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Doctoral Thesis

Coherent states and classical radiative observables in the S-matrix formalism

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Declaration of Authorship

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work. I agree to deposit this thesis in the University's open access institutional repository or allow the Library to do so on my behalf, subject to Irish Copyright Legislation and Trinity College Library conditions of use and acknowledgement.

Some of the research leading to this thesis, in particular [1–7], has appeared previously in the following publications

[1] R. Gonzo, T. Mc Loughlin, D. Medrano, and A. Spiering. "Asymptotic charges and coherent states in QCD". In: *Phys. Rev. D* 104.2 (2021), p. 025019. arXiv: 1906.11763 [hep-th].

[2] S. Banerjee, S. Ghosh, and R. Gonzo. "BMS symmetry of celestial OPE". In: *JHEP* 04 (2020), p. 130. arXiv: 2002.00975 [hep-th].

[3] R. Gonzo and A. Pokraka. "Light-ray operators, detectors and gravitational event shapes". In: *JHEP* 05 (2021), p. 015. arXiv: 2012.01406 [hep-th].

[4] A. Cristofoli, R. Gonzo, D. A. Kosower, and D. O'Connell. "Waveforms from Amplitudes". In: (July 2021). arXiv: 2107.10193 [hep-th].

[5] R. Gonzo and C. Shi. "Geodesics from classical double copy". In: *Phys. Rev. D* 104.10 (2021), p. 105012. arXiv: 2109.01072 [hep-th].

[6] A. Cristofoli, R. Gonzo, N. Moynihan, D. O'Connell, A. Ross, M. Sergola, and C. D. White. "The Uncertainty Principle and Classical Amplitudes". In: (Dec. 2021). arXiv: 2112.07556 [hep-th].

[7] R. Britto, R. Gonzo, and G. R. Jehu. "Graviton particle statistics and coherent states from classical scattering amplitudes". In: (Dec. 2021). arXiv: 2112.07036 [hep-th].

Where the thesis is based on work done by myself jointly with others, the source will be always given and clearly stated.

Summary of the thesis

In this thesis, we study classical radiative observables perturbatively in terms of on-shell scattering amplitudes. In particular, we focus primarily on the two-body problem in gauge and gravitational theories by using an effective field theory approach. The Kosower-Maybee-O'Connell (KMOC) approach, which follows from the classical on-shell reduction of the in-in formalism by using appropriate massive particle wavefunctions, is extended to include classical waves which are naturally described by coherent states. Global observables like the impulse and localized observables like the waveform and gravitational event shapes are then studied in the amplitude approach, making contact also with asymptotic symmetries and light-ray operators defined near null infinity. The classical factorization of radiative observables from the uncertainty principle is proved to be equivalent to a Poisson distribution in the Fock space, and this provides new evidence in favor of a representation of the classical S-matrix in terms of an eikonal phase and a coherent state of gravitons.

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Contents

Dedicated to my family, for all their love and support

Chapter 1

Introduction

1.1 The S-matrix

One of the greatest achievements of high-energy theoretical physics is the surprising ability to describe complex phenomena with a simple theory. While experiments are the guide we trust to understand the wild nature of physics, experience has shown that insights on future predictions of our theory can be driven by pure intellectual developments.

The Standard Model (SM) of particle physics is a successful quantum field theory describing fundamental particles and their interactions at a subatomic scale, which beautifully combines together decades of effort to unify electromagnetism, weak and strong interactions. While the predictions of such model are in agreement with most of the up-to-date experiments carried over at high-energy colliders like LHC, this cannot be the end of the story. In particular, the spacetime on which of all particles are propagating has to be dynamical, as predicted by Einstein theory of General Relativity (GR). Those theories can be combined together in an effective field theory (EFT) valid below the Planck mass, which is the low-energy description of all fundamental forces in our universe. Ultimately we would like to embed this effective description in a UV complete theory like string theory, but many of the questions can be addressed by using the EFT itself. How can we understand dark matter and dark energy? What is the mechanism behind baryon asymmetry and neutrino oscillations? Can we solve the hierarchy problem? Are there higher-derivative corrections to general relativity? The answer requires new physics beyond our current knowledge.

A Lagrangian formulation exists for the SM as a gauge theory with a local $SU(3)\times SU(2)\times U(1)$ gauge group and for GR in terms of the Einstein-Hilbert Lagrangian, which is invariant under diffeomorphism. There is a natural extension of these theories to the most general EFT which is compatible with their symmetries: we can concretely translate our ignorance about the full quantum gravity theory into a set of Wilson coefficients of higher-dimensional operators in the EFT, which can be constrained also by the internal consistency of the theory (causality, unitarity) and testable predictions for measurable observables. While is extremely helpful to have an off-shell formulation of the theory in terms of the quantum fields, this is highly redundant at the level of the on-shell observables.

A unique insight into observables in a quantum gravity theory in asymptotically flat spacetimes is offered by scattering amplitudes, which are perturbatively defined matrix elements of the time evolution operator from the far past to the far future

$$S = U(+\infty, -\infty). \tag{1.1}$$

There are several reasons why these can be considered “the most perfect microscopic structures in the universe” [8]:

- In quantum gravity, there are no local gauge-invariant operators which can give physical observables because of the invariance under diffeomorphism at the semi-classical level [9]. Therefore it is natural to study the S-matrix and other gauge-invariant observables defined from it which are living at the boundary of an asymptotically flat spacetime;

- In the context of AdS/CFT, the holographic principle allows to study gravitational physics from a boundary perspective and viceversa. The enormous progress in understanding concrete examples of the duality made possible to answer deep questions in high-energy physics. The S-matrix has by definition an holographic interpretation in asymptotically flat spacetimes: indeed, the recently developed celestial holography program [10] aims to recast our knowledge of bulk physics into a set of correlators in an exotic celestial CFT;

- The type of numbers and functions which appear in scattering amplitudes have a deep connection to mathematics: from number theory (theory of motives and coaction principle) to positive geometries (amplituhedron, EFThedron,..), representation theory (of the gauge groups), combinatorics and statistics (of the particle distribution);

- Relations between different theories become manifest in compact gauge-invariant expressions like amplitudes and related observables: the double copy is an example of such powerful relations [11], which allows to bypass the complexity of perturbative gravity by studying gauge theory amplitudes and can also be extended to non-perturbative solutions;

- Classical physics, which consists of a set of classical fields, waves and particles obeying a set of differential equations, is hidden inside the quantum S-matrix picture. The way that the classical emerges involve a subtle $\hbar \to 0$ limit, which will be the one of the main topic of this thesis;

- Finally the most crucial aspects of the S-matrix: its close relation to experiment. From subatomic scales with the high-energy scattering of beam of protons at colliders like LHC to the astronomic scales of compact binaries objects emitting gravitational waves, scattering amplitudes have proven to be crucial to understand the physics behind them, at least in the perturbative regime.

1.2 Classical gravitational physics from the S-matrix

Theoretical waveforms play an important role in the LIGO/Virgo Collaboration's observational program of gravitational-wave events from binary mergers [12–19]. These waveforms provide templates that enable the detection of events against otherwise overwhelming noise backgrounds. They also allow observers to extract the masses and spins of the binaries' constituents. To date, theorists have computed waveforms (or equivalently, spectral functions for decaying binaries) using long-established effective-one-body (EOB) methods [20–23] and numerical-relativity approaches [24, 25]. In particular, the classical Hamiltonian for the binary has been derived with the the 'traditional' Arnowitt-Deser-Misner Hamiltonian formalism [26–33], as well as by computations in the effective-field theory approach pioneered by Goldberger and Rothstein [34–61] and in the tradition post-Newtonian approach [62–69].

The start of the gravitational-wave observational era has spurred theorists to explore new approaches to computing classical observables for the two-body problem in gravity, in particular using quantum scattering amplitudes. The connection between the quantum S-matrix and observables in classical General Relativity (GR) was first explored nearly fifty years ago by Iwasaki [70]. Earlier investigations included extraction of the two-body potential from amplitudes and the study of quantum corrections to gravity [71–77], considering gravity as an effective field theory valid below the Planck scale [72, 78].

An important step was taken by Cheung, Solon, and Rothstein (CRS) [79], building on earlier work by Neill and Rothstein [80]. They showed how to match effective field theories (EFTs) efficiently to scattering amplitudes above threshold in order to extract a classical potential. The classical potential can then be used in the EOB or other frameworks to make predictions for bound-state quantities. Bern, Cheung, Roiban, Shen, Solon, and Zeng used [81, 82] this approach to compute the third-order corrections (G^3) to the conservative potential. This milestone computation went beyond what had been known from direct classical GR calculations, and provided the first concrete fulfillment of the promise of the scattering-amplitudes class of methods. On-shell scattering amplitudes techniques, powered by locality, unitarity and double copy [83–106], have been used to get compact analytic expressions for the state-of-art binary dynamics for spinless pointlike bodies at 3 PM order and at 4 PM order [79–81, 107–109]. A handful of alternative and complementary approaches have also been developed in recent years. The relativistic eikonal expansion [110–119], the heavy particle effective theory [120–122] and semiclassical worldline tools [123–126] offered many insights on the binary problem, both at the conceptual and at the practical computational level. Moreover, the formalism can be extended to include both spinning bodies [82, 127, 128] and finite size effects [129, 130] in terms of additional higher-dimensional operators.

The dynamics of the binary in the presence of radiation is much less understood compared to the conservative case. This is very important, for example to establish a direct connection with the waveforms [4, 131–134]. Unitarity dictates that, even at the classical level, observables are IR-finite only when we include both real and virtual radiation, as stressed in [135, 136]. This is crucial to obtain a well-behaved scattering angle at high energies [137], as was proven by a direct calculation of radiation reaction effects [115, 116, 138–140].

There are other interesting less inclusive IR-safe observables like energy event shapes [3]. Event shapes or weighted cross-sections describe the distribution of outgoing particles and their properties such as charge or energy-momentum in a particular direction as determined by a detector. They are important tools for analyzing jets produced in collider experiments [141–143]. In the '90s, Korchemsky and Sterman[1] discovered that such event shapes can be equivalently described by the expectation value of some non-local operators – line integrals of local conserved currents – inserted at different locations on the celestial sphere [146, 147]. While these objects are well-known in perturbative QCD, where they are used to study hadron scattering at high energies [148], it is only recently that Maldacena and Hofman initiated a systematic investigation of such operators in conformal field theory [149]. It is now well understood how to extract event shapes from OPE data and symmetries in conformal field theories (especially for $\mathcal{N} = 4$ SYM) [150, 151]. Moreover, the connection between energy event shapes and the ANEC operator establishes bounds on the a and c coefficients characterizing the conformal anomaly [149, 152, 153]. Of course, such developments led to increased interest in CFT light-ray operators which culminated in the systematic analysis of [154–157]. We can also study the correlation between

[1] See [144, 145] for earlier approaches.

event shapes by considering more than one detector (ie., multiple insertions of the corresponding non-local operators). Of particular interest is the so called energy-energy correlator, which is an infrared-finite observable that can be measured in collider experiments and used to provide a measure of the QCD coupling constant. The leading order QCD prediction for energy-energy correlators was computed in the late '70s [142, 143]. More recently, analytical expressions at NLO [158, 159] as well as numerical expressions at NNLO [160] have become available. In $\mathcal{N} = 4$ SYM, there are even analytic results at NNLO [161].

In the conservative case, a dictionary has been found [162, 163] which enables the analytic continuation of observables from hyperbolic-like scattering orbits to bound orbits, which ultimately are of direct relevance to LIGO. This is even more crucial when radiation is included, and indeed there are promising results in this direction for local-in-time contributions [66, 164, 165]. The amazing correspondence between these different type of solutions is due to the fact that the differential equations at the classical level are the same for both systems, and only the boundary conditions are changing. This becomes clear when the solution is expressed in terms of the conserved charges, as emphasized in [5] as well as in the original work [162, 163].

All these amplitude approaches share the need for careful analysis of the classical limit, as done in seminal work by Kosower, Maybee and O'Connell (KMOC) [166]. In this thesis, we will make use of this formalism mainly to describe classical waves and radiative observables in a scattering problem, extending the original approach in several directions. In particular, the outline of the thesis is as follows.

In chapter 2, we will discuss the derivation of the KMOC formalism from the in-in approach at zero temperature and we will extend how the formalism to include classical waves described in terms of coherent states. In chapter 3, we will study how localized detectors can be represented in terms of a system of light-ray operators defined near null infinity from Einstein equations, as well as their relation with event shapes. In chapter 4 we will study both global and local observables for radiation, focussing on the impulse, the waveform and the gravitational energy event shapes. In chapter 5, we will discuss how coherent states arise from many different perspectives: soft theorems, asymptotic symmetries, the uncertainty principle and the analysis of the particle distribution. In chapter 6, we will explicit compute some of the amplitudes relevant for the classical radiative observables in the two-body problem from tree-level amplitudes, showing that coherence is obeyed explicitly. In chapter 7, we finally extend the eikonal formulation for the two-body problem to the emission of real radiation with the use of a coherent state of gravitons. We briefly conclude in chapter 8.

Most of this thesis is based on published work, we will make that explicit here. We have adapted some material from [6] for section 2.1 and section 2.2 of chapter 2, and for section 4.1, section 4.3 and section 4.5 of chapter 4. Chapter 3, section 4.4 of chapter 4, appendix B and appendix C are entirely based on [3]. Section 5.1, section 4.2 of chapter 4, section 5.2 and section 5.3 of chapter 5, chapter 7, appendix A, appendix E, appendix F, appendix G and appendix H are mostly based on [4]. Section 5.4 of chapter 5, chapter 6 and appendix D come from [7]. Section 4.6 and section 5.5 are completely original.

We will use relativistic units, retaining $c = 1$, even as we restore $\hbar$ explicitly. We will use the mostly minus convention, except otherwise stated. We define the symmetrized (resp. antisymmetrized) product for any tensorial expression T as $T_{(\mu\nu)} = T_{\mu\nu} + T_{\nu\mu}$ (resp. $T_{[\mu\nu]} = T_{\mu\nu} - T_{\nu\mu}$).

