote all the fields of the theory. Nevertheless, what we are really interested in is the correlation function that reproduces physically the generating function of the ramified Donaldson invariants. This comes in a straight forward manner by following the analysis of Chapter 4. Therefore, we want include to our path integral the low energy surface observable $\widetilde{I}_-(\boldsymbol{x})$ from Equation (2.69) as well as the $\mathcal{Q}$-exact surface deformation $\widetilde{I}_+$ from Equation (2.71). In both operations we have to take into account substitutions (5.13). Due to the presence of the surface defect $\boldsymbol{S}$, the first operator gets an extra contribution. Therefore,

[6]There is a clash of notation here. In the Lagrangian by η we denote the Grassman valued zero-form of the theory and by η_{m} we denote the "magnetic charge" of the surface defect that we denoted as η earlier. We will eventually integrate out the zero-form η and thus we will be able to return to the previous notation for the magnetic charge.

overall it reads

$$\widetilde{I}_-(\boldsymbol{x},\boldsymbol{S}) = \frac{\mathring{\imath}}{\sqrt{2}\pi}\int_{\boldsymbol{x}}\left(\frac{1}{32}\frac{d^2u}{da^2}\psi\wedge\psi - \frac{\sqrt{2}}{4}\frac{du}{da}(\mathscr{F}_- + D)\right) + \frac{\mathring{\imath}}{2}\int_{\boldsymbol{S}}\alpha\frac{du}{da}\delta_{\boldsymbol{y}}^- . \tag{5.14}$$

Here $\boldsymbol{y}\in H_2(M)$ is an arbitrary cycle and $\delta_{\boldsymbol{y}}$ corresponds to its Poincaré dual in $H^2(M)$. The new contribution is $\mathcal{Q}$-invariant, as the first summand in (5.14) whose terms arise from the descent procedure. In the limit $\alpha\to 0$ we return to the theory without the surface operator. Furthermore, just like in the ordinary theory, the ramified theory develops a contact term operator dependence as well due to the UV to IR map

$$I_2(\boldsymbol{x})I_2(\boldsymbol{y}) \to \widetilde{I}_-(\boldsymbol{x})\widetilde{I}_-(\boldsymbol{y}) + \text{ contact term.} \tag{5.15}$$

Except for the term $\boldsymbol{x}^2G(u)$ we encountered in Equation (2.72) there exists also a term of the form $\boldsymbol{S}^2H(u)$ for a holomorphic function $H\in\mathscr{O}_{\mathbb{C}}$. In principle we should also include a term proportional to the intersection of $\boldsymbol{x}$ and $\boldsymbol{S}$. But, such a term would vanish since the ramified Donaldson invariants are defined for $\boldsymbol{x}\in H_2(\bar{X}) = H_2(X\backslash\boldsymbol{S})$. Because $\boldsymbol{x}$ is homologous to $\boldsymbol{y}$ we realize that as a result $\boldsymbol{y}\cap\boldsymbol{S} = \boldsymbol{x}\cap\boldsymbol{S} = 0.$ Taking all the above into account, as well as the fact we will be evaluating the theory on simply connected manifolds, $\pi_1(X)=0$, we conclude that the correlation function we want to evaluate is [79, equation 5.8]

$$\left\langle e^{2pu - \frac{\mathring{\imath}}{4\pi}\int_{\boldsymbol{x}}\frac{du}{da}(\mathscr{F}_- + D) + \boldsymbol{x}^2G(u) + \tilde{\boldsymbol{S}}^2H(u)}\right\rangle, \tag{5.16}$$

where $\tilde{\boldsymbol{S}} = \frac{\pi\mathring{\imath}\alpha}{2}\boldsymbol{S}$. We already recognize two contact terms, unlike in [44, 52, 58, 67] where there is only one, with the second one here being precisely due to the presence of the surface defect as we described previously. The contact terms are given by

$$G(u) = \frac{1}{24}\left(8u - E_2(\tau)\left(\frac{du}{da}\right)^2\right), \tag{5.17}$$

$$H(u) = up_2(u), \tag{5.18}$$

where the polynomial $p_2(u) = \sum_{n\in 2\mathbb{Z}_{\geq 1}} a_n u^{-n} \in \mathbb{Q}[u^{-1}]$ is chosen such that it vanishes at the classical limit, $\lim_{u\to\infty} p_2(u) = 0$, and $p_2(-u) = p_2(u)$. In [79] the author chooses to use the simplest term, u^{-2} with $a_2 = \frac{1}{4}$. We can leave this polynomial arbitrary for the purposes of this paper. Note that none of the contact terms $G(u)$ and $H(u)$ have any singularities at the SW points.

In the presence of the surface operator $\boldsymbol{S}$, it would seem natural though to add a second term, the one corresponding to the anti-holomorphic part of the last term in Equation (5.14). In order such an inclusion to not destroy the topological nature of the theory it has to be $\mathcal{Q}$-exact but such an operator cannot be constructed. As a result, our $\overline{\mathcal{Q}}$-exact "deformation" operator is

$$\widetilde{I}_+(\boldsymbol{x},\boldsymbol{S}) = -\frac{i}{\sqrt{2}\pi}\int_{\boldsymbol{x}}\left(\frac{1}{2}\frac{d^2\bar{u}}{d\bar{a}^2}\eta\chi + \frac{\sqrt{2}}{4}\frac{d\bar{u}}{d\bar{a}}(\mathscr{F}_+ - D)\right). \tag{5.19}$$

All in all, we see that the analysis is identical to the one of the unramified u-plane integral, the only differences being:

- that we have to consider the (extended) vector bundle $\mathscr{E}'$ instead of $\mathscr{E}$ by making a choice of a lift of α from $\mathbb{T}$ to $\mathfrak{t}$ (this point will become clear when we study the photon path integral),
- the presence of the additional contact term $H(u)$.

We are in position now to write down the precise form of the path integral together with the insertions that give a physical formulation of the generating function of the ramified Donaldson invariants. In analogy to Chapter 4 we switch notation and denote the Z_u, the u-plane integral, as $\Phi^J_{\boldsymbol{\mu}}$. Then, the correlation function we are interested in can be written as

$$\Phi^J_{\boldsymbol{\mu}} = \int [\mathscr{D}X]\; e^{-\int_M \mathscr{L}+\mathscr{O}[X]}, \tag{5.20}$$

where

$$\mathscr{O}[X] = 2pu + \boldsymbol{x}^2 G(u) + \tilde{\boldsymbol{S}}^2 H(u) + \widetilde{I}_-(\boldsymbol{x},\boldsymbol{S}) + \widetilde{I}_+(\boldsymbol{x},\boldsymbol{S}). \tag{5.21}$$

The way to proceed is identical to the one in 2.4.2. To this end, let us restrict to Kähler surfaces and denote the Kähler form by J and by $\underline{J}$ we denote its normalization with respect to the quadratic form. Also recall the variables

$$\boldsymbol{\rho} = \frac{\boldsymbol{x}}{2\pi}\frac{du}{da}, \tag{5.22}$$

which is a a modular form with weight $(1,0)$ under $\Gamma^0(4)$ as well as

$$\boldsymbol{b} = \frac{\mathrm{Im}(\boldsymbol{\rho})}{y}. \tag{5.23}$$

The real change compared to the usual Donaldson-Witten theory amounts to the photon path integral. The rest of the terms, modulo the new contact term, remain the same. Therefore let us discuss the photon path integral of the ramified theory which gives a Siegel-Narain theta function that reads

$$\begin{aligned}\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho};\alpha) = e^{-2\pi\tau_2 \boldsymbol{b}_+^2} & \sum_{\tilde{\boldsymbol{k}}\in\Lambda+\boldsymbol{\mu}} \partial_{\bar{\tau}}\left(\sqrt{2\tau_2}B(\tilde{\boldsymbol{k}}+\boldsymbol{b},\underline{J})\right)(-1)^{B(\tilde{\boldsymbol{k}},K_X)} \\ & \times \exp\Big(-\pi \mathring{i}\,\bar{\tau}\tilde{\boldsymbol{k}}_+^2 - \pi \mathring{i}\,\tau\tilde{\boldsymbol{k}}_-^2 - 2\pi \mathring{i}\,B(\tilde{\boldsymbol{k}}_+,\bar{\boldsymbol{\rho}}) - 2\pi i B(\tilde{\boldsymbol{k}}_-,\boldsymbol{\rho})\Big) \\ & \times \exp\Big(-2\pi \mathring{i}\,B(\tilde{\boldsymbol{k}},\frac{\eta}{2}\delta_{\boldsymbol{S}})\Big),\end{aligned} \tag{5.24}$$

Here $\tilde{\boldsymbol{k}} = [\mathscr{F}]/4\pi$, the flux of the extended bundle. It is related to the flux of the usual gauge bundle as

$$\tilde{\boldsymbol{k}} = \boldsymbol{k} - \frac{\alpha}{2}\delta_{\boldsymbol{S}}. \tag{5.25}$$

In general, unless we want to stress the dependence of the $\mathfrak{t}$-lift (choice of α), we will omit it from the functions it appears. It is straight forward to see the relation of the Siegel-Narain function for the ramified theory to the unramified one from Equation (4.42). The two are related simply as

$$\lim_{(\alpha,\eta)\to(0,0)} \tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}) = \Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}). \tag{5.26}$$

Note that the η that appears in (5.24) is the "magnetic charge" associated to α (denoted as η_{m} in the Lagrangian (5.12)) and not the Grassmann valued scalar field of course that has been integrated out.

