Unifying the Irish Vertical Datum with the Normaal Amsterdams Peil (NAP)

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by
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August 27, 2020.

Summary

Orthometric heights play a fundamental role in a wide range of applications in engineering and scientific activities such as monitoring floods, evacuation route planning, crustal motion, subsidence, surface deformations due to seismic events, surface deformations due to mining, storm surges and coastal inundation, sea level rise and other events. Activities like floods (at a regional scale) or sea level rise (at a global scale) demand accurate orthometric height information, and this information needs to refer to a unified vertical datum, that is to say the same zero-height reference surface. Therefore, vertical datum unification has been investigated extensively in geodetic literature, because the adoption of a conventional Vertical Reference System (VRS) is a necessary condition for a future simplification in data harmonisation and interoperability. The vertical datum adopted as the zero level for the United European Levelling Network (UELN), is the tide gauge in Amsterdam in the Netherlands referred to as the Normaal Amsterdams Peil (NAP). The differences between UELN, in the NAP, and the vertical datum of national height systems, have been computed for most countries in Europe. However, Ireland and Northern Ireland levelling networks have not been connected to the UELN due to the lack of a physical connection between Ireland and Europe. The aim of this study is to relate the Irish vertical datum to the NAP datum. There are basically three different strategies for the unification of the local vertical datum (LVD) namely: (1) *Precise Levelling*, i.e., traditional spirit levelling or Global Navigation Satellite System levelling (GNSS-levelling); (2) *Ocean Levelling*; (3) *use of the Geodetic Boundary Value Problem (GBVP) method.*

However, the first two strategies are not practical in Ireland. In the first method, there is a lack of a physical connection across the Irish sea, and a lack of an appropriate gravimetric geoid model to cover Ireland and any of the many countries which have already been connected to the NAP datum. The second method is not practical due to the low qualities of altimetric sea surface heights close to the coastline. On the other hand, GBVP is the rigorous approach for LVD unification that makes use of GNSS ellipsoidal heights, levelling heights and gravimetric geoid heights computed from global geopotential models (GGMs) and

local gravity anomaly data following the Remove-Compute-Restore approach.

The objective of this thesis is to unite Irish vertical datum to the NAP datum by applying the GBVP method. The GBVP is solved by determining the long-wavelength geoid undulations from a Global Geopotential Model (GGM) combined with the short-wavelength gravity signals from terrestrial and EGM2008 gravity data.

The thesis is based on four papers, which form Chapter 2 to Chapter 5. Chapter 1 is a brief introduction. Chapter 2 (Paper 1) deals with terrestrial gravity signals and systematic biases and errors associated with gravity data in Ireland-Northern Ireland. In Chapter 3 (Paper 2), the terrain corrections and correlations of topographical effects on gravity and potential are computed and analysed. Downward continuation of topography-reduced gravity anomalies (that is the results of computations from papers 1 and 2) is investigated in Chapter 4 (paper 3). The overall aim of this research, relating the Irish vertical datum to the NAP datum, is achieved in Chapter 5 (paper 4) using the GBVP method. Finally, Chapter 6 comprises conclusions and some recommendations for future work.

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I remember my dear parents, Habiba and Tavcol. Though you were many miles away from me, you were always an inspiration. I especially remember my father, who left us unexpectedly in 2016. May the road rise to meet you, Dad.

Dedicated to wife Louise and
to my children Jack, Hannah and Eva

Contents

Publications

PAPERS IN PREPARATION:

- **Sajjadi, S.** Martinec, Z., et al., Topographical effects over Ireland and their contributions to a 1-cm geoid over Ireland. Submitted to *Journal of Geodesy.*
- **Sajjadi, S.** Martinec, Z., et al., Downward continuation of gravity disturbance and the precision of the geoid over Ireland. Submitted to *Journal of Geodesy.*
- **Sajjadi, S.** and, Martinec, Z. Relating the Irish vertical datum to the Normaal Amsterdams Peil (NAP). To be submitted to *Studia Geophysica et Geodetica.*

MISCELLANEOUS PEER-REVIEWED PUBLICATIONS:

- **Sajjadi, S.**, Martinec, Z., Prendergast, P., Hagedoorn, J., Sachl, L., Readman, P., Edwards, R., OReilly, B., Horan, C., 2020. ***The unification of gravity data for Ireland-Northern Ireland.*** The Leading Edge 39, 135 – 143.

List of Symbols

d	Analytical expression of the Poisson Kernel
ψ	Angular distance between the geocentric directions Ω and Ω'
T^h	Anomalous gravitational potential harmonic outside the geoid
T	Anomalous gravitational potential
δg^{BP}	Bouguer plate reduction
Δg^B	Bouguer gravity anomaly
V^ω	Centrifugal potential
ϑ	Co-latitude
V^ϱ	Compensation potential
$P(r, \Omega)$	Computation point
R_{te}	Correlation coefficient between topographical effects and height of computation points
ϱ	Crustal density
$\sigma(\Omega')$	Density of a condensation layer
δA	Direct topographical effect on gravity
$\delta D^{\ell,\psi_0}$	Direct topographical effects of high-degree components of gravity anomalies on geoid undulation
δD	Direct topographical effect on geoid undulation
${}^*\Delta g_h$	Downward continued gravity anomalies on the geoid
h	Ellipsoidal height
$\Delta\gamma_0$	Error in reference normal gravity due to coordinates displacements
e	First eccentricity
Δg^{FA}	Free-air gravity anomaly
Ω_0	Full solid angle

N	Geoidal height/undulation
$N_{geo,MH}$	Geometric geoid undulation at Malin Head
N_{geo}	Geometric geoid undulation
$N_{grav,MH}$	Gravimetric geoid undulation at Malin Head
N_{grav}	Gravimetric geoid undulation
V^B	Gravitational potential of the spherical Bouguer layer
V^c	Gravitational potential of compensated/condensed masses
V^g	Gravitational potential of masses below the geoid
V^R	Gravitational potential of the terrain roughness term
V^t	Gravitational potential of topographical masses
V	Gravitational potential of the Earth
δg	Gravity disturbance vector
Δg_{geoid}	Gravity anomalies on the geoid
ΔW	Gravity potential discrepancies or gravity potential parameters
g_{MH}	Gravity at Malin Head tide-gauge station
g	Gravity of the Earth
W_{local}	Gravity potential of local vertical datum
W_{MH}	Gravity potential at Malin Head tide-gauge station
W_{NAP}	Gravity potential of Normaal Amsterdams Peil (European Vertical Datum)
W	Gravity potential of the Earth
Δg^{obs}	Harmonic gravity anomaly at observation point
Δg^h	Harmonic gravity anomaly
ΔW	Height datum discrepancies or vertical datum parameters
V^σ	Helmerts 2nd condensation potential
$\widetilde{L^{-1}}$	Indefinite radial integral of the Newton Kernel
K^c	Integration Kernel of Helmerts condensation layer
K^t	Integration kernel of gravitational potential
$P'(r', \Omega')$	Integration point
$\Delta\phi$	Latitudinal displacement
N^ℓ_{grav}	Long-wavelength gravimetric geoid undulation
φ	Longitude
ϱ_0 or ρ_0	Mean crustal density
R	Mean radius of the Earth
χ	Misfit between observables $f_i := f(\Omega_i)$ and the model values $f_i(\vec{\tau})$

G	Gravitational constant
γ_0	Normal gravity of the reference ellipsoid
γ_Q	Normal gravity of the ellipsoid
γ	Normal gravity
U	Normal gravity potential
H	Orthometric height
Ω	Pair of angular spherical coordinates (ϑ, φ)
P_g	Point on the geoid
P_t	Point on the topography
P	Point on the Earth surface
Q	Point on reference ellipsoid
K	Poissons Kernel
δN	Primary indirect topographical effect on geoid
δV_{P_g}	Primary indirect topographical effect on potential
ΔN	Primary indirect effect on the geoid
r	Radial distance
ν	Radius of curvature in the prime vertical
r_g	Radius of the geoid
r_g	Radius of geoid
r_t	Radius of the topography
W_0	Reference value of gravity potential on the geoid
E_r	Relative error with respect to integration Kernels
δV	Residual topographical potential
g_{res}	Residual gravity anomalies
δS	Secondary indirect topographical effect on gravity
N^s_{grav}	Short-wavelength gravimetric geoid undulation
L	Spherical distance between point (r, Ω) and point (r', Ω')
$S(\psi)$	Stokes function
S_g	Surface of the geoid
S_t	Surface of the topography
$\Delta g^{h,FA}$	Topography-reduced free-air gravity anomalies
N^h	Undulation of the co-geoid

List of Acronyms

BVP	Boundary Value Problem
BGS	British Geological Survey
CGVD2013	Canadian Geodetic Vertical Datum of 2013
CPU	Central Processing Unit
CHAMP	Challenging Minisatellite Payload
DEM	Digital Elevation Model
DTM	Digital Terrain Model
DTE	Direct Topographic Effect
DWC	Downward Continuation
DIAS	Dublin Institute for Advanced Studies
EGM	Earth Gravity Model: EGG97 Earth Gravitational Geoid 1997, EGM2008 Earth Gravitational Model 2008, EGG2015 European Gravimetric (Quasi) Geoid 2015.
ED50	European Datum 1950
ESA	European Space Agency
ETRF89	European Terrestrial Reference Frame 1989
FBM	Funemental Bench Mark
GMT	Generic Mapping Tools
GBVP	Geodetic Boundary Value Problem
GRS67	Geodetic Reference System 1967
GRS89	Geodetic Reference System 1989
GSNI	Geological Survey Northern Ireland

GGM	Global Geopotential Model
GPS	Global Position System
GNSS	Gobal Navigation Satellite System
GOCE	Gravity fields and steady-state Ocean Circulation Explorer
GOCO05S	The combined satellite gravity field model: GOCE (42 months), GRACE (10.5 yrs.), CHAMP (8 yrs.), SLR (5 yrs.)
GRACE	Gravity Recovery and Climate Experiment
GridInQuest	Ordnance Survey Ireland-Northern Ireland Transformation Software GridInQuestI publication 2002 GridInQuestII publication 2015
IERS	International Earth Rotation and Reference Systems Service
IGSN71	International Gravity Standardisation Network 1971
IHRS	International Height Reference Systems
IG	Irish Grid
Ireland	Two jurisdictions of Ireland and Northern Ireland
ITM	Irish Transverse Mercator
LPSNI	Land and Property Services Northern Ireland
LVD	Local Vertical Datum
MDT	Mean Dynamic Topography
MSL	Mean Sea Level
m	Meter unit of distance
m/s	Meter per second unit of velocity
mGal	Milligal unit of acceleration
NASA	National Aeronautics and Space Administration (USA)
NAP	Normaal Amsterdams Peil (the Europe Vertical Datum)
ODD	Ordnance Datum Dublin also known as Poolbeg Lighthous Datum
OSGB	Ordnance Survey Great Britain
OSGM02	Ordnance Survey Geoid Model 2002
OSGM2015	Ordnance Survey Geoid Model 2015
OSi	Ordnance Survey Ireland
PITE	The primary indirect topographical effect on potential
QGR	Quadrangle Grid Resolution
RCR	Remove-Compute-Restore
RTM	Residual Topography/Terrain model

SGG	Satellite Gravity Gradiometry
SLR	Satellite Laser Ranging
SSTop	Sea Surface Topography
SITE	Secondary indirect topographical effect on gravity
SRTM	Shuttle Radar Topography Mission
TG	Tide Gauge
TG-MH	Tide Gauge station at Malin Head
TE	Topographic Effect
TM	Topographic Masses
UELN	United European Levelling Network
UK	United Kingdom of Great Britain and Northern Ireland
UTM	Universal Transverse Mercator
VRS/EVRS	Vertical Reference System/European Vertical Reference System
VLBI	Very-Long-Baseline Interferometry

1

Introduction

The geoid is defined as an equipotential surface along which the Earth's gravity potential (W) is constant and equal to a reference value W_0. This datum is chosen such that the geoid coincides with a mean level of the oceans, and can be mathematically extended over the continents. As a result of unequal distribution of masses in the Earth's interior, the geoid is irregularly shaped. It describes the figure of the Earth by a physical quantity, the gravity potential, in contrast to the idealized geometrical figure of a reference *ellipsoid*. The separations between the two surfaces is called the geoid *undulation* N, or geoidal heights.

The distance between the levelling point at the Earth's surface and the geoid counted along the plumb line is the so-called *orthometric height* H. Hence, the geoid is geometrically considered as the reference surface, or the "level 0"of orthometric heights. Orthometric heights play a fundamental role in a wide range of applications in engineering, and scientific activities, such as monitoring floods, evacuation route planning, crustal motion, subsidence, surface deformations due to seismic, surface deformations due to mining, storm surges and coastal inundation, sea-level rise or other events. In events such as floods (at regional scale) or sea level rise (at global scale) not only are there demands for accurate orthometric height information, but also that such information needs to refer to the same zero-height reference surface (vertical datum).

There are several hundred local vertical datums all around the world that are defined by the local Mean Sea Level (MSL). However, due to the existence of the continents, oceanic currents, atmospheric pressure effects, and external gravity forces, these datums are biased in comparison with each other (they are not at the same level surface) and to a global geoid determined using gravimetric satellite observations. Height datum unification is the process of determining these biases between vertical datums, and it has been at the centre of the discussions by many researchers during the last few decades; see, e.g., (*Amjadiparvar et al.*, 2013, 2016; *Ardalan et al.*, 2010; *Colombo*, 1980; *Ebadi et al.*, 2019; *Hayden et al.*, 2012; *Moritz*, 2015; *Rapp and Balasubramania*, 1992; *Rummel*, 2012; *Rummel and Teunissen*, 1988; *Sánchez and Sideris*, 2017; *Sánchez et al.*, 2016; *Sanso and Venuti*, 2002; *Véronneau and Huang*, 2016), and many others.

The global unification of national height datums is an important research topic in geodesy because the adoption of a conventional Vertical Reference System is a necessary condition for data harmonisation and interoperability. Accordingly, the United European Levelling Network (UELN) (*Sacher et al.*, 2003) was resumed in 1994 under the title UELN-94, with the objectives of establishing a unified vertical datum for Europe, at one decimetre level, with the simultaneous enlargement of UELN as far as possible to include Central and Eastern European countries. The vertical datum adopted as the zero level for the UELN is the tide gauge in Amsterdam in the Netherlands referred to as the Normaal Amsterdams Peil (NAP).

The vertical datum of the European Vertical Reference System (EVRS) is defined as the zero level of this tide gauge. There are currently three official pan-European realizations of EVRS:

1. In 2000, the first definition of the EVRS was given, i.e., ***EVRF2000***, and the extended UELN-95/98 was used as the basis for its first realization (*Augath and Ihde*, 2002).

2. In 2004, a need for an improved common EVRS was recognized, due to

extending the UELN, the appearance of new levelling data, and the replacement of national levelling data, which were contained in the network of 1994. Therefore in 2009, ***EVRF2007*** was adopted at the EUREF symposium in Brussels (*Sacher et al.*, 2009). In the realization of EVRF2007, the datum was realized by 13 datum points with their heights in EVRF2000, and in principle every point of EVRF2007 was a representative of the datum NAP. The measurements of EVRF2007 had been reduced to the epoch 2000 using the land uplift model $NKG2005LU$ (*Ågren and Svensson*, 2007).

3. In 2015, due to extending the UELN far to the East, updating and improving the accuracies of existing data, the EUREF resolution No. 4 considered the need for a new realization of EVRS. A new set of 13 datum points is used for height datum realization of the ***EVRF2019*** (see (*Sacher and Liebsch*, 2019a, Table 1: Datum points in EVRF2019)). The difference between the heights in EVRF2007 and EVRF2019 does not exceed ±15mm, and according to its definitions, the datum of EVRF2019 is at the level of NAP. Furthermore, EVRF2019 is a zero-tide system, where the tide-generating potential is eliminated, but the deformation potential of the Earth is retained see, e.g., RH 2000 adopted in Sweden in 2005 (*Ågren and Svensson*, 2007) and the N2000 adopted in Finland in 2007 (*Saaranen et al.*, 2009). In 2016, the $NKG2005LU$ model was replaced with the new model $NKG2016LU - lev$, and all the levelling data had been reduced to epoch 2000 similar to that of EVRF2007 (*Vestøl et al.*, 2016).

The differences between UELN, in the NAP, and the vertical datum of national height systems, have been computed for most countries in Europe. However, Ireland and Northern Ireland (island of Ireland) levelling networks have not been connected to the UELN due to its separation from Great Britain to its east by the North Channel, the Irish Sea, and St George's Channel. Great Britain itself (an island) was connected to the UELN via the channel tunnel by adopting the mean value of two observations taken in 1994.

The connections in EVRF2000 or EVRF2007 between Great Britain and Europe were derived from hydrodynamic levelling (Ocean Levelling) between Great Britain and France. It seems there is no documentation is published, that describes the details of the determination and the used hydrodynamic model. The EVRF2019 height of the British tunnel endpoint is determined from the measurement through the channel tunnel (see (*Sacher and Liebsch*, 2019a)).

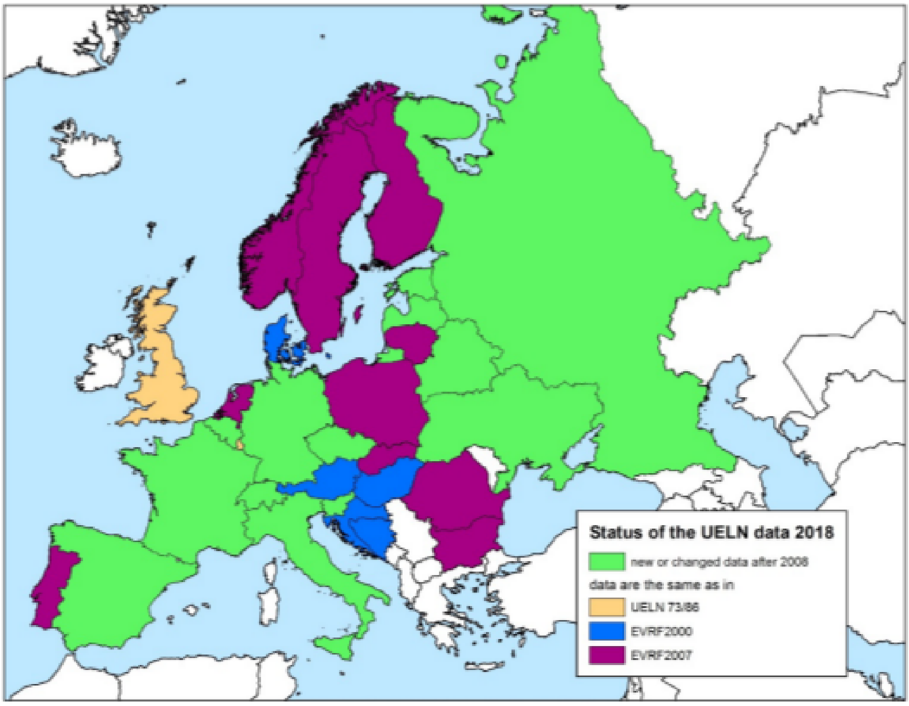

Figure 1.1: The status of the data in the UELN 2018 (*Sacher and Liebsch*, 2019a).

There are three alternative methods to connect independent height reference frames (*Rummel*, 2001):

1. *Precise Levelling*: Theoretically, to address the problem of height unification in Ireland, one could extend the UELN network to Ireland from the closest UELN stations in the UK using Spirit levelling. Due to the lack of physical connection across the Irish sea, the use of spirit levelling is not possible.

2. *Ocean Levelling*: This method is based on using precisely determined mean dynamic sea surface topography (SSTop), at tide gauges (co-located at datum points), i.e., the height of the mean sea surface above the geoid. The SSTop difference between different gauges would correspond to the height offset between the datum zones. Information on the SSTop obtained from different

sources, ocean models (dynamic levelling) or a combination of satellite altimetry and a precise gravimetric geoid (altimetric levelling) have different accuracies. The accuracy of oceanographic methods depends on data availability and quality, as well as on the spatial resolution of the ocean model in use. The altimetric method, which does not require oceanographic data, faces problems at coastlines because of the low quality of altimetric sea surface heights close to the coastline. Therefore although this method is possible, it is not suitable because of its low accuracy.

3. *Geodetic Boundary Value Problem (GBVP)*: The GBVP is the rigorous approach for LVD unification that makes use of GNSS ellipsoidal heights, levelling heights and gravimetric geoid heights computed from global geopotential models (GGMs) and local gravity anomaly data (*Rummel and Teunissen*, 1988). The theory of height unification by the GBVP approach is well developed, and has been intensively discussed under different titles, e.g., *Global or World Vertical Datum* (*Balasubramania*, 1994; *Rapp*, 1983), *Global Vertical Network* (*Colombo*, 1980), *Global Height Datum Unification* (*Ardalan and Safari*, 2005), *Global Unification of Height Systems* (*Rummel*, 2001), *Height or Vertical Datum Problem* (*Heck and Rummel*, 1990; *Sacerdote and Sanso*, 2001; *Sacerdote et al.*, 2004), *Vertical Datum Connection* (*Van Onselen*, 1998; *Xu*, 1992), *Global Unified Height Reference System* (*Ihde and Sánchez*, 2005; *Kutterer et al.*, 2012; *Sanchez*, 2007), etc. Even though different names have been used, the fundamental quantities of interest are the local geopotential value differences with that of the reference value ($\Delta W = W_0 - W_{local}$), which are referred to here as gravity potential discrepancies or gravity potential parameters.

The objective of this thesis is to relate the Irish vertical datum to the Normaal Amsterdams Peil (NAP) by applying the GBVP method. The GBVP is solved by determining the long-wavelength geoid undulations from a Global Geopotential Model (GGM) combined with the short-wavelength gravity signals from terrestrial and EGM2008 gravity data. The long-wavelength components of geoid undulation are determined by adopting the European Gravimetric (Quasi)Geoid EGG2015 potential value, $W_0^{EGG2015} = 62,636,857.91 \mathrm{m}^2\ \mathrm{s}^{-2}$ (*Denker et al.*, 2018).

In contrast, the short-wavelength components of geoid undulation are computed from terrestrial gravity data combined with EGM2008 gravity signals to extend missing data along the coast, lakes, areas in high elevated topography, and offshore areas.

The thesis is based on four papers (three of which have been submitted to refereed journals), which form Chapter 2 to Chapter 5.

Paper 1 (Chapter 2), analyses the systematic biases and errors associated with gravity data in Ireland-Northern Ireland and the conversion of gravity to a consistent and unified system, which is essential in geodetic applications.

The short-wavelength component of geoid undulation is computed using the so-called Remove-Compute-Restore (RCR) approach. The RCR procedure is a method that fulfils Stokes's requirements[1] for computing the geoid. The summary of this procedure explained here in three steps, reveals the purpose of the last three papers and also shows the interrelationships between these papers.

Step 1: *Removing the gravitational effect of the residual topographical masses*:

Subtracting the gravitational effect of the residual topographical masses, δV, from the anomalous gravitational potential T creates the potential T^h that is harmonic outside the geoid,

$$T^h = T - \delta V \ . \tag{1.1}$$

The gravity attraction of the residual topographical masses are then defined by

$$\delta A := \frac{\partial \delta V}{\partial r} \tag{1.2}$$

at the point of the gravity measurements. To make the potential harmonic in a space above the geoid, these effects have to be calculated and removed

[1]No masses outside the geoid and the measurements are referred to the geoid.

from the observations, see (*Sajjadi et al.*, 2018a):

$$\Delta g^h = \Delta g^{obs} - \delta A \ . \tag{1.3}$$

In paper 2 (Chapter 3), the computational method and the contribution of the topographical effects in geoid determination, have been investigated numerically in three test areas located in Ireland. Topographical effects for Ireland-Northern Ireland in several quadrangle grid resolutions (QGR) are computed. The results of the computations are implemented in paper 3 (Chapter 4), for determination of topography-reduced free-air gravity anomalies Δg^h prior to Downward Continuation (DWC).

Step 2: *Computation of the residual geoid, or co-geoid*:

Downward continuation of Δg^h is part of the second step of the RCR procedure. At the first phase of the computation, Δg^h is continued from the surface down to the geoid; the method of computation is analysed and discussed in paper 3 (Chapter 4). The second phase of the computation proceeds when the two basic requirements of Stokes' formula have been met. In this phase we compute the geoidal undulations N^h using Stokes' formula

$$N^h = \frac{R}{4\pi\gamma_Q} \int\limits_{\Omega_0} \Delta g^h S(\psi) d\Omega, \tag{1.4}$$

where ψ is the spherical solid angle between the computation point and an integration point and $S(\psi)$ is the Stokes function (*Vaníček and Kleusberg*, 1987) integrated over the full solid angle Ω_0.

Step 3: *Adding the contribution of the topography to the solution*:

The geoid undulations computed from Stokes' integral gives the height N^h of the equipotential surface of T^h that is called the *co-geoid*. The actual geoid undulations, which account for the short-wavelength components, are obtained by adding the primary indirect effect on the geoid to the co-geoid heights. This is explained and numerically analysed in paper 4 (Chapter 5).

The long-wavelength components of geoid undulations are computed by adopting the IERS potential value. Then the gravimetric geoid undulations values N_{grav}, which are the sum of the short-wavelength components and the corresponding long-wavelength components, are determined.