Chapter 2

KMOC: classical on-shell reduction of the in-in formalism

In this chapter, we will introduce the KMOC formalism to study the classical two-body scattering problem and we will briefly discuss how to derive it from first principles in quantum field theory. We will start from massive point particles as in the seminal work [166], and then we will include classical waves by introducing coherent states. At the end, we will incorporate spin in the description in terms of spin coherent states.

2.1 Scattering of point particles

We wish to describe here massive point particle in the quantum field theory formalism. We must distinguish units of energy and length, which we denote by $[M]$ and $[L]$ respectively. We use the standard normalization for the annihilation and creation operators of the scalar field such that,

$$[a(p), a^\dagger(p')] = (2\pi)^3 2E_p\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')\,. \tag{2.1}$$

Accordingly, n-point scattering amplitudes continue to have dimension $[M]^{4-n}$.

We keep $[M]^{-1}$ as the dimension of single-particle states $|p\rangle$,

$$|p\rangle \equiv a^\dagger(p)|0\rangle\,, \tag{2.2}$$

with the vacuum state being dimensionless. We define n-particle plane-wave states as simply the tensor product of normalized single-particle states. The state $|p\rangle$ represents a particle of momentum p and positive energy, while $\langle p| = \langle 0|a(p)$ is the conjugate state.

The scattering matrix S and the transition matrix T are both dimensionless. Scattering amplitudes are matrix elements of the latter between plane-wave states,

$$\langle p'_1 \cdots p'_m|T|p_1 \cdots p_n\rangle = \mathcal{A}_{n+m}(p_1, \cdots, p_n \to p'_1, \cdots, p'_m)\delta^{(4)}(p_1 + \cdots p_n - p'_1 - \cdots - p'_m)\,. \tag{2.3}$$

As our formalism encompasses both QED and gravity, as well as other theories with massless force carriers, we denote the coupling by g. In electrodynamics, it corresponds to e, while in gravity to $\kappa = \sqrt{32\pi G}$. It is $g/\sqrt{\hbar}$ that is the dimensionless coupling in electrodynamics, and similarly in gravity $\kappa/\sqrt{\hbar}$ has the correct dimension of the inverse mass. We will denote the generic amplitude with $\mathcal{A}_n^{(L)}$, where n is the number of legs and L is the number of loops, and more in general we will use the notation $\mathcal{M}_n^{(L)}$ for the amplitudes in a gravitational theory.

We start by taking the momenta of all particles as the primary variables; for most massless momenta, wavenumbers are the variables of interest. We introduce a notation for the wavenumber $\bar{q}$ associated to the momentum q,

$$\bar{q} \equiv q/\hbar\,. \tag{2.4}$$

We use the notation of ref. [166] for the on-shell phase-space measure of a massive particle,

$$\mathrm{d}\Phi(p) = \hat{\mathrm{d}}^4 p\,\hat{\delta}(p^2 - m^2)\,\theta(p^0)\,. \tag{2.5}$$

where p^0 is the energy component of the four-vector and the carets indicate factors of 2π:

$$\hat{\mathrm{d}}^4 p \equiv \frac{\mathrm{d}^4 p}{(2\pi)^4}, \qquad \hat{\delta}(\cdot) \equiv (2\pi)\delta(\cdot)\,. \tag{2.6}$$

Given our convention for normalizing single-particle states, their inner product is,

$$\begin{aligned}\langle p'|p\rangle &= (2\pi)^3 2E_p \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \\ &=: \delta_\Phi(p - p').\end{aligned} \tag{2.7}$$

With this notation, we can also rewrite the normalization of creation and annihilation operations of eq. (2.1) in a natural form,

$$[a(p), a^\dagger(p')] = \delta_\Phi(p - p')\,. \tag{2.8}$$

We will continue to use the notation of ref. [166] for initial states involving only massive particles: we take the initial momenta to be p_1 and p_2, initially separated by a transverse impact parameter b. The latter is transverse in that $p_1 \cdot b = 0 = p_2 \cdot b$. In the quantum theory, the system of massive particles is described by wave functions, which we build out of plane waves. In the classical limit, these wave functions must localize the two point-like particles, and must separate them clearly. We describe the incoming particles in the far past by wave functions $\psi_j(p)$, which we take to have reasonably well-defined positions and momenta. We will review the requirements on the wave packets, discussed in detail in sect. 4 of ref. [166], below.

We express the initial state in terms of plane waves $|p_1\,p_2\rangle$,

$$|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)\;\psi_A(p_1)\psi_B(p_2)\,e^{ib\cdot p_1/\hbar}|p_1\,p_2\rangle\,. \tag{2.9}$$

We require each wave function $\psi_j(p)$ to be normalized to unity,

$$\int \mathrm{d}\Phi(p)\;|\psi_j(p)|^2 = 1 \quad j = A, B, \tag{2.10}$$

so that our incoming state is also normalized to unity,

$$\begin{aligned}\langle\psi_{\text{in}}|\psi_{\text{in}}\rangle = \int &\mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)\mathrm{d}\Phi(p_1')\mathrm{d}\Phi(p_2')e^{ib\cdot(p_1 - p_1')/\hbar} \\ &\times \psi_A(p_1)\psi_A^*(p_1')\,\psi_B(p_2)\psi_B^*(p_2')\,\delta_\Phi(p_1 - p_1')\,\delta_\Phi(p_2 - p_2') \\ = \int &\mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)\;|\psi_A(p_1)|^2|\psi_B(p_2)|^2 \\ = 1\,.&\end{aligned} \tag{2.11}$$

The wavefunctions $\psi_A(p_1), \psi_B(p_2)$ are defined as

$$\psi_A(p_1) := \mathcal{N} m_A^{-1} \exp\left[-\frac{p_1 \cdot v_A}{\hbar \ell_c^A/\ell_w^2}\right], \qquad \psi_B(p_2) := \mathcal{N} m_B^{-1} \exp\left[-\frac{p_2 \cdot v_B}{\hbar \ell_c^B/\ell_w^2}\right], \tag{2.12}$$

where $\mathcal{N}$ is a normalization factor. The wavefunctions $\psi_j(p)$ for each particle have the property that they localise the particle's position with an uncertainty ℓ_w which is characteristic of the wavepacket[1]. At the same time, the wavepackets must localise the momenta of the particles to uncertainty Δp of order $\hbar/\ell_w$. In the classical limit, we require that these uncertainties are negligible compared to the distance, of order b, between the particles

$$\ell_w \ll b\,, \tag{2.13}$$

and also require that the momentum-space uncertainty is negligible compared to the masses of the particles:

$$\frac{\hbar}{\ell_w} \ll m_j \Rightarrow \ell_c^j \ll \ell_w\,, \tag{2.14}$$

where $\ell_c^j \equiv \hbar/m_j$ is a measure of the Compton wavelength of the particle. More generally, if ℓ_s is the distance of closest approach of the particles during scattering, we require that

$$\ell_c^j \ll \ell_w \ll \ell_s\,. \tag{2.15}$$

Small-angle scattering has the property that $\ell_s \simeq b$. Therefore taking the classical limit requires that we impose the ‘

$$\ell_c^j \ll \ell_w \ll b \qquad \text{for} \quad j = A, B, \tag{2.16}$$

which ensure that wavefunctions such as those in eq. (5.70) effectively localize the massive particles on their classical trajectories as $\hbar \to 0$.

Since we will encounter these wavepackets rather frequently, it is sometimes convenient to write the two-particle momentum-space wavefunction as

$$\psi_b(p_1, p_2) \equiv \psi(p_1, p_2) e^{ib\cdot p_1/\hbar} \quad , \quad \psi(p_1, p_2) \equiv \psi_1(p_1)\psi_2(p_2)\,. \tag{2.17}$$

We will further use the short-hand notation

$$\mathrm{d}\Phi(p_1, p_2, p_3, \ldots) \equiv \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)\mathrm{d}\Phi(p_3)\cdots\,. \tag{2.18}$$

for integrals over the phase space of various particles.

Now that we have an incoming state, we may write the outgoing state in terms of the time evolution operator $U(+\infty, -\infty)$, which is equal to the S matrix:

$$\begin{aligned} |\psi_{\text{out}}\rangle &= S|\psi_{\text{in}}\rangle \\ &= \int \mathrm{d}\Phi(p_1, p_2)\, \psi_b(p_1, p_2)\, S|p_1, p_2\rangle\,. \end{aligned} \tag{2.19}$$

Since the S matrix can be written in terms of scattering amplitudes, the outgoing state itself can be written in terms of integrals over amplitudes and the incoming state.

[1]We could choose different uncertainties $\ell_w^{(i)}$ for each particle at the expense of a slightly more complicated notation. There is no reason for us to exploit this freedom here, so we choose the simpler case of one common ℓ_w.

In order to expand in the $\hbar \to 0$ limit and extract the leading, classical, term for any observable, as mentioned above we must make the powers of $\hbar$ explicit. These arise from two sources: powers ordinarily hidden inside electromagnetic or gravitational couplings; and powers arising from keeping the wavenumbers of massless particles fixed rather than their momenta. This is true both for emitted and virtual particles, when considering quantities such as the total emitted radiation.

In order to control the $\hbar$ expansion of scattering amplitudes, it is useful to introduce some further notation. Let us write the amplitudes as explicit Laurent series in $\hbar$

$$\mathcal{A}_n^{(L)}(i \to f) = \hbar^{-C(n,L)} \left(\mathcal{A}_{n,0}^{(L)}(i \to f) + \hbar \mathcal{A}_{n,1}^{(L)}(i \to f) + \cdots \right), \tag{2.20}$$

where the quantities $\mathcal{A}_{n,p}^{(L)}$ are $\hbar$-independent gauge-invariant sub-amplitudes and we have isolated the leading power in $\hbar$ which we call $C(n,L)$. This expansion defines an infinite set of objects $\mathcal{A}_{n,p}^{(L)}$ which could in principle be reassembled into the full amplitude. They are a kind of partial amplitude, but distinct from the usual use of this term. We will therefore refer to them as "fragmentary amplitudes", or simply as "fragments."

Since $\hbar$ is dimensionful, it is useful to view these fragmentary amplitudes in a slightly different way. Amplitudes are functions of Mandelstam invariants; in the semi-classical region, we are expanding in powers of momentum transfers, such as $q^2 = \hbar^2 \bar{q}^2$ at four points, divided by Mandelstam $s = (p_1 + p_2)^2$. The semiclassical expansion is an expansion in powers of $\hbar\sqrt{-\bar{q}^2/s}$. More general amplitudes involve a richer set of momentum transfers $\bar{q}_{ij}^2$; our expansion is in powers of $\hbar\sqrt{-\bar{q}_{ij}^2/s}$. We only consider amplitudes with two incoming massive particles.

We may also view the expansion as being in (inverse) powers of the large mass of the scattering particles [95, 105, 120, 121]. This makes contact with effective field theory, especially heavy quark effective theory or, more generally, heavy particle effective theories as has been emphasised in references [105, 120, 121]. Our fragmentary amplitudes correspond in this context to the standard HQET expansion in inverse powers of heavy masses. It seems likely that a study of the properties of amplitudes in these theories would illuminate the structure of the fragmentary amplitudes.

Notice that this expansion is analogous, but different, to a soft expansion. In the soft expansion we take the momentum of an individual particle soft. In this transfer expansion we take the momenta in all messenger lines to be of the same order, and small compared to the incoming centre of mass energy. It is possible to perform the transfer expansion and then, in a second stage, to single out some line, say an outgoing photon, and take its momentum to be softer than all other messenger lines. This yields the soft limit of the transfer expansion. It corresponds to the low-frequency limit in the classical approximation. Interesting classical physics, including memory effects, appear in this region [167–175].

For the total emitted radiation, we can define the operator

$$\mathbb{K}^\mu := \sum_{\sigma=\pm 2} \int \mathrm{d}\Phi(k) k^\mu a_\sigma^\dagger(k) a_\sigma(k), \tag{2.21}$$

where $a_\sigma^\dagger(k)$ (resp. $a_\sigma(k)$) is the graviton creation (resp. annihilation) operators of a definite helicity $\sigma = \pm 2$. In eq. (3.33) of ref. [166], there is an expression for time-averaged radiated momentum,

$$R^\mu \equiv \langle k^\mu \rangle = \langle \psi_{\text{in}} | \, S^\dagger \mathbb{K}^\mu S \, | \psi_{\text{in}} \rangle \tag{2.22}$$

Inspired by [176], we will then compute perturbatively the expectation value of

$$\langle 0_{\rm in}|\, k^\mu a^\dagger_\sigma(k) a_\sigma(k) |0_{\rm in}\rangle \tag{2.23}$$

purely from the Schwinger-Keldysh (SK) perspective,[2] where $|0_{\rm in}\rangle$ is the initial graviton state at $t = -\infty$. We will start from simplicity in pure Einstein gravity and later we will include matter coupled with gravity, in order to take the appropriate classical limit using the KMOC formalism.

One can express eq. (2.23) with the LSZ reduction as

$$\begin{aligned}\langle 0_{\rm in}|\, k^\mu a^\dagger_\sigma(k) a_\sigma(k) |0_{\rm in}\rangle &= k^\mu \varepsilon^{\alpha\beta}_\sigma(k)\varepsilon^{\rho\xi}_\sigma(k)\\ &\times \int \mathrm{d}^4x \int \mathrm{d}^4y\, e^{ik\cdot(x-y)/\hbar}\, \Box_x \Box_y \langle 0_{\rm in}|\, h_{\alpha\beta}(x) h_{\rho\xi}(y)|0_{\rm in}\rangle .\end{aligned} \tag{2.24}$$

where the polarization vectors satisfy,

$$\left[\varepsilon^\sigma_{\mu\nu}(k)\right]^* = \varepsilon^{-\sigma}_{\mu\nu}(k)\,. \tag{2.25}$$

More generally, $a^\dagger_\sigma(k)$ creates a single-messenger state of momentum k and helicity σ, while $a_\sigma(k)$ destroys such a state. Equivalently, the latter operator creates a conjugate state of momentum k and helicity σ. The commutation relations are

$$\left[a_\sigma(k), a^\dagger_{\sigma'}(k')\right] = \delta_{\sigma,\sigma'}\delta_\Phi(k-k')\,. \tag{2.26}$$

For example, a single-particle positive-helicity state is

$$|k^+\rangle \equiv a^\dagger_+(k)|0\rangle = \left[a_+(k)\right]^\dagger |0\rangle\,. \tag{2.27}$$

The conjugate state is $\langle k^+|$.