Taking a closer look at $\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})$ and requiring that it has the correct modular behavior (the discussion of which we postpone for Subsection 5.3.4) in order the integrand of the ramified u-plane integral to be modular invariant, forces $\alpha \in \mathbb{Z}$ for $SO(3)$ gauge bundles and $\alpha \in 2\mathbb{Z}$ for $SU(2)$ gauge bundles. The last term in (5.24) is equal to one therefore and as a result of requiring that it has the correct modular properties, we pick specific allowed surface operators, that is pairs (α,η), shown in Figure 5.2 below.

5.3.3 The ramified u-plane integral

We can now write down the ramified u-plane integral $\tilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x})$ for the theory in the presence of surface defects.Taking into account the measure factor $\nu(\tau)$, the point and contact term operators associated to $p,\boldsymbol{x},\boldsymbol{S}$ and the Siegel-Narain theta function we discussed in the

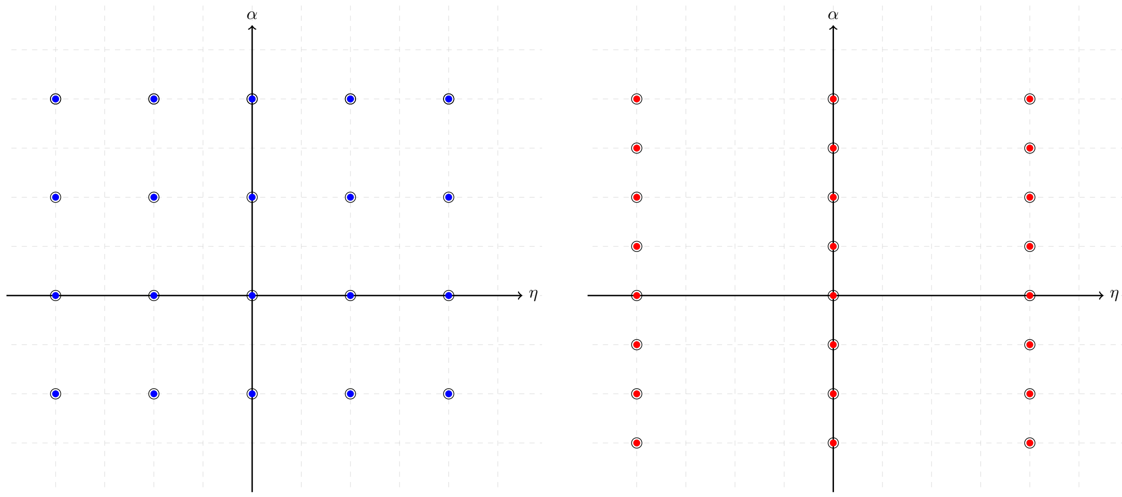

Figure 5.2 Electric and magnetic charges of the surface operator for gauge group $SU(2)$ in blue and $SO(3)$ in red. Both lattices are integer.

previous subsection we have

$$\tilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = \int_{\mathscr{B}} da\wedge d\bar{a}\;\nu(\tau)e^{2pu+\boldsymbol{x}^2G(u)+\tilde{\boldsymbol{S}}^2H(u)}\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}). \tag{5.27}$$

The integration domain of the integral above is the Coulomb branch $\mathscr{B}$ which is identified with $\mathbb{H}/\Gamma^0(4)\cong\mathbb{CP}^1\backslash\{\pm1,\infty\}$. It is more natural to make a coordinate transformation for the measure therefore and write it in terms of the complexified gauge coupling. Using the standard by now change of variables (2.97)

$$\tilde{\nu}(\tau) = \frac{da}{d\tau}\nu(\tau),$$

the ramified u-plane integral (5.27) takes the following convenient form of a modular integral

$$\tilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = \int_{\mathbb{H}/\Gamma^0(4)} d\tau\wedge d\bar{\tau}\;\tilde{\nu}(\tau)e^{2pu+\boldsymbol{x}^2G(u)+\tilde{\boldsymbol{S}}^2H(u)}\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}). \tag{5.28}$$

5.3.4 Modularity of the ramified u-plane integrand

In order the ramified u-plane integral (5.28) to make sense it has to be modular invariant under $\Gamma^0(4)$. In (5.28) the measure $d\tau\wedge d\bar{\tau}$ has modular weight $(-2,-2)$ under $\Gamma^0(4)$. As a result we have to require that the integrand transforms as a modular form of weight $(2,2)$ in order to obtain a single-valued quantity. Earlier, we characterized $\boldsymbol{\rho}$ as an elliptic variable and this is due to the fact that under the congruent subgroup at hand $\frac{du}{da}$ is a modular form of weight one. In specific, under an S transformation $\tau \mapsto \frac{\tau}{\tau+1}$ we have

$$\frac{du}{da} \mapsto \frac{1}{\tau+1}\frac{du}{da}, \tag{5.29}$$

as we also explained in Chapter 4. Using the standard properties of Jacobi theta functions we showed in [44] that $\boldsymbol{\rho}$ transforms under the two generators of $\Gamma^0(4)$ as (4.45)

$$\begin{aligned} \boldsymbol{\rho}(\tau+4) &= -\boldsymbol{\rho}(\tau) \\ \boldsymbol{\rho}\left(\frac{\tau}{\tau+1}\right) &= \frac{\boldsymbol{\rho}(\tau)}{\tau+1} \end{aligned}$$

The T transformation

In order to find the T transformation $\tau \mapsto \tau+4$ for $\Gamma^0(4)$ we will apply a $\tau \mapsto \tau+1$ transformation four times. We will also allow a generic shift $\boldsymbol{\mu} \mapsto \boldsymbol{\mu} + \frac{K_X}{2}$. Therefore we have

$$\tilde{\Psi}^J_{\boldsymbol{\mu}+\frac{K_X}{2}}(\tau+1,\boldsymbol{\rho}) = e^{\pi \imath(\boldsymbol{\mu}^2-\frac{K_X^2}{4})}\tilde{\Psi}^J_{\boldsymbol{\mu}+\frac{K_X}{2}}(\tau,\boldsymbol{\rho}+\boldsymbol{\mu}), \tag{5.30}$$

and repeating four times yields

$$\tilde{\Psi}^J_{\boldsymbol{\mu}+\frac{K_X}{2}}(\tau+4,\boldsymbol{\rho}) = e^{4\pi \imath(\boldsymbol{\mu}^2-\frac{K_X^2}{4})}\tilde{\Psi}^J_{\boldsymbol{\mu}+\frac{K_X}{2}}(\tau,\boldsymbol{\rho}+4\boldsymbol{\mu}). \tag{5.31}$$

We can get rid of the shift $\frac{K_X}{2}$ and also by noting that $B(\boldsymbol{k},4\boldsymbol{\mu}) \in 2\mathbb{Z}$ we finally obtain

$$\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau+4,\boldsymbol{\rho}) = e^{2\pi \imath B(\boldsymbol{\mu},K_X)}\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}). \tag{5.32}$$

Note here that we have not treated the transformation of $\boldsymbol{\rho}(\tau)$ yet. Since K_X is a characteristic vector, we have that $\boldsymbol{l}^2 + B(l,K_X) \in 2\mathbb{Z}$ for any vector $\boldsymbol{l} \in \Lambda$. Therefore, the exponential we see in Equation (5.32) can be written as

$$(-1)^{Q(2\boldsymbol{\mu})+B(\boldsymbol{\mu},K_X)-3B(2\boldsymbol{\mu},K_X)} = e^{-6\pi \imath B(\boldsymbol{\mu},K_X)}, \tag{5.33}$$

since $Q(2\boldsymbol{\mu}) + B(\boldsymbol{\mu},K_X) \in 2\mathbb{Z}$. As a result we can write (5.32) taking into account the transformation of $\boldsymbol{\rho}$ under $\Gamma^0(4)$ we obtain

$$\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau+4,-\boldsymbol{\rho}) = -\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}). \tag{5.34}$$

The S transformation

Similarly, for the S transformation we can first perform a $\tau \mapsto -\frac{1}{\tau}$ transformation and then generalize. We have therefore

$$\tilde{\Psi}^J_{\boldsymbol{\mu}+\frac{K_M}{2}}\left(-\frac{1}{\tau},\frac{\boldsymbol{\rho}}{\tau}\right) = -\mathring{\imath}(-\mathring{\imath}\tau)^{\frac{n}{2}}(i\bar{\tau})^2 e^{-\frac{\pi\mathring{\imath}\boldsymbol{\rho}^2}{\tau}+\pi\mathring{\imath}\frac{K_X^2}{2}}(-1)^{B(\boldsymbol{\mu},K_X)}\tilde{\Psi}^J_{\frac{K_X}{2}}(\tau,\boldsymbol{\rho}-\boldsymbol{\mu}), \tag{5.35}$$

and similarly as for the T transformation, by repeating four times this procedure we obtain

$$\tilde{\Psi}^J_{\boldsymbol{\mu}+\frac{K_X}{2}}\left(\frac{\tau}{\tau+1},\frac{\boldsymbol{\rho}}{\tau+1}\right) = (\bar{\tau}+1)^2(\tau+1)^{\frac{b_2}{2}}e^{-\frac{\pi\mathring{\imath}\boldsymbol{\rho}^2}{\tau+1}+\frac{\pi\mathring{\imath}\sigma(X)}{4}}\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}), \tag{5.36}$$

where we use the fact that for simply connected four-manifolds we have $K_M^2 = \sigma(X)+8$. Both the T and S transformations can be derived analogously to the transformations of Appendix A.1.5