The local gravity potential value at Malin-Head tide gauge station W_{MH} is estimated by varying the adopted potential value to fit gravimetric geoid undulation at Malin-Head tide gauge station $N_{grav,MH}$ to that of geometric geoid undulation $N_{geo,MH}$.

Then the so-called height datum discrepancies, or vertical datum parameters between the tide gauge station at Malin-Head and NAP or other regions, are computed by

$$\frac{W_{MH} - W_{NAP}}{g_{MH}} = \Delta N \ , \tag{1.5}$$

where W_{NAP} is the gravity potential value of Normaal Amsterdams Peil (NAP) and g_{MH} is the actual gravity value at the Malin Head Tide Guage Station.

Finally, Chapter 6 comprises conclusions and recommendations for future work. In summary, in high level terms, the main contributions of the thesis are as follows.

- The Bouguer and free-air gravity anomalies of Ireland-Northern Ireland have been unified into a consistent system, with the same co-ordinate system, the same gravity base station and the same vertical datum (Paper 1, Chapter 2).

- Terrain effects in several QGR's in Ireland-Northern Ireland are computed and made available for the use in geophysical and geodetical applications (Paper 2, Chapter 3).

- The problem with the instability of downward-continuation with respect to the regularisation of gravity data in Ireland is solved and presented (Paper 3, Chapter 4).

- A conventional potential value as a reference vertical datum, in Ireland-Northern Ireland, is defined, which provides monitoring of the solid and fluid Earth in Ireland and enables connection to the European Vertical Datum (NAP) as required by the aim of this study, but also with the world height system (Paper 4, Chapter 5).

Diagramatic Summary of Structure of Approach

Figure 1.2 summarises the structure of the work in this dissertation described above. The terrestial gravity data (top left of figure) is used in the determination of the free-air gravity anomaly; this process is described in Chapter 2 (Paper 1). The topographical effects on the gravity anomalies (δA) are computed, and removed from the free-air gravity anomaly, resulting in the topography-reduced free-air gravity anomaly, as described in Chapter 3 (Paper 2). The latter quantity is downward-continued to the geoid (Chapter 4, Paper 3), and then the co-geoid is computed N^h. The primary indirect topographic effect on the geoid δN is added to the co-geoid to determine the short-wavelength geoid undulation N^s. This whole process is depicted in the light boxes with black connectors in the figure.

The long-wavelength geoid N^l is computed, making use of the gravity potential formula at the top of the figure, as described in Chapter 5 (Paper 4). The computations of the different terms in the equation are represented with the grey boxes and dashed connectors. The Stokes co-efficient V_{jm} is computed from GOCE satellite data up to degree and order 240. The centrifugal potential V^{ω} is considered fixed in this study. The initial gravity potential value W_0 is adopted from the IERS gravity potential value. The above terms and the radius r_{ref} of the reference ellispoid GRS80 are used to compute the radius $r_{g,lw}$ of the long-wavelength geoid. The long-wavelength geoid N^l is computed from the differences between r_{ref} and $r_{g,lw}$.

The gravimetric geoid undulation N_{grav} is computed as the sum of the short-wavelength N^s and the long-wavelength N^l (shown with the blue connector). The geometric geoid undulation N_{geo} (pale blue oval in the figure) is computed from the ellipsoidal height and the orthometric height. The adopted gravity potential value IERS W_0 is varied in such a way that N_{grav} becomes equal to N_{geo}, up to a high degree of precision (this loop is represented by the red connectors). The value then of W_0 is the returned value of the local gravity potential W_{MH} at Malin Head tide gauge station (light orange box).

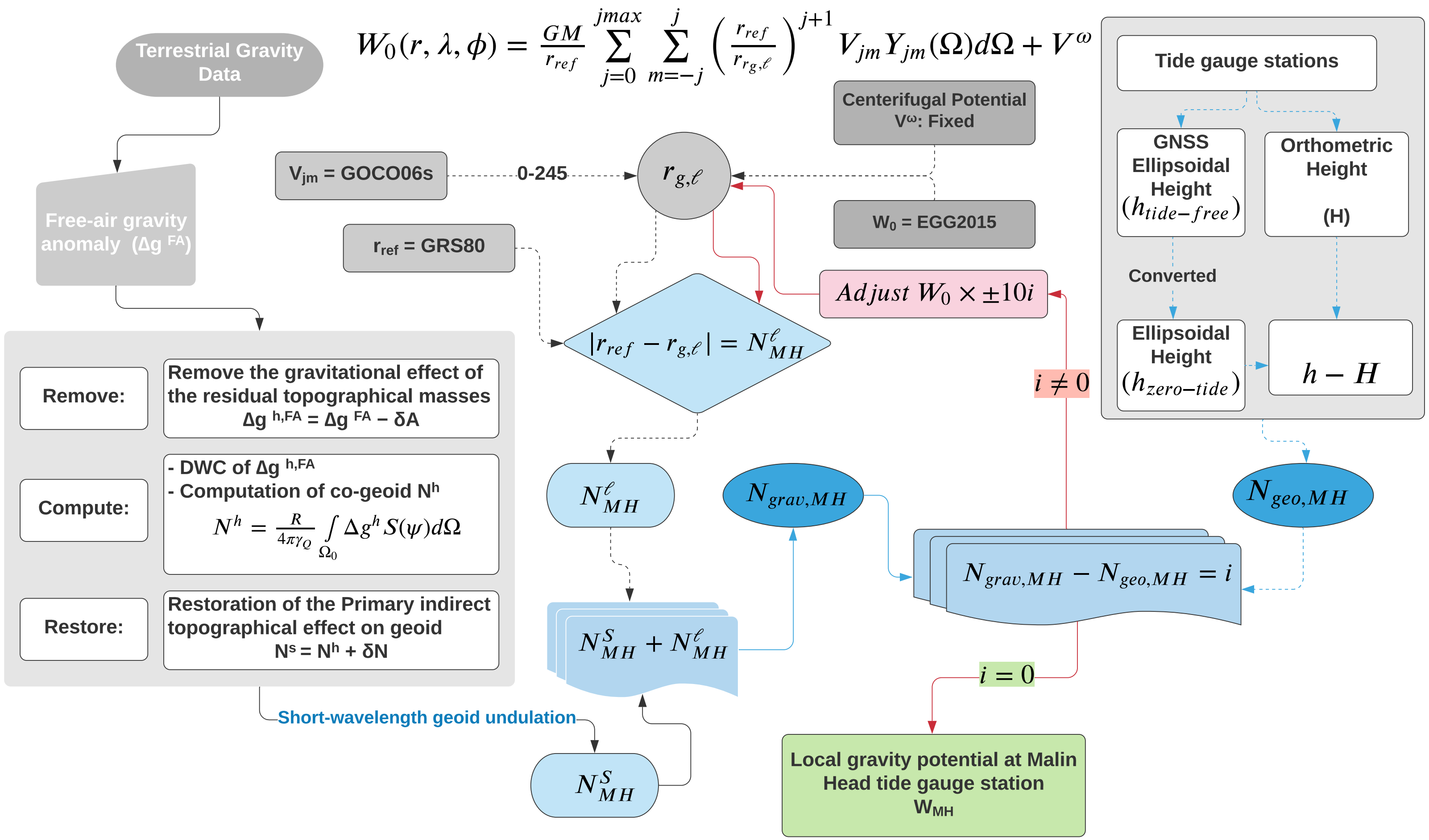

Figure 1.2: Summary of the determination of short -wavelength components of geoid undulation through the process of RCR procedure (black connectors); determination of long-wavelength components of geoid using gravity potential formula (dashed-black connectors); the gravimetric and geometric geoid undulation (blue and dashed-blue connectors respectively); the gravity potential value at Malin-Head GNSS tide-gauge station represented in the loop (red connectors).

2

Paper 1: The Unification of Gravity Data for Ireland-Northern Ireland

Our knowledge of gravity data and their accuracy are essential in the precise determination of the geoid undulations. Since topographic heights are determined from the surface of the geoid, many applications in geophysics, geodesy, oceanography and engineering practices requires physically defined heights related to mean sea level, i.e., the geoid. Due to technological advancements in GNSS technology, it has become a standard tool for solving many tasks. However, because of the non-practical nature of geodetic heights obtained from GNSS observations, these heights are mostly being abandoned for practical applications in engineering. A precise geoid model can be employed to convert the geodetic (GNSS-derived) heights into orthometric (i.e. sea level-related) height values, by subtracting the geoidal height from the geodetic height. Consequently, the geoid model must be known to an accuracy comparable to the accuracy of geodetic heights and traditional levelling heights.

The accuracy of the geoid models is strongly dependent on gravity data used in the computations. Therefore, this paper (paper 1) analyses the systematic biases and errors associated with gravity data in Ireland-Northern Ireland and its conversion into a consistent and unified system.

The gravity data in Ireland-Northern Ireland is given in different coordinate systems (IG[2] and ITM[3]), different gravity base stations (Dunsink and Cambridge) and different vertical datums (Malin Head and Belfast tide gauge). The conversion of the gravity data to a consistent system, which refers to unified coordinates, base station and vertical datum, is essential especially in geoid determination. Furthermore, in this paper, a new standardised and unified data format is computed and proposed for the supply of gravity data for Ireland-Northern Ireland to minimise the potential of misinterpreting the data. As part of this study, simple Bouguer and free-air gravity anomaly maps are produced for Ireland-Northern Ireland as an example of how to integrate the data correctly.

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[2] Irish Grid
[3] Irish Transverse Mercator

The unification of gravity data for Ireland-Northern Ireland

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Abstract

The systematic biases and errors associated with gravity data in Ireland and Northern Ireland and the conversion of gravity to a consistent and unified system are analyzed. The gravity data in Ireland and Northern Ireland are given in different coordinate systems (Irish Grid and Irish Transverse Mercator), different gravity base stations (Dunsink and Cambridge), and different vertical datums (Malin Head and Belfast tide gauge). The conversion of the gravity data to a consistent system, which refers to unified coordinates, base station, and vertical datum, is essential in geophysics and geodesy, especially in geoid determination. A new standardized and unified data format is computed and proposed for the supply of gravity data for Ireland and Northern Ireland to minimize the potential of misinterpreting the data. As part of this study, simple Bouguer and free-air gravity anomaly maps are produced for Ireland and Northern Ireland to give an example of how to integrate the data.

Introduction

Note: In this paper, italic text for *Ireland* represents the two jurisdictions of Ireland and Northern Ireland. Therefore, Ireland (with no italics) refers to the 26 counties in the south, and *Ireland* (with italics) refers to the 32 counties and includes Northern Ireland.

The conversion of gravity data to a consistent system is a well-known task in gravity field modeling and is documented by many authors (Keller et al., 2002; Pavlis et al., 2012). High-precision studies in gravity field modeling require a careful treatment of gravity data to ensure a coherent framework for data analysis and to avoid datum-related biases. Consistent transformations between different geodetic reference systems are essential. This paper discusses combining the gravity data of Ireland, available from the Dublin Institute for Advanced Studies (DIAS), and that of Northern Ireland, available from the Geological Survey Northern Ireland (GSNI). These gravity data in *Ireland* were collected during 1950–1980, a period of significant change in spatial reference systems, vertical datums, surveying and mapping technology, new processing algorithms, and computational capabilities. Consequently, care is needed when combining the data into a unified format to ensure consistency and modernization of spatial parameters.

The purpose of gravity surveys in *Ireland* was to determine the local gravity anomaly for geophysical studies, e.g., Murphy (1962), Readman et al. (1995, 1997), O'Reilly et al. (1999), etc. However, geodesists desire gravity data to compute centimeter-scale resolution geoid models for measuring coordinates of features to an ever-increasing accuracy level. Therefore, a secondary aim of this paper is to examine the accuracy and suitability of gravity data for use in geodetic applications.

Bouguer and free-air gravity anomaly maps of *Ireland* are produced by subtracting normal gravity values (computed using Gravity Formula 1980 [Moritz, 1980]) from the absolute (observed gravity values plus value of reference datum) gravity values. The deliverables of the present work include new Bouguer and free-air gravity anomaly maps for *Ireland*. This paper also describes the existing data formats in detail and proposes a new format for future data delivery.

Gravity data

The gravity data set for Ireland and Northern Ireland is shown in Figure 1 and is currently available from DIAS and GSNI, respectively.

The density of the data averages approximately one station per 1 km^2 in Northern Ireland and one station per 3 km^2 in Ireland, as illustrated in Figure 2a. Additionally, the gravity stations in the southwest have a somewhat sparse distribution, especially in the upland areas (Figure 2b).

***Gravity data for* Ireland.** Gravity data for Ireland were collected over an extended period of time from 1949 to the 1980s. The format of gravity data provided by DIAS is illustrated in Table 1. The easting and northing in Table 1 are 1975 Irish Grid realization (IG75). The latitude and longitude values are computed from the transformation formula given in Murphy (1982).

The height of each gravity station was determined by leveling the station to the nearest Ordnance Survey Ireland (OSi) benchmark or interpolated from spot heights of *Ireland* 6-inch maps. All heights are computed in feet using the Ordnance Datum Dublin (ODD), also known as Poolbeg Lighthouse Datum, which is the level that the tide fell on 8 April 1837 in Dublin Bay. This datum corresponds to 8.218 ft below mean sea level (MSL) defined during the summer of 1842 (Airy et al., 1845), and supplied heights added 8.2 ft to correct heights to MSL. Although MSL was defined in 1842, it was not used widely until it was redefined at Malin Head in 1970, which is the MSL of the tide gauge at Malin Head, County Donegal (Prendergast, 2004b).

The values of gravity at Dunsink Observatory (Dublin), Sligo (Courthouse), Galway (University College), and Cork (University College) were initially established around Ireland in March and April, 1949, by pendulum observations and compared with the value at Cambridge (Pendulum House). Dunsink Observatory

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Figure 1. Distribution of gravity data in *Ireland*. Each data point represents a measurement collected by the DIAS (23,285 data points) and GSNI (11,256 data points).

(latitude = 53.392°, longitude = −6.351°), about 4 miles northwest of Dublin, was adopted as the gravity base station for Ireland. The pendulum values and the differences of gravity from Cambridge found by least squares are (Cook, 1950):

Dunsink	+120.84 mGal	±0.56
Sligo	+197.28 mGal	±0.82
Galway	+96.91 mGal	±0.64
Cork	−22.51 mGal	±0.87

where the value of gravity at Cambridge has been taken as 981265.0 mGal. This network was densified by Murphy during the 1950s (Murphy, 1950, 1957). The observations were taken in a network of one-day closed loops to correct for instrumental drift and to compute closing errors and corrections. Gravity stations were observed every 2 miles, and the standard deviation of a single observation was estimated to be ±0.29 mGal (Thirlaway, 1951).

The accuracies of the gravity station heights are estimated at ±0.05 m if leveled to an OSi benchmark and ±0.5 m if interpolated from spot heights (heights shown in the map) at OSi's 6-inch maps (Murphy, 1987). Since a 1 m error in elevation is equivalent to an error of ≈ 0.3 mGal in the free-air gravity anomaly and ≈ 0.1 mGal in the Bouguer gravity anomaly, a ±0.5 m error in elevation introduces an error of ±0.15 mGal and ±0.05 mGal in free-air gravity anomaly and Bouguer gravity anomaly, respectively.

The gravity values are relative and tied to the base station at the Dunsink Observatory. The final column in Table 1 lists the calculated Bouguer gravity anomalies, obtained from Gravity Formula 1980, using European Datum 1950 latitudes.

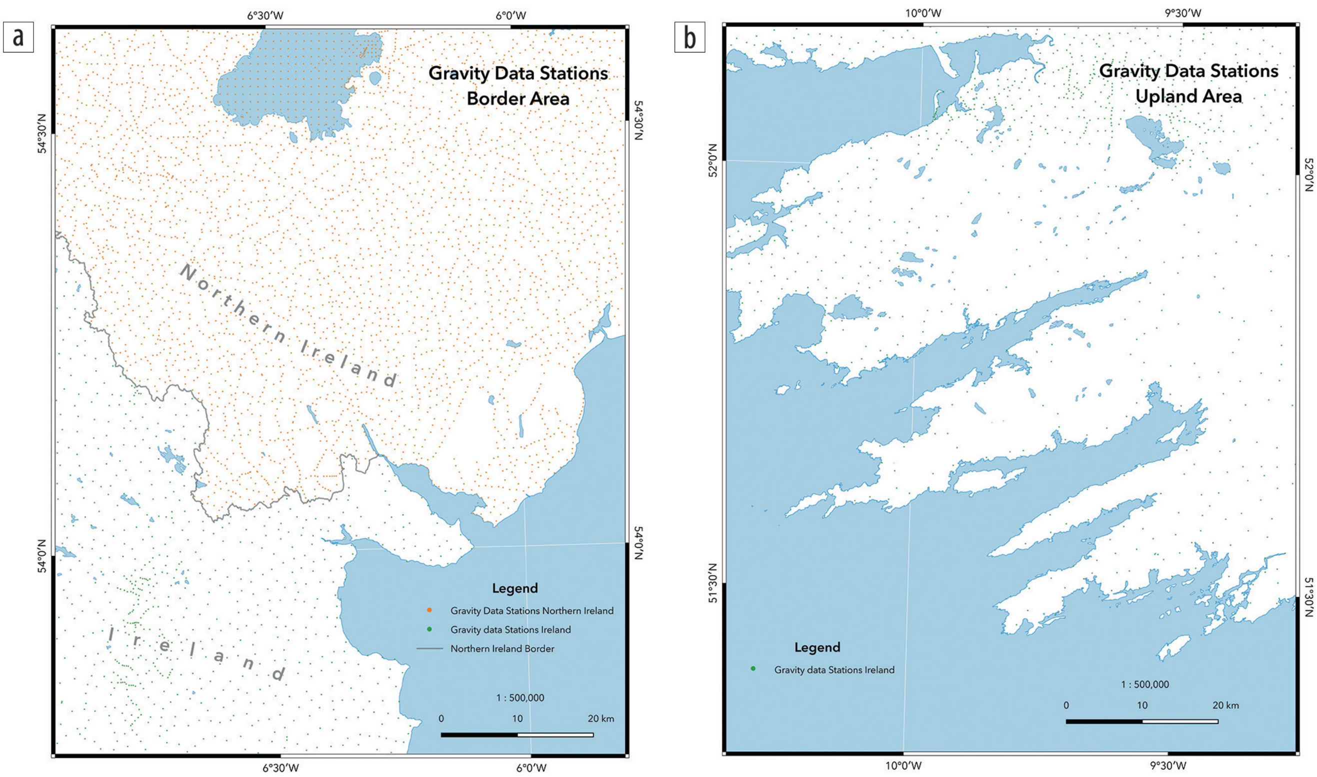

Figure 2. (a) Comparing data along the border between Ireland and Northern Ireland. (b) Illustration of missing data in upland area in southern Ireland.

The accuracy of the coordinates is evaluated as follows. The 6-inch mapping of Ireland was produced during the 19th century using separate Cassini projections for each of the 32 counties. IG75 coordinates for the gravity stations in Ireland were computed in the following manner. First, latitude and longitude values for the corners of the 6-inch maps on the county indexes were measured. Second, IG75 coordinates were computed for these corners. Finally, IG75 values were derived for the gravity stations. The accuracy of the IG75 coordinates for the gravity stations in Ireland was estimated by Murphy (1982) at approximately 60 m. The accuracy of the gravity data will be dealt with later in this paper.

Gravity data for Northern Ireland. The gravity data for Northern Ireland were collected over an extended period of time from 1949 to the 1990s. The format of the gravity data set for Northern Ireland, supplied by GSNI, is summarized in Table 2. A value of –99 in base station (column 10) represents a value of 981253.921 mGal for Pendulum House, Cambridge (the gravity base station for United Kingdom). The elevation of observed gravity (column 11) is given above Belfast Lough vertical datum (column 8). The free-air and Bouguer gravity anomalies (given in columns 12 and 14, respectively) are both computed using Gravity Formula 1967 (Heiskanen and Moritz, 1967). The terrain corrections (in column 13) were computed using the Hammer method (Hammer, 1939). Transformation of IG75 values (in columns 6 and 7) using Grid InQuest II software shows that the latitude and longitude of the reference ellipsoid (in columns 2–5) is not European Terrestrial Reference Framework 1989 (ETRF89). For example, when the IG75 coordinates (easting = 311950 m, northing = 451260 m) for gravity point U 1 1A04 are transformed using Grid InQuest II, a horizontal displacement of ≈ 70 m is found, thereby proving the reference ellipsoid used is not ETRF89. It should be noted that the differences between the Airy and the Airy modified systems are small and in most cases quite negligible. Therefore, it is assumed that the reference ellipsoid is either the Airy modified ellipsoid, or the Airy ellipsoid, similar to the geographic coordinates for Ireland.

Transformation of reference frameworks

Height of datums. Ordnance Survey mapping in *Ireland* has used a number of different height datums since 1824 when the Ordnance Survey was established in *Ireland*. The original height datum was defined in feet using ODD and was used for all the imperial scale maps from 1837 until 1970. A new *Ireland* height datum (MSL in meters) was established at Malin Head, County Donegal between January 1960 and December 1969 and was adopted in 1970. These 10 years of observations were not long enough to calculate a proper MSL, which requires a minimum period of 18.61 years (Haigh et al., 2011). As such, a third height datum was established at Belfast Lough by Ordnance Survey of Northern Ireland (OSNI) in 1957 from which the large-scale mapping of Northern Ireland is referenced. The relationship between these three height datums is shown in Figure 3.

Ordnance Survey Great Britain, Land and Property Services Northern Ireland (LPSNI, which includes OSNI), and OSi jointly developed a geoid model for the United Kingdom and *Ireland* (OSGM02) in 2002. This geoid model includes two surfaces — a scientific geoid model, which is a pure gravimetric geoid, and a corrector surface, which modifies the GPS heights to correspond to the leveling network in *Ireland* (Iliffe et al., 2003). The OSGM02 was updated in 2016 by Greaves et al. (2016) with a newer geoid model which amends both surfaces. The geoid model was revised using the new European Space Agency satellite-only gravity field model, and the corrector surface was revised to compensate for lower accuracy height data mainly in the west of *Ireland*. The effects of these changes to both surfaces combine to produce significant deviations between the two geoid models — OSGM02 and OSGM15 —so users have been advised to use OSGM15 since August 2016 (Greaves et al., 2016).

Gravity data positioning. The original geoid model OSGM02 was published by the three Ordnance Surveys using a software package called Grid InQuest in 2002 to supply transformations between the main reference frameworks used. In Ireland these reference frameworks include ETRF89, IG75, Irish Transverse Mercator (ITM), and Universal Transverse Mercator Zone 29. This software is available on the Ordnance Survey website for single-point conversions, or it can be downloaded free of charge for multipoint conversions. The web converter on the OSi website (https://www.osi.ie/services/geodetic-services/coordinate-converter) was updated to use the OSGM15 geoid model in 2016 (Greaves et al., 2016), and a new version of the software Grid InQuest II, which uses the OSGM15 geoid model, was also released.

Table 1. Format of gravity data supplied for Ireland by DIAS.

Station ID	Easting (m)	Northing (m)	Longitude (degree)	Latitude (degree)	Elevation (feet)	Observed gravity (mGal)	Bouguer anomaly (mGal)
5128	286190	184970	-6.7188	52.9087	444.6	-95.97	-21.7
5122	272060	180040	-6.9299	52.8665	173.0	-82.87	-21.2
22972	273610	179770	-6.9069	52.8639	186.7	-86.06	-23.3
⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱
2227	298280	161490	-6.5461	52.6957	265.6	-70.20	11.9
23039	297080	160660	-6.5641	52.6885	413.0	-80.68	10.9
23038	299050	160340	-6.5351	52.6852	304.0	-70.35	15.0

Table 2. Format of gravity data supplied for Northern Ireland by GSNI.

Station ID	Latitude			Longitude			Latitude	Longitude
	degrees	minutes	seconds	degrees	minutes	seconds	Decimal degrees	Decimal degrees
U1 1A02	55	17	76	-6	12	41	55.2960	-6.2068
U1 1A03	55	17	73	-6	14	24	55.2955	-6.2373
U1 1A04	55	17	98	-6	15	23	55.2996	-6.2538
⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱
U7 79A33	54	47	58	-6	29	21	54.7930	-6.4868
U7 79A34	54	48	1	-6	28	81	54.8002	-6.4802
U7 79A35	54	48	75	-6	28	40	54.8125	-6.4733

Table 3. Absolute gravity values published for Cambridge and Dunsink between 1950 and 2017.

Determinations of gravity	Cambridge	Correction Cambridge to Dunsink	Dunsink value
	mGal	mGal	mGal
1950 (Cook)	981265.0	+120.84	981385.84
1963 (Wollard and Rose)	981268.80	+120.30	981389.1
1980 (Woolard and Godley)	981253.94	+120.84	981374.78
1987 (Murphy)			981374.920
2017 (BGS)	981253.921		981374.761

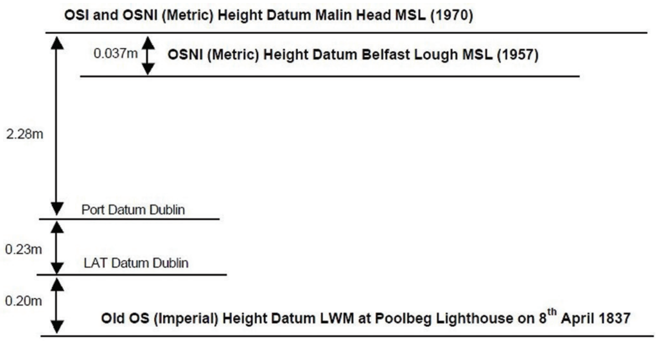

Figure 3. Relationship between the three height datums (in bold) used for mapping in *Ireland* (Prendergast, 2004a).