We follow the usual amplitudes convention of representing an outgoing positive-helicity graviton of momentum k by $\varepsilon^+_{\mu\nu}(k)$, which also corresponds to an incoming negative-helicity graviton of the opposite momentum. To understand the helicity flip for an incoming state, note that we can analytically continue an incoming momentum k to an outgoing momentum $k' = -k$. The energy component k'^0 of the outgoing momentum is now negative. Thus, in an all-outgoing convention, positive-helicity gravitons of momentum k with $k^0 > 0$ are represented by the polarization vector $\varepsilon^+_{\mu\nu}(k)$, while positive-helicity gravitons of momentum k with $k^0 < 0$ are represented by the polarization vector $\varepsilon^-_{\mu\nu}(k)$.

Notice that there is no (time) ordering in the correlator function. We now need to make contact with a generating functional to be able to compute this expression in perturbation theory. The idea is to introduce a new complex contour, called the Keldysh contour, which is made of two branches called $+$ and $-$ running parallel to the usual time axis (see Fig. 2.1) and to formally double the set of fields $h^{(\pm)}_{\mu\nu}$ involved in the path integral. Each copy of the fields will be labelled by the index $+$ or $-$ according to the branch of the contour $\mathcal{C}$ they belong to.

[2] This is also called the in-in formalism at zero temperature.

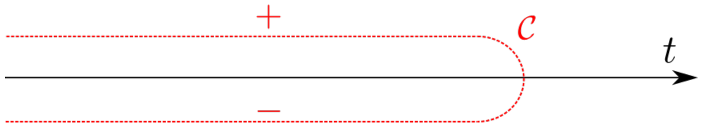


FIGURE 2.1: The two branches of the Schwinger-Keldysh contour $\mathcal{C}$ run from above (+) to below (-) the real time axis

Using the interaction representation for the quantum fields, we can write[3]

$$\langle 0_{\rm in}|\, h_{\alpha\beta}(x)h_{\rho\xi}(y)|0_{\rm in}\rangle = \int \mathcal{D}h^{(+)}\mathcal{D}h^{(-)}\, h^{(-)}_{\alpha\beta}(x)h^{(+)}_{\rho\xi}(y) \times e^{\frac{i}{\hbar}\int_{\mathbb{R}\times\mathbb{R}^3} \mathrm{d}^4x\left(\mathcal{L}^{(+)}_{\mathrm{GR},int}[h^{(+)}]-\mathcal{L}^{(-)}_{\mathrm{GR},int}[h^{(-)}]\right)}, \tag{2.28}$$

where $\{\mathcal{L}^{(+)}_{\mathrm{GR},int}[h^{(+)}], \mathcal{L}^{(-)}_{\mathrm{GR},int}[h^{(-)}]\}$ is a set of two copies of the interaction Lagrangian in the pure gravity theory where all the fields belong the same branch of the contour $\mathcal{C}$. At this point we can rewrite the initial expression as

$$\langle 0_{\rm in}|\, k^\mu a^\dagger_\sigma(k)a_\sigma(k)|0_{\rm in}\rangle = k^\mu \varepsilon^{\alpha\beta}_\sigma(k)\varepsilon^{\rho\xi}_\sigma(k) \times \int \mathrm{d}^4x\int \mathrm{d}^4y\, e^{ik\cdot(x-y)/\hbar}\,\Box_x\,\Box_y \langle 0_{\rm in}|\, P h^{(-)}_{\alpha\beta}(x)h^{(+)}_{\rho\xi}(y)e^{i\int_{\mathcal{C}\times\mathbb{R}^3}\mathrm{d}^4x\mathcal{L}_{\mathrm{GR},int}[h])/\hbar}|0_{\rm in}\rangle, \tag{2.29}$$

where the ordering P corresponds to

$$P h_{\alpha\beta}(x)h_{\rho\xi}(y) = \begin{cases} T h_{\alpha\beta}(x)h_{\rho\xi}(y) & \text{if} \quad x_0, y_0 \in \mathcal{C}_+ \\ \bar{T} h_{\alpha\beta}(x)h_{\rho\xi}(y) & \text{if} \quad x_0, y_0 \in \mathcal{C}_- \\ h_{\alpha\beta}(x)h_{\rho\xi}(y) & \text{if} \quad x_0 \in \mathcal{C}_-, y_0 \in \mathcal{C}_+ \\ h_{\rho\xi}(y)h_{\alpha\beta}(x) & \text{if} \quad x_0 \in \mathcal{C}_+, y_0 \in \mathcal{C}_-. \end{cases} \tag{2.30}$$

The ordering P acts on the space of both copies of the fields $(+)$ and $(-)$, and in eq.(2.30) we have implicitly identified each of them with one branch of the contour around the time axis,

$$\begin{aligned} h_{\alpha\beta}(x) &= h^{(+)}_{\alpha\beta}(x) \quad \text{if} \quad x_0 \in \mathcal{C}_+ \ , \\ h_{\alpha\beta}(x) &= h^{(-)}_{\alpha\beta}(x) \quad \text{if} \quad x_0 \in \mathcal{C}_- \ . \end{aligned} \tag{2.31}$$

We have therefore unified the treatment of the two time orderings in terms of the contour $\mathcal{C}$ depicted in Fig. 2.1, which allows to reformulate the perturbation theory calculation in terms of a simple path integral representation for the general in-in expectation value. Indeed, one can write a generating functional

$$\mathcal{Z}^{\rm SK}[j^{(+)}, j^{(-)}] := \langle 0_{\rm in}|e^{i\int_{\mathcal{C}\times\mathbb{R}^3}\mathrm{d}^4x(\mathcal{L}_{\mathrm{GR},int}[h]+j^{\mu\nu}h_{\mu\nu})/\hbar}|0_{\rm in}\rangle \tag{2.32}$$

[3]For simplicity, we have suppressed the spacetime indices in the path integral variables and the boundary conditions of the path integral which should force the state to be $|0_{\rm in}\rangle$ at $t = -\infty$.

in terms of which eq. (2.29) can be written as

$$\langle 0_{\text{in}}|\, k^\mu a^\dagger_\sigma(k) a_\sigma(k)|0_{\text{in}}\rangle \\ = k^\mu \varepsilon^{\alpha\beta}_\sigma(k)\varepsilon^{\rho\xi}_\sigma(k)\int \mathrm{d}^4x \int \mathrm{d}^4y\, e^{ik\cdot(x-y)/\hbar}\,\Box_x\,\Box_y \frac{\delta \mathcal{Z}^{\text{SK}}[j^{(+)},j^{(-)}]}{i\delta j^{\alpha\beta,(+)}(x)i\delta j^{\rho\xi,(-)}(y)}\bigg|_{j^{(\pm)}=0}. \tag{2.33}$$

The generic SK propagator in the $(+)/(-)$ basis can be written as

$$\int \mathrm{d}^4x e^{ik\cdot x/\hbar}\langle 0_{\text{in}}|\, Ph^{(w)}_{\alpha\beta}(x)h^{(w')}_{\rho\xi}(0)|0_{\text{in}}\rangle = \begin{pmatrix} G^{++}_{\alpha\beta\rho\xi}(k) & G^{-+}_{\alpha\beta\rho\xi}(k) \\ G^{+-}_{\alpha\beta\rho\xi}(k) & G^{--}_{\alpha\beta\rho\xi}(k)\end{pmatrix} \\ = \begin{pmatrix} \frac{i}{\hbar^2\bar{k}^2+i\epsilon} & 2\pi\theta(-\bar{k}^0)\delta(\hbar^2\bar{k}^2) \\ 2\pi\theta(\bar{k}^0)\delta(\hbar^2\bar{k}^2) & -\frac{i}{\hbar^2\bar{k}^2-i\epsilon}\end{pmatrix}\mathcal{P}_{\alpha\beta\rho\xi}, \tag{2.34}$$

where w, w' can take values ±1, and $\mathcal{P}_{\alpha\beta\rho\xi} := \frac{1}{2}\left(\eta_{\alpha\rho}\eta_{\beta\xi}+\eta_{\alpha\xi}\eta_{\beta\rho}-\eta_{\alpha\beta}\eta_{\rho\xi}\right)$ is the standard numerator for the graviton propagator. It is interesting to notice that, consistently with causality, different $i\epsilon$ prescriptions are related to fields living on different branches of the contour $\mathcal{C}$. It is manifest that we can choose any basis for the SK formulation, for example the time-ordered/anti time-ordered (also called $(+)/(-)$) basis as in the previous calculations or the retarded/advanced basis, and the result will be independent of that choice.

The direct connection with the standard Feynman integral perturbative expansion can be seen directly at the level of the generating functional. We can express the SK generating functional in terms of the Feynman generating functional and its conjugate

$$\mathcal{Z}^{\text{SK}}[j^{(+)},j^{(-)}] = e^{\int \mathrm{d}^4x\mathrm{d}^4y G^{\mu\nu\rho\xi,+-}(x,y)\,\Box_x\,\Box_y\frac{\delta^2}{i\delta j^{\mu\nu,(+)}(x)i\delta j^{\rho\xi,(-)}(y)}}\mathcal{Z}[j^{(+)}]\mathcal{Z}^*[j^{(-)}]. \tag{2.35}$$

To make the connection with the KMOC formalism more precise, we need to add matter coupled with gravity and to consider as our initial state $|\psi_{\text{in}}\rangle$. Essentially all the previous arguments go through by extending the discussion for a correlator of a set of massive scalar and graviton fields. Then we have

$$\langle \psi_{\text{in}}|\, k^\mu a^\dagger_\sigma(k)a_\sigma(k)|\psi_{\text{in}}\rangle \\ = k^\mu\varepsilon^{\alpha\beta}_\sigma(k)\varepsilon^{\rho\xi}_\sigma(k)\int \mathrm{d}^4x\int \mathrm{d}^4y\, e^{ik\cdot(x-y)/\hbar}\,\Box_x\,\Box_y\langle\psi_{\text{in}}|\, h_{\alpha\beta}(x)h_{\rho\xi}(y)|\psi_{\text{in}}\rangle, \tag{2.36}$$

and when we connect this with the interaction representation,

$$\langle \psi_{\text{in}}|\, Ph^{(-)}_{\alpha\beta}(x)h^{(+)}_{\rho\sigma}(y)e^{i\int_{\mathcal{C}\times\mathbb{R}^3}\mathrm{d}^4x\mathcal{L}_{\text{GR+matter,int}}[h])/\hbar}|\psi_{\text{in}}\rangle, \tag{2.37}$$

we must take the LSZ reduction for the massive external states with the appropriate KMOC wavefunction $\psi_A(p_1)$ and $\psi_B(p_2)$ as defined in eq. (5.70),

$$\int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)\,\psi_A(p_1)\psi_B(p_2)e^{i\frac{p_1\cdot b}{\hbar}}\prod_{i=1}^{2}\left[\int \mathrm{d}^4x_i\, e^{i\frac{p_i\cdot x_i}{\hbar}}\left(\Box_{x_i}+\frac{m_i^2}{\hbar^2}\right)\right]\langle\ldots\phi(x_1)\phi(x_2)\ldots\rangle, \tag{2.38}$$

which in the limit $\hbar\to0$ will effectively localize the massive particles on their classical trajectories characterized by a 4-velocity v_A and v_B and by an impact parameter b^μ.

The in-in formalism is a set of off-shell techniques in QFT which in principle can be used to compute the expectation value of any quantum field or polynomial thereof, including for example the stress tensor and its conserved charges. Here we have shown that taking an appropriate LSZ reduction on the external legs and using appropriate wavefunctions for the massive particles, we naturally obtain the KMOC formalism. Under LSZ reduction, the contraction arising from time-ordered $(+)$ or anti-time ordered $(-)$ correlators of fields $\{h_{\mu\nu}, \phi\}$ in the Schwinger-Keldysh formalism maps to S-matrix elements (with the $+i\epsilon$ prescription) and their conjugates (with the $-i\epsilon$ prescription). Moreover, the contraction of fields belonging to different branches of the contour ($(+)$ and $(-)$ or vice versa) gives the unitarity cut contributions. See Fig. 2.2 for a pictorial representation of these different contributions. This helps also to address some concerns raised in [177, 178] on getting classical observables from scattering amplitudes with a definite $i\epsilon$ prescription.

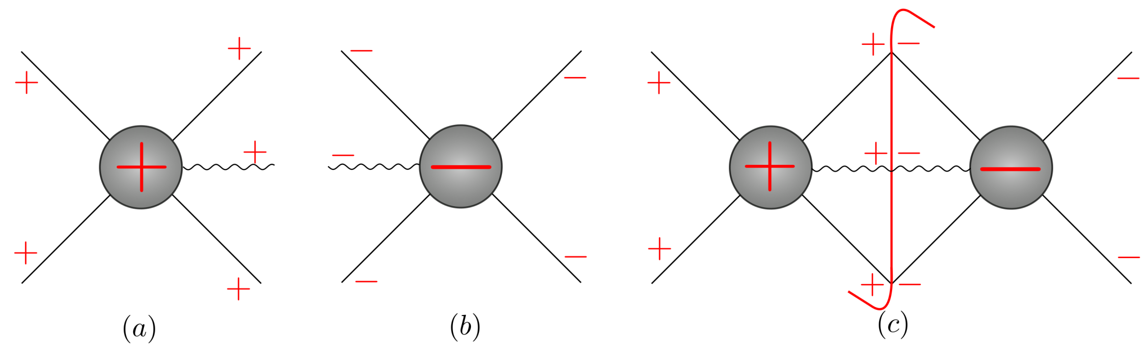


FIGURE 2.2: The contributions of the type (a) (resp. (b)) arise from purely time-ordered fields (resp. anti-time ordered) and correspond, under LSZ reduction for the external legs, to on-shell contributions which are linear in the amplitudes. On the other hand, terms of the type (c) mix fields on different branches of the Schwinger-Keldysh contour, which corresponds in unitarity cut contributions between one amplitude and its conjugate in the on-shell formalism.