The rest of the terms

As for the rest of the terms, that is the contact terms as well as the measure factor the analysis is identical to [44] without any modifications. The contact term transforms as

$$\begin{aligned} e^{\boldsymbol{x}^2 G(\tau+4)} &= e^{\boldsymbol{x}^2 G(\tau)}, \\ e^{\boldsymbol{x}^2 G(\frac{\tau}{\tau+1})} &= e^{\boldsymbol{x}^2 G(\tau)+\frac{\pi\mathring{\imath}}{\tau+1}\boldsymbol{\rho}^2}. \end{aligned} \tag{5.37}$$

Note that the function $H(u)$ is modular invariant and thus its transformations are trivial. As a result we see that indeed the integrand has the desired (2,2) weight under $\Gamma^0(4)$. This is in perfect agreement with the proof of Tan that the ramified u-plane integrand without the Ω insertion is indeed modular invariant. The fact that our integrand is modular invariant is no surprise since the insertion of the supersymmetric surface operator supported on $\boldsymbol{S}$ does not contribute to the modularity properties of the integrand.

5.3.5 The ramified u-plane integral as a total derivative

In [44, Section 4] and also in Section 4.7 we expressed the integrand of the u-plane integral $\Phi^J_{\boldsymbol{\mu}}(p,\boldsymbol{x})$ of the Donaldson-Witten theory without surface operators in terms of the total $\bar{\tau}$ derivative of a non-holomorphic function $\widehat{\mathscr{H}}$. This function is the modular completion of a mock modular form $\mathscr{H}(\tau)$ whose shadow is the Siegel-Narain theta function $\Psi^J_{\boldsymbol{\mu}}$. As we will see, these ideas can be applied for the ramified u-plane integral as well in a straight forward manner .

Recall the domain of integration $\mathbb{H}/\Gamma^0(4)$ which is the union of six images of the fundamental domain of $SL_2(\mathbb{Z})$. As we discussed in detail in Section 4.7 integrals over $\mathbb{H}/\Gamma^0(4)$ of modular invariant integrands of the form $d\tau \wedge d\bar{\tau}\ h(\tau,\bar{\tau})$ can be evaluated, in special cases in a quite straight forward way. These cases involve integrands that can be expressed as the total anti-holomorphic derivative to $\bar{\tau}$ of very specific function $\widehat{\mathscr{H}}$ with the property (4.54)

$$\frac{d}{d\bar{\tau}}\widehat{\mathscr{H}}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}) = \tilde{f}(\tau)\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}), \tag{5.38}$$

for some holomorphic multiplicative function $\tilde{f}(\tau)$ (that as we will see is almost identical to the one of the unramified theory). As in Chapter 4, in the case of the ramified theory $\widehat{\mathscr{H}}(\tau,\bar{\tau})$ is also a modular form of $(2,0)$ and it is the modular completion of some (holomorphic) mock modular form of weight $(b_2/2,0)$. As we have already seen, integrals such as the ramified u-plane integral (5.28) can be evaluated by relating them to integrals over $\mathscr{F}_\infty$ and as we will see they localize to the cusps of $\mathbb{H}/\Gamma^0(4)$ [44, Appendix C]. To ease the notation let us define

$$\tilde{f}(\tau) = \tilde{\nu}(\tau)e^{2pu+\boldsymbol{x}^2G(u)+\tilde{\boldsymbol{S}}^2H(u)}. \tag{5.39}$$

Then, the ramified u-plane integral can be written as

$$\tilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = \int_{\mathbb{H}/\Gamma^0(4)} d\tau \wedge d\bar{\tau}\ \tilde{f}(\tau)\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}). \tag{5.40}$$

Recall that for the theory with surface defects, the Siegel-Narain theta function is defined with respect to the gauge bundle $\mathscr{E}' \to X$ (it also contained an extra term that depends on $\boldsymbol{S}$ but we showed this term to be equal to one due to the fact that α has to be an integer for $\tilde{\boldsymbol{k}}$ to belong to the lattice). Using this fact we are now able to rewrite the integrand of (5.28) replacing $\tilde{\Psi}^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})$ with the anti-holomorphic derivative of the indefinite theta function

$$\begin{aligned}\widehat{\Theta}^{JJ'}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}) = \sum_{\boldsymbol{k}\in\Lambda+\mu} \tfrac{1}{2}\Big(E(\sqrt{2y}B(\tilde{\boldsymbol{k}}+\boldsymbol{b},\underline{J})) - \mathrm{sgn}(\sqrt{2y}B(\tilde{\boldsymbol{k}}+\boldsymbol{b},J'))\Big) \\ \times (-1)^{B(\tilde{\boldsymbol{k}},K_M)}q^{-\frac{\tilde{\boldsymbol{k}}^2}{2}}e^{-2\pi i B(\tilde{\boldsymbol{k}},\boldsymbol{\rho})}.\end{aligned} \tag{5.41}$$

Note that this function explicitly depends on the choice of lift of the connection to $\mathfrak{t}$ via α. As a result, the ramified u-plane integral (5.28) can be written as

$$\tilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x},\boldsymbol{S}) = \sum_i \oint_{\partial_i\mathscr{B}} du\ \left(\frac{d\tau}{du}\right)\widehat{\mathscr{H}}^J(\tau,\boldsymbol{\rho};\alpha), \tag{5.42}$$

with $i \in \{0, 2, i\infty\}$ and where the function $\widehat{\mathscr{H}}$ we were seeking reads for the ramified theory as follows (note that we stress the dependence on α),

$$\widehat{\mathscr{H}}^J(\tau, \boldsymbol{\rho}; \alpha) = \tilde{f}(\tau; \alpha)\, \widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau, \boldsymbol{\rho}; \alpha). \tag{5.43}$$

Therefore the ramified u-plane integral can be evaluated by the following formula

$$\tilde{\Phi}_{\boldsymbol{\mu}}^J(p, \boldsymbol{x}, \boldsymbol{S}) = 4\Big[\tilde{f}(\tau; \alpha)\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau, \boldsymbol{\rho}; \alpha)\Big]_{q^0} + \Big[\tau \to -\frac{1}{\tau}\Big]_{q^0} + \Big[\tau \to \frac{\tau}{\tau+4}\Big]_{q^0}. \tag{5.44}$$

At the $\tau \to i\infty$ limit the error function becomes the sign function and as a result we can substitute the non-holomorphic indefinite theta function $\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau, \boldsymbol{\rho})$ with the holomorphic indefinite theta function $\Theta_{\boldsymbol{\mu}}^{JJ'}(\tau, \boldsymbol{\rho})$. See appendix A.3 and [44, Appendix B] for details. Note that we have written these functions in italics in order to stress that they are (slightly) different that the functions defined in [44] due to the extended gauge bundle and the dependence on α and $\boldsymbol{S}$. For four-manifolds which admit a metric of positive scalar curvature formula (5.44) reduces to just the first summand which in turn completely determines the ramified Donaldson invariants. For example, this is valid for specific examples of Kähler surfaces of Kodaira dimension $-\infty$ that (in addition) are simply connected. Such surfaces include the Hirzebruch surfaces $\mathbb{F}_l$, the projective plane $\mathbb{CP}^2$ and some blow-ups of it. Actually, even for the computation of the usual Donaldson invariants for $\mathbb{CP}^2$ for which $b_2 = 1$ and the class of the period point is proportional to the hyperplane class H, as we saw in Section 4.7.3 one needs to use the blow up $\widehat{\mathbb{CP}^2}$ in order to apply the indefinite theta functions in the evaluation of the u-plane integral and of course this is also true for the ramified theory with the embedded surfaces. For four-manifolds that do not satisfy this criterion we should also take into account the contributions $Z_{\rm SW}$ from the Seiberg-Witten points of the Coulomb branch $\mathscr{B}$ as we mentioned earlier and these contributions were derived in [79].

5.3.6 Wall-crossing formula

It will not come as a surprise that the ramified Donaldson invariants for four-manifolds with $b_2^+ = 1$ are only piece-wise invariants [47, 79]. This means that they invariants jump discontinuously as we move across *walls* that divide the space of self-dual two-forms into various chambers. In each of those chambers Donaldson invariants are constant under smooth variations of the metric. The wall-crossing formula was derived in the context of the u-plane integral in [67]. Note that similar behaviour has recently been observed for the u-plane integral of the AD3 theory [66]. Let us explain the wall-crossing formula slightly better in order to complement the last paragraph of Section 4.7.1.