There are many methods available to transform coordinates between coordinate systems. OSi and LPSNI considered the following three options when creating the Grid InQuest software, and they tested their results against their network of GPS stations (Chen and Hill, 2005).

A Helmert transformation is normally used when converting coordinates from one reference surface to another and may provide an accuracy of 0.5 m. (The Helmert transformation, named after Friedrich Robert Helmert, is used in geodesy, which is the science of the measurement and mapping of the earth's surface.) A polynomial transformation uses 35 parameters (Chen and Hill, 2005) for latitude and longitude and another 16 reference parameters for height transformations and can provide an accuracy of 0.37 m. A third method involves a grid look-up method that uses 1656 parameters (Chen and Hill, 2005) and may supply accuracies of 0.33 m, but it requires a large amount of storage and computational power (Ordnance Survey of Ireland, 1999). The OSi and LPSNI used the polynomial transformation within their Grid InQuest software.

The coordinates of the gravity stations were derived from the 6-inch maps to an accuracy of 60 m (Murphy, 1982). Therefore, any one of these three transformation methods would be sufficient to transform the IG75 coordinates for the gravity data supplied. Further computations for unification of the vertical datum in *Ireland* with Europe requires data with spherical coordinates, which are not supplied by Grid InQuest II, so a new program was written with a Helmert transformation method using the formulas published by OSi (Ordnance Survey of Ireland, 1999). The ITM and ETRF89 values computed using the new program were checked against values computed using Grid InQuest II, and the results were within the decimeter level.

Combining the data for *Ireland*

Height data. The adjustment of imperial heights from the 6-inch Ordnance Survey maps in *Ireland* (ODD) to MSL (Malin Head) height datum of 1970 for Ireland requires two corrections (see Figure 3). First, convert the height in feet (foot of Bar O_1) (Ordnance Survey of Ireland, 1996) to meters by $h_1 \times 0.304\,800\,749\,1 = h_2$, with h_1 in feet and h_2 in meters. Second, correct for vertical offsets between datum by $h_2 - 2.71$ = height related to MSL at Malin Head.

Gravity data. The gravity data supplied for Ireland are referenced to the value at the pendulum base station at Dunsink Observatory in Dublin, which is itself tied to Pendulum House in Cambridge. The gravity data for Northern Ireland are related to Pendulum House in Cambridge (Masson-Smith et al., 1974), for which there have been a number of determinations of gravity values, so care must be taken when integrating the data. Initially, Cook (1950) published a value of 981265.0 mGal for Cambridge (Table 3). Then, 13 years later, Woollard and Rose (1963) published an amended value of 981268.8 mGal for Cambridge, yielding a value of 981389.1 mGal for Dunsink. Then, 17 years later, Woollard and Godley (1980) published new values for all national gravity base stations due to the adoption of the GRS67

Irish Grid easting	Irish Grid northing	Elevation	Bouguer density	Base station	Observed gravity	Free-air anomaly	Terrain correction	Bouguer anomaly
(meters)	(meters)	(decimeters)	($Mg=m^3$)		(hundredths of mGal)	(hundredths of mGal)	(hundredths of mGal)	(hundredths of mGal)
313880	451370	960	270	-99	25070	251	134	-700
311950	451260	841	270	-99	25347	166	127	-658
310890	451710	887	270	-99	25618	542	74	-387
⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱
297310	394970	115	270	-99	24605	1460	5	1335
297720	395780	115	270	-99	24765	1559	5	1434
298130	397160	115	270	-99	24847	1536	5	1411

normal gravity formula and the creation of the International Gravity Standardisation Network 1971 (IGSN 71). A Potsdam datum correction of -14 mGal (Woollard, 1979) was incorporated in the IGSN 71 values. These new values were 981253.94 mGal for Cambridge and 981374.78 mGal for Dunsink. Finally, the value of 981374.761 mGal used for Dunsink in this research was computed using the 2017 value for Pendulum House in Cambridge 981253.921 mGal and a correction value of 120.84 mGal between Dunsink and Cambridge.

Error estimate of normal gravity due to coordinate displacements. There is another source of errors in free-air gravity anomalies. The coordinates of the corners of the 6-inch maps in *Ireland* were computed by the DIAS (Murphy, 1956). These latitudes and longitudes were derived on the Airy ellipsoid using the coordinates of the county origins and the county indexes for the 6-inch maps. Their accuracy is estimated at one second in both latitude and longitude (≈ 30 m), but at the county boundaries these errors can be as large as 60 m (Murphy, 1982). Errors in the normal gravity introduced by latitudinal displacement of 60 m gives an error of ≈ 0.3 mGal in gravity, that is computed from:

$$\Delta\gamma_0 = \frac{\partial\gamma_0}{\partial\phi}\Delta\phi\,, \tag{1}$$

where $\frac{\partial\gamma_0}{\partial\phi}$ is given by

$$\frac{\partial\gamma_0}{\partial\phi} = -\frac{\gamma_a\cos(\phi)\sin(\phi)\left(e^2 v\sin^2(\phi) - 2v - e^2\right)}{\left(1-e^2\sin^2(\phi)\right)^{\frac{3}{2}}}, \tag{2}$$

where γ_0 is the normal gravity on the surface of the ellipsoid, which can be computed using the formula of Somigliana Green (Heiskanen and Moritz, 1967; Moritz, 1980); $\Delta\phi$ is the latitudinal displacement; v is the radius of curvature in the prime vertical; and e is the first eccentricity (Heiskanen and Moritz, 1967). The gravity surveys carried out in Ireland derived IG75 coordinates for the gravity stations from the coordinates of the corners of the 6-inch maps, and it is assumed a similar process was used in Northern Ireland. The accuracies of the coordinates for the corners of the 6-inch maps are reflected in the accuracies of IG75 coordinates supplied for the gravity stations. The gravity data sets for both Northern Ireland and Ireland provide IG75 coordinates (easting and northing), which should be transformed into ETRF89 coordinates using the polynomial transformation in Grid InQuest II.

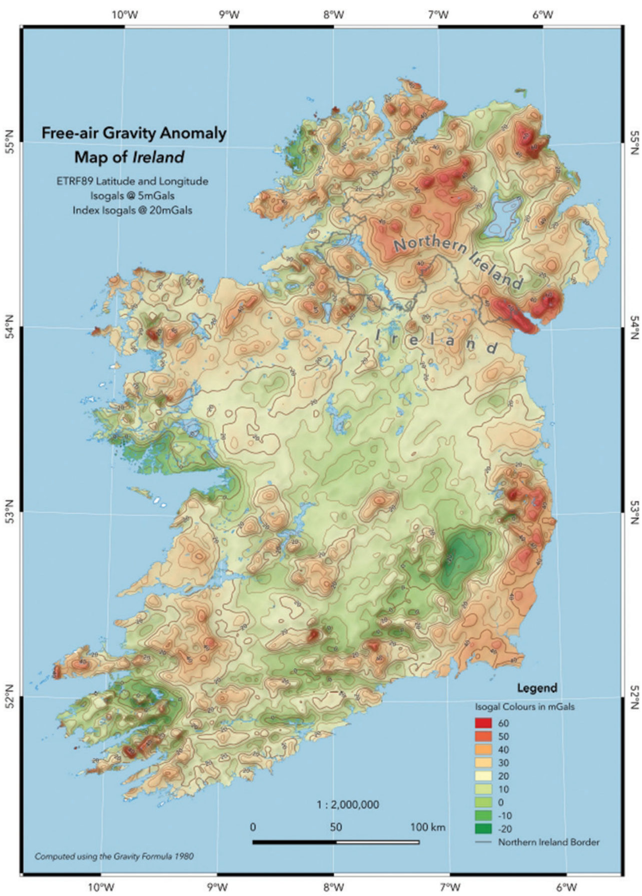

Figure 4. New free-air gravity anomaly map for *Ireland.*

Gravity anomaly mapping and data format

The purpose of a gravity survey is to provide information on local gravity anomalies, which is the difference between observed gravity on the earth's surface and its corresponding value from a gravity field model. Since lateral variations in gravity anomalies are related to rock density distributions within the earth, gravity measurements help us understand the internal structure of the earth. A number of corrections must be applied to the measured gravity values to predict the nature of the subsurface.

The elevation of the stations where each gravity measurement is carried out must be referred to a reference datum. The simplest theory for reduction of gravity to the geoid is to neglect all the topographic masses between observation points on the surface of the earth and their corresponding points on the geoid. This is called the free-air correction or reduction, $\delta g^{FA}(\Omega)$, and when combined with the removal of normal gravity $\gamma_0(\phi)$, leaves the free-air gravity anomaly $\Delta g^{FA}(\Omega)$ (Torge, 1989)

$$\Delta g^{FA}(\Omega) = g(\Omega) - \gamma_0(\phi) + 0.3886 \times H(\Omega), \qquad (3)$$

where $\Omega = (\theta, \phi)$ is the horizontal locations of computation point, and H is the orthometric height of computation point. A new free-air gravity anomaly map is published for *Ireland* (Figure 4). Two maps at enlarged scales are also illustrated along the border between Ireland and Northern Ireland and in an upland area in the southwest (Figure 5).

The free-air gravity correction accounts only for the variation in the elevation of the gravity stations with respect to the center of the earth, and the gravitational attraction of topographical masses is not considered. As a result, the free-air gravity anomalies display a strong correlation with the topography. Therefore, the gravitational attraction of topographical masses must be considered in a different way. The simplest way is to approximate the topography surrounding the gravity point by an infinite plate of thickness H with constant density ρ_0 equal to mean topographical density ρ_0 = 2670 kg.m^{-3}. This does not imply that a whole region is modeled by such a plate. On the contrary, an individual plate is considered for each gravity point.

Then, the Bouguer gravity anomalies of a plate, $\Delta g^{B}(\Omega)$, with topographical density ρ_0 and elevation $H(\Omega)$, are given by the equation (Heiskanen and Moritz, 1967)

$$\Delta g^{B}(\Omega) = \Delta g^{FA}(\Omega) + \delta g^{BP}(\Omega), \qquad (4)$$

in which $\delta g^{BP}(\Omega)$ denotes the Bouguer plate reduction (Heiskanen and Moritz, 1967) as

$$\begin{aligned} \Delta g^{BP}(\Omega) &= -2\pi G \rho_0 \, H(\Omega) \\ &= -0.1119 H \,[\text{mGal}], \end{aligned} \qquad (5)$$

with standard density ρ_0 = 2.67g cm^{-3}, the Newton's gravitational constant (Mohr and Taylor, 2005) $G = (6.6742 \pm 0.001) \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, and H in meters. Figure 6 shows a new simple Bouguer gravity anomaly for *Ireland*. Two maps in Figure 7 show the Bouguer gravity anomalies along the border between Ireland and Northern Ireland and the upland area in the southwest of Ireland on an enlarged scale, respectively.

The format outlined in Table 4 is proposed for the supply of gravity data for *Ireland* in the future. The data contain ITM coordinates, which is the spatial reference system currently recommended by the National Mapping Agency and the professional

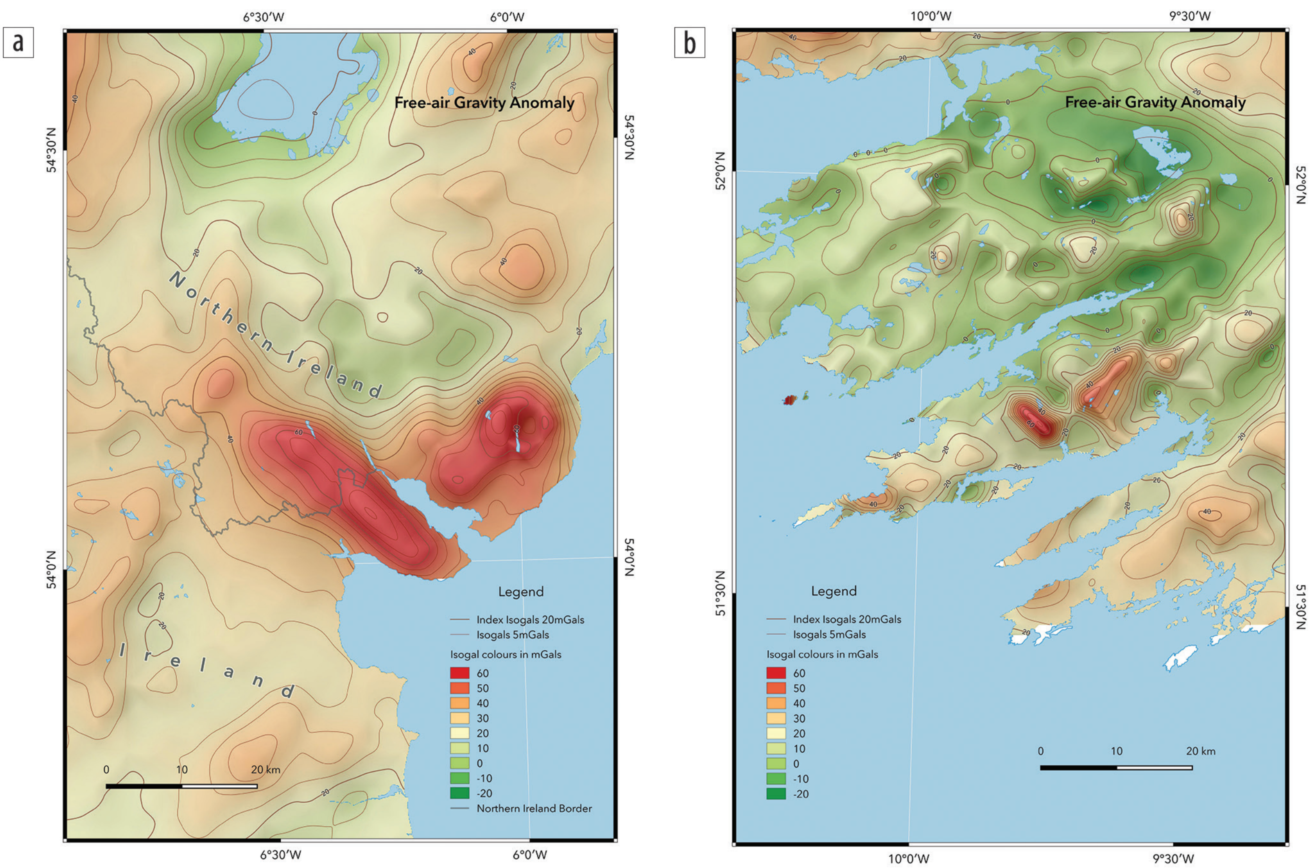

Figure 5. (a) Illustration of free-air gravity anomalies in the upland area and (b) along the border between Ireland and Northern Ireland.

bodies for surveyors in *Ireland*. The ETRF89 coordinates computed using Helmert transformation formula (Ordnance Survey of Ireland, 1999) from IG and ITM coordinates in Ireland and Northern Ireland data, respectively. The height of gravity stations is computed above the MSL at Malin Head. Free-air and Bouguer gravity anomalies are computed using Gravity Formula 1980.

Conclusions

Gravity data were collected over an extended period of time from 1949 until the 1980s in Ireland and until the 1990s in Northern Ireland. During this time, the absolute gravity values for the base stations in Cambridge and Dunsink were amended on a number of occasions as well as the correction values between these base stations. Similarly, there are three height datums in *Ireland* (one historical — Poolbeg), and care must be taken to interpret the data supplied correctly to allow them to be properly integrated. Finally, data are supplied using old spatial reference systems (Airy ellipsoid, and IG75 coordinate system), which predate the modern system used today (ETRF89 and ITM). Consequently, a new format for publishing the gravity data is proposed.

Users of the data should be cognizant of the survey methods used to collect the gravity data and height data (combination of survey and estimation) and how coordinates of the gravity stations were obtained (using graphical methods). The gravity data for *Ireland* contain an error of 0.15 mGal due to inaccuracies of elevation and errors of 0.3 mGal due to inaccuracies of position. Understanding the quality of the data allows users to use the data

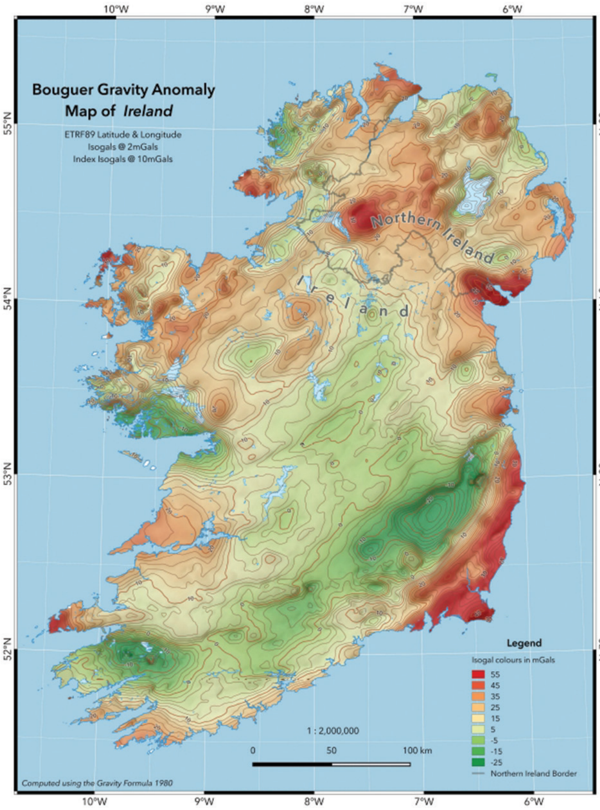

Figure 6. New Bouguer gravity anomaly map for *Ireland*.

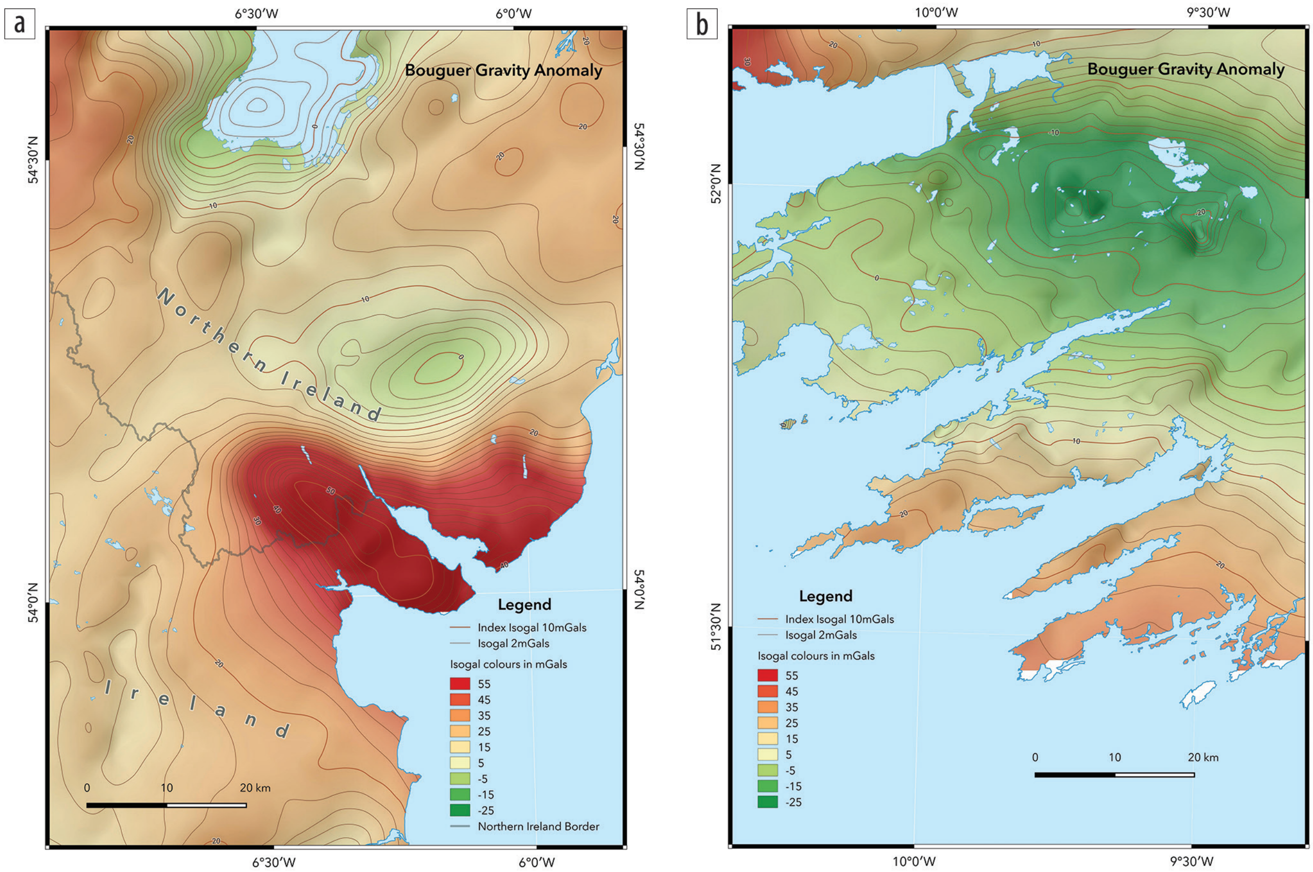

Figure 7. (a) Illustration of Bouguer gravity anomalies in the upland area and (b) along the border between Ireland and Northern Ireland.

Table 4. Proposed new format for supply of gravity data for *Ireland*.

ITM		ETRF89		Height	Absolute gravity	Free-air anomaly	Bouguer anomaly
Easting(m)	Northing(m)	Longitude	Latitude	MSL Malin-Head	Unit: mGal	Unit: mGal	Unit: mGal
686121.73	685007.15	-6.71965864	52.90900131	132.804	981278.791	-7.369	-22.230
671994.81	680078.28	-6.93072519	52.86678780	50.020	981291.891	-16.135	-21.732
673544.47	679808.34	-6.90777491	52.86415260	54.196	981288.701	-17.807	-23.871
676703.78	679728.34	-6.86088408	52.86299301	65.931	981283.031	-19.754	-27.131
675514.04	679128.48	-6.87868799	52.85777082	68.979	981282.741	-18.648	-26.366
⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱
708308.04	818978.02	-6.34367552	54.10845447	128.354	981455.731	64.259	49.897
707228.26	818498.13	-6.36035084	54.10437010	99.764	981465.271	65.328	54.165
707238.25	817668.30	-6.36049222	54.09691447	92.205	981466.061	64.427	54.110
712307.20	819347.92	-6.28240992	54.11091985	0.948	981473.201	42.200	42.094
711887.28	818178.18	-6.28926153	54.10050491	63.127	981460.101	49.185	42.121

appropriately, and this knowledge is essential if terrestrial data are to be combined with satellite gravity data.

In the near future, it is intended that the Bouguer and free-air gravity anomalies maps published here for *Ireland* will be supplied to the Geological Survey of Ireland for dissemination. It is also proposed that the DIAS transfers its gravity data for Ireland to the Geological Survey of Ireland for dissemination free of charge in the new format. TLE

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Data and materials availability

Data for Ireland are available from the Dublin Institute for Advanced Studies, and data for Northern Ireland are available from the Geological Survey of Northern Ireland.

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Conclusion

Unified absolute gravity, free-air gravity anomaly, and Bouguer gravity anomaly in ITM and ETRF89 coordinates for Ireland-Northern Ireland have been computed and presented. This unified dataset enables the delivery of accurate information for many potential scientific uses in geophysics, geodesy, oceanography and engineering applications in Ireland. The quality assessment of the Ireland-Northern Ireland terrestrial gravity data (supplied by DIAS and GSNI) has been analysed and presented. Since different institutions have collected the data over many decades, it was given in different coordinate systems, with different gravity base stations and different vertical datums. In the 1970's the worldwide gravity system IGSN71 was introduced in Ireland and Northern Ireland, and a Potsdam datum correction of −14 mGals was incorporated in base stations in Ireland for Dunsink (981 374.78 mGal) and Northern Ireland for Cambridge (981 253.94 mGal). The historical gravity survey was converted to a current gravity system.

Absolute gravity values calculated from base stations with their absolute positions allow a new method of analysis and data usage in Ireland-Northern Ireland. For instance, these allow geological mapping of the whole country by introducing minimum errors along the borderlines between Ireland and Northern Ireland data sets; or the determination of the geoid from the gravity disturbance instead of the traditional method using free-air gravity anomalies (note that in this study, the traditional method is used).

The free-air gravity anomalies computed in paper 1 (Chapter 2), are adjusted (topographical effects are determined and removed from gravity data) in Paper 2 (Chapter 3), producing the topography-reduced free-air gravity anomalies on the Earth's surface. Then the corrected free-air gravity anomalies, in Paper 3 (Chapter 4), are downward continued to the geoid for the determination of topography-reduced free-air gravity anomalies on the geoid. Furthermore, these are used to unify the Ireland vertical datum with the NAP vertical datum in Amsterdam in Paper 4 (Chapter 5).