This derivation gives some insight into the relation between the SK formalism and the KMOC formalism relevant to fully on-shell calculations, like the radiated energy, angular momentum, or more localized observables like the waveform and gravitational event shapes (essentially by considering only the on-shell radiative contribution of the fields arising in the large r limit). But it also extends beyond this. In particular, it explains some recent derivations of off-shell metric configurations from "amplitudes" with one off-shell graviton leg [179]. In that case one avoids taking the LSZ reduction of the graviton field whose expectation value is taken. A simple example is given by the (linearized) metric generated by on-shell matter particles coupled to gravity. For example, this justifies the results obtained in [179] for the derivation of gravitational shock wave configurations from the 3-point function with two massless on-shell scalars and one off-shell graviton. The same argument can be repeated for any other on-shell matter configuration coupled to one off-shell graviton, essentially making use of the (linearized) stress tensor [180, 181]. Alternatively, one can work fully on-shell but in $(2, 2)$ signature, as shown in [106, 182, 183].

2.2 Scattering of classical waves

We are now ready to to include initial-state massless classical waves in the formalism of ref. [166]. A naive extension of the considerations of ref. [166] to massless particles is clearly impossible. A particle's Compton wavelength diverges when its mass goes to zero, making it impossible to satisfy the required conditions (see eq. (2.16)). It doesn't make sense to treat messengers (photons or gravitons) as point-like particles. Indeed, Newton and Wigner [184] and Wightman [185] proved rigorously long ago that a strict localization of known massless particles in position space is impossible[4]. A proper treatment instead relies on coherent states. We begin such a treatment by discussing general aspects of coherent states, focusing on the electromagnetic case. We then describe the kind of coherent states of interest to us.

We can write the electromagnetic field operator as,

$$\mathbb{A}_\mu(x) = \frac{1}{\sqrt{\hbar}} \sum_{\sigma=\pm 1} \int \mathrm{d}\Phi(k) \left[a_\sigma(k) \varepsilon_\mu^{\sigma *}(k)\, e^{-ik\cdot x/\hbar} + a_\sigma^\dagger(k) \varepsilon_\mu^{\sigma}(k)\, e^{+ik\cdot x/\hbar} \right], \tag{2.39}$$

We are using the same symbol (namely, a_σ) for annihilation operators in both electromagnetism and gravity; we hope context will make clear which operator is relevant. Using the form of the electromagnetic field in eq. (2.39), the electromagnetic field strength operator is,

$$\mathbb{F}_{\mu\nu}(x) = -\frac{i}{\hbar^{3/2}} \sum_{\sigma=\pm 1} \int \mathrm{d}\Phi(k) \left[a_\sigma(k)\, k_{[\mu} \varepsilon_{\nu]}^{\sigma *}(k)\, e^{-ik\cdot x/\hbar} - a_\sigma^\dagger(k)\, k_{[\mu} \varepsilon_{\nu]}^{\sigma}(k)\, e^{+ik\cdot x/\hbar} \right]. \tag{2.40}$$

Introduce the coherent-state operator,

$$\mathbb{C}_{\alpha,\sigma} \equiv \mathcal{N}_\alpha \exp\left[\int \mathrm{d}\Phi(k)\, \alpha(k) a_\sigma^\dagger(k) \right], \tag{2.41}$$

where the normalization $\mathcal{N}_\alpha$ will be given below. We can build coherent states of the electromagnetic field using this operator, such as a positive-helicity one,

$$|\alpha^+\rangle = \mathbb{C}_{\alpha,(+)}|0\rangle\,. \tag{2.42}$$

More generally, we could consider coherent states containing both helicities. Since coherent-state operators for different helicities commute and every polarization vector can be decomposed in the helicity basis, there is no loss of generality in making a specific helicity choice for the coherent states we consider. The coherent state operators are unitary,

$$(\mathbb{C}_{\alpha,\sigma})^\dagger = (\mathbb{C}_{\alpha,\sigma})^{-1}\,. \tag{2.43}$$

The normalization factor $\mathcal{N}_\alpha$ is determined by the condition $\langle\alpha^+|\alpha^+\rangle = 1$, that is,

$$\mathcal{N}_\alpha = \exp\left[-\frac{1}{2} \int \mathrm{d}\Phi(k)\, |\alpha(k)|^2 \right], \tag{2.44}$$

as can be seen by using the Baker–Campbell–Hausdorff formula.

At this stage, the function $\alpha(k)$ is quite general, however in specific examples, we may take it to be real. We will see below that it is subject to certain restrictions in

[4]The proof holds for vector bosons and gravitons.

the classical limit. We will also see that its functional form will determine the physical shape of the corresponding state, so we will call it the 'waveshape' function.

The coherent-state creation operator acting on the vacuum can be rewritten using the Baker-Campbell-Hausdorff identity as a displacement operator [186, 187] yielding

$$\mathbb{C}_{\alpha,\sigma}|0\rangle = \exp\left[\int \mathrm{d}\Phi(k)(\alpha(k)a_\sigma^\dagger(k) - \alpha^*(k)a_\sigma(k))\right]|0\rangle\,. \tag{2.45}$$

Its action on creation and annihilation operators is given by,

$$\begin{aligned}\mathbb{C}^\dagger_{\alpha,\sigma}a_\rho(k)\mathbb{C}_{\alpha,\sigma} &= a_\rho(k) + \delta_{\sigma\rho}\,\alpha(k)\,,\\ \mathbb{C}^\dagger_{\alpha,\sigma}a^\dagger_\rho(k)\mathbb{C}_{\alpha,\sigma} &= a^\dagger_\rho(k) + \delta_{\sigma\rho}\,\alpha^*(k)\,.\end{aligned} \tag{2.46}$$

To interpret the state, let us compute $\langle\alpha^+|\mathbb{A}^\mu(x)|\alpha^+\rangle$. It is useful to note,

$$\begin{aligned}a_+(k)|\alpha^+\rangle &= \alpha(k)|\alpha^+\rangle\,,\\ a_-(k)|\alpha^+\rangle &= 0\,,\\ \langle\alpha^+|a^\dagger_+(k) &= \langle\alpha^+|\alpha^*(k)\,,\\ \langle\alpha^+|a^\dagger_-(k) &= 0\,,\end{aligned} \tag{2.47}$$

which incidentally imply that the dimension of $\alpha(k)$ is the same as the dimension of the annihilation operator: inverse mass. It is then easy to see that,

$$\begin{aligned}\langle\alpha^+|\mathbb{A}_\mu(x)|\alpha^+\rangle &= \frac{1}{\sqrt{\hbar}}\int \mathrm{d}\Phi(k)\,\left[\alpha(k)\varepsilon^{+*}_\mu(k)e^{-ik\cdot x/\hbar} + \alpha^*(k)\varepsilon^+_\mu(k)e^{+ik\cdot x/\hbar}\right]\\ &= \int \mathrm{d}\Phi(\bar{k})\,\left[\bar{\alpha}(\bar{k})\varepsilon^{+*}_\mu(\bar{k})e^{-i\bar{k}\cdot x} + \bar{\alpha}^*(\bar{k})\varepsilon^+_\mu(\bar{k})e^{+i\bar{k}\cdot x}\right]\\ &\equiv A^{\mathrm{cl}}_\mu(x)\,,\end{aligned} \tag{2.48}$$

where we define

$$\bar{\alpha}(\bar{k}) \equiv \hbar^{3/2}\alpha(k)\,. \tag{2.49}$$

Additional constraints on $\bar{\alpha}$ will emerge below from the consideration of correlators in the classical limit. Note that the polarization vector is invariant under the rescaling from a momentum to a wavevector: $\varepsilon_\sigma(\bar{k}) = \varepsilon_\sigma(k)$ is independent of $\hbar$.

The most general solution of the classical Maxwell equation in empty space is,

$$\sum_{\sigma=\pm1} A^{\mathrm{cl},\sigma}_\mu(x) = \sum_{\sigma=\pm1}\int \mathrm{d}\Phi(\bar{k})\big[\widetilde{A}_\sigma(\bar{k})\varepsilon^{\sigma*}_\mu(\bar{k})e^{-i\bar{k}\cdot x} + \widetilde{A}^*_\sigma(\bar{k})\varepsilon^\sigma_\mu(\bar{k})e^{+i\bar{k}\cdot x}\big]\,, \tag{2.50}$$

in terms of Fourier coefficients $\widetilde{A}_\sigma(\bar{k})$, which we can identify as $\bar{\alpha}(\bar{k})$. Evidently our state $|\alpha^+\rangle$ contributes only the terms of positive helicity ($\sigma = +1$); a more general coherent state involving creation operators of both helicities would generate this most general solution of the free Maxwell equations.

To further illuminate the meaning of coherent states, we may consider scattering amplitudes in the presence of a non-trivial background field $A^{\mathrm{cl}}(x)$. The scattering matrix in the presence of this background field depends on it. We denote this dependence by $S(A^{\mathrm{cl}})$. Using the properties of the coherent state operator it can be shown that,

$$\mathbb{C}^{\dagger}_{\alpha,\sigma} S(A) \mathbb{C}_{\alpha,\sigma} = S(A + A^{\sigma}_{\mathrm{cl}})\,. \tag{2.51}$$

Coherent states thus allow us to capture the physics of a specific background field based on vacuum scattering amplitudes:

$$\mathbb{C}^{\dagger}_{\alpha,\sigma} S(0) \mathbb{C}_{\alpha,\sigma} = S(A^{\sigma}_{\mathrm{cl}})\,. \tag{2.52}$$

The formulation of the perturbation theory in a fixed background is particularly convenient when the Feynman rules — or the scattering amplitudes — in the background are known exactly [102, 188–192].

As usual, we define the operator measuring the number of photons to be,

$$\mathbb{N}_{\gamma} = \sum_{\sigma=\pm 1} \int \mathrm{d}\Phi(k)\, a^{\dagger}_{\sigma}(k) a_{\sigma}(k)\,. \tag{2.53}$$

A short computation shows that the expectation number N_γ of photons in our coherent state is,

$$N_{\gamma} = \langle \alpha^{+} | \mathbb{N}_{\gamma} | \alpha^{+} \rangle = \int \mathrm{d}\Phi(k) |\alpha(k)|^2 = \frac{1}{\hbar} \int \mathrm{d}\Phi(\bar{k}) |\bar{\alpha}(\bar{k})|^2\,. \tag{2.54}$$

The classical limit $\hbar \to 0$ thus corresponds to the limit of a large number of photons, that is a limit of large occupation number [193]. Therefore as a consequence we will have,

$$N_{\gamma} \gg 1\,. \tag{2.55}$$

We must choose the waveshape α such that the integral in the last line of eq. (2.54) is not parametrically small as $\hbar \to 0$. A simple way to do so is to choose $\bar{\alpha}$ independent of $\hbar$. Similarly, the momentum carried by the coherent state is,

$$\bar{k}^{\mu}_{\gamma} = \langle \alpha^{+} | \mathbb{K}^{\mu}_{\gamma} | \alpha^{+} \rangle = \int \mathrm{d}\Phi(k) |\alpha(k)|^2\, k^{\mu} = \int \mathrm{d}\Phi(\bar{k}) |\bar{\alpha}(\bar{k})|^2\, \bar{k}^{\mu}\,. \tag{2.56}$$

This quantity ("K beam") is finite in the classical limit, as expected. We emphasize that this coherent-state construction and its connection to classical states generalizes in a straightforward way to any massless particle, including gravitons. In particular, In classical GR, we define the spacetime metric to be

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,. \tag{2.57}$$

The constant κ is given in terms of Newton's constant G by[5]

$$\kappa \equiv \sqrt{32\pi G}\,. \tag{2.58}$$

In a quantum description, the corresponding field operator is

$$h_{\mu\nu}(x) = \frac{1}{\sqrt{\hbar}} \sum_{\sigma=\pm 2} \int \mathrm{d}\Phi(k) \left[a_{\sigma}(k)\, \varepsilon^{*\sigma}_{\mu}(k) \varepsilon^{*\sigma}_{\nu}(k)\, e^{-i\bar{k}\cdot x} + \mathrm{h.c.} \right]\,. \tag{2.59}$$

[5] We retain this definition even in units where $\hbar \neq 1$. In that case the dimensions of the fields A_μ and $h_{\mu\nu}$ are both $\sqrt{\text{mass/length}}$.

Notice that we have written the polarisation tensor that conventionally appears in the graviton operator as an explicit outer product of two polarisation vectors; this is always possible. The linearised Riemann tensor operator follows from conventional definitions and is given by[6]

$$\begin{aligned}\mathbb{R}_{\mu\nu\rho\xi}(x) &= \frac{\kappa}{2}\left(\partial_\xi\partial_{[\mu}h_{\nu]\,\rho} - \partial_\rho\partial_{[\mu}h_{\nu]\,\xi}\right)\\ &= -\frac{\kappa}{2}\frac{1}{\sqrt{\hbar}}\sum_{\sigma=\pm2}\int \mathrm{d}\Phi(k)\left[a_\sigma(k)\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\,\bar{k}_{[\sigma}\varepsilon^{*\sigma}_{\rho]}(k)\,e^{-i\bar{k}\cdot x} + \text{h.c.}\right].\end{aligned} \tag{2.60}$$

It is worth starting from the familiar case of geometric optics. This is a purely classical approximation to wave phenomena, valid in situations where the wavelength is negligible in comparison to other physical scales. One of our foci will be on phenomena associated with scattering light from a point-like object. For problems of this type to be well-defined, the incoming wave must be spatially separated from the incoming particle in the far past. Consequently, we need to understand how to describe a localized incoming beam of light. We can choose the beam to be moving in the z direction, localized around the origin of the x–y plane. To see how to do this, let's consider some examples.