The wall-crossing formula prescribes this discontinuous change of $\tilde{\Phi}^J_{\boldsymbol{\mu}}$ under the variation of a metric with period point $J_0 \in H^2(X,\mathbb{R})$ to another metric with period point $J_1 \in H^2(X,\mathbb{R})$. If these two period points belong to the same chamber the result vanishes of course. Let us recall that a wall is defined as follows. First we consider the "forward" positive cone $V_+ := \{J \in H^2(X,\mathbb{R}) | Q(J) > 0\}$. Then, any $\xi \in H^2(X,\mathbb{Z})$ such that $Q(\xi) < 0$ defines a wall in V_+ by

$$W_\xi := \{J \in V_+ \mid B(\xi, J) = 0\}. \tag{5.45}$$

The complement of the walls in the positive cone are the chambers. Due to the presence of the surface defect the walls are defined as follows for the theory with the defects

$$W_{\tilde{\boldsymbol{k}};\alpha} := \{J \in V_+ \mid B(\tilde{\boldsymbol{k}}, J) = 0\}, \tag{5.46}$$

and when comparing to the unramified theory this tells us that the walls are shifted in $H^2(X,\mathbb{R})$ and this shifting explicitly depends on the choice of lift of the maximal torus $\mathbb{T}$ to $\mathtt{t}$ via α. Using the same argumentation as in [44], but for the ramified theory, the difference of the Coulomb branch between two neighboring chambers is given by a term $\Delta\tilde{\Phi}^{J_1J_0}_{\boldsymbol{\mu}} = \tilde{\Phi}^{J_1}_{\boldsymbol{\mu}} - \tilde{\Phi}^{J_0}_{\boldsymbol{\mu}}$ which reads

$$\Delta\tilde{\Phi}^{J_1J_0}_{\boldsymbol{\mu}} = \int_{\mathbb{H}/\Gamma^0(4)} d\tau \wedge d\bar{\tau}\, \tilde{f}\,(\tilde{\Psi}^{J_1}_{\boldsymbol{\mu}} - \tilde{\Psi}^{J_0}_{\boldsymbol{\mu}}), \tag{5.47}$$

with the contribution from the cusp at $i\infty$ giving the following result

$$\Delta\tilde{\Phi}^{J_1J_0}_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = 4\Big[\tilde{\nu}(\tau) e^{2pu+\boldsymbol{x}^2G(u)+\tilde{\boldsymbol{S}}^2H(u)}\widehat{\Theta}^{J_1J_0}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho};\alpha)\Big]_{q^0}. \tag{5.48}$$

This can be seen as the difference of the ramified u-plane integral for two metrics corresponding to J_0 and J_1. Note that here both J_0 and J_1 are period points in $H^2(X,\mathbb{R})$ and as a result the indefinite theta function $\widehat{\Theta}$ contains an error function for both J_0 and J_1 (see Equation A.24). It is trivial to show that this formula reduces to formula (4.11) of [44] in the limit $(\alpha,\eta) \to 0$. Finally note that in [79] it is also shown that the wall-crossing formula of $Z_u = \Phi^J_{\boldsymbol{\mu}}$ for the SW points $+1$ and -1 of the Coulomb branch $\mathscr{B}$ cancel the contribution that can arise from the call-crossing of the ramified Seiberg-Witten invariants Z_{SW}.

5.4 Discussion and summary

In this chapter we have given a fresh look on the determination of the u-plane integral of the Donaldson-Witten theory on a four-manifold M in the presence of a surface defect $\boldsymbol{S}$ inspired by [44, 67, 79].

We considered the insertion of a $\mathcal{Q}$-exact surface operator $\widetilde{I}_+$ to the path integral of the low energy effective theory as in Chapter 4. This operator couples to the self-dual part of the curvature $\mathscr{F}$ of the extended bundle $\mathscr{E}' \to X$. After some manipulations the ramified u-plane integral localizes to the cusps of the Coulomb branch and as a result of our considerations, the determination of the ramified u-plane integral simplifies drastically since there is no need to use the cumbersome techniques of lattice reduction. The modularity of the integrand is preserved, as expected, and computation of the ramified Donaldson invariants follows from a very simple formula. We stress that this comes in parallel to what was found in Chapter 4 where we showed that the usual Donaldson invariants can be obtained by a very similar simple formula as well and at the limit of vanishing volume for the embedded surface our result reduces to the one of the usual Donaldson-Witten theory. Therefore Chapters 4 and 5 have shown that the relation between Donaldson-Witten theory and the theory of indefinite theta functions and mock modular forms is much stronger and deeper than what it was initially thought after the publication of the fundamental papers [26, 27]. More generally, mock modular forms arise in increasing frequency in physical theories and appear to be of importance in low dimensional topology.

Returning to the u-plane integral, we would like to mention that it should not be so astonishing or surprising that a localization formula such as the one of Equation (5.42) appears in this context. This is a very generic feature of topological gauge theories. Similar integrals have often appeared in the literature for such theories.

An interesting direction to go forward would be to consider relating ramified u-plane integrals and indefinite theta functions for theories with higher rank gauge groups where the duality group lifts to $Sp(2r,\mathbb{Z})$ where r denotes the dimension of the Coulomb branch $\mathscr{B}$. We already presented a (far from complete) discussion towards this direction, for the usual Donaldson-Witten theory, at Section 4.8.

Another direction worth of investigating is to generalize the results of this paper for four-manifolds that are not simply connected. Four-manifolds of the form $\mathbb{R} \times Y$ (where Y is a three-manifold), and $\mathbb{R}^2 \times \Sigma_g$ (where Σ_g is a genus g Riemann surface) are of particular interest since they could relate our result, and especially mock modular forms, to the instanton Floer homology of Y and the quantum cohomology of the moduli space of flat connection $\mathscr{M}_{\text{flat}}$ of Σ_g respectively. See [30] for a discussion of Donaldson-Witten theory and its reductions to such four-manifolds and the connections to surface operators.

These tools, although computationaly strong, are not expected to yield some new information about four-manifolds, at least not directly for "conventional" operators and/or "conventional" theories. The use of indefinite theta functions though, and mock modular forms more generally, in the world of topological gauge theories, might be useful towards finding new four-manifold invariants, for example by studying topological versions of superconformal theories [66] and even topological class-$\mathscr{S}$ theories. It is of great curiosity of ours to see if such tools can somehow be employed in the very much unexplored world of (maybe topological) $\mathscr{N}=3$ theories and if such theories can provide any alternate roots for four-manifold invariants. Nevertheless, simplifying the evaluation of such Coulomb branch integrals, especially from the point of view of supersymmetric gauge theories, is quite important regardless of the mathematical problem of finding new invariants.

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Appendix A

Modular forms and theta functions

A.1 Modular forms and theta functions

In this appendix we would like to collect some important notions from the theory of modular forms for the convenience of the reader. For a comprehensive exposition the reader is reffered to the plethora of available literature such as [38, 76, 92].

A.1.1 Modular groups

The modular group $SL_2(\mathbb{Z})$, is the group of integer matrices with unit determinant

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z};\, ad-bc=1 \right\}. \tag{A.1}$$

which acts naturally on the Lobachevsky or upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau+b}{c\tau+d}.$$

We introduce moreover the congruence subgroup $\Gamma^0(n)$ which is defined as

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \middle| b = 0 \bmod n \right\}. \tag{A.2}$$

We say that a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k for any congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ if for any $\gamma \in \Gamma$ it satisfies

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \tag{A.3}$$

and it is holomorphic at the cusp at infinity $\tau \to \mathring{\imath}\infty$. In the following subsections we define various kinds of modular forms. There also exist mixed modular forms that are functions $f : \mathbb{H} \times \bar{\mathbb{H}}$ which transform as

$$f\left(\frac{a\tau + b}{c\tau + d}, \frac{a\sigma + b}{c\sigma + d}\right) = (c\tau + d)^k (c\sigma + d)^l f(\tau, \sigma). \tag{A.4}$$

The space of mixed modular forms for a modular subgroup Γ is denoted as $\mathbb{M}_{(k,l)}(\Gamma)$.

A.1.2 Eisenstein series

We let $\tau \in \mathbb{H}$ and define $q = e^{2\pi \mathring{\imath} \tau}$. Then the Eisenstein series $E_k : \mathbb{H} \to \mathbb{C}$ for even $k \geq 2$ are defined as the q-series

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n, \tag{A.5}$$

with $\sigma_k(n) = \sum_{d|n} d^k$ the divisor sum. For $k \geq 4$, E_k is a modular form of $SL_2(\mathbb{Z})$ of weight k. In other words, it transforms under $SL_2(\mathbb{Z})$ as

$$E_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k E_k(\tau). \tag{A.6}$$

On the other hand E_2 is a quasi-modular form, which means that although it is a holomorphic function in the upper-half plane, the $SL_2(\mathbb{Z})$ transformation of E_2 includes a shift in addition to the weight for any $\tau \in \mathbb{H}$,

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6\mathring{\imath}}{\pi} c(c\tau + d). \tag{A.7}$$

Eisenstein series $E_4(\tau)$ and $E_6(\tau)$ are somewhat special since they generate the ring of modular forms of $SL_2(\mathbb{Z})$. On the other hand the ring of quasi-modular forms is generated by $E_2(\tau), E_4(\tau)$ and $E_6(\tau)$.