3

Paper 2: The optimal topographic grid resolution for 1-cm geoid determination over Ireland

The term *terrain effect* is used to express the gravitational effects of topographical masses on gravity anomalies, the deflection of the vertical, and other observed quantities. It can be classified according to the location of anomalous masses. Topographical effects are the direct influence of the visible topography in mountains areas; isostatical effects account for a hypothesised isostatic compensation, while the residual terrain model (RTM) effects account for short-wavelength topographic irregularities referring topographic elevations to a smooth mean elevation surface, which may be defined, for instance, by the spherical harmonic expansion of topographic heights.

Before the determination of residual geoid undulation, the terrain corrections have to be determined and removed from free-air gravity anomalies.

Digital Elevation Models (DEM) are commonly used to compute the topographical effects on gravity and potential. Computation of topographical effects on geoid determination depends on the resolution of the input data, and can often be very time-consuming. High spatial resolution terrain data are crucial for gravity field modelling, especially in the areas with high elevated topography. Usually, the method of computing topographic effect from DEM's is applied using different approximations, and the results of computations are area-dependent.

The purpose of paper 2 is to compute topographical effects on gravity and potential. It is also intended to show the correlation between the spatial resolution of DEM's, and the elevation of computation points, and also the correlation with elevation of the data surrounding the computation points. These correlations are expected to show the most suitable spatial resolution of DEM's to provide intended geodetic accuracies.

In this study, the formulation of Helmert's second condensation method in the determination of topographical effects is reviewed. The topographical effects of several different spatial resolutions using Helmert's second condensation method over three different elevated topographies in Ireland are investigated numerically. Correlations between topographical effects and elevation of data points are analysed and displayed.

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The optimal topographic grid resolution for 1-cm geoid determination over Ireland

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Abstract

The topographical effect on geoid height determination in Ireland, using Helmert's second method of condensation, is computed. The size and spatial content of topographical effect depends on the topographical heights and their spatial resolution. We determine the optimal topography grid resolution for 1cm geoid determination over Ireland. The numerical results with several different quadrangle grid resolutions, show that it is not guaranteed to compute a 1cm geoid with the finest available spatial grid resolutions. To reach the accuracy of the order of ± 1 mm in value of primary indirect effect on the geoid height determination, it is sufficient to use 1000 m quadrangle grid resolutions (QGR) in Ireland. In contrast, unpredictable outcomes in the determination of direct topographical effects on gravity suggest that topographical surface may not be a deterministic function.

Keywords: Digital Elevation Model, Topographical Effect, Stokes' Integral

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1. Introduction

The classical solution of the geodetic boundary value problem (BVP) using Stokes's formula for gravimetric geoid determination assumes that there are no masses outside the geoid and the gravity measurements are referred to the geoid. However, the presence of the topography violates these assumptions because: firstly the geoid is a surface extended through the continents, so there are topographic masses of continents between the geoid and the land surface (the so-called Topographic Mass, TM); secondly, the geoid is a mathematical surface, on which gravity measurements cannot be directly carried out. One way to adjust Stokes' requirements is to replace the TMs with an auxiliary body situated below or on the geoid and determine the changes in the gravitational potential due to this replacement.

In the intervening period of the theory of physical geodesy, a wide range of methods has been developed, each optimised for different use cases in many areas worldwide. To name a few common methods:

- the Airy-isostatic reduction method, where the topographical masses are removed to fill roots of the continents bringing the density from its constant value to that of the upper mantle (Heiskanen and Moritz, 1967);

- the Residual Terrain Model (RTM) scheme (Forsberg, 1984), which evaluates the gravitational effects of the mass anomalies relative to a mean elevation surface;

- Helmert's first and second method of condensation, where in the first method the condensation layer is situated 21 km below sea-level, while in the second method the masses are condensed onto the geoid Heiskanen and Moritz (1967); Martinec and Vanicek (1994a,b); Heck (2003).

The choice of each method is area-dependent; hence, the optimal reduction method can be chosen relative to the accuracy of computed gravimetric geoid heights compared to that of geometric geoid heights obtained from GPS/levelling data. The use of Helmert's second method of condensation is one of the common gravimetric reduction methods in the context of Topographical Effects (TE) for the determination of geoid height, having been applied by many scientists, e.g., (Moritz, 1968; Wang, 1990; Heck, 1993, 2003; Martinec et al., 1993; Vanicek and Martinec, 1994; Martinec and Vanicek, 1994a,b; Martinec, 1998; Nahavandchi, 1998; Sjöberg, 2000; Tziavos and Sideris, 2013).

However, this study is not intending to choose the optimal reduction method for Ireland, but to establish a correlation between TEs and the choice of grid resolution as a function of topographical height using Helmert's second method of condensation. The computation of TEs by numerical integration is a time-consuming process. In the last decade, an ever-growing array of DEM datasets has become available across a range of resolutions and spatial scales. For large areas, such as a country or a continent, the power of today's computers is insufficient for computing TE from new DEMs (e.g., SRTM 30 m satellite datasets, lidar/ stereo-photogrammetry/ Structure from Motion (SfM) datasets). Therefore, this study aims to study not only the correlation of TEs with topography to select an optimum grid resolution for test areas but also to consider the computational considerations to decrease computation time. For this purpose, TEs computed for three possible attributes of topography (low/high/mixture of both high and low elevation) with various grid resolutions are numerically analysed.

2. Some Preliminaries

The topographical masses are the masses outside the geoid and below the topographical surface (see Figure 2). The gravitational potential V^t generated by the topographical masses is

$$V^t(r,\Omega) = G\int_{\Omega_0}\int_{r'=r_g(\Omega')}^{r_t(\Omega')}\frac{\rho_0(r',\Omega')}{L(r,\psi,r')}r'^2 dr' d\Omega' . \quad (1)$$

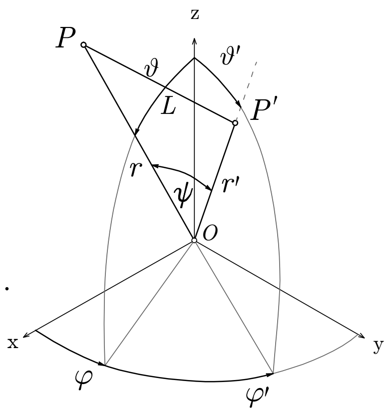


Figure 1: Spherical coordinates of the computation point $P(r,\Omega)$, an integration point $P'(r',\Omega')$, the distance L and solid angle ψ between P and P'.

where G is the Newton's gravitational constant, $G = 6.67 \cdot 10^{-11}$ m^3.kg^{-1}.s^{-2}, $\rho_0(r',\Omega')$ is the mass density inside the Earth's interior located at $P'(r',\Omega')$, $L(r,\psi,r')$ is the distance between P and P' and ψ is the angular distance (spherical solid angle) between the geocentric directions $\Omega = (\vartheta,\varphi)$ and $\Omega' = (\vartheta',\varphi')$ (Figure 1), where

$$\cos\psi = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi-\varphi') , \quad (2)$$

with Ω_0 being the full solid angle and $d\Omega' = \sin\vartheta' d\vartheta' d\lambda'$. The argument notation in $L(r,\psi,r')$ is used to emphasize the fact that L depends on radial distances r and r', and the angular distance ψ.

$$L(r,\psi,r') := \sqrt{r'^2 - 2rr'\cos\psi + r^2} . \quad (3)$$

In the limiting case, the topographical masses may be compensated by a thin mass layer located on the geoid somewhat like a glass sphere made over very thin but very robust glass (Hofmann-Wellenhof and Moritz, 2006) (see Figure 2). This

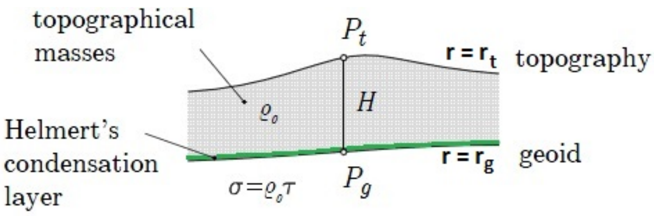


Figure 2: Helmert condensation layer of density σ

kind of compensation is called *Helmert 2nd condensation* approximating the actual potential of the topographical masses V^t by the potential of a single layer V^c described by Newton's surface integral as

$$V^c(r,\Omega) = GR^2 \int_{\Omega_0} \sigma(\Omega')L^{-1}(r,\psi,R)d\Omega' \;, \tag{4}$$

where L^{-1} is the reciprocal distance $1/L$ and $\sigma(\Omega)$ is a surface density of Helmert's condensation layer,

$$\sigma(\Omega) = \rho_0 \tau(\Omega) \;, \tag{5}$$

with

$$\tau(\Omega) = H(\Omega)\left(1 + \frac{H(\Omega)}{R} + \frac{H^2(\Omega)}{3R^2}\right) \;. \tag{6}$$

The difference between the gravitational potential V^t of TMs and the gravitational potential V^c of compensation masses, is defined as the residual topographical potential δV,

$$\delta V := V^t - V^c. \tag{7}$$

Martinec (1998) describes the effect of TMs on geoid height determination by utilising δV in the following three terms:

1. *The primary indirect topographical effect (PITE) on potential.*

   Assuming the density of topographic masses is constant ($\rho_0 = 2.67$ g/cm^3)

and the geoid is a sphere of radius $r_g(\Omega) = R = 6371\ km$[1], in which the radius on the topography, $r_t(\Omega)$, is

$$r_t(\Omega) = R + H(\Omega), \tag{8}$$

then the PITE is given by (Martinec, 1998, Eqn. 3.51):

$$\begin{aligned}\delta V(\Omega) = &- 2\pi G\rho_0\, H^2(\Omega)\left(1+\frac{2}{3}\frac{H(\Omega)}{R}\right)+ \\ &+ G\rho_0 \int_{\Omega_0}\left[\widetilde{L^{-1}}(r,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')} - \frac{R^2[\tau(\Omega')-\tau(\Omega)]}{L(R,\psi,R)}\right] d\Omega'\ ,\end{aligned} \tag{9}$$

where he first term in Eqn. (9) is the Bouguer term and the second term is the terrain roughness term. We introduced the symbol $\widetilde{L^{-1}}(r,\psi,r')$ for an indefinite radial integral of the Newton kernel,

$$\widetilde{L^{-1}}(r,\psi,r') := \int_{r'} \frac{r'^2}{L(r,\psi,r')} dr'\ , \tag{10}$$

which can be evaluated analytically (Gradshteyn, 1979)

$$\begin{aligned}\widetilde{L^{-1}}(r,\psi,r') =& \frac{1}{2}\left(r'+3r\cos\psi\right)L(r,\psi,r')+ \\ &+\frac{r^2}{2}(3\cos^2\psi-1)\ln\left|r'-r\cos\psi+L(r,\psi,r')\right|+C\ ,\end{aligned} \tag{11}$$

where the constant C may depend on the variables r and ψ only.

The unit of PITE is m^2/s^2. Note that to correct the geoidal heights N by this effect, δV is divided by the normal gravity γ_Q,

$$\delta N(R,\Omega) = \frac{\delta V(R,\Omega)}{\gamma_Q}\ , \tag{12}$$

[1]The relative error introduced by approximating r_g is of the order of 3×10^{-3} in the classical problems (Heiskanen and Moritz, 1967), which at most causes a long-wavelength error of 0.5 meters in geoidal heights.

which is the primary indirect topographical effect on the geoid. The unit of δN is meters.

2. *The direct topographical effect (DTE) on gravity.*

   The DTE is the radial derivative of the residual potential δV taken at a point on topography $(R+H(\Omega),\Omega)$ (Martinec, 1998, Eqn. 3.45):

$$\delta A(\Omega) = G\rho_0 \int_{\Omega_0} \left[\left. \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \right|_{r'=R}^{r_t(\Omega')} - \left. \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \right|_{r'=R}^{r_t(\Omega)} \right.$$
$$\left. - R^2[\tau(\Omega') - \tau(\Omega)] \frac{\partial L^{-1}(r,\psi,R)}{\partial r} \right]_{r=r_t(\Omega)} d\Omega' \ . \quad (13)$$

   The unit of DTE is mGal.

3. *The secondary indirect topographical effect (SITE) on gravity.*

   This effect is expressed by means of the PITE, δV_{P_g}, at a point on the geoid multiplied by $2/R$,

$$\delta S(\Omega) = \frac{2}{R} \delta V_{P_g}(\Omega) \ . \quad (14)$$

   The unit of this effect is mGal.

Notice that the radial derivative of the Newton surface and volume integrals, which is required for computing $\delta A(\Omega)$, is

$$\frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} = \int_{r'} r'^2 \frac{\partial L^{-1}(r,\psi,r')}{\partial r} dr' \ , \quad (15)$$

and using the derivation of $(u^\alpha)' = \alpha u' u^{\alpha-1}$ in Eqn. (3),

$$\frac{\partial L^{-1}(r,\psi,r')}{\partial r} = -\frac{r - r'\cos\psi}{(r'^2 - 2rr'\cos\psi + r^2)^{3/2}} \ , \quad (16)$$

where substituting the expression (16) in (15), the analytical expression is given by,

$$\frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} = \left(3r^2\cos\psi + r(r' - 6\cos\psi^2 r') + \cos\psi r'^2\right) L^{-1}(r,\psi,r') \\ + r(3\cos\psi^2 - 1)\ln|L(r,\psi,r') - r\cos\psi + r'| + C \; . \quad (17)$$

3. Digital Elevation Models in Ireland

Ordnance Survey Ireland (OSi) and Land Property Services of Northern Ireland (LPS-NI), formerly Ordnance Survey Northern Ireland (OSNI), provide a regular grid DEM for their respective areas at a 10 m grid resolutions. The DEM data acquired for this paper were re-sampled by the providers and supplied at a 50 m grid resolution. The DEM data supplied for the Ireland are provided in the Irish Transverse Mercator (ITM) spatial reference system (the geographic coordinate system for Ireland); heights are related to the Malin Head vertical datum. These data are derived from aerial photographs for the 10 m contours on the discovery map series at $1:50,000$ during the 1990's. The first half of the production used direct plotting of the 10 m contours and the second half of production used autocorrelation to create a regular grid at 10 m intervals from which the 10 m contours were derived. The accuracy of these height data are quoted as ± 2.5 m (OSi, 2015).

The DEM data for Northern Ireland are derived from 1:10,000 orthophotography between 2003 and 2006 (OSNI, 2008). The DEM is provided in the Irish Grid (1975) spatial reference system; the heights are referenced to the vertical datum at Belfast Lough. The relationship between the two vertical datums is as follows

(Sajjadi et al., 2020):

$$\text{Belfast Lough height} + 0.037\ \text{m} = \text{Malin Head Height} \tag{18}$$

OSNI (2008) quotes the accuracy of the Northern Ireland data as:

— 65% of the DEM data is within ± 1.0 metre accuracy,

— 95% of the DEM data is within ± 2.0 metre accuracy,

— 99% of the DEM data is within ± 3.0 metre accuracy.

In order to obtain a unified spherical coordinate system for *Ireland*[1], firstly, the vertical components of Northern Ireland's DEM were adjusted to that of Ireland vertical datum by applying constant vertical offsets (0.037 m) between datums. Then the horizontal components of DEMs in IG/ITM were converted to the geodetic coordinate using seven-parameter transformation (see for example formulas published by OSi (Ireland, 1999)). Furthermore, the geodetic coordinates were converted into spherical coordinates using the relationship between geodetic and spherical coordinates (see for example (Hofmann-Wellenhof and Moritz, 2006)). Finally, QGR at 50 m, 100 m, 200 m, 300 m, 400 m, 500 m, 1000 m, 1100 m, 1200 m, 1300 m, 1400 m, 1500 m, 2000 m and 3000 m were interpolated for numerical investigations in this study.

4. Computational Considerations

The determination of topographical effects from DEMs is a very time consuming process, particularly when computations are required for large areas, such

[1] It should be noted that in this paper, italic text for *Ireland* represents the two jurisdictions of Ireland and Northern Ireland. Therefore Ireland refers to the 26 counties in the south, and *Ireland* refers to the 32 counties of the whole island.

as a country or a continent. With a fine grid resolution for instance, 50 m QGR, computations are beyond what a multi-processor computer can accomplish within a reasonable time-frame for an area such as Ireland. Tests carried out in this paper have resulted that increasing the spatial resolution of DEM by a factor of two increases CPU computational time by a factor of fourteen. Although the computational time is a factor to be taken into account, it is less important since it is the spatial resolution of DEM which is critical for improving accuracy required when a precise geoid is determined.

Modern DEMs created using Lidar technology have QGR's as fine as 1m (see for example Ordnance Survey Ireland Lidar data captured between 2006 and 2008), so the power of today's computers is insufficient for computing TE for such a fine resolution.

The Fast Fourier Transform (FFT), which relies on linearisation and series expansions of the non-linear terrain effect integrals, provides a reduction in computational time by several orders of magnitude, compared to space domain integration methods (Forsberg, 1985). In the FFT, the higher-order terms of the radially integrated Newton kernel expressed by the Taylor series expansion are neglected. In addition, the reciprocal distance $1/L$ is approximated by the planar distance $1/\ell_0$ i.e., $\ell_0 = L(R, \psi, R)$ (Omang and Forsberg, 2000, Eqn. (15)), which is a sufficiently good only if $\ell_0 > H(\Omega')$. This condition is violated for computing topographical effects in rough terrain. From this reason, the FFT has not been used in this study.

4.1. Three areas considered

To be able to accomplish the investigations of topographical effects in view of precise geoid determination over Ireland within a reasonable time-frame, three

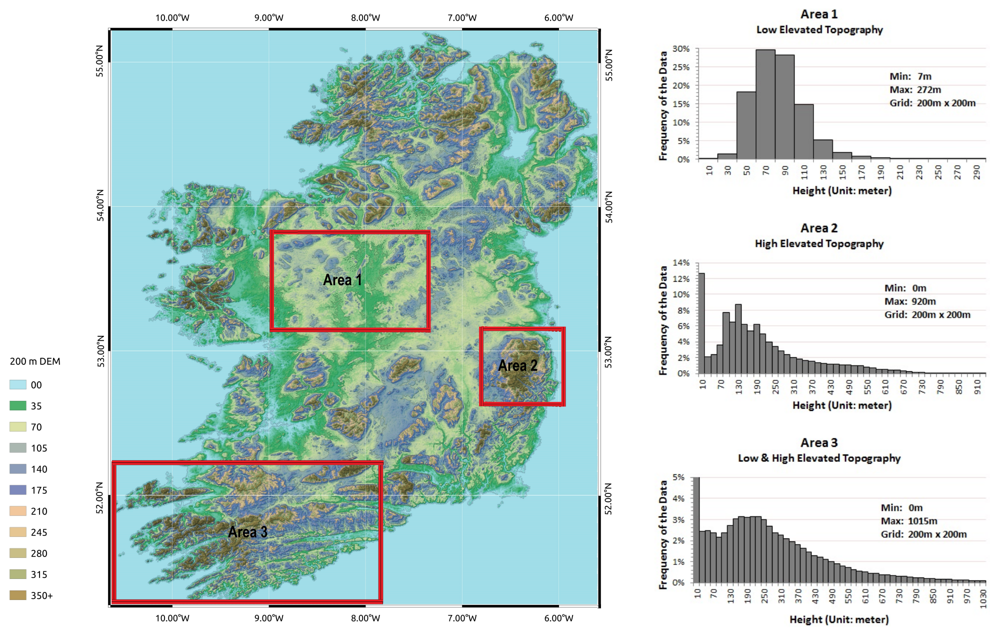


Figure 3: Left - DEM for Ireland and Northern Ireland in 200 m QGR shows the three computation areas. Right - The histograms show the distributions of topographic heights over the three test areas.

test areas were selected for numerical investigations (see Figure 3):

1. Low elevated topography (Area-1 in Counties Roscommon and Galway, Longitude 08.99°W to 07.36°W and Latitude 53.13°N to 53.81°N),
2. High elevated topography (Area-2 in Counties Dublin and Wicklow, Longitude 6.79° W - 5.97° W and Latitude 52.64° N - 53.13° N),
3. Combination of both low & high elevated topographies (Area-3 in Counties Cork and Kerry, Longitude 10.30° W - 7.54° W and Latitude 51.15° N - 52.12°).

4.2. *Integration area restricted*

The gravitational potential of TMs of finite thicknesses behaves like the potential of a thin layer when it is observed from a larger distance. This is explained by the behaviour of integration kernels generating the potential of the gravitational potential $V^t(r,\Omega)$ and $V^c(r,\Omega)$. Figure 4 illustrates the behaviour of the two kernels for the determination of PITE and DTE relative to maximum elevation in Ireland, which is the *Carrauntoohil* elevation of 1039 m (OSi, 1995).

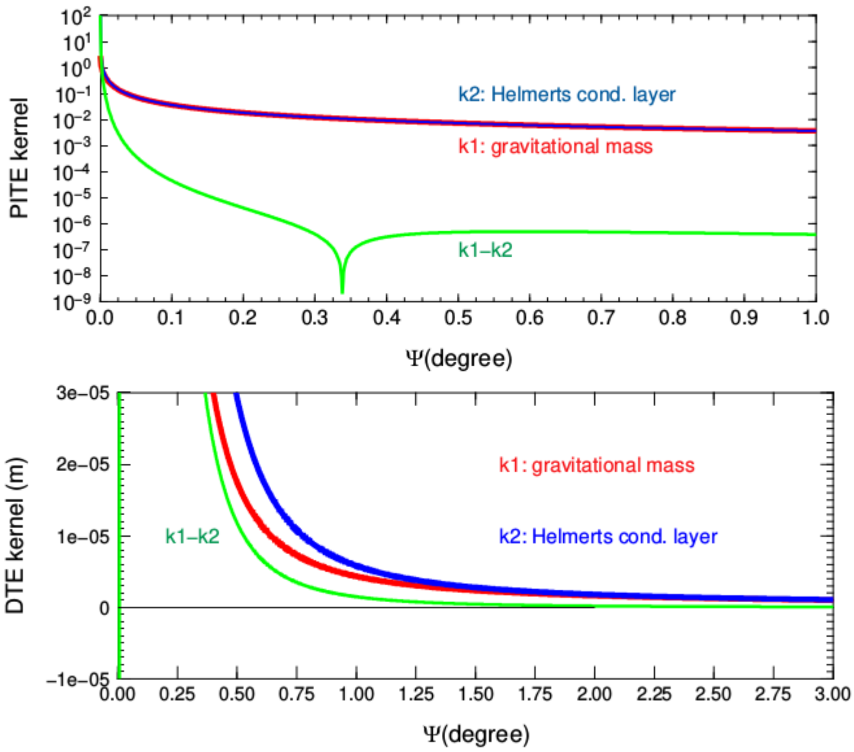


Figure 4: The behaviour of K_t and K_c in the dependence of varying the angular distance ψ in the determination of PITE and DTE in Ireland. The elevation of computation and integration point are fixed to 1039 m and 1 m respectively.

In the computation of PITE Eqn. (9), when the angular distance ψ between the computation point and integration point increases, the integration kernel

$$K^{c}_{pite}(R,\psi,R) = R^2\left(\frac{\tau(\Omega')}{L(R,\psi,R)}\right) \tag{19}$$

generating the potential of Helmert's condensation layer $V^c(r,\Omega)$ approaches the integration kernel

$$K^{t}_{pite}(R,\psi,H(\Omega')) = \widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H(\Omega')} \tag{20}$$

generating the gravitational potential $V^t(R,\Omega)$. Thus, the differences between two kernels $K^t_{pite} - K^c_{pite} = \delta K_{pite}$ decreases (see Figure 4-PITE kernel). In determination of DTE Eqn. (13) a similar decrease occurs for the difference between two kernels ($K^t_{dte} - K^c_{dte} = \delta K_{dte}$), when the integration kernel

$$K^{c}_{dte}(H(\Omega),\psi,R) = R^2\tau(\Omega)\frac{\partial L^{-1}(r,\psi,R)}{\partial r}\ , \tag{21}$$

generating the potential of Helmert's condensation layer $V^c(r,\Omega)$ approaches the integration kernel

$$K^{t}_{dte}(H(\Omega),\psi,H(\Omega')) = \frac{\partial\widetilde{L^{-1}}(r,\psi,r')}{\partial r}\Bigg|_{r'=R}^{R+H(\Omega')}\ , \tag{22}$$

generating the gravitational potential $V^t(r,\Omega)$ of the TMs (see Figure 4-DTE). This also means that the magnitudes of δK_{pite} or δK_{dte} are largest in the immediate neighbourhood of the computation point. The choice of varying angular distance and fixing the elevation of integration point to 1039 m and computation point to 1 m or vice versa, enables us to determine the most attainable differences (maximum or minimum) between the kernels in question.