The simplest option for the waveshape is,

$$\alpha(k) = \alpha_\odot\delta^3(k - \hbar\bar{k}_\odot)\,, \tag{2.61}$$

where $\bar{k}_\odot$ ("k-bar beam") is the overall wavevector of the wave, and $\alpha_\odot$ ("α beam") is a constant which scales like $\sqrt{\hbar}$. Defining $\bar{\alpha}_\odot = \hbar^{-1/2}\alpha_\odot$, this choice implies that,

$$\bar{\alpha}(\bar{k}) = \bar{\alpha}_\odot\delta^3(\bar{k} - \bar{k}_\odot)\,, \tag{2.62}$$

and that the classical field takes the form,

$$A^\mu_{\text{cl}}(x) = 2\Re\bar{\alpha}_\odot\,\varepsilon^{*\mu}_\odot(\bar{k}_\odot)e^{-i\bar{k}_\odot\cdot x}\,. \tag{2.63}$$

It is worth pointing out here that the expectation value of the gauge potential between coherent states is always a real quantity: a physical field which can be measured. We can choose

$$\begin{aligned}\bar{k}^\mu_\odot &= (\omega, 0, 0, \omega)\\ \varepsilon^\mu_\odot &= \frac{1}{\sqrt{2}}(0, 1, i, 0)\,,\end{aligned} \tag{2.64}$$

to provide an explicit example. If we pick the normalization of $\bar{\alpha}$ to be given by $\bar{\alpha}_\odot = A_\odot/\sqrt{2}$ with $A_\odot$ real, the classical field for this example is,

$$A^\mu_{\text{cl}}(x) = A_\odot\,(0, \cos\omega(t-z), -\sin\omega(t-z), 0)\,, \tag{2.65}$$

which is a plane wave of circular polarization[7] moving in the z-direction with angular frequency ω. This wave is completely delocalized, which is a disadvantage for our purposes: we wish to have a clean separation between the incoming wave and point-like particle states.

[6] It could be worth remarking that the Riemann tensor is gauge-invariant in linearised theory.

[7] The wave $\langle\alpha^-|\mathbb{A}^\mu|\alpha^-\rangle$ is circularly polarized in the opposite sense.

To localize the wave, we may "broaden" the delta function in eq. (2.61). Define,

$$\delta_\sigma(\bar{k}) \equiv \frac{1}{\sigma\sqrt{\pi}} \exp\left[-\frac{\bar{k}^2}{\sigma^2}\right], \tag{2.66}$$

which is normalized so that

$$\int_{-\infty}^{\infty} \mathrm{d}\bar{k}\, \delta_\sigma(\bar{k}) = 1\,. \tag{2.67}$$

The peak width is controlled by the parameter σ. As $\bar{k}$ is a wavenumber, σ has dimensions of inverse length. We may choose our incoming wave, moving along the z-axis, to be symmetric under a rotation about that axis. Consider the choice,

$$\alpha(k) = \frac{1}{\hbar^3}|\boldsymbol{k}|(2\pi)^3 A_\odot \sqrt{2\hbar}\, \delta_{\sigma_\parallel}(\omega - k^z/\hbar)\delta_{\sigma_\perp}(k^x/\hbar)\delta_{\sigma_\perp}(k^y/\hbar)\,; \tag{2.68}$$

or equivalently,

$$\bar{\alpha}(\bar{k}) = \sqrt{2}|\bar{\boldsymbol{k}}|(2\pi)^3 A_\odot\, \delta_{\sigma_\parallel}(\omega - \bar{k}^z)\delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y)\,, \tag{2.69}$$

with $A_\odot$ real. We have introduced two measures of beam spread, $\sigma_\parallel$ and $\sigma_\perp$, along and transverse to the wave direction respectively. The corresponding classical field is,

$$\begin{aligned} A^\mu_{\mathrm{cl}}(x) = \sqrt{2}A_\odot \Re \int \mathrm{d}^3\bar{k}\, \varepsilon^{*\mu}_\odot(\bar{k})\delta_{\sigma_\parallel}(\omega - \bar{k}^z) \\ \times\, \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y)e^{-i\bar{k}\cdot x}\Big|_{\bar{k}^0=\sqrt{(\bar{k}^x)^2+(\bar{k}^y)^2+(\bar{k}^z)^2}}\,. \end{aligned} \tag{2.70}$$

We emphasize that other choices of wave shape are available in the classical theory: the only constraint is that N_γ must be large.

Let us further refine our example by taking $\sigma_\parallel$ to be very small compared to the other two scales, $\sigma_\perp$ and $\omega = \bar{k}^0_\odot$. We are thus considering a monochromatic beam, for which we can replace $\delta_{\sigma_\parallel}$ by a Dirac delta distribution. Doing so, we obtain,

$$A^\mu_{\mathrm{cl}}(x) = \sqrt{2}A_\odot \Re \int \mathrm{d}^2\bar{k}_\perp\, \varepsilon^{*\mu}_\odot(\bar{k})\, \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y)\, e^{-it\sqrt{\omega^2+(\bar{k}^x)^2+(\bar{k}^y)^2}} e^{i\omega z} e^{i\bar{k}^x x} e^{i\bar{k}^y y}\,. \tag{2.71}$$

We can simplify this expression with the following considerations. For the beam to be moving in the z-direction, the photons in the beam should dominantly have their momenta, or equivalently their wavenumbers, aligned in the z-direction. However, the broadened distribution $\delta_{\sigma_\perp}$ does allow small components of momentum in the x and y directions. These components should be subdominant. The corresponding x and y wavenumbers are of order $\sigma_\perp$ while the wavenumber in the z direction is of order ω. Let us define the (reduced) wavelength $\lambda\!\!\!^{-} \equiv \omega^{-1}$. We must thus require,

$$\lambda\!\!\!^{-}{}^{-1} \gg \sigma_\perp\,. \tag{2.72}$$

We can also define a transverse size of the beam,

$$\ell_\perp = \sigma_\perp^{-1}\,, \tag{2.73}$$

along with a 'pulse length',

$$\ell_\parallel = \sigma_\parallel^{-1} \,. \tag{2.74}$$

We see that we must require,

$$\lambdabar \ll \ell_\perp \,. \tag{2.75}$$

In other words, a collimated monochromatic beam must have a transverse size which is large in units of the beam's wavelength. The requirement in eq. (2.75) is in some respects analogous to the first part of the 'Goldilocks' condition of eq. (2.16). However, we emphasize that eq. (2.75) arises from our desire to localize the wave in the far past. In particular, waves violating the requirement in eq. (2.75) may still be classical.

Turning back to eq. (2.71), we may now simplify the time-dependent exponential factor. The broadened delta distribution $\delta_{\sigma_\perp}$ forces,

$$(\bar{k}^x)^2 + (\bar{k}^y)^2 \lesssim \sigma_\perp^2 = \ell_\perp^{-2} \,, \tag{2.76}$$

so that,

$$\sqrt{\omega^2 + (\bar{k}^x)^2 + (\bar{k}^y)^2} \lesssim \sqrt{\omega^2 + \ell_\perp^{-2}} \simeq \omega + \mathcal{O}(\ell_\perp^{-2}\omega^{-2}) \simeq \omega \,. \tag{2.77}$$

For the wave's field, we obtain, in this approximation,

$$\begin{aligned} A^\mu_{\rm cl}(x) &= \sqrt{2}A_\odot \Re\bigg\{ e^{-i\omega(t-z)} \int d^2\bar{k}_\perp \, \varepsilon^{*\mu}_\odot(\bar{k}) \, \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y) e^{i\bar{k}^x x} e^{i\bar{k}^y y} \bigg\} \\ &= \sqrt{2}A_\odot \Re\bigg\{ e^{-i\omega(t-z)} \, \varepsilon^{*\mu}_\odot(\bar{k}^\mu_\odot) \int d^2\bar{k}_\perp \, \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y) e^{i\bar{k}^x x} e^{i\bar{k}^y y} \bigg\} \,, \end{aligned} \tag{2.78}$$

where we can replace $\varepsilon^\mu_\odot(\bar{k})$ by $\varepsilon^\mu_\odot(\bar{k}^\mu_\odot)$ because of the smallness of the transverse components of $\bar{k}$. (Recall that $\bar{k}^\mu_\odot = (\omega, 0, 0, \omega)$.) To continue, we may note that the integral,

$$\int_{-\infty}^{\infty} d\bar{q} \, e^{i\bar{q}x} \delta_\sigma(\bar{q}) = e^{-x^2\sigma^2/4} \,, \tag{2.79}$$

so that we finally obtain,

$$A^\mu_{\rm cl}(x) = \sqrt{2}A_\odot \Re\Big[e^{-i\omega(t-z)} \varepsilon^{*\mu}_\odot(\bar{k}^\mu_\odot) \, e^{-(x^2+y^2)/(4\ell_\perp^2)} \Big] \,. \tag{2.80}$$

This is indeed a wave of circular polarization along the z-axis, with finite size in the x–y plane.

Our approximation that $\sigma_\parallel$ is negligible gives us a beam of *infinite* spatial extent along the direction of propagation (here, the z axis). Were we to stop short of the $\sigma_\parallel \to 0$ limit, we would find a finite size in this direction too. The occupation number, which is divergent for infinite extent in the z-direction, would also become finite for nonvanishing $\sigma_\parallel$.

The classical field in eq. (2.80) describes a beam of light that does not spread in the transverse direction, in apparent contradiction to the non-zero transverse momenta the integral contains. This seeming contradiction is lifted when we compute the field of eq. (2.71) to the next order in $1/(\omega\ell_\perp)$ and $t/\ell_\perp$. The result for short enough times is,

$$
\begin{aligned}
A^{\mu}_{\rm cl}(x) = \sqrt{2}A_{\odot}\Re\Bigg\{ & \frac{\exp[-i\omega(t-z)]}{1+i\frac{t}{2\omega\ell_\perp^2}}\varepsilon_{\odot}^{*\mu}(\bar{k}_{\odot}^{\mu})\,\exp\left[-\frac{(x^2+y^2)}{4\ell_\perp^2[1+it/(2\omega\ell_\perp^2)]}\right]\Bigg\} \\
& + \frac{A_{\odot}}{\sqrt{2}}\Re\Bigg\{\exp[-i\omega(t-z)]\left[i\frac{x}{\ell_\perp^2}\partial_{\bar{k}^x}\varepsilon_{\odot}^{*\mu}(\bar{k})\Big|_{\bar{k}=\bar{k}_\odot} + i\frac{y}{\ell_\perp^2}\partial_{\bar{k}^y}\varepsilon_{\odot}^{*\mu}(\bar{k})\Big|_{\bar{k}=\bar{k}_\odot}\right] \\
& \times \exp\left[-\frac{(x^2+y^2)}{4\ell_\perp^2}\right]\Bigg\} + \cdots .
\end{aligned}
\tag{2.81}
$$

In the classical limit, the Compton wavelength ℓ_c of a point-like particle must be unobservably small. However, there is (in general) no need for the wavelength of massless waves to be small. On the contrary, finite-wavelength classical waves are quotidian phenomena, and propagate along the pages of many classical-physics textbooks.

In the scattering of two point-like particles, this requirement on ℓ_c would be violated if the particles approach at distances smaller than (or of order of) their Compton wavelength, because then the underlying wave nature of the particles becomes important. Thus we arrive at the conclusion that classical scattering of two particles obtains only when the impact parameter $b \neq 0$.

In contrast, for a wave of wavelength $\lambda\!\!\!^{-}$ interacting with a particle, we simply require that $\lambda\!\!\!^{-}$ be much larger than the Compton wavelength ℓ_c of the particle. When this is the case, the messengers comprising the wave cannot resolve the quantum structure of the particle. For the classical point-particle approximation to be valid, we further require that $\lambda\!\!\!^{-}$ should be large compared to the finite size ℓ_w of the particle's wave packet. We thus have the requirement,

$$
\ell_c \ll \ell_w \ll \lambda\!\!\!^{-}\,,
\tag{2.82}
$$

for classical interactions of a wave with a particle of Compton wavelength ℓ_c. There is no *a priori* constraint on the impact parameter b.

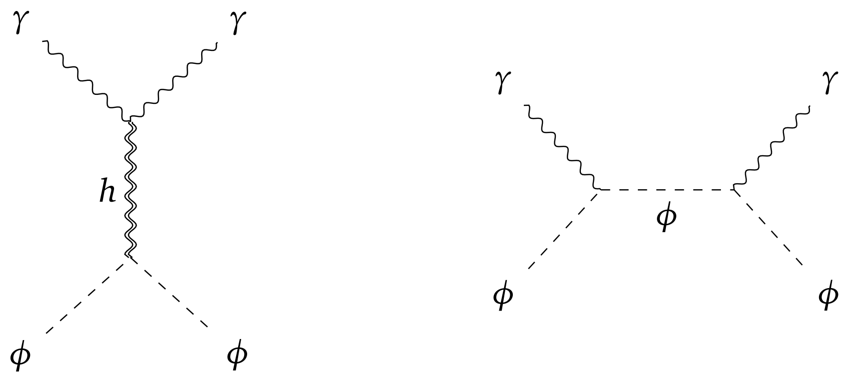


FIGURE 2.3: While the t-channel graviton exchange contribution exists for a photon interacting gravitationally with a scalar, this is not true in electromagnetic case

As exemplified in Fig. 2.3, in the electromagnetic scattering of a photon off a charged particle, there is no t-channel contribution. Correspondingly we are primarily interested in the $b \simeq 0$ case (More precisely, we are interested in b smaller than the transverse size of the beam). We will explore this in more detail below. In contrast, in the gravitational scattering of a photon off a neutral particle, there are both s- and t-channel contributions. In this case, we are interested in general b.

The interaction between our particle and our wave introduces another length scale to consider, namely the scattering length ℓ_s. Let $q = \hbar \bar{q}$ be a characteristic momentum exchange associated with the interaction; then the scattering length is defined to be,

$$\ell_s = \frac{1}{\sqrt{|\bar{q}^2|}} \,. \tag{2.83}$$

The value of the scattering length depends on the details of the scattering process. In the case where two point-like particles scatter, for instance, one finds that $\ell_s \sim b$. In the case at hand where a particle interacts with a wave this need not be the case. Indeed for an s channel processes it is more natural to expect ℓ_s to be determined by the off-shellness of intermediate propagators such as $s - m^2$. For definiteness let us take the momentum of the incoming particle to be $p_1 = m_A v_A$ while the incoming wave has characteristic wavenumber $\bar{k}_\odot$. Then $s - m_A^2 = 2\hbar \bar{k}_\odot \cdot p_1$, so that the scattering length should be,

$$\ell_s \sim \frac{1}{\bar{k}_\odot \cdot v_A} \,. \tag{2.84}$$

This is simply of the order of the wavelength of the incoming wave.