A.1.3 Dedekind eta function

The Dedekind eta function $\eta : \mathbb{H} \to \mathbb{C}$ is defined as

$$\begin{aligned}\eta(\tau) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \\ &= q^{\frac{1}{24}} (q)_{\infty}.\end{aligned} \tag{A.8}$$

It is a modular form of weight $\frac{1}{2}$ under $SL_2(\mathbb{Z})$ with a non-trivial multiplier system. It transforms under the generators of $SL_2(\mathbb{Z})$ as

$$\begin{aligned}\eta(-1/\tau) &= \sqrt{-i\tau}\,\eta(\tau), \\ \eta(\tau+1) &= e^{\frac{\pi i}{12}}\,\eta(\tau).\end{aligned} \tag{A.9}$$

A.1.4 Jacobi theta functions

The classical Jacobi theta functions $\vartheta_j : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$, $j = 1, \ldots, 4$, are defined as

$$\begin{aligned}\vartheta_1(\tau, v) &= i \sum_{r \in \mathbb{Z}+\frac{1}{2}} (-1)^{r-\frac{1}{2}} q^{r^2/2} e^{2\pi i r v}, \\ \vartheta_2(\tau, v) &= \sum_{r \in \mathbb{Z}+\frac{1}{2}} q^{r^2/2} e^{2\pi i r v}, \\ \vartheta_3(\tau, v) &= \sum_{r \in \mathbb{Z}} q^{r^2/2} e^{2\pi i r v}, \\ \vartheta_4(\tau, v) &= \sum_{r \in \mathbb{Z}} (-1)^r q^{r^2/2} e^{2\pi i r v}.\end{aligned} \tag{A.10}$$

We let $\vartheta_j(\tau, 0) = \vartheta_j(\tau)$ for $j = 2, 3, 4$. Their transformations under the generators of $\Gamma^0(4)$ are

$$\begin{aligned}\vartheta_2(\tau+4) &= -\vartheta_2(\tau), & \vartheta_2\left(\frac{\tau}{\tau+1}\right) &= \sqrt{\tau+1}\,\vartheta_3(\tau), \\ \vartheta_3(\tau+4) &= \vartheta_3(\tau), & \vartheta_3\left(\frac{\tau}{\tau+1}\right) &= \sqrt{\tau+1}\,\vartheta_2(\tau), \\ \vartheta_4(\tau+4) &= \vartheta_4(\tau), & \vartheta_4\left(\frac{\tau}{\tau+1}\right) &= e^{-\frac{\pi i}{4}}\sqrt{\tau+1}\,\vartheta_4(\tau).\end{aligned} \tag{A.11}$$

A.1.5 Siegel-Narain theta functions

Siegel-Narain theta functions form a large class of theta functions for indefinite theta lattices that only depend on the lattice data. The classical Jacobi theta functions of Section A.1.4 are special cases of Siegel-Narain theta functions. We restrict to indefinite theta lattices of signature $(1,n-1)$, that is Lorentzian lattices, equipped with a bi-linear form $B(\boldsymbol{k},\boldsymbol{l})$ for $\boldsymbol{k},\boldsymbol{l}\in\Lambda$ and a quadratic form $Q(\boldsymbol{k})=B(\boldsymbol{k},\boldsymbol{k})=\boldsymbol{k}^2$ (compare also to Equations (4.14) and (4.12)). Furthermore we denote by K the characteristic vector of the lattice Λ such that for any vector $\boldsymbol{k}\in\Lambda$ we have $Q(\boldsymbol{k})+B(\boldsymbol{k},K)\in 2\mathbb{Z}$. Then, given an element $J\in\Lambda\otimes\mathbb{R}$ with positive norm, $Q(J)>0$, it is possible to decompose the space $\Lambda\otimes\mathbb{R}$ to a positive definite subspace $\Lambda_+=\mathrm{span}\{J\}$ as well as an orthogonal to it negative subspace Λ_-. The normalization of J is defined as $\underline{J}:=\frac{J}{Q(J)}$ and we can use it to define projection of an arbitrary vector $\boldsymbol{k}$ to the positive and negative definite subspaces of Λ as

$$\begin{aligned}\boldsymbol{k}_+ &:= B(\boldsymbol{k},\underline{J})\underline{J},\\ \boldsymbol{k}_- &:= \boldsymbol{k}-\boldsymbol{k}_+.\end{aligned} \tag{A.12}$$

With the definitions given above, the Siegel-Narain theta function that is of interest to the present thesis and has appeared a few times in the main text is a map $\Psi^J_{\boldsymbol{\mu}}:\mathbb{H}\times\mathbb{C}\to\mathbb{C}$. The second argument of the map is an *elliptic variable*. For a J as the one discussed previously and for a conjugacy class $\boldsymbol{\mu}\in\Lambda\otimes\mathbb{R}$ the Siegel-Narain theta function reads

$$\begin{aligned}\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}](\tau,\bar\tau,z,\bar z) = \sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}} &\mathscr{K}(\boldsymbol{k})(-1)^{B(\boldsymbol{k},K)}q^{-\frac{\boldsymbol{k}_-^2}{2}}\bar q^{\frac{\boldsymbol{k}_+^2}{2}}\\ &\times e^{-2\pi i B(\boldsymbol{k}_-,z)-2\pi i B(\boldsymbol{k}_+,\bar z)},\end{aligned} \tag{A.13}$$

where $\boldsymbol{\mu}=\Lambda/2$ and $\mathscr{K}:\Lambda\to\mathbb{C}$ is a summation kernel. The modular properties of $\Psi^J_{\boldsymbol{\mu}}[\mathscr{K}]$ depend on the kernel $\mathscr{K}$ (as we have seen in Chapters 4 and 5). For the trivial kernel $\mathscr{K}=1$ the transformations under $SL_2(\mathbb{Z})$ transformations are

$$\begin{aligned}\Psi^J_{\boldsymbol{\mu}+K/2}[1](\tau+1,\bar\tau+1,z,\bar z) &= e^{\pi i(\boldsymbol{\mu}^2-K^2/4)}\Psi_{\boldsymbol{\mu}+K/2}[1](\tau,\bar\tau,z+\boldsymbol{\mu},\bar z+\boldsymbol{\mu}),\\ \Psi^J_{\boldsymbol{\mu}+K/2}[1](-1/\tau,-1/\bar\tau,z/\tau,\bar z/\bar\tau) &= (-i\tau)^{\frac{n-1}{2}}(i\bar\tau)^{\frac{1}{2}}\exp(-\pi i z^2/\tau+\pi i K^2/2)\\ &\quad\times(-1)^{B(\boldsymbol{\mu},K)}\Psi^J_{K/2}[1](\tau,\bar\tau,z-\boldsymbol{\mu},\bar z-\boldsymbol{\mu}).\end{aligned} \tag{A.14}$$

For the case of the partition function in Chapter 3, we set the elliptic variables $z,\bar z$ to zero. Using the above $SL_2(\mathbb{Z})$ transformations and Poisson resummation we can verify that $\Psi^J_{\boldsymbol{\mu}}[1]$

is a modular form for the congruence subgroup $\Gamma^0(4)$. The transformations under the generators of this group read

$$\begin{aligned}&\Psi^J_{\boldsymbol{\mu}}[1]\left(\frac{\tau}{\tau+1},\frac{\bar\tau}{\bar\tau+1}\right)=(\tau+1)^{\frac{n-1}{2}}(\bar\tau+1)^{\frac{1}{2}}\exp\left(\tfrac{\pi \mathring{\imath}}{4}K^2\right)\Psi^J_{\boldsymbol{\mu}}[1](\tau,\bar\tau),\\&\Psi^J_{\boldsymbol{\mu}}[1](\tau+4,\bar\tau+4)=e^{2\pi \mathring{\imath} B(\boldsymbol{\mu},K)}\Psi_{\boldsymbol{\mu}}[1](\tau,\bar\tau),\end{aligned}\tag{A.15}$$

where we have set $\boldsymbol{z}=\bar{\boldsymbol{z}}=0$. Transformations for other kernels appearing in the main text can be easily determined from these expressions.

A.2 Mock modular forms

In this subsection we will briefly introduce the notion of mock modular forms and for the shake of simplicity and brevity we will restrict to the modular group $SL_2(\mathbb{Z})$. A beautiful exposition can be found at [93].