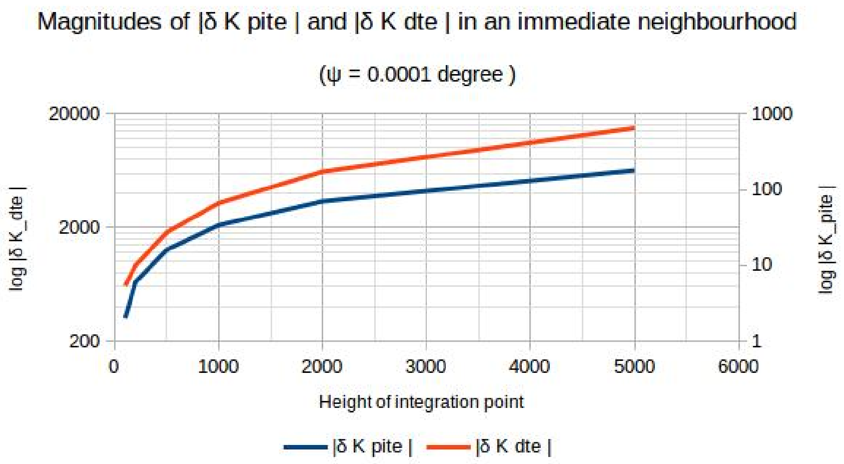


Figure 5: Magnitudes of $|\delta K_{pite}|$ and $|\delta K_{dte}|$ in an immediate neighbourhood ($\psi = 0.0001°$) of the computation point with $H(\Omega) = 1$ and varying height of integration point.

Furthermore, a numerical examination of δK_{pite} and δK_{dte}, controlled by varying topographic height $H(\Omega')$ at a fixed angular distance $\psi = 0.0001°$, in the immediate neighbourhood of the computation point shows that the larger the height of the integration point, the larger the difference between these kernels, and therefore the topographical effects are stronger (see Figure 5).

Assuming that the condensation density of Helmert's layer is the same for all computation points, (i.e., this means that $\tau(\Omega)$ in Eqn. (6) is constant), then relative errors E_r with respect to the integration kernel, K^c (or K^t) can be computed by

$$E_r = \left| \frac{K^t - K^c}{K^c} \right| \times 100\% \ , \tag{23}$$

where K^c and K^t given by either Eqns. (19) and (20) or, Eqns. (21) and (22). Inspecting Figure (6) shows that in order to reach overall accuracies of PITE better than 1% of relative errors in Ireland, it is sufficient to integrate over a spherical cap

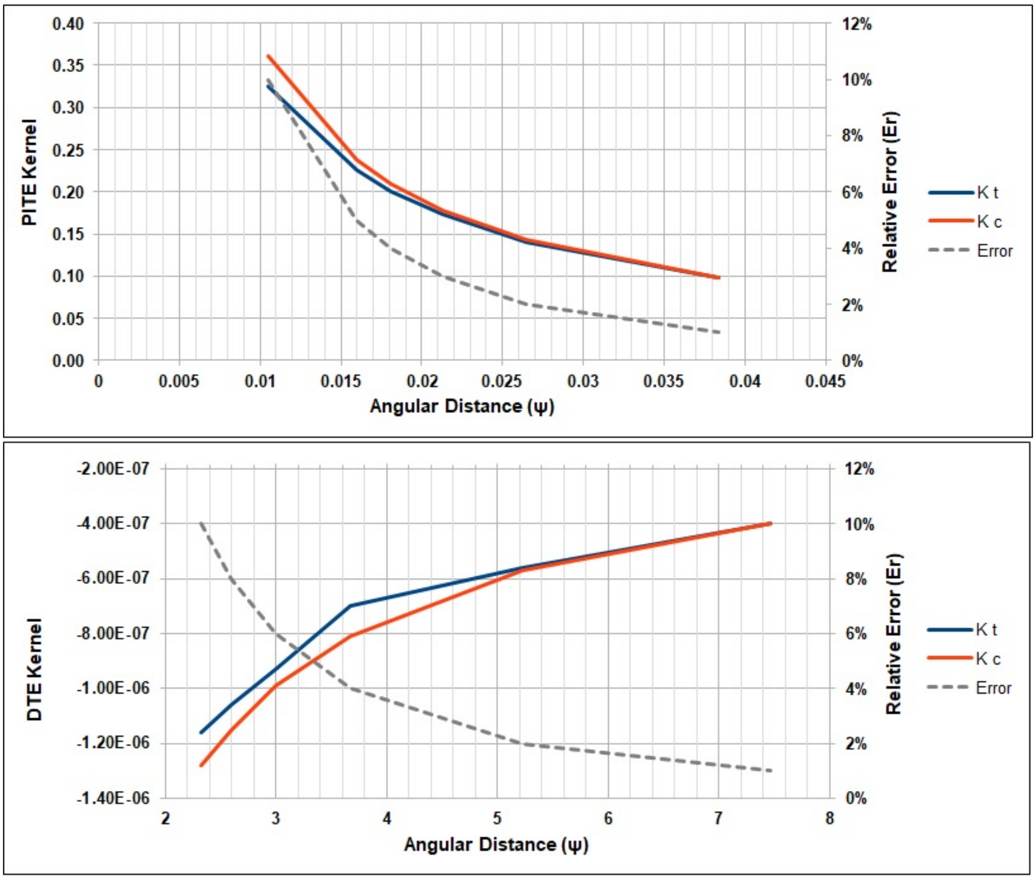


Figure 6: Approximate estimation of cut off degree in the determination of PITE/1.d10 and DTE/1.d14 where $H(\Omega) = 0$ and $H(\Omega') = 1039$ m.

with a radius of $\psi_0 = 0.0384°$. Furthermore, in order to reach overall accuracies of DTE better than 10% of relative errors in Ireland, where differences between the kernels δK_{DTE} reach six orders of magnitude (0.175×10^{-6}), it is sufficient to integrate over a spherical cap with a radius of $\psi_0 = 2.036°$.

5. Numerical Investigations of Topographical Effects in Ireland

Topographical effects in three areas are computed using Eqns: (9), (13) and (14)[2]. The minimum and maximum of the results illustrated graphically in Figure (7) show significant correlations between topographical effects and a spatial resolution of DEMs. Notice that the changes in topographical effects from the previous grid resolution are shown as percentages at each grid resolution. This is explained by the correlation coefficient defined as (see for example (Ahlgren et al., 2003))

$$R_{te} = \frac{n\left[\sum(H \times TE)\right] - (\sum H)(\sum TE)}{\sqrt{\left[n\sum H^2 - (\sum H)^2\right]\left[n\sum TE^2 - (\sum TE)^2\right]}} \,, \tag{24}$$

where n is the sample size. The results of the computed correlation coefficients for PITE and DTE, summarised in Table (1), shows a negative correlation between topography and topographical effects. This means that as the height of computation points is increasing the TEs are decreasing towards the strongest negative correlation.

[2]Notifying that the effect of SITE on geoid heights (expressed utilising the PITE, Eqn. (14) determinations are as low as seven orders of magnitude due to low elevation of topography in Ireland. Therefore, the results of determination of SITE, are not further discussed in this study.

Table 1: Pearsons correlation between the elevation of computation points and TEs for six different grid resolutions.

	Area-1 Low Topography		**Area-2 High Topography**		**Area-3 Low & High Topography**	
Grid (m)	R_{dte}	R_{pite}	R_{dte}	R_{pite}	R_{dte}	R_{pite}
1000×1000	-0.60	-0.96	-0.49	-0.93	-0.51	-0.88
500×500	-0.53	-0.96	-0.43	-0.94	-0.43	-0.88
250×250	-0.48	-0.96	-0.40	-0.94	-0.38	-0.88
200×200	-0.44	-0.96	-0.38	-0.94	-0.36	-0.88
100×100	-0.45	-0.96	-0.38	-0.94	-0.35	-0.88
50×50	-0.45	-0.96	-0.38	-0.94	-0.34	-0.88

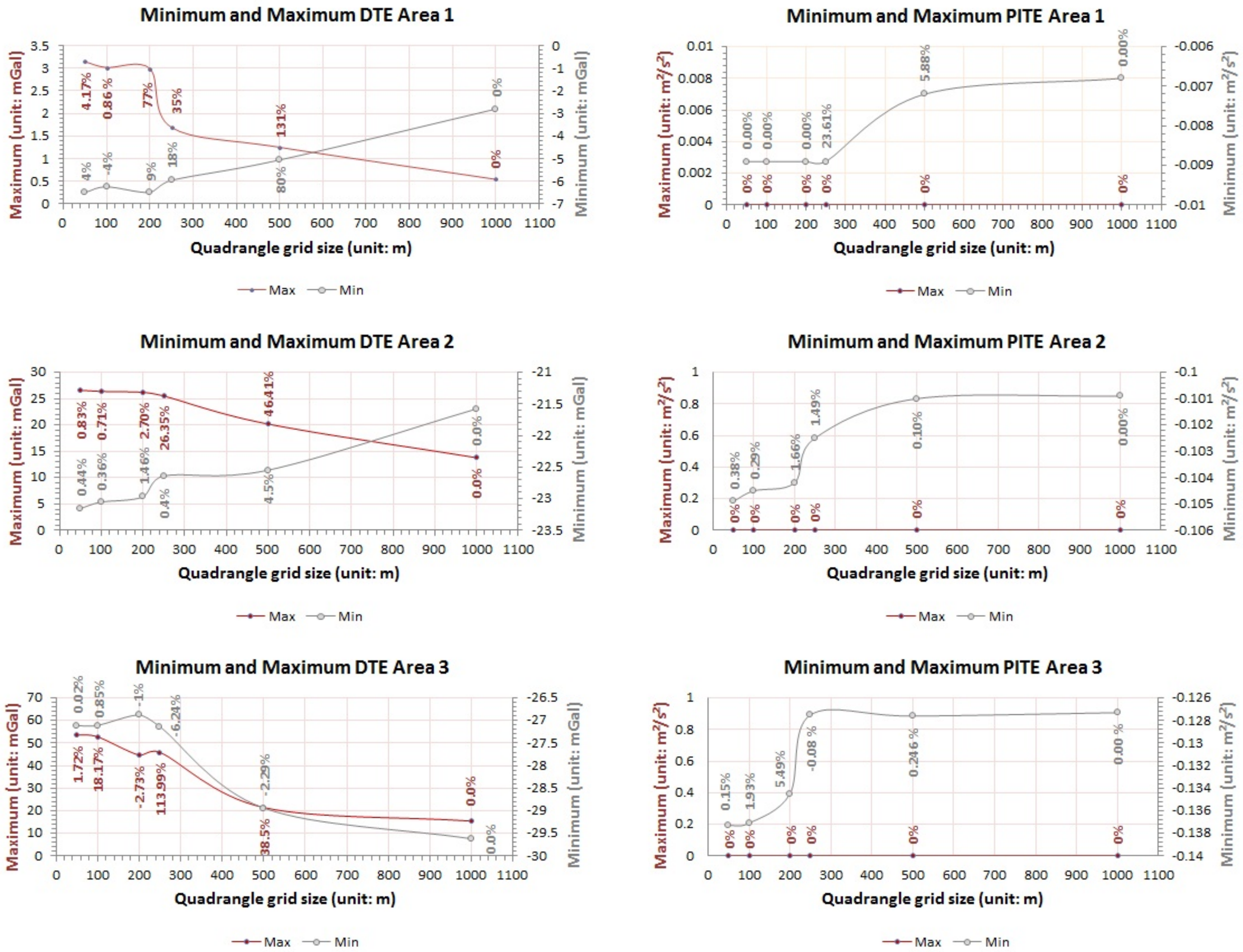


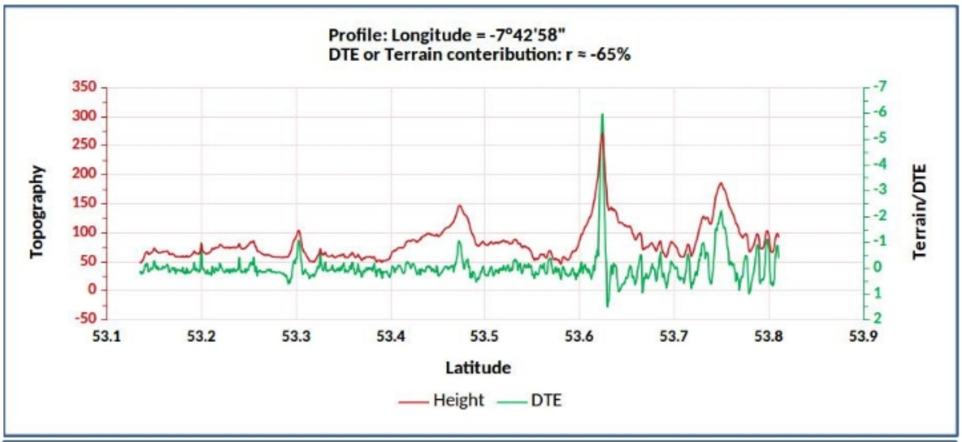
Figure 7: Primary Indirect and Direct Topographical Effects on potential and gravity in three test areas on 6 different grid resolutions.

Equations (9) and (13) assume the compensation is strictly local (Martinec, 1998), which means $\delta A(\Omega)$ consists of the terrain roughness contribution only. The longitudinal profiles plotted for Area-1 (see Figure 8) show that the Bouguer components of PITE has a larger contribution to topographical effects than the terrain roughness components. However, for a DEM with a tiny grid step size the magnitude of the Bouguer component becomes comparable with that of the terrain roughness component which reduces the correlation between DTE and PITE. Since DTE does not contain a Bouguer component, the correlation between DTE and DEM is in general smaller than that for PITE. Therefore, the correlations of -0.92 and -0.54 (on average) between PITE and DEM, and DTE and DEM has been observed, respectively.

Even though Area-3 contains the largest elevation points of the three test areas, it shows less correlation on average ($\bar{R}_{dte} \approx -0.39$, $\bar{R}_{pite} \approx -0.88$) compared to Area-2 ($\bar{R}_{dte} \approx -0.41$, $\bar{R}_{pite} \approx -0.94$) and Area-1 ($\bar{R}_{dte} \approx -0.49$, $\bar{R}_{pite} \approx -0.96$). This is explained by the roughness of the terrain in Area-3 compared to smoother changes in height over areas 1 and 2.

Making the grid resolution finer shows significant changes in amplitude of DTE and PITE as illustrated in Figures (10) and (9) respectively. This is also graphically illustrated for Area-2 (the results of PITE) in Figure 9 and for Area-3 (the results of DTE) in Figure 10 in 1000 m and 50 m grid resolutions.

A question arises here is what grid resolutions should be chosen to compute TEs with a prescribed accuracy; in other words, what is the effect of the change in DTM resolution on the geoid determination? To answer this question, the topographical effects on geoid heights are computed and analysed.



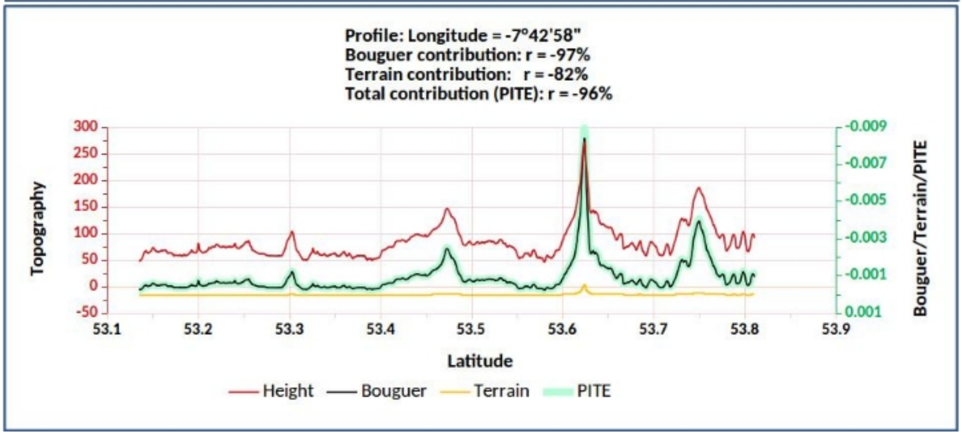


Figure 8: Correlation between topographical effects and elevation of topography from 50 m^2 QGRs.

5.1. The effect of PITE on geoid heights

The primary indirect topographical effects on the geoid height determination are computed by Eqn. (12). Inspecting the results in Figure (11), it can be observed:

- In Area-1, when the grid resolution is densified from 1000 m to 250 m, the PITE increases at most by 0.3 mm. However, further densification of grid

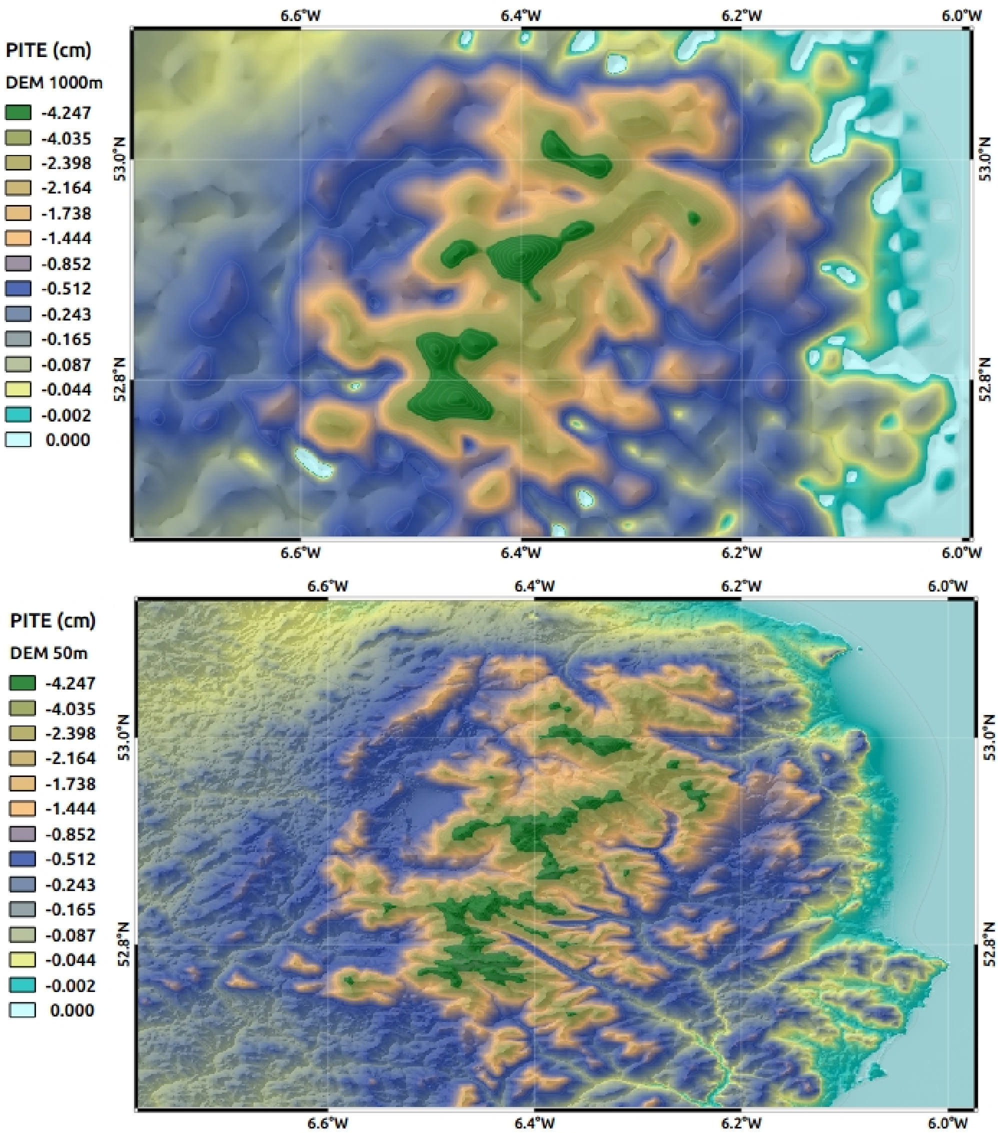


Figure 9: Illustration of PITE on High elevated topography in Area-2 for 1000 m and 50 m grid resolutions.

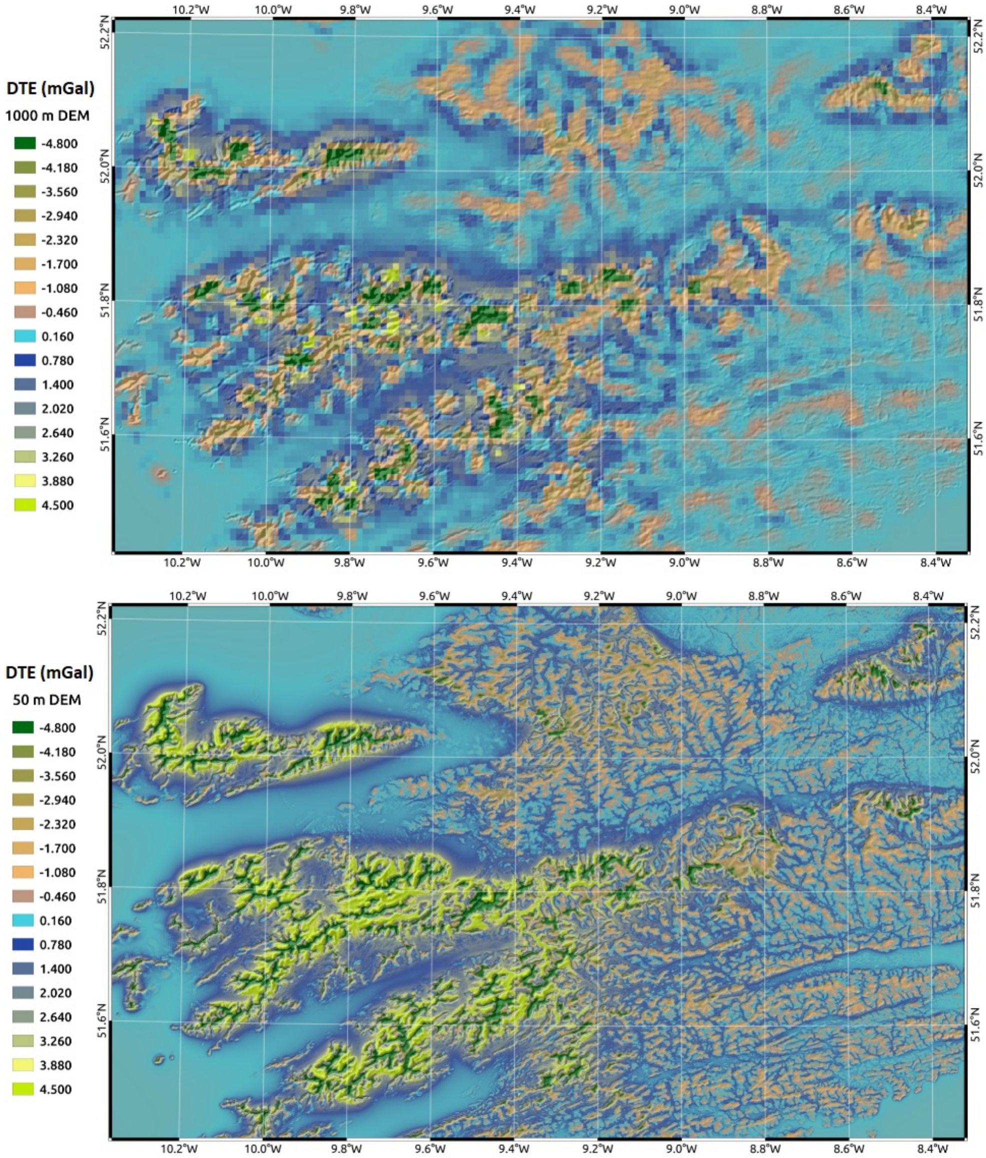


Figure 10: Illustration of DTE on High and low elevated topography in Area-3 for 1000 and 50 meters grided resolutions.

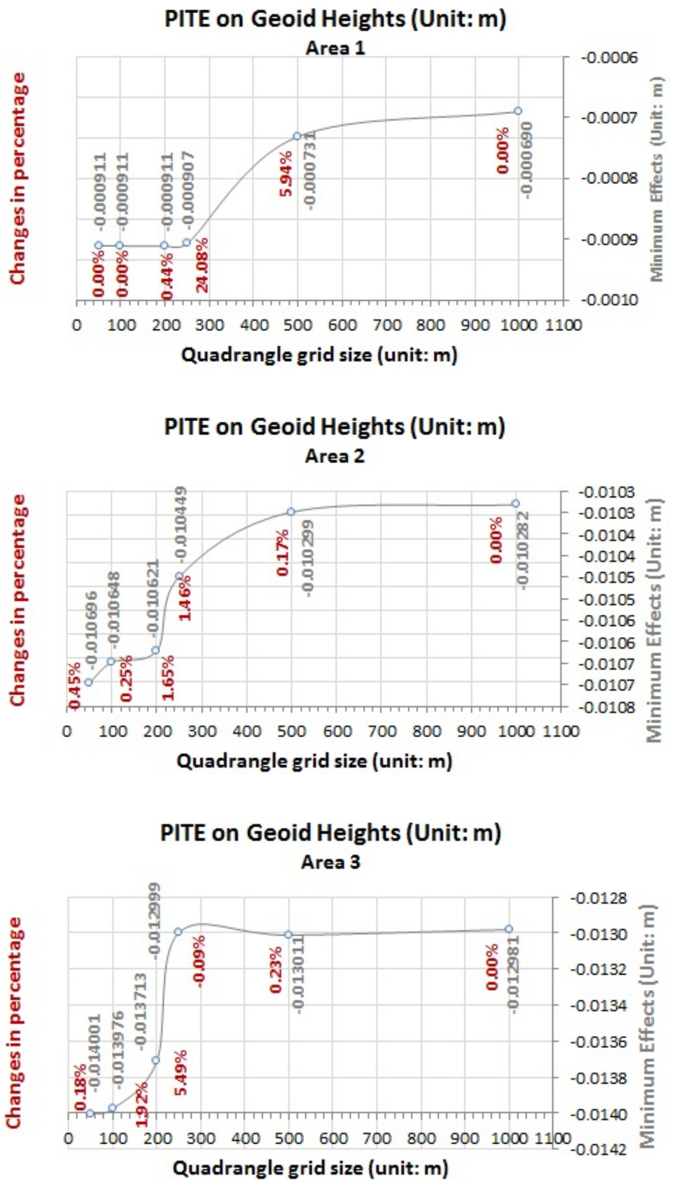


Figure 11: Primary Indirect Topographical Effect on geoid height determination in three test areas.

resolution does not changes the results at 0.01 mm level (not shown here).