We turn next to the construction of the incoming state. As in ref. [166] and in eq. (2.9), we write the point particle as a superposition of plane-wave states weighted by a wavefunction $\psi(p)$. We can then write the messenger wave as a coherent state of helicity σ characterized by the waveshape $\alpha(k)$. We start with a basis of states constructed out of coherent states of definite helicity $|\alpha^\sigma\rangle$ for the messenger and plane-wave states for the massive particle,

$$|p_1\, \alpha_2^\sigma\rangle = |p_1\rangle |\alpha_2^\sigma\rangle \,. \tag{2.85}$$

Our initial state then takes the form,

$$|\psi_{w,\text{in}}\rangle = \int \mathrm{d}\Phi(p_1)\; \psi_A(p_1)\, e^{ib\cdot p_1/\hbar} |p_1\, \alpha_2^\sigma\rangle \,. \tag{2.86}$$

The impact parameter b now separates the particle from the center of the beam in the far past. As in the earlier discussion, the state is normalized to unity, $\langle \psi_{w,\text{in}} | \psi_{w,\text{in}} \rangle = 1$. We will leave the 'in' subscript implicit in the case of wave scattering, so that we can also avoid confusion with the incoming two-particle state for the two-body problem denoted as $|\psi_\text{in}\rangle$.

Information about the classical four-velocity of the point particle is hidden inside $\psi_A(p)$. The explicit example studied in ref. [166] made use of a linear exponential (which slightly counter-intuitively reduces to a Gaussian in the nonrelativistic limit). In the same way, the information about the overall momentum $K_\odot$ of the messenger wave is hidden inside $\alpha(k)$. For the wave scattering, we will make use of the coherent wave shape $\alpha(k)$ chosen in eq. (2.68) corresponding to the choice of $\bar{\alpha}(k)$ of eq. (2.69), independent of $\hbar$ as desired. We will elucidate inequalities between the various parameters defining the beam below, where relevant.

2.3 Extension with classical spinning particles

We will comment here briefly on how to extend the previous construction to massive external particles which have a large classical angular momentum, following the Schwinger construction [194] developed in the KMOC formalism in [128]. We can define a set of $2s+1$ harmonic oscillators for the massive little group SU(2), in terms of which a generic spin vector can be expressed

$$[a^a, a_b^\dagger] = \delta_b^a \qquad \boldsymbol{S} = \frac{\hbar}{2} a_a^\dagger \boldsymbol{\sigma}_b^a a^b, \tag{2.87}$$

where $\boldsymbol{\sigma}_b^a$ are the standard Pauli matrices. Using eq. (2.87), it is easy to see that the spin vector $\boldsymbol{S}$ obeys the expected algebra

$$[S^i, S^j] = i\hbar\epsilon^{ijk} S^k \,. \tag{2.88}$$

The arbitrariness in the choice of the quantization axis is related to the spinorial rotation $U \in \mathrm{SU}(2)$, which is related to the vector rotation $O \in \mathrm{SO}(3)$ for the spin vector

$$a^a \to U_b^a a^b, \quad a_a^\dagger \to U_a^b a_b^\dagger = a_b^\dagger (U^\dagger)_a^b \quad \Rightarrow \quad S^i \to O^{ij} S^j \tag{2.89}$$

Crucially, eq. (2.88) is invariant under this transformation. A generic spin s state is then defined as

$$|s, \{a\}\rangle \equiv |s, \{a_1 \dots a_{2s}\}\rangle = \frac{1}{\sqrt{(2s)!}} a_{a_1}^\dagger a_{a_2}^\dagger \dots a_{a_{2s}}^\dagger |0\rangle \equiv \frac{(a_a^\dagger)^{\odot 2s}}{\sqrt{(2s)!}} |0\rangle. \tag{2.90}$$

Spin coherent states are then defined [195–197] as eigenstates of a_a:

$$|\alpha^S\rangle := e^{-\frac{1}{2}\alpha_a^* \alpha^a} e^{\alpha^a a_a^\dagger} |0\rangle \to a_a |\alpha^S\rangle = \alpha_a |\alpha^S\rangle \,. \tag{2.91}$$

where α_a^* is the complex conjugation of the spinor-valued "spin shape" α_a. The classical limit is obtained when

$$\alpha_a = \frac{\bar{\alpha}^S}{\hbar} \to \|\alpha\| \equiv \sqrt{\tilde{\alpha}_a \alpha^a} \overset{\hbar \to 0}{\sim} \hbar^{-1/2} \,, \tag{2.92}$$

since this implies the expectation value for spin operator gives back the classical value

$$\langle S^i \rangle_{\alpha^S} = \frac{\hbar}{2} (\alpha^* \sigma^i \alpha) = \frac{1}{2} (\bar{\alpha}^* \sigma^i \bar{\alpha}) \,. \tag{2.93}$$

Moreover, this implies that the uncertainty principle is obeyed in the spin space

$$\langle S^i S^j \rangle_{\alpha^S} \overset{\hbar \to 0}{\sim} \langle S^i \rangle_{\alpha^S} \langle S^j \rangle_{\alpha^S} \,, \tag{2.94}$$

as we will also discuss later in section 5.3.

Chapter 3

Event shapes and light-ray operators

We would like to revisit here the derivation of event shapes in light of the recent developments in classical gravitational physics. In particular, we would like to understand how the energy carried away by gravitational waves, which is collected by a detector localized in a direction $\hat{\boldsymbol{n}}$, is described at the quantum level. We will see that the corresponding operator is related to the Isaacson effective stress tensor [198, 199] by the integration over the retarded working time of the detector, and its action can be computed explicitly using standard techniques in the asymptotic expansion. Using the Bondi gauge framework for asymptotically flat spacetimes, we will show that the energy flow operator can be also written in terms of the Bondi news squared term and this provides an extension of ANEC operator at null infinity to a shear inclusive ANEC. With our detector interpretation, the latter naturally provides the sum of energies of all massless quanta (i.e. radiation) emitted in a direction $\hat{\boldsymbol{n}}$ when acting on on-shell states. Motivated by that, we will define a new system of light-ray operators for linearized gravity, which will naturally combine with the standard stress tensor definition in scalar and gauge theories providing a unified treatment of all massless particles at $\mathcal{I}^{\pm}$.

3.1 Introduction to event shapes in collider physics

Let's consider the process of electron-positron annihilation in QCD: for a generic outcome of the scattering process

$$e^{+}\, e^{-} \to X, \tag{3.1}$$

we would like to understand the properties of the emitted particles as captured by one (or many) physical detectors located at spatial infinity. To make contact with the structure at null infinity, it is convenient to work with flat null coordinates for Minkowski spacetime

$$\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}r - r^2\mathrm{d}z\mathrm{d}\bar{z} \tag{3.2}$$

so that the null boundaries are located at $r \to +\infty$ while keeping $(u, z, \bar{z})$ fixed. On this hypersurface $(z, \bar{z})$ are *stereographic coordinates* on the celestial sphere. This set of coordinates correspond to wrapping up the transverse spatial coordinates of the light-sheet at infinity

$$\mathrm{d}s^2 = \mathrm{d}x^-\mathrm{d}x^+ - \mathrm{d}x^1\mathrm{d}x^1 - \mathrm{d}x^2\mathrm{d}x^2 \tag{3.3}$$

via the transformation

$$x^- \to u + rz\bar{z} \qquad x^+ \to r \qquad x^1 \to \frac{1}{2}r(z+\bar{z}) \qquad x^2 \to -\frac{1}{2}ir(z-\bar{z}). \tag{3.4}$$

Instead of considering the total cross section for the process in eq. (3.1)

$$\sigma_{\text{tot}}(p_1,p_2) := \sum_X (2\pi)^4 \delta^4(p_1+p_2-k_X)|\mathcal{M}_{e^+e^-\to X}|^2, \tag{3.5}$$

where p_1 and p_2 are the incoming momenta of e^+ and e^- and k_X is the sum of the outgoing momenta, we introduce a less inclusive observable by defining a weight $w(X)$ such that

$$\begin{aligned}\sigma_W(p_1,p_2) &:= \frac{1}{\sigma_{\text{tot}}(p_1,p_2)} \sum_X (2\pi)^4 \delta^4(p_1+p_2-k_X) w(X) |\mathcal{M}_{e^+e^-\to X}|^2 \\ &= \frac{1}{\sigma_{\text{tot}}(p_1,p_2)} \sum_X \langle p_1 p_2 | S^\dagger | X \rangle w(X) \langle X | S | p_1 p_2 \rangle,\end{aligned} \tag{3.6}$$

which is the so-called "weighted cross section". Different choices of the weight factors $w(X)$ can tell us about different properties of the distribution of outgoing particles, like the flow of quantum number or of energy and momentum in a particular direction on the celestial sphere.

Of particular interest is the to the choice of $w(X)$ corresponding to the energy flow. Given an external on-shell state $|X\rangle = |k_1 \dots k_n\rangle$ of n massless particles with $k_X^\mu = \sum_{i=1}^n k_i^\mu$, we define

$$w_{\tilde{\mathcal{E}}}(X)(k_1,...,k_n) := \sum_{i=1}^n E_i \delta^2(\Omega_{\hat{\boldsymbol{k}}_i} - \Omega_{\hat{\boldsymbol{n}}}), \tag{3.7}$$

where $k_i^\mu = (E_i, \boldsymbol{k}_i)$ is the 4-momentum of the individual particles and $\Omega_{\hat{\boldsymbol{k}}_i} = \frac{\boldsymbol{k}_i}{|\boldsymbol{k}_i|}$ is the solid angle in the direction of $\boldsymbol{k}_i$ [150, 151]. The weighted cross section measures the distribution of energy in the final state that flows in the direction of $\hat{\boldsymbol{n}}$, and as such we can interpret $w_{\mathcal{E}}(X)$ as an eigenvalue of the average null energy operator (ANEC) at *spatial infinity*

$$\tilde{\mathcal{E}}(\hat{\boldsymbol{n}}) = \int_0^{+\infty} \mathrm{d}t \lim_{r\to+\infty} r^2\, \hat{n}^i T_{0i}(t, r\hat{\boldsymbol{n}}), \tag{3.8}$$

where the energy momentum tensor is always understood to be normal ordered and t is the physical working time of the detector. Looking at the energy flow operator in eq. (3.8) it is easy to generalize it to a momentum flow operator (which is well-known in the jet literature [200, 201])

$$\tilde{\mathcal{P}}_\mu(\hat{\boldsymbol{n}}) = \int_0^{+\infty} \mathrm{d}t \lim_{r\to+\infty} r^2\, \hat{n}^i T_{\mu i}(t, r\hat{\boldsymbol{n}}), \tag{3.9}$$

whose action on a state gives the linear momentum flowing the particular direction $\hat{\boldsymbol{n}}$

$$w_{\tilde{\mathcal{P}}_j}(X)(k_1,...,k_n) := \sum_{i=1}^n (\boldsymbol{k}_j)_i \delta^2(\Omega_{\hat{\boldsymbol{k}}_i} - \Omega_{\hat{\boldsymbol{n}}}). \tag{3.10}$$

One can define also the expectation value of the ANEC operator at infinity, namely the 1-point energy event shape

$$\langle\tilde{\mathcal{E}}(\hat{\boldsymbol{n}})\rangle := \frac{1}{\sigma_{\text{tot}}(p_1,p_2)}\langle p_1p_2|S^\dagger\tilde{\mathcal{E}}(\hat{\boldsymbol{n}})S|p_1p_2\rangle, \tag{3.11}$$

where it is always understood that $\tilde{\mathcal{E}}(\hat{\boldsymbol{n}})$ acts on the set of outgoing states $|X\rangle$ inserted via the completeness relation, as usual in the Schwinger-Keldysh formalism.

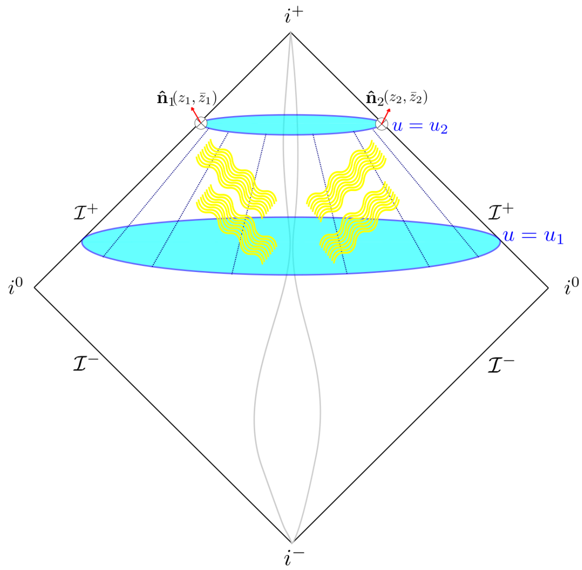


FIGURE 3.1: The radiation due to the scattering of two massive objects is captured by two detectors located at spatial infinity in the directions $\hat{\boldsymbol{n}}_1(z_1,\bar{z}_1)$ and $\hat{\boldsymbol{n}}_2(z_2,\bar{z}_2)$.

In the case of two (or more) detectors, as previously mentioned, we can also define the energy-energy correlator which is related to the 2-pt *Wightman* function of two ANEC operators at infinity inserted at different points on the celestial sphere

$$\langle\tilde{\mathcal{E}}(\hat{\boldsymbol{n}})\tilde{\mathcal{E}}(\hat{\boldsymbol{n}}')\rangle := \frac{1}{\sigma_{\text{tot}}(p_1,p_2)}\langle p_1p_2|S^\dagger\tilde{\mathcal{E}}(\hat{\boldsymbol{n}})\tilde{\mathcal{E}}(\hat{\boldsymbol{n}}')S|p_1p_2\rangle. \tag{3.12}$$

It is quite interesting to analyze this 2-pt correlator for the gravitational radiation in the classical limit, where in the most general case a superposition of coherent states represents the radiation at the quantum level.