A central element in the definition of a mock modular form is the notion of the *shadow map*. This is a map whose argument is a non-holomorphic modular form $h:\mathbb{H}\times\bar{\mathbb{H}}\to\mathbb{C}$ of mixed weight $(\ell,2-k+\ell)$. In other words, under the generators of $SL_2(\mathbb{Z})$, h transforms as following

$$h\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar\sigma+b}{c\sigma+d}\right)=(c\tau+d)^{\ell}(c\sigma+d)^{2-k+\ell}h(\tau,\sigma).\tag{A.16}$$

Then the shadow map is defined as a map that send h to the non-holomorphic period integral h^* which can be written as

$$h^*(\tau,\bar\tau)=-2^{1-k}\mathring{\imath}\int_{-\bar\tau}^{\mathring{\imath}\infty}\frac{h(\tau,-v)}{(-\mathring{\imath}(\tau+v))^{k-\ell}}dv.\tag{A.17}$$

What it crucial about this function is that it almost transforms as a modular form of weight k (so it is holomorphic) under $SL_2(\mathbb{Z})$ but this transformation produces a shift by a holomorphic period integral (which is the reason of the lack of full modularity)

$$h^*\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar\tau+b}{c\sigma+d}\right)=(c\tau+d)^k\left(h^*(\tau,\bar\tau)+2^{1-k}\mathring{\imath}\int_{\frac{d}{c}}^{\mathring{\imath}\infty}\frac{h(\tau,-v)}{(-\mathring{\imath}(\tau+v))^{k-\ell}}dv\right).\tag{A.18}$$

Using the shadow maps of such functions we can define the completion and the shadow of a mock modular form. Let f be some holomorphic q-series which itself also transforms almost as a modular form. The sum

$$\widehat{f}(\tau,\bar\tau)=f(\tau)+g^*(\tau,\bar\tau),\tag{A.19}$$

is called the *completion* of f if and only if $\widehat{f}$ is a (non-holomorphic) modular form of weight k for some $k \in \mathbb{Z}$. Then, the function h^* is called the *shadow* of f. As we have indirectly seen in chapter 4 and 5, the non-holomorphic shadow h can be obtained by acting with $y^{k-\ell}\partial_{\bar{\tau}}$ on $\hat{f}$,

$$y^{k-\ell}\partial_{\bar{\tau}}\widehat{f}(\tau,\bar{\tau}) = g(\tau,\bar{\tau}). \tag{A.20}$$

Of course, most of q series do not have such modular completions because their modular transformations do not necessarily get eliminated by the contribution of some h^* transformation.

With the definitions above we define a *mock modular form* as a q-series f whose shadow $h \in \mathbb{M}_\ell \otimes \overline{\mathbb{M}}_{2-k+\ell}$ necessarily factors in the form

$$h = h_1\bar{h}_2. \tag{A.21}$$

Here h_1 is strictly a weight ℓ holomorphic modular for and h_2 is a weight $(2-k+\ell)$ holomorphic modular form. Mock modular forms can be split to the two classes shown below.

- Pure mock modular forms are mock modular forms with $\ell = 0$ and $h_1 =$ constant.
- Mixed mock modular forms are mock modular forms with arbitrary ℓ and $h_1 \neq= h_1(\tau)$.

There is also the notion of *depth* of a mock modular form f which is an integer equal to the number of period integrals involved in the transformation properties of f. For example a depth zero mock modular form coincides with a usual modular form while a depth one mock modular form fall into the discussion above. There exist higher depth mock modular forms but we will not discuss them in this thesis as they do not play a role in the rank one theories.

Role of mock modular forms in physics

As we have explained earlier mock modular forms appear everywhere in theoretical physics. Most prominently, mock modular forms started acquiring attention by the physics community with the discovery of Mathieu moonshine [21]. In this reference that authors noticed that a topological invariant of the $K3$ surface called the *elliptic genus* (which can be thought of a generalization of the Witten index) admits a character decomposition of the thesuperconformal algebra of the $\mathcal{N} = (4,4)$ CFT defined on $K3$ with the property that the degeneracy of the massive states of the theory coincides with simple combinations of irreducible representations of the Mathieu group M_{24}, the second largest of the sporadic finite groups. This observation can be interpreted as a SCFT living on $K3$ whose symmetry group is M_{24}. Then it was suggested by Cheng et. al. [11], in analogy to the McKay-Thompson series that the elliptic genus actually is a mock modular form. After these observations the literature has grown towards finding CFTs with other sporadic groups as symmetry groups.

Another example, is of course, the partition function of the $\mathcal{N}=4$ topologically twisted SYM theory with $G=SU(2)$ and $SO(3)$, the Vafa-Witten theory on $\mathbb{CP}^2$. Although when it was discovered the theory of mock modular forms was not known, by now we know the nature of these partition functions which are given in terms of mock modular forms, their modularity properties and even that higher rank Vafa-Witten theory gives rise to higher-depth mock modular forms [56].

In this thesis we saw how mock modular forms appear in the context of Donaldson-Witten theory (via the indefinite theta functions to be discussed analytically below). There are many other instances that mock modular forms appear. All the above should convince us on the deep role that mock modular forms have in theoretical physics and should encourage further investigation.

A.3 Indefinite theta functions

Indefinite theta functions (sometimes also called *indefinite theta series*) are theta functions associated to an indefinite lattice Λ. Such functions are special cases of mock modular forms, as mentioned in the introduction, and they have been getting a lot of attention since Zwegers' fundamental thesis [92] (for a very recent exposition see [7]). The relation of indefinite theta functions to the usual theta series (like the classical Jacobi theta functions) is very similar to the relation between mock modular forms and classical modular forms (see [56, Section 3.3] for details). For our purposes we specialize to unimodular latices of signature $(1,n-1)$. It is clear that for such a lattice there will exist vectors that have negative definite norm and the sum, which can be divergent in general, schematically will read as

$$\sum_{\boldsymbol{v}\in\Lambda} q^{-\pi y\frac{Q(\boldsymbol{v})}{2}}. \tag{A.22}$$

Therefore we need to somehow regularize the sum such that we get a convergent series. This is done by summing only positive definite vectors with the ceveat that the series loses its modularity properties. For the purposes of this paper, and for the quadratic form Q and bilinear form B, as well as $J,J'\in\Lambda\otimes\mathbb{R}$ such that $B(J,J')>0$, $\underline{J}$ is the normalization of J, $\tau\in\mathbb{H}$, K a characteristic vector for Λ, $\boldsymbol{z}\in\Lambda\otimes\mathbb{C}$, $\boldsymbol{\mu}\in\Lambda\otimes\mathbb{R}$ and $\boldsymbol{b}=\frac{1}{y}\mathrm{Im}(\boldsymbol{z})\in\Lambda\otimes\mathbb{R}$, the indefinite theta series is defined as

$$\Theta_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{z}) := \sum_{\boldsymbol{\nu}\in\Lambda+\boldsymbol{\mu}} \frac{1}{2}\Big\{\mathrm{sgn}(B(\boldsymbol{\nu}+\boldsymbol{b},\underline{J})) - \mathrm{sgn}(B(\boldsymbol{\nu}+\boldsymbol{b},\underline{J}'))\Big\} \times (-1)^{B(\boldsymbol{\nu},K)} q^{-\frac{Q(\boldsymbol{\nu})}{2}} e^{-2\pi \mathring{\imath} B(\boldsymbol{\nu},\boldsymbol{z})}. \tag{A.23}$$

This sum is convergent but not modular therefore [94]. Still, modular properties can be recovered by including modifying slightly the kernel $\mathrm{sgn}(B(\boldsymbol{\nu}+\boldsymbol{b},J)) - \mathrm{sgn}(B(\boldsymbol{\nu}+\boldsymbol{b},J'))$ which amounts to adding to it some non-holomorphic terms. As it is explained in full detail in [93, 94] there exists a modular completion $\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}$ of $\Theta_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{z})$. This amounts to substituting the sign functions of (A.23) with rescaled error functions. The completion reads

$$\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{z}) := \sum_{\boldsymbol{\nu}\in\Lambda+\boldsymbol{\mu}} \frac{1}{2}\Big\{E(B(\boldsymbol{\nu}+\boldsymbol{b},\underline{J})) - E(B(\boldsymbol{\nu}+\boldsymbol{b},\underline{J}'))\Big\} \times (-1)^{B(\boldsymbol{\nu},K)} q^{-\frac{Q(\boldsymbol{\nu})}{2}} e^{-2\pi \mathring{\imath} B(\boldsymbol{\nu},\boldsymbol{z})}, \tag{A.24}$$

where, as explained in Section 5.3.5 as well, the (rescaled) error function is the map $E : \mathbb{R} \to [-1,1]$ and it is defined as

$$E(u) = 2\int_0^u e^{-\pi t^2} dt = \mathrm{Erf}(\sqrt{\pi}u), \tag{A.25}$$

and note that when $y \to \infty$ the function $E(u)$ from (A.24) reduces to the sign function of (A.23), that is

$$\lim_{y\to\infty} E(\sqrt{2y}u) = \mathrm{sgn}(u). \tag{A.26}$$

Analytical continuation of E (in order to be complex valued) makes it convergent only for

$$-\frac{\pi}{4} < \mathrm{Arg}(u) < \frac{\pi}{4}.$$

The modular transformation properties of such indefinite theta functions under $SL_2(\mathbb{Z})$ are explicitly derived in Zweger's thesis [94, chapter 2] and also in [85] by Vignéras. The generators T and S of $SL_2(\mathbb{Z})$ act on $\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{z})$ as

$$\begin{aligned} \widehat{\Theta}^{JJ'}_{\boldsymbol{\mu}+\frac{K}{2}}(\tau+1,\boldsymbol{z}) &= e^{\pi \mathring{\imath}(\boldsymbol{\mu}^2-\frac{K^2}{4})} \widehat{\Theta}^{JJ'}_{\boldsymbol{\mu}+\frac{K}{2}}(\tau,\boldsymbol{z}+\boldsymbol{\mu}) \\ \widehat{\Theta}^{JJ'}_{\boldsymbol{\mu}+\frac{K}{2}}\left(-\frac{1}{\tau},\frac{\boldsymbol{z}}{\tau}\right) &= \mathring{\imath}(-\mathring{\imath}\tau)^{\frac{n}{2}}(-1)^{B(K,\boldsymbol{\mu})} e^{-\frac{\pi \mathring{\imath} \boldsymbol{z}^2}{\tau}+\frac{\pi i K^2}{2}} \widehat{\Theta}^{JJ'}_{\frac{K}{2}}(\tau,\boldsymbol{z}-\boldsymbol{\mu}). \end{aligned} \tag{A.27}$$