- In Area-2, when the grid resolution is densified from 1000 m to 200 m, the PITE increases at the most 0.5 mm, and similarly to the previous case, further densification changes the results at the 0.01 mm level at most.

- In Area-3, when the grid resolution is densified from 1000 m to 200 m, the PITE increases at the most 0.8 mm and further densification of grid resolution does not change the results significantly.

Therefore, it can be conclude that:
*"**To reach the accuracy of the order of** 1 **mm in value of** $\delta N(\Omega)$**, it is sufficient to use** 1000 **m QGR in the determination of PITE in Ireland**"*.

5.2. The effect of DTE on geoid heights

Several numerical tests were carried out to compute DTEs on geoid heights in dependence of varying topographic heights and grid resolutions. The DTEs on geoid heights, δD, is computed by applying Stokes' integral to DTE as a known function $f(\Omega)$ distributed in the near-zone spherical cap C_{ψ_0}

$$\delta D^{\ell,\psi_0}(\Omega) = \frac{R}{4\pi} \int_{C_{\psi_0}} f^{\ell}(\Omega') S^{\ell}(\psi) d\Omega' \,, \tag{25}$$

where $\delta D^{\ell,\psi_0}$ is a higher-degree (presented as superscript ℓ) correction to the geoid height, C_{ψ_0} is a spherical cap of radius ψ_0 (for this study $\psi_0 = 1^\circ$ is chosen) and $S^{\ell}(\psi)$ is the spheroidal Stokes function, Vaníček and Kleusberg (1987). The graphical presentations of the results for Area-2 are shown in Figure 12. However, an examination of the results summarized in Figure 13, shows them to be unpredictable.

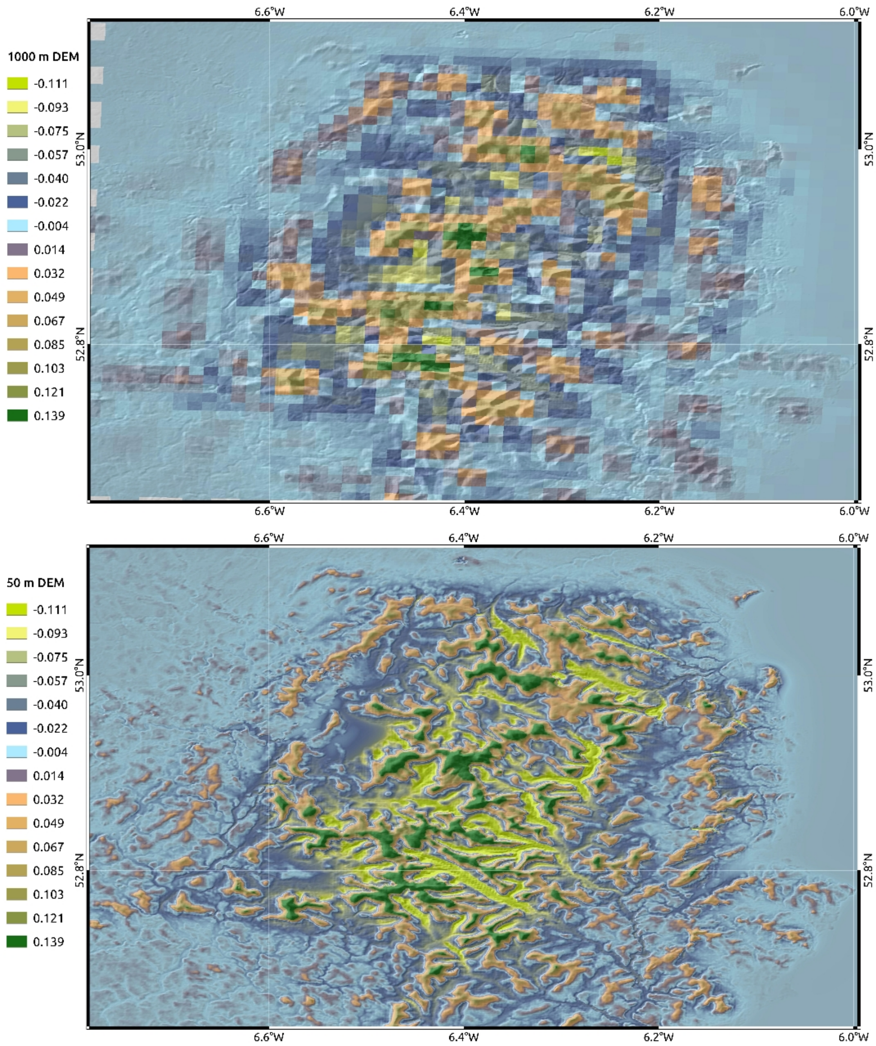


Figure 12: Illustration of DTE on geoid undulation for High elevated topography (Area-2) at 1000 and 50 meter grid resolutions.

In low elevated topography, densifying grid resolutions from 1000m to 100m QGR shows that the minimal amplitude of $\delta D^{\ell,\psi_0}$ does not converge at any stage of densifications.

In high elevated topography densifying grid resolutions from 1000m to 100m QGR shows convergence of the results. But the maximal value of $\delta D^{\ell,\psi_0}$ is when grid resolution is densified from 100m to 50m and QGR increases from 0.413m to 0.415m. This raises the question if convergence has been achieved at this QGR or if a finer QGR is required, or whether the results suggest that topographic surface is not a deterministic function for determination of DTE.

In area with rough topography (high-and-low elevations), densifying grid resolutions from 1000m to 100m QGR also shows that the minimal and maximal amplitude of δD does not converge at any stage of grid desification.

It should be declared that, the minimum/maximum norm values depending on the resolution of the DEMs studied here differs from the standard error (1 sigma) and that studying the maximum norm might give too pessimistic conclusions. However, based on the above analysis, it can be concluded that:
The determination of the DTE on geoid height shows that topographic surface may not be a deterministic function.

6. Conclusions

Computing topographical effects on potential and gravity for large areas is a very time consuming processes. Increasing the resolution of sampled DEM by a factor of 2 (e.g. from 100 m to 50 m quadrangle) increases the number of data by a factor of 4, and increases the computational time by a factor of approximately 14. Thus, it is suggested to restrict the integration area to a small area of radius ψ_0 around the computation point.

A sparse grid size, particularly in rugged areas, is not sufficient to express the irregularities of the terrain and thus does not reveal properly the contribution to

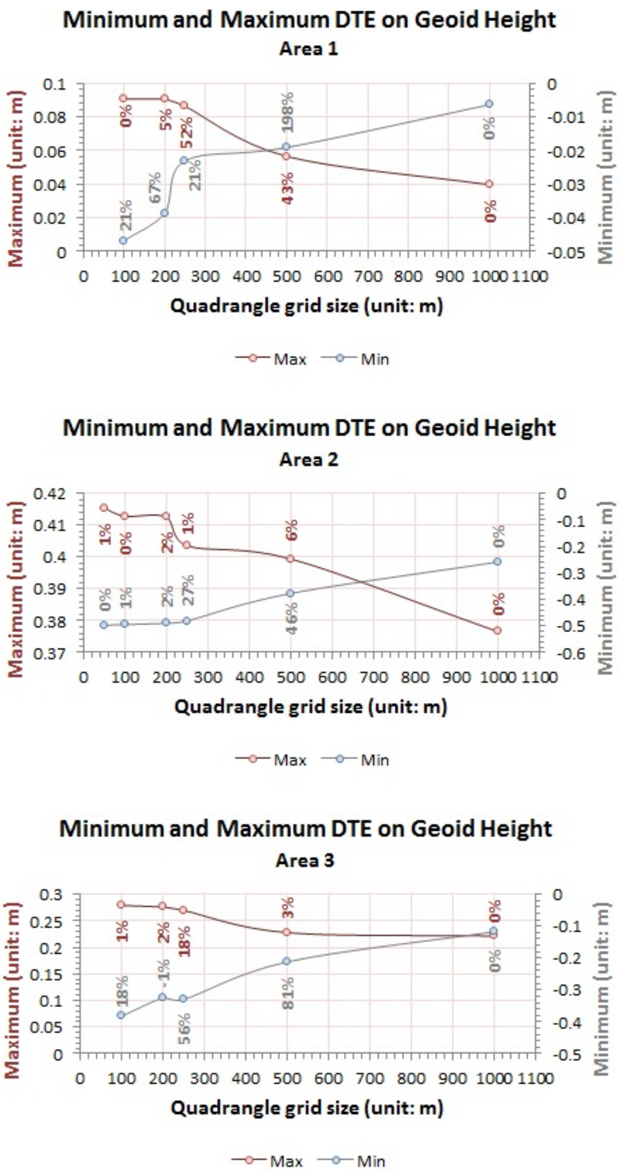


Figure 13: Direct Topographical Effects on geoid heights in three test areas.

geoidal height due to terrain height variations.

With regards to determination of PITE, Ireland can be considered a low elevated topography, and it is sufficient to use a 1000 m QGR.

The determination of the DTE on geoid height shows unpredictable results, which may suggest that the topographic surface is not a deterministic function. One way to extend this analysis is not to consider the topographic surface as a deterministic function, but rather as a fractal function, and carry out the integration over topography in a fractal sense. This research purposes of investigating the use of fractals for computing DTE and PITE in the future.

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Conclusions

The residual topographical potential, δV, is defined as the difference between gravitational potential V^c of compensating masses and gravitational potential V^t of topographical masses ($\delta V = V^t - V^c$). In this study, the effect of topographical masses through δV on the determination of the geoid undulations expressed by three terms:

- the direct topographical effect on gravity δA
- the primary indirect topographical effect on potential δV_{P_g}
- the secondary indirect topographic effect on gravity δS.

These effects are computed using Helmert's second condensation method at three test areas and also over the whole of Ireland-Northern Ireland.

In subsection 4.2 of this study, it is shown that the gravitational potential of topographical masses of a finite thickness behave like the potential of a thin layer when it is observed from a considerable distance. So, as a result, in this study the integration over the full solid angle Ω' restricted to a small area of radius ψ_0 surrounding the computation point.

Numerical investigations were conducted to determine the degree of cut off points with respect to relative errors introduced in this study (see Eq. 16). Approximate estimation of the cut-off degree in the determination of PITE or DTE in Ireland-Northern Ireland, with the maximum topographical elevation of 1039 m at Carrauntoohil, East Cork, shows that

(i) in order to reach overall accuracies of δV_{P_g} better than 1% of relative error in Ireland it is sufficient to integrate over a spherical cap with the radius of 0.0384°;

(ii) in order to reach overall accuracies of δA better than 1% of relative error in Ireland it is sufficient to integrate over a spherical cap with the radius of 7.4640°.

The results of computing DTE (δA) show that it is not guaranteed to compute a 1cm geoid with the 50 m quadrangle grid resolutions provided by OSi and

GSNI. The convergence issue with computations for data below 200 m QGR may correspond to suffering from numerical instabilities in solving the volume integral as given in Eq. (7). Also, this could be corresponding to errors in input data, i.e., in the derivation of DEM from $1:10,000$ orthophotography. However, an interesting potential future research direction could be to consider the topographic surface as a fractal function, rather than a deterministic function, and carry out the integration over topography in a fractal sense.

Furthermore, the computation of DTE (δA) shows the consistency of the results up to a spatial resolution of 200 m QGR. Therefore in this study, the topographical effect computed at 200 m QGR for removing the effects of topography at free-air gravity anomaly stations *Sajjadi et al.* (2020) is used. This is determined, i.e., topography-reduced free-air gravity anomaly, prior to the downward continuation process in Paper 3 (Chapter 4).

4

Paper 3: The stability criterion for DWC of surface gravity data with various spatial resolutions over Ireland

In papers 1 and 2, the topography-reduced free-air gravity anomaly was computed, and in the current chapter, this is downward continued to the geoid by employing the Poisson downward continuation technique.

The solution of linear algebraic equations with the discretised Poisson's integral, is iteratively solved by the conjugate gradient method. To find an approximate solution, in this study, we defined a *misfit* (a degree of difference) between the observables and the model[1] values.

[1]An initial guess that generates a sequence of improving approximate solutions.

Then the linear system of algebraic equations (see Eq.14) is solved iteratively minimising the misfit function (see Eq.17). Since the iterative method successively approximates the solution of a linear system, a tolerance condition is introduced to end the computations.

Instability of solutions to the Poisson downward continuation technique, numerically over several regularised data files, is analysed. Thus, the best spatial range for Irish gravity data that can be used for stable downward continuation of surface gravity data in Ireland-Northern Ireland is derived.

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The stability criterion for DWC of surface gravity data with various spatial resolutions over Ireland

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Abstract

The stability of Poisson's Downward Continuation (DWC) technique is analysed, with respect to varying the spatial resolution of surface gravity data in Ireland. The solution of linear algebraic equations with the discretized Poisson integral are iteratively solved by the conjugate gradient method. The topography reduced free-air gravity anomalies computed by removing the Direct Topographical Effects on gravity (DTE) from free-air gravity anomalies are continued from the surface down to the geoid. Numerical investigations show that when the spatial resolution of surface gravity data is finer than a certain threshold, the downward continuation becomes numerically unstable, and must be regularized. The results of computations for several spatial resolutions show that the distance500 m Regulated Spatial Resolution (RSR) is the minimum range of surface gravity data in Ireland, for which the DWC is unconditionally stable. In this case, DWC continued data contribute from -0.018 m to 0.039 m to geoid heights.

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Keywords: Downward Continuation, Stokes Integral, Poisson Formula

1. Introduction

The geoid determination using the Stokes integral requires the gravity to be known on the geoid, but the gravity observations are only available on the Earth's surface that significantly differs from the geoid for continental areas. To obtain the boundary values for the Stokes' formula, the gravity at the Earth's surface has to be reduced onto the geoid. This reduction is known as *downward continuation*.

The main difficulty with presenting gravity on the geoid is a non-zero mass density distribution betweenthe geoid and Earth's surface, which makes the disturbing gravitational potential non-harmonic outside the geoid. To continue the gravity observations from the Earth's surface to the geoid is an ill-posed problem in the sense that the continued gravity does not continuously depend on the observations. The consequence of this is that even small measurement errors result in high-frequency oscillations in the solution. Thus,to obtain a stable solution, some regularization approach should be applied to reduce these oscillations. There is an extended list of papers in geodetic literature studying the DWC problem including (Bjerhammar, 1965; Cruz, 1985; Vaníček et al., 1996; Martinec, 1996; Novak, 2000; Novák et al., 2001; Sjöberg, 2003, 1975, 2007; Moritz, 1980a; Ilk, 1987; Wang, 1988, 1990; Engels et al., 1993; Ågren, 2004a,b; Ågren and Sjöberg, 2014; Sebera et al., 2014, 2015; Michel and Telschow, 2016; Farahani et al., 2017) etc.

In this study, we aim to analyse the stability of the Poisson DWC technique with various regularised surface gravity data in Ireland. Therefore at first, the Poisson DWC technique, in the form of the Fredholm integral equation of the 1^{st} kind, is solved using the Nyström method. Secondly, six sets of regularised surface gravity data were chosen by eliminating the close-range data with a spatial distance of 10 m, 50 m, 100 m, 400 m, 500 m and 1000 m respectively from the original data. Then, the topography-reduced free-air gravity anomalies are computed at these data points. These were then combined with that of EGM2008 up to degree/order 2190 to fill the gaps on the data (gaps in high elevated topography, lakes, along the coastlines, a 1° stretch, enclosing the offshore area).

Furthermore, the combined gravity data are numerically analysed to identify a minimum range of gravity data for which the DWC is stable. Finally, the contribution of regularised data to geoid heights in Ireland is determined.

2. Downward continuation of topography-reduced free-air gravity anomalies

The Poisson formula (Kellogg, 1929) that solves Dirichlet's Boundary Value Problem (see for example (Sneeuw, 2006)), relates the harmonic function $\tau(r,\Omega)$ on the Earth's surface with $\tau(R,\Omega')$ on the geoid:

$$\tau(r,\Omega) = \frac{1}{4\pi}\int_{\Omega_0} \tau(R,\Omega')K(r,\psi,R)d\Omega' \ . \tag{1}$$

where Ω stands for the pair of angular spherical coordinates (the longitude and latitude); Ω_0 is full solid angle; r is the radius of the computation point with known height ($H(\Omega) > 0$) of the Earth's surface above the geoid of radius r_g that is approximated by the mean radius of the Earth ($r_g = R$), ψ is the angular distance between the geocentric directions of computation point Ω and the integration point Ω', and $K(r,\psi,R)$ is the Poisson's kernel. Note that when $R = r$, i.e., $H(\Omega) = 0$ (for example in tide-gauge stations), the Poisson's kernel becomes the Dirac delta function, and Eqn.1 has a simple solution $\tau(r,\Omega) = \tau(R,\Omega)$. The Poisson's kernel

$K(r,\psi,R)$ given by:

$$K(r,\psi,R) = \frac{R(r^2-R^2)}{L^3}\ , \tag{2}$$

rapidly decreases with increasing angular distance ψ from the computation point (see Fig.1), where $L = \sqrt{r^2+R^2-2rR\cos\psi}$ is the spatial distance between computation points (r,Ω) and an integration point (R,Ω'). This enables for the regional

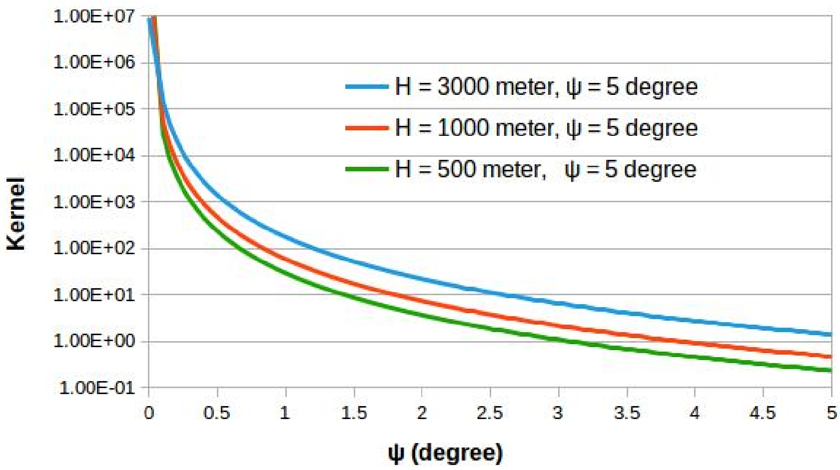

Figure 1: Logarithmic behaviour of the Poisson kernel for three different elevations (fixed at H = 3000m, 1000m, and 500m) and the angular distance up to 5°.

geoid determination to integrate over a small spherical cap ψ_0 instead of the whole Earth,

$$\tau(r,\Omega) \simeq \frac{1}{4\pi}\int_{\psi_0} \tau(R,\Omega')K(r,\psi,R)d\Omega'\ . \tag{3}$$

The radius ψ_0 of the spherical cap may be chosen by various methods. In this study a spherical cap of the radius of 1° is used, which assures the contribution from far zones is sufficiently small (Vaníček et al., 1996; Nahavandchi, 1998).

A significant increase of Poisson's kernel when $\psi \to 0$ makes numerical integration in Eqn. (3) difficult. That is why we remove a small neighbourhood of the point $\psi = 0$ from the integration domain and evaluate this contribution analytically. Formally, we can write

$$\tau(r,\Omega) = \frac{1}{4\pi} \int\limits_{\psi_0} \left[\tau(R,\Omega') - \tau(R,\Omega) \right] K(r,\psi,R) d\Omega' + d(r,\psi_0)\tau(R,\Omega) \;, \qquad (4)$$

where

$$d(r,\psi_0) = \frac{1}{4\pi} \int_{\psi_0} k(r,\psi,R) d\Omega' \;. \qquad (5)$$

$d(r,\psi_0)$ is the integral of Poisson's kernel which can be evaluated analytically (Martinec, 1996),

$$d(r,\psi_0) = \frac{r+R}{r} \left(1 - \frac{r-R}{\sqrt{r^2+R^2-2rR\cos\psi_0}} \right) \;. \qquad (6)$$

In order to transform the integral Eqn. (4) into a system of linear algebraic equations, we use the Nyström method, also called quadrature method (Nyström, 1930). This method is based on an approximation of the integral in Eqn. (4) by a numerical quadrature,

$$\tau(r_i,\Omega) = \frac{1}{4\pi} \sum_{j=1}^{jmax} W_j K(r,\psi_j,R) \left[\tau(R,\Omega'_j) - \tau(R,\Omega) \right] + d(r,\psi_0)\tau(R,\Omega) \;, \qquad (7)$$

where W_j are the weights of the quadrature rule, which are equally sized for infinitesimal integration area (area divided by the number of data). This is valid for any observation point of Ω. Hence, for the ith data point, we have

$$\tau(r_i,\Omega_i) = \sum_{j=1}^{jmax} A_{ij} \tau(R,\Omega_j) \qquad \begin{cases} \forall \;\; i \in [1,...,imax] \\ \forall \;\; j \in [1,...,jmax] \end{cases} \qquad (8)$$

where the elements of matrix A are

$$A_{ii} := d(r_i,\psi_0) - \frac{1}{4\pi}\sum_{\substack{j=1\\ j\neq i}}^{jmax} W_j K(r_i,\psi_{ij},R)\ ,$$

$$A_{ij} := \frac{1}{4\pi} W_j K(r_i,\psi_{ij},R) \qquad\qquad i\neq j\ \ .$$

The system Eqn. (8) of linear algebraic equations can be expressed in matrix form as:

$$A.\vec{\tau} = \vec{f} \tag{9}$$

with

$$\vec{f} := [f(r_1,\Omega_1),\ f(r_2,\Omega_2),\ f(r_3,\Omega_3)\ \ ,\ \ \ldots\ \ ,\ \ f(r_{imax},\Omega_{imax})]^T \tag{10}$$

$$\vec{\tau} := \left[\tau(R,\Omega'_1)\ ,\tau(R,\Omega'_2)\ ,\tau(R,\Omega'_3)\ \ ,\ \ \ldots\ \ ,\ \ \tau(R,\Omega'_{jmax})\right]^T. \tag{11}$$

2.1. Minimization

In general, the number of data points *imax*, differs from the number of computation points, *jmax* ($imax \neq jmax$). Hence, A is in general a non-quadratic matrix and an exact solution of the system of linear algebraic equations (9) does not exist. While the observation points are irregularly distributed, here the computation points are considered equidistantly distributed such that $jmax < imax$. To find an approximate solution of (9), we define a misfit between observables $f_i := f(r_i,\Omega_i)$ and the model values $f_i(\vec{\tau})$ by

$$\chi^2 = \sum_{i=1}^{imax} [f_i - f_i(\vec{\tau})]^2. \tag{12}$$

In Eqn. (12) we compare the observations (stored in the vector $\vec{f}$) on the Earth's surface for given observation locations and the model observation vector $\vec{f}(\vec{\tau})$

containing the upward continued model values obtained from the values on the geoid. The sensitivity of the misfit χ^2 with respect to model parameters τ_j, $\tau_j = \tau(R, \Omega'_j)$, is given by partial derivatives of χ^2 with respect to τ_j:

$$\frac{\partial \chi^2}{\partial \tau_j} = -2 \sum_{i=1}^{imax} [f_i - f_i(\vec{\tau})] \frac{\partial f_i}{\partial \tau_j} \;, \tag{13}$$

where $j = 1, 2, ..., jmax$, and

$$\frac{\partial f_i}{\partial \tau_j} = A_{ij} \;. \tag{14}$$

In vector notation, we have

$$\nabla_{\vec{\tau}} \chi^2 = -2A^T [\vec{f} - \vec{f}(\vec{\tau})] \;. \tag{15}$$

The stability of the DWC solution will be examined by its L_2 norm:

$$L_2(\vec{\tau}) = \sqrt{\sum_{j=1}^{jmax} \tau_j^2} \;. \tag{16}$$

3. Data sets

3.1. Gravity data

The gravity observations in Ireland have been made for geophysical rather than geodetic purposes in lowland areas along roads and tracks that do not extend into the surrounding land areas. Hence, information on the gravity signals are missing in areas with higher elevations, lakes, coastlines and the offshore regions. Moreover, the spatial resolution (distance between data points) varies from a few metres to hundreds of metres in some areas. The density of the data averages approximately one station per 1 km^2 in Northern Ireland and one station per 3 km^2 in Ireland see (Sajjadi et al., 2020). The gravity data points throughout Ireland are illustrated in Fig. 2.