In flat null coordinates the momentum flow operator can be written directly in terms of the energy flow

$$\mathcal{P}^\mu(\hat{\boldsymbol{n}}) = n^\mu\mathcal{E}(\hat{\boldsymbol{n}}) = n^\mu\int_{-\infty}^{+\infty}\mathrm{d}u\,\lim_{r\to\infty} r^2T_{uu}(u,r,z_{\hat{\boldsymbol{n}}},\bar{z}_{\hat{\boldsymbol{n}}}), \tag{3.13}$$

where u is the retarded time and identify $\hat{\boldsymbol{n}}$ with the related coordinates on the celestial sphere

$$\hat{\boldsymbol{n}} = \big(z+\bar{z},-i(z-\bar{z}),1-|z|^2\big) \longrightarrow \hat{\boldsymbol{n}} \Leftrightarrow (z_{\hat{\boldsymbol{n}}},\bar{z}_{\hat{\boldsymbol{n}}}). \tag{3.14}$$

Here, we emphasize that we use $\mathcal{E}(\hat{\boldsymbol{n}})$ in place of $\tilde{\mathcal{E}}(\hat{\boldsymbol{n}})$ for technical convenience. One can work in Bondi gauge where the detector would register the energy flow as given by

$\tilde{\mathcal{E}}(\hat{\boldsymbol{n}})$ (i.e. the standard local energy factor). We will do this later on for gravitational event shapes in section 4.4.

3.2 Light-ray operators in QFT

In this section, we define the family of light-ray operators under consideration and make contact with their physical significance and in particular with the corresponding surface charges of null-sheet symmetry generators. Light-ray operators depend implicitly on a choice of null-sheet and we choose future null infinity.

In flat-null coordinates, our family of light-ray operators is defined by

$$\begin{aligned}
\mathcal{E}(\hat{\boldsymbol{n}}) &= \int_{-\infty}^{+\infty} \mathrm{d}u \lim_{r\to\infty} r^2 T_{uu}(u, r, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}}), \\
\mathcal{K}(\hat{\boldsymbol{n}}) &= \int_{-\infty}^{+\infty} \mathrm{d}u\, u \lim_{r\to\infty} r^2 T_{uu}(u, r, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}}), \\
\mathcal{N}_z(\hat{\boldsymbol{n}}) &= \int_{-\infty}^{+\infty} \mathrm{d}u \lim_{r\to\infty} r^2 T_{uz}(u, r, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}}), \\
\mathcal{N}_{\bar{z}}(\hat{\boldsymbol{n}}) &= \int_{-\infty}^{+\infty} \mathrm{d}u \lim_{r\to\infty} r^2 T_{u\bar{z}}(u, r, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}}).
\end{aligned} \tag{3.15}$$

We will collectively denote these operators by $L(\hat{\boldsymbol{n}}) = \{\mathcal{E}(\hat{\boldsymbol{n}}), \mathcal{K}(\hat{\boldsymbol{n}}), \mathcal{N}_z(\hat{\boldsymbol{n}}), \mathcal{N}_{\bar{z}}(\hat{\boldsymbol{n}})\}$.

This family of light-ray operators can be understood as *surface densities*[1] for the corresponding conserved charges of the future null boundary of Minkowski spacetime. To see this, consider an affine transformation of the null surface generator [202]

$$\delta u = A + B\, u, \tag{3.16}$$

where A and B are arbitrary constants. Since $\lim_{r\to\infty} r^2 T_{uu}(u, r, z, \bar{z})$ corresponds to the generator of null diffeomorphisms, we can integrate it to obtain either the generator of null translation $\delta u = A$

$$\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}u \int \mathrm{d}^2 z \lim_{r\to\infty} r^2 T_{uu}(u, r, z, \bar{z}) \tag{3.17}$$

or of dilatations $\delta u = B\, u$

$$\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}u \int \mathrm{d}^2 z\, u \lim_{r\to\infty} r^2 T_{uu}(u, r, z, \bar{z}), \tag{3.18}$$

which is an equivalent definition of $\int \mathrm{d}^2 z\, \mathcal{K}(z, \bar{z})$. The latter is called "boost mass" M_{boost} in the literature and it is usually defined for null-like horizons[2] using the

[1] There are various subtleties in such definition as we will discuss explicitly here. Technically surface densities are defined up to boundary terms that do not change the total charge. Thus, only *the equivalence class* of a light-ray operator is uniquely defined. Moreover we are interested in the standard (hard) charge coming from contracting the stress tensor $T_{\mu\nu}$ with a Killing vector ξ^μ, which might differ from the isometry charge in the covariant phase space formulation.

[2] More generically, choosing a spatial slice of the null surface $u^0 = \Lambda(z, \bar{z})$, a canonical "boost energy" can be defined [202]

$$\frac{1}{2}\int \mathrm{d}^2 z \int_{\Lambda(z,\bar{z})}^{+\infty} \mathrm{d}u\, (u - \Lambda(z, \bar{z})) \lim_{r\to\infty} r^2 T_{uu}(u, r, z, \bar{z}), \tag{3.19}$$

which turns out to be connected with the notion of "area operator", providing thus a link to the concept of generalized entropy and to the generalized second law.

boost Killing vector of a causal wedge in Minkowski spacetime [202–204]. From an entanglement entropy perspective in conformal field theory, it was found that the modular hamiltonian for spacetime regions which have a future null-like horizon has a natural local expression [205] in terms of $\mathcal{E}(\hat{\boldsymbol{n}})$ and $\mathcal{K}(\hat{\boldsymbol{n}})$.

It is interesting to discuss the boost mass, associated to an asymptotic boost symmetry, from a physical perspective [206]. While for empty Minkowski spacetime the boost mass is exactly zero, for more general non-asymptotically flat spacetimes it might be relevant for a proper formulation of the first law of black hole thermodynamics. In particular, it turns out that while the ADM mass M_{ADM} measures the total "monopole" distribution of the matter in the spacetime, the boost mass M_{boost} measures the total "dipole" moment of the mass distribution at infinity [206]. Since we are only considering QFTs on a flat Minkowski background, we will not delve deeper into this topic. However, as we will see we will find the monopole versus dipole distinction relevant for understanding the action of $\mathcal{K}(\hat{\boldsymbol{n}})$ on on-shell states from our detector perspective. From now on, we will call $\mathcal{K}(\hat{\boldsymbol{n}})$ the *boost energy (surface) density*. In the same spirit, the ANEC at infinity can be called *null energy (surface) density*.

Following a similar line of reasoning one finds that $\mathcal{N}_z(\hat{\boldsymbol{n}})$ and $\mathcal{N}_{\bar{z}}(\hat{\boldsymbol{n}})$ are associated to the local vector fields ∂_z and $\partial_{\bar{z}}$, i.e. with the angular momentum flux charge

$$\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}u \int \mathrm{d}^2 z \lim_{r\to\infty} r^2 T_{uz}(u,r,z,\bar{z})\,, \qquad \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}u \int \mathrm{d}^2 z \lim_{r\to\infty} r^2 T_{u\bar{z}}(u,r,z,\bar{z}). \tag{3.20}$$

Therefore $\mathcal{N}_z(\hat{\boldsymbol{n}})$ and $\mathcal{N}_{\bar{z}}(\hat{\boldsymbol{n}})$ can be thought as *angular momentum flux (surface) densities*. More generally, this family of non-local light-ray operators appear naturally in Einstein equations solved near null infinity and in particular in (BMS and Poincaré) balance flux laws [207], where they represent the radiative contribution to the fluxes.

Here we interpret physically the insertion of our light-ray operators as an insertion of a physical detector on the boundary of Minkowski spacetime. We will be interested in how these operators act on on-shell states and we will then compute their expectation values in perturbation theory, as in the standard event shapes literature.

3.3 The Spin-0 light-ray operators

Here, we consider a theory of self-interacting massless scalars with a potential $Q(\phi)$ that is polynomial in the fields and without derivative interactions. We derive explicit expressions for our family of light-ray operators in terms of the scalar creation and annihilation operators.

The Hilbert stress tensor $T^{\mathrm{matter}}_{\mu\nu} = -\frac{2}{\sqrt{|g|}}\frac{\delta S^{\mathrm{matter}}}{\delta g^{\mu\nu}}$ for our theory of scalars is simply

$$T^{\mathrm{scalar}}_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \eta_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - Q(\phi)\right). \tag{3.21}$$

The scalar stress tensor components which are relevant for the light-ray operators are[3]

$$T^{\mathrm{scalar}}_{uu}(x)=(\partial_u\phi)(\partial_u\phi), \qquad T^{\mathrm{scalar}}_{uz}(x)=(\partial_u\phi)(\partial_z\phi), \qquad T^{\mathrm{scalar}}_{u\bar{z}}(x)=(\partial_u\phi)(\partial_{\bar{z}}\phi). \tag{3.22}$$

[3]One can easily work in a more general setting with derivative interactions. However, such terms will drop off due to their scaling in $\frac{1}{r}$ in the limit that the null-sheet is pushed out to null-infinity. Instead, we prefer the interaction terms drop off directly from the structure of stress tensor components since this more closely resembles scalar light-ray operators in the bulk (see appendix C).

The next step is to substitute the leading term in the large r expansion of the scalar field into the expressions above. However, to do this, we need to impose some boundary conditions on the field.

Since we require the energy flux across $\mathcal{I}^+$ to be finite, we impose the following falloff condition for the scalar field[4]

$$\phi(u,r,z,\bar{z}) = \frac{\phi^{(0)}(u,z,\bar{z})}{r} + \mathcal{O}\left(\frac{1}{r}\right), \qquad \lim_{u\to\pm\infty} \phi^{(0)}(u,z,\bar{z}) = 0. \tag{3.23}$$

Far away from interactions we can safely use the free field mode expansion

$$\phi(x) = \int \mathrm{d}\Phi(p) \left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right] \tag{3.24}$$

and evaluate the field with the large r saddle-point estimate. Physically, r is the distance from the detector to the region where the interaction is localized and it should be much larger than the inverse of the smallest momentum scale that appears in the S-matrix [201]. Note that in our coordinate conventions the canonical commutators read

$$[a(p), a^\dagger(q)] = \frac{4(2\pi)^3}{\omega_p}\delta(\omega_p - \omega_q)\delta^2(z_{\hat{\boldsymbol{p}}} - z_{\hat{\boldsymbol{q}}}). \tag{3.25}$$

The large r limit of such a Fourier integral is controlled by the exponents of the exponentials

$$f(z,w) := ip\cdot x = i\frac{\omega_p u}{2} + i\frac{\omega_p r}{2}|z-w|^2, \tag{3.26}$$

where we use

$$\begin{aligned} x^\mu &= \frac{r}{2}\left(1 + z\bar{z} + \frac{u}{r}, z+\bar{z}, -i(z-\bar{z}), 1 - z\bar{z} - \frac{u}{r}\right), \\ p^\mu &= \frac{\omega_p}{2}\left(1 + w\bar{w}, w+\bar{w}, -i(w-\bar{w}), 1 - w\bar{w}\right). \end{aligned} \tag{3.27}$$

as our parametrization for the coordinates and on-shell momentum [208, 209]. The only saddle point is at $(z,\bar{z}) = (w,\bar{w})$. Making the following change of variables $(y_1, y_2) = \left(\frac{1}{2}(z+\bar{z}), -\frac{i}{2}(z-\bar{z})\right)$ brings the exponentials into Gaussian form, which integrates to

$$\frac{1}{2}\int \mathrm{d}^2z\, e^{-i\frac{\omega_p r}{2}(z-w)(\bar{z}-\bar{w})} = \int \mathrm{d}y_1\mathrm{d}y_2\, e^{-i\frac{\omega_p r}{2}(y_1^2+y_2^2)} = \frac{(2\pi)(-i)}{r\omega}. \tag{3.28}$$

Substituting into the mode expansion for $\phi(x)$ yields the leading term in the large r limit

$$\phi^{(0)}(u,\hat{\boldsymbol{n}}) = \frac{i}{(8\pi^2)}\int_0^{+\infty} \mathrm{d}\omega \left[a^\dagger(\omega, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}})e^{i\frac{\omega u}{2}} - a(\omega, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}})e^{-i\frac{\omega u}{2}}\right]. \tag{3.29}$$

[4] Soft scalar modes are beyond the scope of this work, so we do not allow the scalar field asymptotic expansion to have a constant piece around $\mathcal{I}^+_\pm$.

When derivatives in z (or $\bar{z}$) acts on field operators expressions, one can write[5]

$$\partial_z \phi^{(0)}(u, \hat{\boldsymbol{n}}) = \frac{i}{(8\pi^2)} \int_0^{+\infty} \mathrm{d}\omega_p \left[\partial_{z_{\hat{\mathbf{n}}}} a^\dagger(\omega_p, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}}) e^{i\frac{\omega_p u}{2}} - \partial_{z_{\hat{\mathbf{n}}}} a(\omega_p, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}}) e^{-i\frac{\omega_p u}{2}} \right] \tag{3.30}$$

for the saddle point estimate of $\partial_z \phi^{(0)}$ where the derivative acts on the creation and annihilation operators localized around $(w, \bar{w}) = (z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}})$.

Inserting the saddle point estimate of $\phi(x)$ into the definition for the scalar ANEC operator at infinity, we find

$$\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}) = \int \mathrm{d}\Phi(p)\, \omega_p \,:a^\dagger(p) a(p): \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}), \tag{3.31}$$

where we have set to zero all contributions proportional to $\delta(\omega_{p_1} + \omega_{p_2})$ since the only cases that these constraints can be satisfied correspond to $\omega_{p_1} \to 0^\pm$ and $\omega_{p_2} \to 0^\mp$, which are unphysical. The action of $\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}})$ on an on-shell state $|X\rangle = |p_1 ... p_n\rangle$ gives

$$\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}})|X\rangle = \sum_{i=1}^n (\omega_i)\, \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}_i})|X\rangle, \tag{3.32}$$

which is natural for an observer located at spatial infinity who is measuring the energy flux along the retarded time u in flat null coordinates. Note that on-shell scattering states are eigenstates of the ANEC operator in the detector limit whose eigenvalues are weight functions similar to eq. (3.7).