As in [44] the object that we are quite interested in is the $\bar{\tau}$ derivative of the modular completed indefinite theta function A.24. This derivative is exactly what we reffered to as the *shadow* in the introductory section of this paper and its modular properties are much easier to determine than those of A.24 (although the notion of the *shadow* is slightly different than the one used in [94] since the indefinite theta functions that appear here are mixed mock modular forms). In specific, we find that

$$\partial_{\bar{\tau}}\widehat{\Theta}^{JJ'}_{\boldsymbol{\mu}}(\tau,z) = \Psi^{J}_{\boldsymbol{\mu}}(\tau,z) - \Psi^{J'}_{\boldsymbol{\mu}}(\tau,z) \tag{A.28}$$

where $\Psi^{J}_{\boldsymbol{\mu}}$ is the Siegel-Narain function associated with Λ and defined in Appendix A.1.

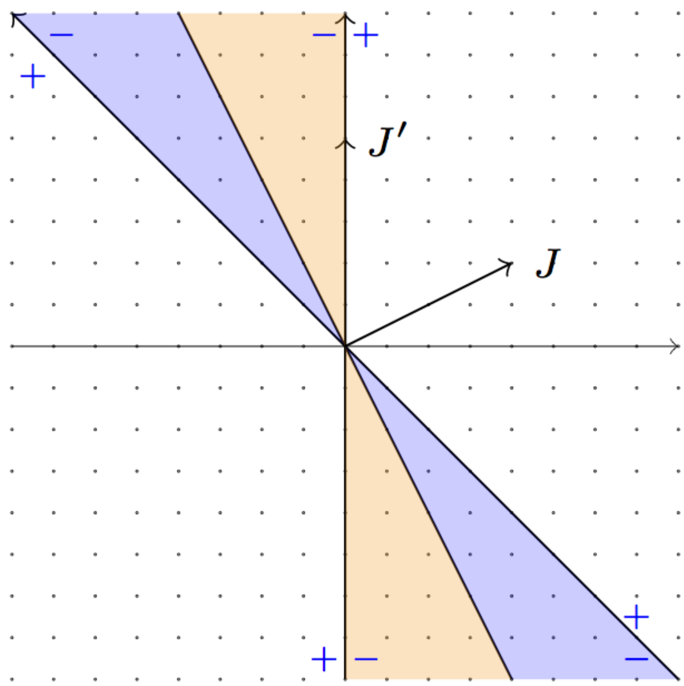

Figure A.1 The positive cone of X is defined for some (for illustrational purposes two-dimensional lattice) lattice Λ which we identify with $H^2(X,\mathbb{R})$ by the "light-cone" drawn via vectors J and J' such that the latter has zero norm, $Q(J')=0$. The vectors we sum over in the indefinite theta function are the ones located in the "upper" yellow cone with negative norm.

If there exists a vector $\boldsymbol{v}_0 \in \Lambda$ such that $Q(\boldsymbol{v}_0) = 0$ then the modular completion of $\Theta^{JJ'}_{\boldsymbol{\mu}}(\tau,z)$ can be simplified because, for such type of lattices, we can choose vectors J (and maybe also J') such that they are identified with the vector $\boldsymbol{v}_0$. Then, as explained in [94], the error function reduces to the sign function. Let us assume that there exists a vector $J' \in \Lambda$ such that $Q(J') = 0$. The series will be convergent by further requiring that $B(\boldsymbol{v}+\boldsymbol{b},J') \neq 0$ (we obviously cannot normalize J' now) for any vector $\boldsymbol{v} \in \Lambda + \boldsymbol{\mu} + \frac{K}{2}$ except if we also have that the other term in the kernel vanishes, i.e., if $B(\boldsymbol{v}+\boldsymbol{b},J) = 0$. The completion of

$\Theta_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{z})$ reads in that case

$$\begin{aligned}\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{\nu}\in\Lambda+\boldsymbol{\mu}+\frac{K}{2}} & \frac{1}{2}\Big\{E(B(\boldsymbol{\nu}+\boldsymbol{b},\underline{J})) - \mathrm{sgn}(B(\boldsymbol{\nu}+\boldsymbol{b},\underline{J}'))\Big\} \\ & \times(-1)^{B(\boldsymbol{\nu},K)} q^{-\frac{Q(\boldsymbol{\nu})}{2}} e^{-2\pi i B(\boldsymbol{\nu},\boldsymbol{z})},\end{aligned} \tag{A.29}$$

the shadow of which exactly corresponds to a Siegel-Narain theta function, in specific we have

$$\partial_{\bar{\tau}}\Theta_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{z}) = \Psi_{\boldsymbol{\mu}}^{J}(\tau,\boldsymbol{z}). \tag{A.30}$$

Finally, let us finish with a remark. It is important that $J' \in \Lambda$ since the modular complete function $\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}$ would not give a convergent series. An example of such a divergence is discussed in [3, Appendix B.3].

Appendix B

Twisted and surface operators

B.1 The self-dual twisted operator

We discuss in this appendix the twisted supersymmetry generators $\mathscr{Q}$, K and L, and we give a formula for $\{\mathscr{Q}, L\}$ for an arbitrary Kähler surface. Recall the global bosonic symmetry group of our theory $G = SU(2)_- \times SU(2)_+ \times SU(2)_R \times U(1)_R$. The first two factors correspond to the global space(time) rotations while the latter two factors correspond to the R-symmetry.

The supersymmety generators $Q_{\alpha A}, \bar{Q}_{\dot{\alpha}}^{B}$, written explicitly, have the following non-zero anticommutator for a local patch given by coordinates x^m such that $m, n = 0, \ldots, 3$

$$\begin{aligned} \{Q_{\alpha A}, \bar{Q}_{\dot{\alpha}}^{B}\} &= 2\,\delta_A^{\ B} (\sigma^m)_{\alpha\dot{\alpha}} P_m, \\ \{Q_{\alpha A}, Q_{\beta B}\} &= 2\sqrt{2}\,\varepsilon_{\alpha\beta}\, Z_{AB}, \end{aligned} \tag{B.1}$$

with $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ is the central charge, Γ is the lattice of electric and magnetic charges of the theory, $P_m \equiv \partial_m$ is the generator of translations, and σ_m the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $\alpha, \dot{\alpha} = 1, 2$ are indices of $\mathrm{SU}(2)_-$ and $\mathrm{SU}(2)_+$ respectively. We define furthermore

$$\sigma_{mn} = \frac{1}{4}(\sigma_m \bar{\sigma}_n - \sigma_n \bar{\sigma}_m), \tag{B.2}$$

with $\bar{\sigma}_m$ the complex conjugate of σ_m.

Topological twisting amounts to redefining the spins of the fields of the vector multiplet and eventually allows to formulate a supersymmetric theory on a compact four-manifold. Our supercharges transform in the $(\mathbf{1}, \mathbf{2}, \mathbf{2})^1 \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2})^{-1}$ representation under the global group G.

Originally, the rotation group is $K' = \mathrm{SU}(2)_- \times \mathrm{SU}(2)_+$ in the untwisted theory. The twist redefines the rotation group of the theory. There are two choices (related by conjugation)

- $K_1' = \mathrm{diag}(SU(2)_- \times SU(2)(2)_R) \times SU(2)(2)_+$,
- $K_2' = \mathrm{diag}(SU(2)(2)_+ \times SU(2)(2)_R) \times SU(2)(2)_-$.