3.2. Topography-reduced free-air gravity anomaly

The topography-reduced free-air gravity anomaly $\Delta g^{h,FA}(\Omega)$ at each observation location is computed by

$$\Delta g^{h,FA}(\Omega) = \Delta g^{FA}(\Omega) - \delta A \ , \tag{17}$$

where superscript h in $\Delta g^{h,FA}(\Omega)$ emphasizes that $\Delta g^{h,FA}$ is harmonic outside the geoid, $\Delta g^{FA}(\Omega)$ is the free-air gravity anomaly (Heiskanen and Moritz, 1967) and δA is the direct topographical effects on gravity. The free-air gravity anomaly is computed from

$$\Delta g^{FA}(\Omega) = g(r,\Omega) - \gamma_0(\phi) + 0.3086 \times H(\Omega) \ , \tag{18}$$

where $g(r,\Omega)$ is the absolute gravity at an observation points, $\gamma_0(\phi)$ is the normal gravity on the surface of the level ellipsoid, which can be computed using the formula of Somigliana Green (Heiskanen and Moritz, 1967; Moritz, 1980b), 0.3086

Figure 2: Distribution of gravity data in Ireland; each data point represents a measurement collected by the DIAS (23285 data points) and GSNI (11256 data points) in geodetic coordinate system.

is the normal gravity gradient in units of mGal/m, and $H(\Omega)$ is the orthometric height of observation points.

The direct topographical effects on gravity, δA, is computed for the 200m quadrangle grid resolution using the 2nd Helmert condensation method (Martinec, 1998, Eqn. 3.45)

$$\delta A(\Omega) = G\rho_0 \int_{\Omega_0} \left[\frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \Bigg|_R^{R+H(\Omega')} - \frac{\partial \widetilde{L^{-1}}(r,\psi,r')}{\partial r} \Bigg|_R^{R+H(\Omega)} - R^2[\upsilon(\Omega') - \upsilon(\Omega)] \frac{\partial L^{-1}(r,\psi,R)}{\partial r} \right]_{R+H(\Omega)} d\Omega' \,. \tag{19}$$

with $R = 6371$ km and

$$\upsilon(\Omega) = H(\Omega)\left(1 + \frac{H(\Omega)}{R} + \frac{H^2(\Omega)}{3R^2}\right) , \tag{20}$$

and Ω' is the angular geocentric spherical coordinates of integration points. Note that we introduced the symbol $\widetilde{L^{-1}}(r, \psi, r')$ for an indefinite radial integral of the Newton kernel,

$$\widetilde{L^{-1}}(r, \psi, r') := \int_{r'} \frac{r'^2}{L(r, \psi, r')} dr' , \tag{21}$$

which can be evaluated analytically (Gradshteyn, 1979)

$$\begin{aligned}\widetilde{L^{-1}}(r, \psi, r') =& \frac{1}{2}\left(r' + 3r\cos\psi\right) L(r, \psi, r') + \\ &+ \frac{r^2}{2}(3\cos^2\psi - 1)\ln\left|r' - r\cos\psi + L(r, \psi, r')\right| + C ,\end{aligned} \tag{22}$$

where the constant C (constant of integration) may depend on the variables r and ψ only.

4. Numerical investigations

Downward continuation of the original gravity dataset shows that the numerical solution of DWC does not converge in the computation process. Therefore, the DWC would have to be regularised (not done here). Nevertheless, removing close-ranged neighbouring input data (here referred to as Regularised Spatial Resolution, RSR)[1] shows that DWC can be stabilised.

The problem with the neighbouring input data is explained in minimising the misfit between the model vector and the observation vector for given observation

[1]Note that for example, 10 m RSR means that the data within the range of ≤ 10 m of the neighbourhood points has been removed from the original gravity dataset.

locations. Duplicated data or data in close range generates a flat platform for the misfit Eqn. (12), and finding the minimum value in such an almost flat valley becomes impossible. Consequently, a small perturbation in input data results in a large impact on the solution (see, for example, (Press et al., 1989, Chapter 10), (Brent, 2013)).

The downward continuation of regularised data above a certain threshold provides the stability in computations, but the results suffer from errors due to gaps in the data set. There are a number of models, for example, EGM2008 (Pavlis et al., 2012) or any combined models based on GOCE, that reach high degree/order of expansion which can be used to fill in the gaps in the dataset. Note that none of these models accuracy has been tested in Ireland yet. Here for no particular reason, the EGM2008 model is used as a fill-in solution. So, gravity anomalies have been computed from EGM2008 to degree/order 2190 at 0.09° grid resolution to a total of 26576 gravity data points. From the 26576 gravity data points 1466 internal (inland) gravity points and 3760 external (coastlines and offshore areas) gravity points were chosen and used to fill-in the gaps in the dataset. Several adopted internal gravity anomalies (EGM2008 data) were compared one to one with terrestrial data at neighbouring points, and they seem to fit exceptionally well with surrounding datasets. Additionally, geoid heights computed from EGM2008 coefficients up to degree/order 2190 were compared with that of independent GPS/Levelling at 14 tide-gauge stations along the coast of Ireland (see Fig.3). The standard error of the discrepancies between geoid undulations is 4.2 cm with the mean value of -33.5 cm. Statistical summary of this comparison is given in Table 1.

Table 1: The Statistical summary of the discrepancies between EGM2008 geoid undulation and GPS/Levelling derived geoid undulations at 14 independent tide-gauge stations (Unit: cm).

Mean	Min	Max
-33.544	-71.770	-15.040

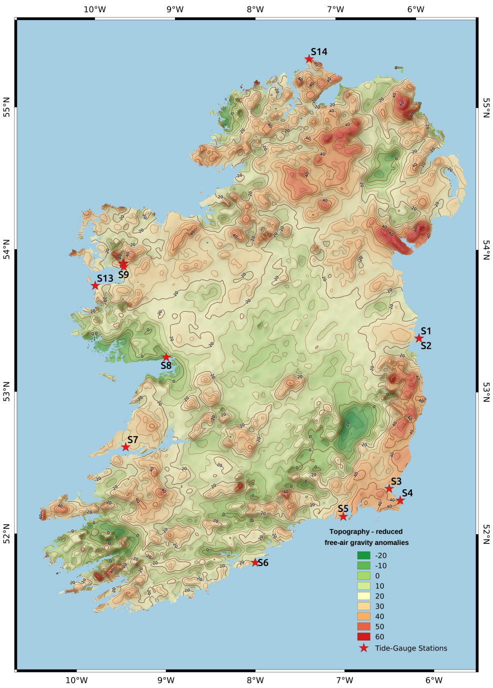

Figure 3: Position of 14 independent GPS/Levelling tide-gauge stations along the coast of Ireland in topography-reduced free-air gravity anomaly map of Ireland.

To identify the minimum RSR where DWC is stable, the linear system of algebraic equations (9) is solved iteratively minimising the misfit function (12) using several different RSRs. Since the iterative method successively approximates the solution of a linear system, a tolerance condition was introduced to end computations via

$$2|\chi^2_{k+1} - \chi^2_k| \leq \chi^2_{tol}(|\chi^2_{k+1}| + |\chi^2_k| + \varepsilon) \,, \tag{23}$$

where χ_k^2 is the value of misfit (12) in the k^{th} iterative step; χ_{k+1}^2 is the value of misfit in the $k+1$ iterative step that should be minimized; $\chi_{tol}^2 = 1.0 \times 10^{-10}$ is the tolerance for the improvement by each iteration and $\varepsilon = 1.0 \times 10^{-10}$ is the small additional offset; these ensure that the computation is terminated if there is insufficient change in the value of misfit from one iteration to the next.

Figure (4), illustrates the summary of computation results for three sets of regularised datasets (regularised spatial resolutions of 1000 m, 500 m, and 400 m). The mean and number of input/output data are also given in Table 2. The misfit χ^2 between observables and the model values was computed using Eqn. (12) as a function of iterations. In Fig, (4), the left-hand y axis represents the logarithmic of

Resolution	imax	Mean	Iteration	jmax	Mean
≥ 1000	36546	19.194	700	34541	19.309
≥ 500	41977	19.581	6000	34541	19.668
≥ 400	42406	19.588	96646	34541	20.062

Table 2: Comparison of the number of data inputs (terrestrial + EGM2008) and data outputs (terrestrial footprints on the geoid) and their mean values for 1000 m, 500 m, and 400 m regularised spatial resolutions.

misfit, while the right-hand y axis shows the L_2 norm of the solution, see Eqn. (16).

We can observe that the computation algorithm for carrying out the DWC process with the data of spatial resolution larger than 500m is stable. The L_2 norm of the solution for 500 m RSR converges after 6000 iterations (with $\chi^2 = 1.03 \times 10^{-4}$, see the green lines), and similarly, for the 1000 m RSR, after 700 iterations (with $\chi^2 = 1.27 \times 10^{-8}$ see the blue lines). For the 400 m RSR (red line) the misfit equals 1.65×10^{11} after 40000 iterations meaning that the downward procedure does not converge. Note that the misfit value of 1.65×10^{11} after

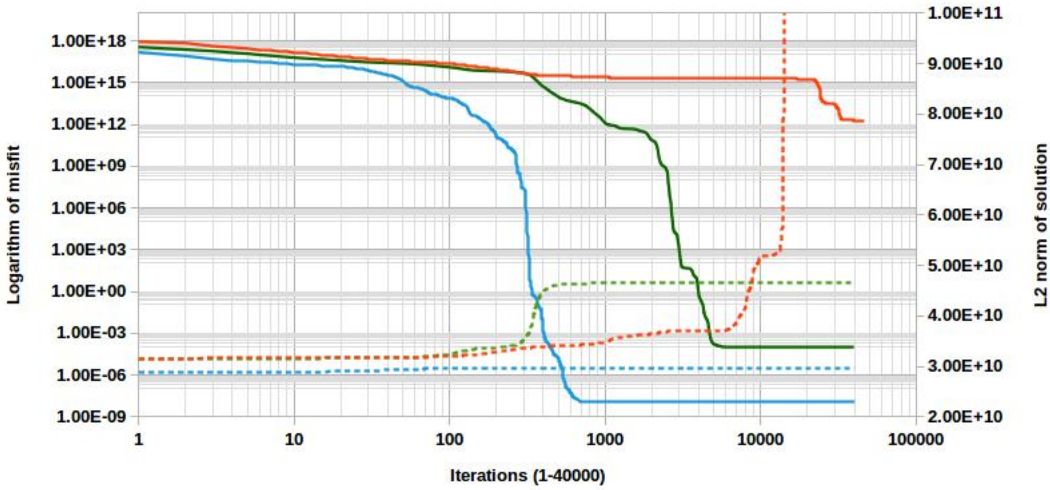

Figure 4: Continuous lines show the misfit χ^2 and dashed lines show the L_2-norm of the solutions as a function of iteration steps for 3 different spatial resolutions, 400 (orange), 500 (green) and 1000 (blue) metres.

96000 iterations did not improve. It is also documented by a larger increase of L_2 norm solution after 5000 iterations. Hence, the 500 m RSR is the finest spatial range for Irish gravity data that can be used for stable downward continuation of surface gravity measurements in Ireland.

To show the effect of various spatial resolutions of gravity data, $\Delta g^{h,FA}(\Omega)$, and their downward continued values (Δg_{geoid}), on geoid determination, we compute the residual gravity anomalies,

$$\Delta g_{res} = \Delta g_{geoid} - \Delta g^{h,FA}(\Omega) \ , \tag{24}$$

and transform them into the residual geoid heights Δg_{res} by Stokes' integral

$$\Delta N_{res} = \frac{R}{4\pi} \int_{C_{\psi_0}} \Delta g_{res}(\Omega') S(\psi) d\Omega' \ , \tag{25}$$

where C_{ψ_0} is a spherical cap of radius ψ_0 (for this study, we choose $\psi_0 = 1°$) and $S(\psi)$ is the spheroidal Stokes function, (Vaníček and Kleusberg, 1987). Figure (5) illustrates the residual free-air gravity anomalies and residual geoid heights with the 500 m RSR, which contributes from -0.018 m to 0.039 m to geoid heights.

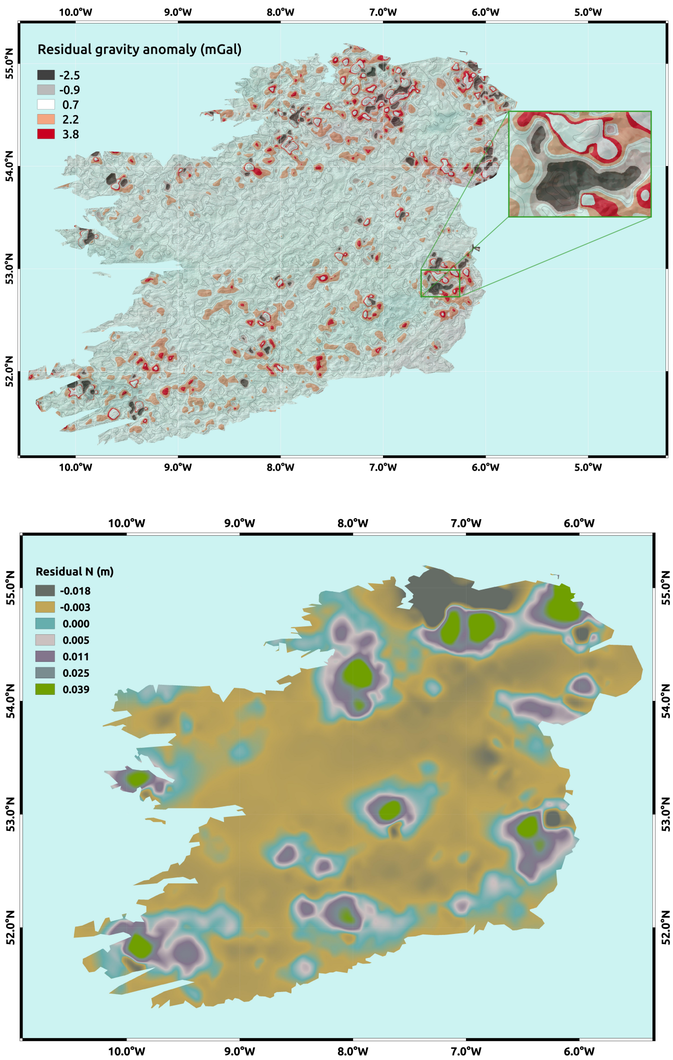

Figure 5: Residual free-air gravity anomaly map of Ireland (top- zoomed on high topography area), and residual geoid height ΔN_{res}, (bottom) for 500m spatial data resolution in *spherical coordinate system*.

5. Conclusions

The topography reduced free-air gravity anomalies in Ireland have been computed by removing the direct topographical effect on gravity from free-air gravity anomalies. Surface gravity data do not cover several regions in Ireland. To be able to perform the DWC of surface gravity to the geoid, we computed the surface gravity values over these regions using the EGM2008 global gravity model. Similarly, gravity data were extended up to 1° in the offshore areas to be able to carry out Poisson's integration along the coastlines.

The results of computations show that a spatial resolution of 500 m is the finest spatial range of gravity data, which provides stable computation results for DWC. For finer spatial data resolutions, the solution of DWC does not converge and must be regularized (not done here). The downward continuation computed for residual gravity anomalies with 500 m spatial resolution contributes from -0.035 m to 0.046 m to geoid heights in Ireland.

Acknowledgment

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Conclusions

The Poisson downward continuation (DWC) technique, in the form of the Fredholm integral equation of the first kind, is employed to compute the topography reduced free-air gravity anomaly on the geoid from its value on the Earth's surface. The linear algebraic equations are solved iteratively, minimizing the misfit function between the observables and the model values. A tolerance condition is introduced to stop the computations when the misfit value is smaller than the tolerance condition.

Numerical investigations show that when spatial resolutions of gravity data are smaller than a specific limit, the downward continuation becomes numerically unstable, and therefore, the regularisation of the spatial resolution of data is introduced.

Downward continuation of regulated data above a certain threshold provides stability in computations, but the results held significant errors surrounding the area with a gap in the dataset. The errors were due to lack of data within the radius ψ_0 of the spherical cap, where we used $\psi_0 = 1°$ in this study. Therefore, the gaps in the data are filled (one point per kilometre) by computing missing signals from EGM2008 coefficients to degree and order 2190. Similarly, along the coastlines, where a cover of $\psi_0 = 1°$ in the integration domain is required, data was extended by 1° using EGM2008 gravity anomalies.

The results of the computations show that the spatial resolution of 500m is the finest spatial range of gravity data that provides stable computation results for DWC. The downward continuation computed for residual gravity anomalies with 500m spatial resolution contributes a range from -0.035m to 0.046m to geoid heights in Ireland.

Since, the data has been continued down to the geoid, the two basic requirements of Stokes' formula are met, and the computation of the geoidal undulations can now be carried out. Determination of geoid undulations is presented and investigated in Paper 4 (Chapter 5).

5

Paper 4: Unifying the Irish Vertical Datum with the Normal Amsterdams Peil (NAP) Vertical Datum

The main objectives of this chapter are to compute a justified value of the gravity potential on the in Malin-Head Tide-Gauge station (MH-TG) Ireland and to determine its vertical offsets from the Normaal Amsterdams Peil (NAP) datum. The geodetic boundary value problem (GBVP) solution is used, based on the global geopotential model (GGM) combined with terrestrial gravity data following the remove-compute-restore approach (RCR) (*Schwarz et al.*, 1990; *Tscherning*, 1986).

To be able to compute the gravity potential of MH-TG as well as its offsets from the NAP datum, the gravimetric geoid undulation is decomposed into long-wavelength N^{ℓ}_{grav} and short-wavelength N^{s}_{grav} components,

$$N_{grav} = N^{\ell}_{grav} + N^{s}_{grav} \ .$$

Short-wavelength components – The downward continued harmonic gravity anomalies from studies presented in paper 1, Chapter 2 (determination of unified gravity anomalies), paper 2, Chapter 3 (determination of harmonic gravity anomalies), paper 3, Chapter 4 (downward continuation of harmoinc gravity anomalies to the geoid) will be used here to compute short-wavelength components of the geoid undulations in Ireland. The determination of the short-wavelength components are computed using the remove-compute-restore approach.

Long-wavelength components – The long-wavelength components of the geoid undulations are computed using GOCO05s (*Mayer-Guerr*, 2015) and GOCO06s (*Kvas et al.*, 2019) coefficients up to degree/order (d/o) 240 and 245 respectively. The unknown radius of the gravimetric geoid is computed by adopting the gravity potential value of the European gravimetric (quasi) geoid EGG2015 value 62 636 857.91 $\mathrm{m}^2\ \mathrm{s}^{-2}$ (*Denker*, 2015).

To fit this gravimetric model surface to the MH-TG, the geometric geoid heights N_{geo} at this station is computed. The value of N_{geo} is computed from GNSS observation, which provides the ellipsoidal height with an accuracy of $h \pm 0.01$m, and orthometric heights derived from Ordnance Survey Ireland's (OSi) corrector surface model with an accuracy of $H \pm 0.030$m

$$N_{geo} = h - H \ .$$

The gravimetric geoid height at the MH-TG is recomputed by varying the adopted EGG2015 gravity potential value such that

$$N_{geo} - N_{grav} = 0 \ .$$

The new value of gravity potential which satisfies this condition is the best-fit gravity potential value of MH-TG W_{MH}, and its offset is given by

$$\Delta N^{i}_{MH} = \frac{W_i - W_{MH}}{g_{MH}} \ ,$$

where $g_{MH} \pm 0.3$mGal (see the accuracy of the gravity data in (*Sajjadi et al.*, 2020)) is the gravity value at MH-TG.

Relating the Irish vertical datum to the Normaal Amsterdams Peil (NAP)

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Abstract

The local gravity potential value on Malin-Head tide gauge station in Ireland is derived from gravimetric and geometric geoid undulations. The geometric geoid undulations are obtained from GNSS and levelling data. In contrast, the gravimetric geoid undulations are computed based upon their long-wavelength components using the GOCE gravity field model, and their short-wavelength components using the terrestrial data through the Remove-Compute-Restore technique. The potential value obtained from the combination of the satellite-only gravity field model *GOCO*06*s* up to degree/order 245, and terrestrial + EGM2008 data for the local tide-gauge station in Malin-Head, is computed as 62 636 858.273 m^2/s^2, and is recommended as the present best estimate value in Ireland. The local gravity potential value on Malin-Head tide gauge station differs by 0.363 m^2/s^2 from the gravity potential value of EGG2015, (62 636 857.91 m^2/s^2), and is equivalent to height differences equal to 0.034 m above Normaal Amsterdams Peil (NAP) datum.

Keywords: Height Unification, Gravity Potential, Tide System

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1. Introduction

The level surface defining a vertical reference system is called the geoid, which is estimated at one or more tide gauge stations. The local tide gauge stations of the national European vertical systems are established at various seas and are biased to each other up to several decimetres. In order to unify the local vertical systems in Europe, in 2000, the European Vertical Reference System, EVRS was defined Ihde et al. (2000). There are currently three official European realizations of EVRS, i.e., EVRF2000 (Augath and Ihde, 2002), EVRF2007 (Sacher et al., 2009), and EVRF2019 (Sacher and Liebsch, 2019). The difference between the heights in EVRF2007 and EVRF2019 does not exceed $\pm$15mm, and according to its definitions, the datum of EVRF2019 is at the same level as NAP. Furthermore, EVRF2019 is a *zero-tide system*, where the tide-generating potential is eliminated, but the deformation potential of the Earth is retained see, e.g., RH 2000; it was adopted in Sweden in 2005 (Ågren and Svensson, 2007) and the N2000 was adopted in Finland in 2007 (Saaranen et al., 2009). Furthermore, according to the definition of $EVRS$, the height component is the difference between the potential W_P of the Earth gravity field in the studied points P and the potential W_0 of the $EVRS$ zero level. The potential difference $-\Delta W_P$ is also called a geopotential number C_P.

The main objective of this study is to perform calculations to generate an accurate value of gravity potential for a local tide-gauge station in Malin-Head Ireland relying on the global gravity field model and terrestrial gravity signals. This is achieved by adopting the gravity potential value of the European gravimetric (quasi) geoid $EGG2015$ value $W_{EGG2015} = 62\ 636\ 857.910$ m^2/s^2 (Denker, 2015). Note that, since the choice of reference potential is arbitrary as far as the determination of the relation to NAP (EVRS) is concerned, any gravity potential values can be adopted to determine the gravity potential value at Malin-Head tide-gauge station (MH-TG). For instance, the International Height Reference System (IHRS), value $W_{IHRS} = 62\ 636\ 853.4$ m^2/s^2 that was defined in IAG resolution 1 at the IUGG general assembly in Prague 2015, see (Sánchez et al., 2015); or International Earth Rotation and Reference Systems Service (IERS) value $W_{IERS} = 62\ 636\ 856.0$ m^2/s^2, which corresponds to the best estimate available in 1998 (Burša et al., 1998). Nevertheless, in this study the gravity potential value $W_{EGG2015}$, is used to determine the gravimetric geoid undulations (N_{grav}) in MH-TG, by decomposing the geoid undulations into long-wavelength ($N^{\ell}_{grav,MH}$) and short-wavelength ($N^{s}_{grav,MH}$) components. The long-wavelength components are computed using the satellite-only gravity field models $GOCO05s$ and $GOCO06s$ up to degree/order (d/o) 240 and 245 respectively. The short-wavelength components are computed with Stoke's method from the combination of EGM2008 and terrestrial gravity data following the remove-compute-restore approach of Tscherning (1986); Schwarz et al. (1990).

The geometric geoid undulations (N_{geo}) at 14 tide-gauge stations are computed from the basic relation between ellipsoidal height h (obtained from $GNSS$ observation), and orthometric heights H,

$$N_{\text{geo}} = h - H\ . \qquad (1)$$

By increasing/decreasing the adopted gravity potential value, the gravimetric geoid undulations (long-wavelength component) can be redetermined iteratively to estimate the gravity potential at MH-TG under the following two conditions:

1. The gravimetric geoid undulation at MH-TG is equal to the geometric geoid

undulation and they are at the same permanent tide system,

$$N_{\mathrm{grav,MH}} = N_{\mathrm{geo,MH}} \;, \tag{2}$$

2. The estimated potential value at MH-TG presents the minimum height components ΔW concerning the $EGG2015$ gravity potential value.

2. Data

The gravity observations in Ireland have been made for geophysical rather than geodetic purposes in lowland areas along roads and tracks (see (Sajjadi et al., 2020)). Hence, information on the gravity signals in areas with higher elevations, lakes, coastlines and the offshore area is missing. The gaps in the data were closed by computing missing signals from EGM2008 coefficients to degree and order 2190 (Pavlis et al., 2012) and extended along the offshore area (at 1° radius) covering in a total area of longitudes: -12.671°/-3.112° and latitudes: 50.298°/56.648°.

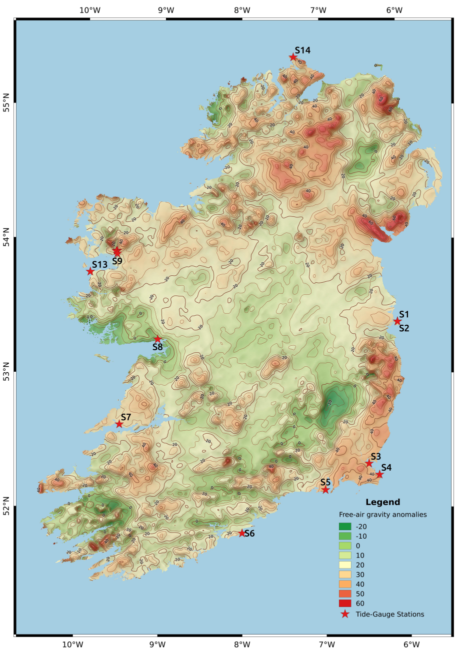

Figure 1: Position of tide guage stations in free-air gravity anomaly map of Ireland.