We choose K_1'. The supercharges transform then under $K_1' \times U(1)_R$ as

$$(\mathbf{2},\mathbf{2})^1 \oplus (\mathbf{1},\mathbf{1})^{-1} \oplus (\mathbf{3},\mathbf{1})^{-1},$$

The three terms combine naturally to the following operators [48, 70]

$$\begin{aligned}
\mathscr{Q} &= \varepsilon^{\dot\alpha\dot\beta}\bar{Q}_{\dot\alpha\dot\beta},\\
K_m &= \frac{i}{4}(\bar\sigma_m)^{\dot\alpha\beta}Q_{\beta\dot\alpha},\\
L_{mn} &= (\bar\sigma_{mn})^{\dot\alpha\dot\beta}\bar{Q}_{\dot\alpha\dot\beta}.
\end{aligned}$$

In terms of differential forms, we define K and L as

$$K = K_m\, dx^m \in \Omega^1(M), \tag{B.3}$$
$$L = L_{mn}\, dx^m \wedge dx^n \in \Omega^2(M). \tag{B.4}$$

The $(\mathbf{2},\mathbf{2})^1$ representation gives thus a 1-form $K \in \Omega^1(M)$, the $(\mathbf{3},\mathbf{1})^{-1}$ representation gives a self-dual two-form $L \in \Omega^2(M)$, while the $(\mathbf{1},\mathbf{1})^{-1}$ representation gives $\mathscr{Q} \in \Omega^0(M)$.

To determine $\{\mathscr{Q}, L\}$, let us first determine the six components $\{\mathscr{Q}, L_{mn}\}$. Using the algebra (B.1) and (B.1), we find for $(m,n) = (0,2)$ and $(1,3)$,

$$\begin{aligned}
\{\mathscr{Q}, L_{02}\} &= 2\sqrt{2}\bar{Z},\\
\{\mathscr{Q}, L_{13}\} &= -2\sqrt{2}\bar{Z},
\end{aligned}$$

while for the other choices of (m,n), $\{\mathscr{Q}, L_{mn}\} = 0$. As a result, the commutator $\{\mathscr{Q}, L\}$ reads on $\mathbb{R}^4$ as

$$\{\mathscr{Q}, L\} = 2\sqrt{2}\bar{Z}\,(dx_0 \wedge dx_2 - dx_1 \wedge dx_3). \tag{B.5}$$

In complex coordinates $z_1 = x_0 + ix_2$, $z_2 = x_1 + ix_3$, we can write this commutator as follows

$$\{\mathscr{Q}, L\} = \sqrt{2}i\bar{Z} \sum_{j=1,2} dz_j \wedge d\bar{z}_j \in \Omega^{1,1}(\mathbb{C}^2), \tag{B.6}$$

We extend to an arbitrary Kähler surface M with Kähler form J, by realizing that $\Omega^{1,1}(M)$ contains a one-dimensional subspace of self-dual forms. Since Equation (B.6) is a $(1,1)$-form and self-dual, this suggests that

$$\{\mathscr{Q}, L\} = \sqrt{2} i \bar{Z} J, \tag{B.7}$$

were $J \in \Omega^{1,1}(M,\mathbb{R})$ is the Kähler form which spans the one-dimensional space of $(1,1)$-forms over M.

B.2 Surface operators, roots and characters

In this appendix we would like to present some well known facts about surface operators and co-root lattices $\Lambda_{\text{cort.}}$ that complements the discussion from Section 5.2.

Let us start with a remark. As we explained in Section 5.2 a surface operator is defined by prescribing a singular behavior for the gauge field along some surface $\boldsymbol{S}$. Nevertheless, there is another way to understand surface operators as a two dimensional theory supported on $\boldsymbol{S}$ whose flavor symmetry group is G, the gauge group of the four dimensional theory over X. Coupling the two dimensional theory to the four-dimensional one amounts to gauging G. For a concrete discussion see [32–34].

In this paper we have focused on the approach of singularities of the gauge field A and our task is to understand what we mean by lifting the bundle with connection one-form A that is $\mathbb{T}$ valued to $\mathfrak{t}$. Recall that to a semi-simple Lie group G we associate a root lattice $\Lambda_{\text{rt.}} \subset \mathrm{g}^{\vee}$. Similarly, for the Langlands dual group ${}^{L}G$ we associate a root lattice that is the co-root lattice of G, $\Lambda_{\text{cort.}} \subset \mathfrak{g}$. For simplicity, assume that G is simply connected. Then the root lattice is embedded in the so-called character lattice $\Lambda_{\text{rt.}} \subset \Lambda_{\text{char.}}$ which simply corresponds to $\text{Hom}(\mathbb{T}, U(1))$. Similarly, $\Lambda_{\text{cort.}} \subset \Lambda_{\text{cochar.}}$ corresponds to $\text{Hom}(\mathbb{T}^{\vee}, U(1)) = \text{Hom}(U(1), \mathbb{T})$. The co-character lattice fits in the following exact sequence

$$0 \to \Lambda_{\text{cochar.}} \to \mathfrak{t} \to \mathbb{T} \to 0. \tag{B.8}$$

Actually it is possible to show that $\Lambda_{\text{cochar.}} = \pi_1(\mathbb{T}) \cong \mathbb{Z}^n$ where n is the dimension of the Cartan subalgebra $\mathfrak{t}$. This can be understood as follows. For any Lie group G we can construct its universal cover $\tilde{G}$ and consider the following exact sequence

$$0 \to \pi_1(G) \to \tilde{G} \to G \to 0. \tag{B.9}$$

In our case we can view the Cartan subalgebra $\mathfrak{t}$ as the universal cover of the maximal torus $\mathbb{T}$ as follows

$$0 \to \pi_1(\mathbb{T}) \xrightarrow{d} \mathfrak{t} \xrightarrow{\exp} \mathbb{T} \to 0, \tag{B.10}$$

or equivalently we can view this configuration as principal $\pi_1(\mathbb{T})$-bundle over $\mathbb{T}$. A fiber of this bundle is exactly $\mathfrak{t}$ as shown in the Figure B.1. Therefore we have a natural identification of $\Lambda_{\text{cochar.}}$ with $\pi_1(\mathbb{T})$.

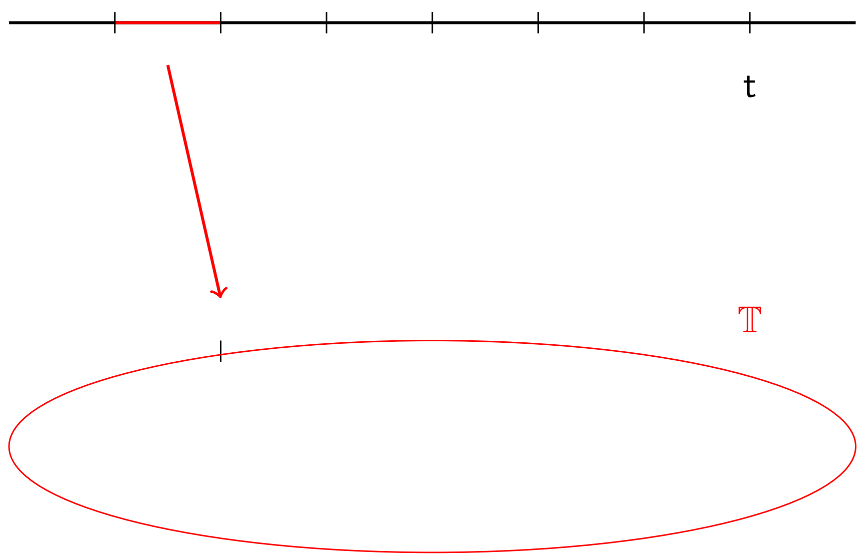

Figure B.1 The red segment in $\mathfrak{t}$ corresponds to $\mathfrak{t}/\pi_1(\mathbb{T})$. Each choice of segment corresponds to a different element of the fundamental group of $\mathbb{T}$. A lift from the base to $\mathfrak{t}$ corresponds to the choice of a surface operator with α prescibed by the choice of line segment.

Now let us present some connections of surface operators with Levi subgroups of G. The surface operators we have discussed are the simplest ones and belong to the so-called “full" surface operators where $\alpha \in \mathbb{T} \cong U(1)^n$. The classification of surface operators has been discussed in detail in [33] (see also [12]) but let us repeat the main idea here. This classification consists of pairs $(\alpha, \mathbb{L})$ where α is the surface operators “electric charge" as in the main part of this paper and $\mathbb{L}$ is a Levi subgroup of G. Let us consider $G = SU(k)$ for example. The Levi subgroups of $SU(k)$ are all possible groups of the form

$$U(1)^{l-1} \times \prod_{i=1}^{l} SU(k_i). \tag{B.11}$$

The minimal Levi subgroup of $SU(k)$ is its maximal torus $\mathbb{T}^{k-1}$ as in the main text of this paper, and the corresponding operator is a full surface operator. For $\mathbb{L} = SU(k-1) \times U(1)$ the corresponding surface operator is called “simple". Of course working with $SU(2)$ and $SO(3)$ restricts a lot the possibilities for surface operators that we can have in our theories.

Finally let us mention that one can consider instead of G its complexification $G_{\mathbb{C}}$ and describe surface operators in terms of parabolic groups. This can be done for theories with $\mathscr{N} \geq 2$ by combining the gauge field with the scalar field defining the complexified connection $\mathscr{A} = A + i\phi$. The surface operators can be described by the the flat connection $\mathscr{A}$ over a $G_{\mathbb{C}}$-bundle that along the embedded surface $\boldsymbol{S}$ its structure group $G_{\mathbb{C}}$ is reduced to a parabolic subgroup $P \subset G_{\mathbb{C}}$ [61]. This point is very useful for understanding surface operators in the context of six-dimensional Donaldson-Thomas theory [12].