A survey company was contracted to provide benchmarks better than 10mm vertical accuracy to Malin-Head datum for the Marine Institute. Fourteen stations, S1-S14 located around the coast of Ireland were surveyed using static GPS observations to 31 stations on the OSi active network to correct for systematic error and bises in GNSS observations. These were post-processed using least squares to produce the latitude, longitude and ellipsoidal heights of these stations. Multiple GPS baselines to the benchmarks ensure that redundancy is built into the observations, with 116 baselines and 306 degrees of freedom.

Ellipsoidal heights obtained from the survey are listed as being of centimetre accuracy, and orthometric heights computed from Ordnance Survey of Ireland corrector surface[1] are ± 3 cm (Greaves et al., 2016). The overall accuracies of N_{geo}, determined from the $GNSS$ real-time kinematic (RTK) network are estimated within $1-3$ cm.

The GNSS-ellipsoidal heights of the tide gauge stations are given in a tide-free ($h_{tide-free}$) system. To be consistent with that of gravimetric geoid undulations, the $h_{tide-free}$ heights were converted into the zero permanent tide system ($h_{zero-tide}$), (see e.g. (Ihde et al., 2008, Eqn.5-7)),

$$h_{zero-tide} = h_{tide-free} - 0.1790\sin^2(\phi) - 0.0019\sin^4(\phi) + 0.0603 \quad [m], \tag{3}$$

where ϕ is the latitude of the computation point in the European Terrestrial Reference System 1989 (ETRS89).

The transformation of ellipsoidal heights from tide-free system to the zero-tide system at MH-TG, has an effect of -0.062 m on the ellipsoidal height, which causes approximately an effect of -0.618 m^2/s^2 in the determination of the gravity potential value.

[1]The Ordnance Survey of Ireland (OSi), Ordnance Surveys of Great Britain, and Land & Property Services (LPS, formally Ordnance Survey Northern Ireland) have collaborated to improve the OSGM02 geoid model Iliffe et al. (2003) covering the United Kingdom and Ireland. The new model is OSGM15 Greaves et al. (2016).

Furthermore, the Malin-Head datum is in contrast to the EVRS in the mean-tide system. Therefore, an offset between the EVRS datum and the Malin-Head datum cannot be used as a constant difference for the whole country. Alternatively, the height of the vertical datum in Malin-Head must be converted to the same zero permanent tide system (see e.g., Ihde et al. (2008)),

$$\begin{aligned} H_{MH,zero-tide} &= H_{MH,mean-tide} \\ &- 0.29541 sin^2\varphi - 0.00042 sin^4\varphi + 0.0994 + 0.1008 \text{ [m]}. \end{aligned} \tag{4}$$

3. Local gravity potential at Malin-Head tide-gauge station

The gravity potential value at MH-TG (W_{MH}), is computed by fixing the gravimetric geoid undulation $N_{\mathrm{grav,MH}}$ to that of the geometric geoid undulation $N_{\mathrm{geo,MH}}$ in MH-TG such that:

$$N_{\mathrm{grav,MH}} - N_{\mathrm{geo,MH}} = 0 \ . \tag{5}$$

The geometric geoid undulations are computed at GNSS stations from ellipsoidal height h and orthometric heights H,

$$N_{\mathrm{geo,MH}} = h - H \ . \tag{6}$$

In order to determine the gravimetric geoid undulation we decompose the gravimetric geoid undulation into its long-wavelength component $N^{\ell}_{\mathrm{grav,MH}}$, and its short-wavelength component $N^{s}_{\mathrm{grav,MH}}$,

$$N_{\mathrm{grav,MH}} = N^{\ell}_{\mathrm{grav,MH}} + N^{s}_{\mathrm{grav,MH}} \ . \tag{7}$$

where long-wavelength components are computed from the spherical harmonic coefficients of a global geopotential model (GGM), and short-wavelength components from a combination of EGM2008 gravity anomalies and terrestrial gravity data.

3.1. Determination of short-wavelength gravimetric geoid undulation N^s_{grav}

The short-wavelength components of the geoid undulation are computed using the Stokes method, where the Stokes's requirements were adjusted using the Remove-Compute-Restore (RCR) procedure summarised as follows:

- The free-air gravity anomalies for Ireland and Northern Ireland are computed, see (Sajjadi et al., 2020), and combined with that of EGM2008 data to close the gaps in the data and also to extend the data along the coastlines and offshore area in Ireland.

- Topographical effects of topography on gravity (DTE) and potential (δV) are computed for 200m quadrangle grid resolutions in Ireland using Helmert's second condensation method see (Sajjadi et al., 2018b).

- The topography-reduced free-air gravity anomalies Δg^h are computed by removing interpolated DTE at gravity stations from combined free-air gravity anomalies in Ireland-Northern Ireland.

- The Poisson downward continuation (DWC) technique, in the form of the Fredholm integral equation of the first kind, is employed to compute the topography reduced free-air gravity anomaly on the co-geoid from its value on the Earth's surface see (Sajjadi et al., 2018a).

- The downward continued data (on the co-geoid surface) were interpolated into regular grids and co-geoid undulations are computed using Stokes' integral.

– The primary indirect effect on the co-geoid are computed

$$\delta N = \frac{\delta V}{\gamma_Q} , \tag{8}$$

and removed from the co-geoid undulations to obtain the short-wavelength geoid undulations

$$N^s_{grav} = N^s_{co-geoid} + \delta N . \tag{9}$$

3.2. Determination of the long-wavelength gravimetric geoid undulation N^{ℓ}_{grav}

The long-wavelength component N^{ℓ}_{grav} of the geoid undulation is the difference between the radius of the geoid at the computation point and the corresponding radius of the reference ellipsoid,

$$N^{\ell}_{\text{grav}} = r_{geoid} - r_{ref} . \tag{10}$$

In this study, the radius of the reference ellipsoid is computed from the Geodetic Reference System 1980 ($r_{ref} = r_{GRS80}$). The unknown long-wavelength radius of the geoid $r^{\ell}_{geoid}(\Omega)$ is computed from the equation relating the spatial and spectral domains of the geopotential as:

$$W_0(r, \lambda, \varphi) = \frac{GM}{r_{ref}} \sum_{j=0}^{jmax} \sum_{m=-j}^{j} \left(\frac{r_{ref}}{r^{\ell}_{geoid}} \right)^{j+1} V_{jm} Y_{jm}(\Omega) + V^{\omega} \tag{11}$$

where

r, λ, φ - spherical geocentric coordinates of computation point (radius, latitude, longitude)

r_{ref} - reference radius

GM - product of gravitational constant and mass of the Earth

j, m - degree, order of spherical harmonic

V_{jm} - Stokes coefficients (fully normalised)

Y_{jm} - is the surface spherical harmonics

$$Y_{jm}(\Omega) = P_{jm}(\sin\varphi) \begin{cases} \cos m\lambda \\ \sin m\lambda \end{cases} , \tag{12}$$

and P_{jm} are the fully normalised Legendre functions,

V^{ω} - is the centrifugal potential of the geoid computed at co-latitude θ,

$$V^{\omega} := \frac{1}{2}\omega^2 r_{g,lw}^2 \sin^2\theta \ . \tag{13}$$

In this study, we assume a rotation of constant angular velocity ω around the rotational axis of the Earth that is furthermore assumed to be fixed.

Under the concept that the reference level can arbitrarily be appointed, and any geopotential value can be assigned to solve the problem, here we adopted the gravity potential value of $W_{EGG2015} = 62\,636\,857.910$ m^2/s^2 as the initial equipotential surface. The unknown radius of the geoid $r^{\ell}_{geoid}(\Omega)$ is computed using the combination of the satellite-only gravity field models *GOCO*05*s* and *GOCO*06*s* (Kvas et al., 2019) with the cut-off d/o of spherical harmonics 240 and 245 respectively (see Sec.3.3).

$$W_{EGG2015} = \frac{GM}{r}\sum_{j=0}^{240}\sum_{m=-j}^{j}\left(\frac{r}{r^{\ell}_{geoid}}\right)^{j+1} V_{jm}^{COCO06s} Y_{jm}(\Omega) + V^{\omega} \ . \tag{14}$$

Furthermore, the effect of the gravitational field generated by topographical masses on gravity potential (see e.g., (Martinec, 1998))

$$\delta V^{t}_{ref} = -2\pi G\rho \sum_{j=0}^{240}\sum_{m=-j}^{j}\left[\left(H^2\right)_{jm} + \frac{2}{3R}\left(H^3\right)_{jm}\right] Y_{jm} \ , \tag{15}$$

at the MH-TG with an elevation of 3.358 m (in zero-tide system) is of -1.09×10^{-5} m^2/s^2, and thus may safely be neglected.

Calling the computed value of r^{ℓ}_{geoid}, the long-wavelength geoid undulation, N^{ℓ}_{grav}, is given by

$$N^{\ell}_{\text{grav}} = r^{\ell}_{geoid} - r_{\text{GRS80}}. \tag{16}$$

Figure 2, illustrates the differences between geometric geoid undulations obtained from GNSS levelling data at the zero-tide system and gravimetric geoid undulations:

(a) with long-wavelength components computed from the adopted potential value of $W_{EGG2015}$ using *GOCO*06*s*, the satellite-only, global gravity field model up to d/o 245 (colour blue);

(b) with the combination of long-wavelength components described at (a) and short-wavelength components of geoid undulation described in Sec. 3.1 (colour orange);

(c) with the adjusted potential value fixed at MH-TG, i.e., to satisfy condition Eqn. 2, computed iteratively from the combination of the long-wavelength and short-wavelength components of geoid undulation using *GOCO*06*s*, global gravity field model up to d/o 245 (colour green).

3.3. The cut-off degree of spherical harmonics

Several numerical tests were carried out to explore the estimation of W_{MH} independence of varying the cut-off degrees of spherical harmonics. In the determination of the unknown radius of the geoid, when the cut-off degrees of spherical

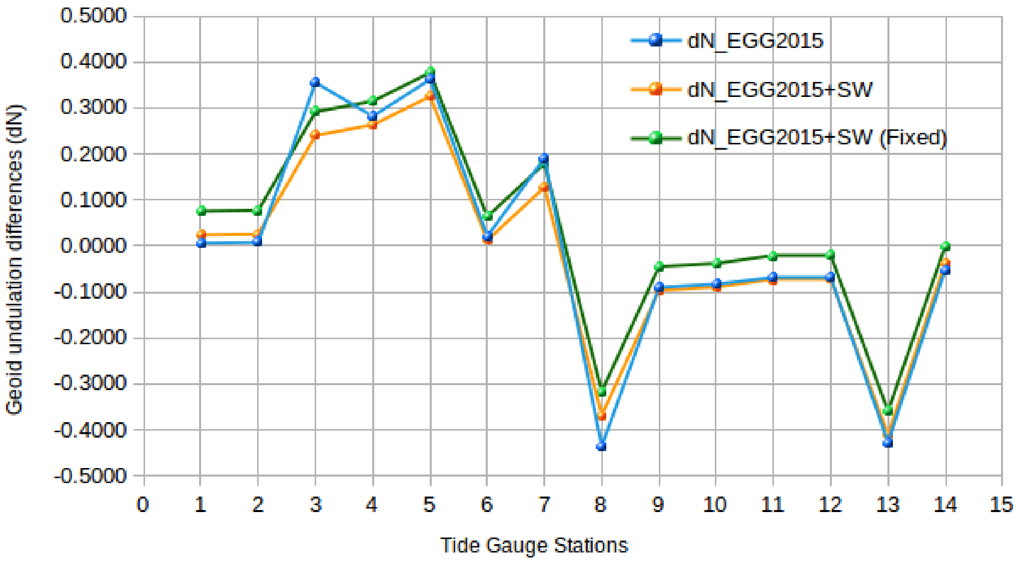

Figure 2: Geoid undulation differences between geometric and gravimetric geoid undulations computed using $GOCO06s$, $GOCO06s+SW$ (**S**hort-**W**avelength) up to d/o 245.

harmonics increases, so as the potential value must be increased/decreased in order to grantee the condition Eqn. 2. Therefore, changes in the cut-off degrees of spherical harmonics change the height components ΔW also. The composition of minimal height components where condition Eqn. 2 is granted, reaches at cut-off degree 240 and 245 using $GOCO05s$ or $GOCO06s$ global gravity field models respectively. Figure 3, illustrates the variation of geoid undulation differences by assigning different cut-off degrees of spherical harmonics using $GOCO05s$ and $GOCO06s$, gravity field models.

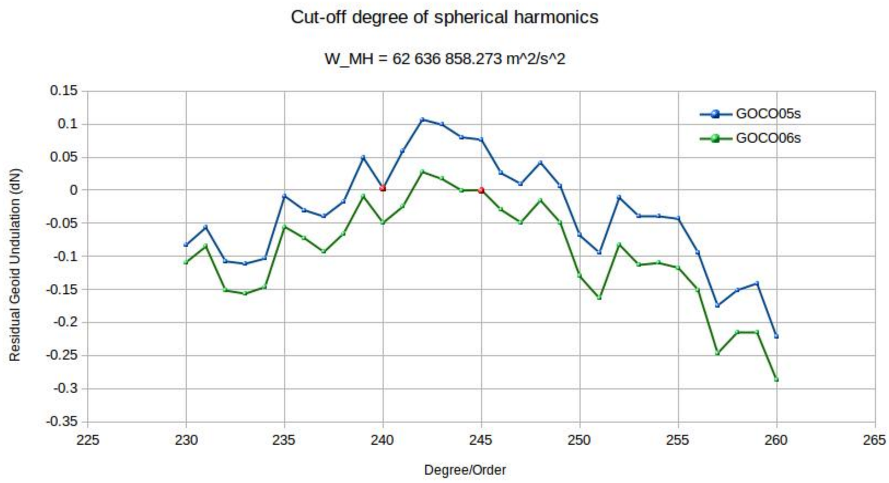

Figure 3: The minimum separation of the equipotential surfaces at MH-TG (coloured in red) for *GOCO*05*s* and *GOCO*06*s* reaches when the cut-off degree of spherical harmonics assigned to d/o 240 and 245 respectively.

4. Vertical datum offset

The latitude independent constant, ΔN_{zero}, can be estimated using

$$\Delta N_{zero} = \frac{W_{MH,zero}(\Omega) - W_0(\Omega)}{g_{MH}} \ , \tag{17}$$

where $g_{\mathrm{MH}} \approx 9.81574319 \ \mathrm{m}^2 \ \mathrm{s}^{-2}$ is the gravity at MH-TG, which corresponds to $\Delta N_{zero} = 0.034$ m.

Since ΔN_{zero} is positive (i.e., $W_{\mathrm{MH,zero}} > W_{\mathrm{EVRS}}$), then the equipotential surface of MH-TG is lying below the equipotential surface of EVRS. Therefore, the height H_{MH} is higher than H_{EVRS}, yielding,

$$\Delta N_{zero} = H_{MH,zero} - H_{EVRS}, \tag{18}$$

which is corresponding to $H_{EVRS} = 3.085$ m. Finally, by reversing the earth tide correction, the latitude dependent difference can be determined,

$$
\begin{aligned}
\Delta N &= H_{MH,mean} - H_{EVRS} \\
&= \Delta N_{zero} + 0.29541 sin^2\varphi + 0.00042 sin^4\varphi - 0.0994 - 0.1008 \ , \quad (19)
\end{aligned}
$$

where ΔN is the final height differences equal to 0.037 m above Normaal Amsterdams Peil (NAP) datum.

The IHRS or IERS coordinates are potential differences referring to the equipotential surface of the Earth's gravity field realised by the conventional value $W_{IHRS} =$ 62 636 853.4 m^2 s^{-2} and $W_{IHRS} = 62\ 636\ 856.0$ m^2 s^{-2}, respectively. A main component of the IHRS or IERS realisation is the integration of the existing height systems into the global height system rather than the European height system as in EGG2015 with the conventional value $W_{EGG2015} = 62\ 636\ 857.91$ m^2 s^{-2}. Hence, the vertical offset between MH-TG ($W_{MH-TG} = 62\ 636\ 858.273$ m^2 s^{-2}) and the IHRS or IERS is 0.496 m and 0.232 m, respectively.

5. Conclusion

The gravity potential of GNSS tide-gauge station at Malin-Head (MH-TG) Ireland W_{MH} is determined as well as the W_{MH} offset from Normaal Amsterdams Peil (NAP) datum. For this purpose, the geodetic boundary value problem (GBVP) solution is used, based on the combination of the global geopotential model, terrestrial and EGM2008 data following the remove-compute-restore approach (RCR).

The geometric geoid undulation, derived from ellipsoidal heights and orthometric heights, was compared to that of gravimetric geoid undulation to estimate the local gravity potential value in MH-TG.

For the first time, a justified value of $W_{MH} = 62\,636\,858.273$ m^2/s^2, has been computed in Ireland, which is 0.037 m above Normaal Amsterdams Peil (NAP) datum.

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Conclusions

The gravimetric geoid undulation in Ireland is determined based on the global geopotential model and the combined terrestrial and EGM2008 data following the Remove-Compute-Restore approach (RCR). Due to the distinct treatment of orthometric heights over test points, i.e., fourteen GNSS tide-gauge stations, the gravity potential value over the gravimetric model is estimated relative to a single test station at the tide-gauge station in Malin Head Ireland.

Therefore, for the first time, a justified value of $W_{MH} = 62\ 636\ 858.273\ \mathrm{m}^2/\mathrm{s}^2$, has been computed in Ireland.

The local gravity potential value on Malin-Head tide gauge station differs by $0.363\ \mathrm{m}^2/\mathrm{s}^2$ from the gravity potential value of $EGG2015$, ($62\ 636\ 857.91\ \mathrm{m}^2/\mathrm{s}^2$), and is equivalent to height differences equal to 0.034 m above Normaal Amsterdams Peil (NAP) datum.

The main conclusions of this study are as follows:

Conclusions of Paper 1

The gravity data for Ireland-Northern Ireland was collected over an extended period of time from 1949 to the 1980s in Ireland and until the 1990s in Northern Ireland. During this time the absolute gravity values for the base stations in Cambridge and Dunsink were amended on a number of occasions as well as the correction values between these base stations. Similarly, there are three height datums in Ireland-Northern Ireland (one historical - Poolbeg) and care must be taken to interpret the data supplied correctly to allow it be properly integrated. Finally, data are supplied using old spatial reference systems (Airy ellipsoid, and Irish Grid 1975 coordinate system), which pre-date the modern system used today (ETRF89 and Irish Transverse Mercator).

Users of the data should be aware of the survey methods used to collect the gravity data and those used to collect the height data (the combination of survey and estimation) and how coordinates of the gravity stations were obtained (using graphical methods). The gravity data for Ireland-Northern Ireland contains an error of 0.15 $mGal$ due to inaccuracies of elevation, and errors of 0.3 $mGal$ due to inaccuracies of position. Understanding the quality of the data allows users to use the data appropriately and this knowledge is essential if terrestrial data is to be combined with satellite gravity data.

The new Bouguer and free-air gravity anomalies maps created during this study for Ireland-Northern Ireland have been supplied to the Geological Survey of Ireland for publication.

Conclusions of Paper 2

Computing topographical effects on potential and gravity are very time-consuming processes. Numerical investigations show that increasing the resolution of sampled DEM by a factor of 2 (e.g. from 100m to 50m quadrangle) increases the number of data by a factor of 4, and increases the computational time by a factor of approximately 14.

Numerical study shows that the gravitational potential of topographical masses behaves like the potential of a thin layer when it is observed from a more considerable distance. Therefore, the integration area, instead of being a full solid angle, is restricted to a small area of radius ψ around the computation point.

A sparse grid size, particularly in rugged areas, is not sufficient to express the irregularities of the terrain and thus does not reveal the contribution accurately to geoidal height due to terrain height variations.

Numerical investigation of the Topographic Effect (TE) on the geoid height determination reflects uncertainty on considering the topographic surface as a deterministic function.

Helmert's second method of condensation was used to compute the topographical effect at the spatial grid resolution of 200m for Ireland-Northern Ireland.

Conclusions of Paper 3

The harmonic gravity anomalies, i.e., the topography reduced free-air gravity anomalies in Ireland, have been computed by removing the direct topographical effect on gravity.

The Poisson downward continuation technique was applied to the harmonic gravity anomaly in Ireland. The solution of the linear algebraic equations with the discretized Poisson integral were iteratively solved by the conjugate gradient method.

Numerical investigations show that when spatial resolutions of gravity data are smaller than a specific limit, the downward continuation becomes numerically unstable, and therefore, the data must be regularised before downward continuation is applied. Results of the computations on several spatial resolutions show that a distance of 500m is the minimum range for gravity data in Ireland, which in turn contributes from 0.035m to 0.046m in geoid determination.

Conclusions of Paper 4

The gravity potential of the GNSS tide-gauge station at Malin Head (TG-MH) Ireland W_{MH} was determined as well as the W_{MH} offset from Normaal Amsterdams Peil (NAP) datum. For this purpose, the geodetic boundary value problem (GBVP) solution is used based on the combination of the global geopotential model, and terrestrial plus EGM2008 data following the Remove-Compute-Restore approach (RCR).

The geometric geoid undulation, derived from ellipsoidal heights and orthometric heights, was compared to that of gravimetric geoid undulation to estimate the local gravity potential value in TG-MH.

For the first time ever, a justified value of W_{MH} = 62 636 858.273 m^2/s^2, has been computed for Ireland.

The local gravity potential value on Malin-Head tide gauge station differs by 0.363 m^2/s^2 from the gravity potential value of $EGG2015$, (62 636 857.91 m^2/s^2), and is equivalent to height differences equal to 0.034 m above Normaal Amsterdams Peil (NAP) datum.

Recommendations for future research

Below are listed some recommendations of potential areas for future research related to this thesis.

- *Determination of the geopotential value of geodetic lines in Ireland.*

  The precise determination of geopotential numbers at geodetic-lines defines a common height reference system in Ireland for the exchange of geoinformation between Ireland and Europe[1]. The observation of the geodetic levelling network connecting the Fundamental Bench Marks (FBM) was conducted in two stages commencing in 1952 in Northern Ireland and completed in 1969 in Ireland. The geodetic network was adjusted as one block in 1970 resulting in 59 geodetic lines observing over 9,000 levelling points.

[1] In Europe three different kinds of height systems are currently being used; orthometric heights, normal heights and normal-orthometric heights.

Adopting a single FBM as a zero point, the potential differences between conjugative points (the geopotential numbers) can be used to determine precise orthometric heights, normal heights, or dynamic heights based on the gravity potential field (*Sansò and Vaníček*, 2006). Then computed heights can be combined with GNSS-levelling to define precise geometric geoid undulations in FBM's.

- *Methods of fractal geometry in the determination of direct topographical effects*

  If a high precision geoid is desired, it is necessary to evaluate terrain corrections more precisely with finer topographical density. In chapter 3, (paper 2) correlations between the elevation of the topography and topographic effect (on the gravity and potential) are determined and illustrated. Finer grid resolutions show significant changes in amplitude of topographical effects, and also computational results show that it is not guaranteed to compute a 1cm geoid with 50m grid resolutions. One way to extend this analysis is not to consider the topographic surface as a deterministic function, but rather as a fractal function, and carry out the integration over topography in a fractal sense.

- *Analysis of direct topographical effect with Helmerts condensation, Pratt-Hayford and the Airy-Heiskanen isostatic compensation models using Radar DEM in Ireland*

  An investigation of different methods, to determine the terrain correction, which also examines suitabilities of the Helmert second condensation method in Ireland, is suggested. This investigation should be carried out with the 90 m resolution SRTM3arc_v4.1 (*Farr et al.*, 2007) as the detailed DTM in order to examine the accuracies of OSi and LPS-NI's digital elevation models. The direct topographical effects on gravity should be computed with the Pratt-Hayford and the Airy-Heiskanen isostatic compensation models and compared to that of the Helmert second condensation method computed in chapter 3 (page 2) of this study.

- *Analysis of downward-continuation of gravity disturbance and free-air gravity anomalies in Ireland*

  Since the accuracies of free-air gravity anomalies and gravity disturbance in Ireland are the same, it is proposed to investigate the accuracy of the co-geoid heights obtained from gravity disturbance with that of the classical method computed in chapter 4 (page 3) of this study.

- *Analysis of the accuracy of EGG15/EGM2008 gravimetric models in Ireland*

  The accuracies of available models for Ireland, for example, the new European Gravimetric (Quasi) Geoid (EGG2015) or the Earth Gravitational Model 2008 (EGM2008), are not determined yet. Therefore, it is suggested to examine the accuracies of these models. Knowing the accuracies of these models allows the examination of our method in the unification of the height datum studied in chapter 5 (paper 4).

- *Analysis of GOCE omission error and its contribution to vertical datum offsets in Ireland.*

  If the accuracies of EGM2008 are shown to be satisfactory in Ireland, then the effect of the omission error can be approximately estimated using this model (*Amjadiparvar et al.*, 2013). The EGM2008 provides reliable evidence on how the effect of omission error should be dealt with in the actual unification of the vertical datums in Ireland-Northern Ireland by the GBVP approach.

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