Inserting eq. (3.29) into the definition of the boost energy density flux $\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}})$ yields

$$\begin{aligned} \mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}) &= \int_{-\infty}^{+\infty} \mathrm{d}u\, (u) \lim_{r\to\infty} r^2 \prod_{i=1}^{2} \left[\int_0^{+\infty} \mathrm{d}\omega_{p_i} \frac{\omega_{p_i}}{2(8\pi^2) r} \right] \\ &\times : \left[a^\dagger(\omega_{p_1}\hat{\boldsymbol{n}}) a(\omega_{p_2}\hat{\boldsymbol{n}}) e^{-i\frac{(\omega_{p_2}-\omega_{p_1})}{2}u} + a(\omega_{p_1}\hat{\boldsymbol{n}}) a(\omega_{p_2}\hat{\boldsymbol{n}}) e^{-i\frac{(\omega_{p_1}+\omega_{p_2})}{2}u} + \text{h.c.} \right] :. \end{aligned} \tag{3.33}$$

It is convenient to change variables

$$\omega_- := \frac{\omega_{p_1} - \omega_{p_2}}{2} \qquad \omega_+ := \frac{\omega_{p_1} + \omega_{p_2}}{2}, \tag{3.34}$$

where $\omega_+ \in [\max\{-\omega_-, \omega_-\}, +\infty[$ and $\omega_- \in]-\infty, +\infty[$ is a convenient slicing of the integration region in the new variables.[6] In the new variables,[7]

$$\begin{aligned} \mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}) = \frac{(-i)}{8} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega_-}{(2\pi)^3} \delta^{(1)}(\omega_-) \int_{\max\{-\omega_-, \omega_-\}}^{+\infty} \mathrm{d}\omega_+ \left((\omega_+)^2 - (\omega_-)^2\right) \\ \times : \left[a^\dagger\left((\omega_+ + \omega_-)\hat{\boldsymbol{n}}\right) a\left((\omega_+ - \omega_-)\hat{\boldsymbol{n}}\right) - \text{h.c.} \right] :, \end{aligned} \tag{3.35}$$

---

[5]Technically, we take first the saddle point estimate of the quantum field so that it will localize on its point particle expression.

[6]The original frequencies are given by $\omega_{p_1} = (\omega_+ + \omega_-)$ and $\omega_{p_2} = (\omega_+ - \omega_-)$ and the integration measure becomes $\mathrm{d}\omega_{p_1} \mathrm{d}\omega_{p_2} = 2\, \mathrm{d}\omega^- \mathrm{d}\omega^+$.

[7]Here we have assumed, as before, that the other contributions dropped because they are unphysical.

where we have used the following distributional identity (for $n = 1$)

$$\int \mathrm{d}u\, (u)^n e^{i\omega_- u} = (2\pi)(-i)^n \delta^{(n)}(\omega_-). \tag{3.36}$$

The derivative acting on the energy delta function might seem troubling at first sight. To develop more intuition, we study the action of this operator on on-shell states $|X\rangle = |p_1...p_n\rangle$:

$$\begin{aligned}\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}})|X\rangle = \frac{(-i)}{4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega_-}{(2\pi)^3} \delta^{(1)}(\omega_-) \int_{\max\{-\omega_-,\omega_-\}}^{+\infty} \mathrm{d}\omega_+ \left((\omega_+)^2 - (\omega_-)^2\right) \\ \times \sum_{i=1}^{n} (2\pi)^3 \frac{4}{\omega_{p_i}} \delta\left((\omega_+ - \omega_-) - \omega_{p_i}\right) \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}_i}) |p_1 ... \underbrace{(\omega_+ + \omega_-)\hat{\boldsymbol{n}}}_{\text{i-th}} ... p_n\rangle.\end{aligned} \tag{3.37}$$

Then one can solve the ω_+ integral

$$\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}})|X\rangle = 2 \sum_{i=1}^{n} \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}_i}) \int_{-\omega_{p_i}}^{+\infty} \mathrm{d}\omega(-i\delta^{(1)}(\omega))\ (\omega_{p_i} + \omega)\, |p_1 ... \underbrace{(\omega_{p_i} + \omega)\hat{\boldsymbol{n}}}_{\text{i-th}} ... p_n\rangle, \tag{3.38}$$

where we have relabelled ω_- as ω. It is also enlightening to compare the simplest matrix elements of $\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}})$ and $\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}})$

$$\begin{aligned}\langle q|\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}})|p\rangle &= 4(2\pi)^3 \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \delta^2(z_{\hat{\boldsymbol{q}}} - z_{\hat{\boldsymbol{n}}}) \delta(\omega_q - \omega_p), \\ \langle q|\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}})|p\rangle &= 2\delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \int_{-\omega_p}^{+\infty} \mathrm{d}\omega(-i\delta^{(1)}(\omega))\ (\omega_p + \omega)\, \langle q\rangle(\omega_p + \omega)\hat{\boldsymbol{n}} \\ &= 8(2\pi)^3 \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \delta^2(z_{\hat{\boldsymbol{q}}} - z_{\hat{\boldsymbol{n}}}) (-i\delta^{(1)}(\omega_q - \omega_p)).\end{aligned} \tag{3.39}$$

It is well-known that in QFT the single contraction $\langle q|p\rangle$ must be interpreted in a distributional sense; in order to properly define such objects one needs to smear them with a well-behaved function (see [210, 211] and references therein). In the S-matrix context, one usually considers outgoing states of the form[8]

$$|\psi_{p,\text{out}}\rangle = \int \mathrm{d}\Phi(\tilde{p})\ \psi_p(\tilde{p}) |\tilde{p}\rangle, \tag{3.40}$$

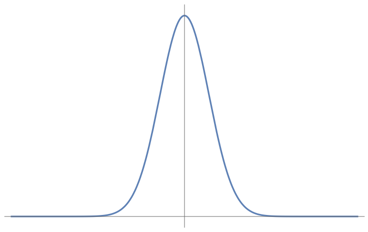
where $\psi_p(\tilde{p})$ is a suitable real momentum wavefunction localized around p. Note that $|\psi_{p,\text{out}}\rangle \to |p\rangle$ when $\psi_p(\tilde{p}) \to (2\pi)^3(2E_p)\delta^3(p - \tilde{p})$. More generally, when $\psi_p(\tilde{p})$ is sufficiently smooth then eq. (3.39) is well defined

$$\begin{aligned}\langle \psi_{q,\text{out}}|\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}})|\psi_{p,\text{out}}\rangle = 4(2\pi)^3 \int \mathrm{d}\Phi(\tilde{p}) \int \mathrm{d}\Phi(\tilde{q})\ \psi_p(\tilde{p})\psi_q(\tilde{q}) \\ \times \delta(\omega_{\tilde{q}} - \omega_{\tilde{p}}) \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\tilde{\boldsymbol{p}}}}) \delta^2(z_{\hat{\tilde{\boldsymbol{q}}}} - z_{\hat{\boldsymbol{n}}}), \\ \langle \psi_{q,\text{out}}|\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}})|\psi_{p,\text{out}}\rangle = 8(2\pi)^3 \int \mathrm{d}\Phi(\tilde{p}) \int \mathrm{d}\Phi(\tilde{q})\ \psi_p(\tilde{p})\psi_q(\tilde{q}) \\ \times (-i\delta^{(1)}(\omega_{\tilde{q}} - \omega_{\tilde{p}})) \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\tilde{\boldsymbol{p}}}}) \delta^2(z_{\hat{\tilde{\boldsymbol{q}}}} - z_{\hat{\boldsymbol{n}}}).\end{aligned} \tag{3.41}$$

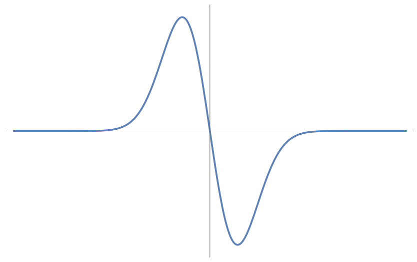
Looking at eq. (3.41), the interpretation of $\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}})$ is also much more clear: while

[8] A similar idea applies to the ingoing case.

$\delta(\omega_{\tilde{q}} - \omega_{\tilde{p}})$ can be considered the limit of a suitable localized function around $\omega_{\tilde{q}} = \omega_{\tilde{p}}$, it turns out that $\delta^{(1)}(\omega_{\tilde{q}} - \omega_{\tilde{p}})$ is probing a dipole-like pattern around $\omega_{\tilde{q}} = \omega_{\tilde{p}}$ (see Fig. 3.2a and Fig. 3.2b respectively).



(A) Standard Gaussian shape peaked around $\omega_{\tilde{q}} = \omega_{\tilde{p}}$, which converges to $\delta(\omega_{\tilde{q}} - \omega_{\tilde{p}})$ in the limit of zero width.



(B) Dipole profile centered around $\omega_{\tilde{q}} = \omega_{\tilde{p}}$, which converges to $\delta^{(1)}(\omega_{\tilde{q}} - \omega_{\tilde{p}})$ in the limit of zero width.

FIGURE 3.2: Comparison of the expectation values of $\langle \psi_{q,\mathrm{out}} | \mathcal{E}_{\mathrm{scalar}}(\hat{\boldsymbol{n}}) | \psi_{p,\mathrm{out}} \rangle$ and $\langle \psi_{q,\mathrm{out}} | \mathcal{K}_{\mathrm{scalar}}(\hat{\boldsymbol{n}}) | \psi_{p,\mathrm{out}} \rangle$ as a function of the energy ω.

The final operator we consider is $\mathcal{N}_{z,\mathrm{scalar}}(\hat{\boldsymbol{n}})$, which after using eq. (3.29) and eq. (3.30) is

$$\begin{aligned}\mathcal{N}_{z,\mathrm{scalar}}(\hat{\boldsymbol{n}}) = \frac{i}{4(2\pi)^3} \int \mathrm{d}\omega_p\, \omega_p : \Big[& a^\dagger\left(\omega_p, z_{\hat{\mathbf{n}}}, \bar{z}_{\hat{\mathbf{n}}}\right) \partial_{z_{\hat{\mathbf{n}}}} a\left(\omega_p, z_{\hat{\mathbf{n}}}, \bar{z}_{\hat{\mathbf{n}}}\right) \\ & - a\left(\omega_p, z_{\hat{\mathbf{n}}}, \bar{z}_{\hat{\mathbf{n}}}\right) \partial_{z_{\hat{\mathbf{n}}}} a^\dagger\left(\omega_p, z_{\hat{\mathbf{n}}}, \bar{z}_{\hat{\mathbf{n}}}\right) \Big] : .\end{aligned} \tag{3.42}$$

More compactly,

$$\mathcal{N}_{z,\mathrm{scalar}}(\hat{\mathbf{n}}) = i \int \mathrm{d}\Phi(p)\, \delta^2\left(z_{\hat{\mathbf{n}}} - z_{\hat{\mathbf{p}}}\right) : \left[a^\dagger(\omega_p \hat{\mathbf{n}}) \overleftrightarrow{\partial}_{z_{\hat{\mathbf{n}}}} a(\omega_p \hat{\mathbf{n}}) \right] :, \tag{3.43}$$

where

$$\frac{i}{2}\partial_z = L_{uz} = \frac{i}{r} \left(\partial_u x^\mu\right) \left(\partial_z x^\nu\right) \left[p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu} \right] \tag{3.44}$$

is the standard orbital angular momentum operator. A similar analysis can be repeated for the antiholomorphic component by flipping $z \leftrightarrow \bar{z}$.

3.4 The Spin-1 light-ray operators

Having found explicit expressions for the light-ray operators for massless scalar theories, we now derive similar results for the light-ray operators for massless spin-1 theories. Using the Hilbert stress tensor $T^{\mathrm{matter}}_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S^{\mathrm{matter}}}{\delta g^{\mu\nu}}$ for pure gauge theories, one obtains

$$T^{\mu\nu}_{\mathrm{photon}} = F^{\mu\alpha} F_\alpha^{\ \nu} + \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \tag{3.45}$$

for Maxwell $U(1)$ theory where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ is the $U(1)$ field strength. The generalization to the non-abelian gauge group $SU(N)$ is straightforward

$$T^{\mu\nu}_{\text{gluon}} = 2\text{Tr}(T^a T^b)\left[(F^a)^{\mu\alpha}(F^b)_\alpha^{\ \nu} + \frac{1}{4}\eta^{\mu\nu}(F^a)^{\alpha\beta}(F^b)_{\alpha\beta}\right], \tag{3.46}$$

where $F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ is the $SU(N)$ field strength. Here, the $SU(N)$ generators are chosen to be hermitian $[T^a, T^b] = i f^{abc} T^c$ and are normalized according to $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

To discuss the large r limit of the stress-tensor, we need the large r limit of the gauge field, which requires making a gauge choice, as well as, specifying boundary conditions. We will adopt the radiation gauge [212]

$$A_u = 0, \qquad r^2 \nabla^\mu A^a_\mu(x)\big|_{\mathcal{I}^-} = 2\left[r^2 \partial_u A^a_r - \partial_z A^a_{\bar z} - \partial_{\bar z} A^a_z\right]\big|_{\mathcal{I}^-} = 0 \tag{3.47}$$

and we require the gauge field components to satisfy the following falloff conditions at infinity

$$A_z(u,r,z,\bar z) = A^{(0)}_z(u,z,\bar z) + \mathcal{O}\left(\frac{1}{r}\right), \qquad A_r(u,r,z,\bar z) = \frac{1}{r^2} A^{(2)}_r(u,z,\bar z) + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{3.48}$$

which are needed to have a finite energy and angular momentum flux at $\mathcal{I}^+$. The stress tensor components relevant for our family of light-ray operators in eq. (3.46) are

$$T_{uu} = \frac{4}{r^2}(\partial_u A^{(0),a}_z)(\partial_u A^{(0),a}_{\bar z}), \tag{3.49}$$
$$T_{uz} = \frac{2}{r^2}(\partial_u A^{(0),a}_z)(2\partial_{[z} A^{(0),a}_{\bar z]} + g f^{abc} A^{(0),b}_z A^{(0),c}_{\bar z}) + \frac{2}{r^2}(\partial_u A^{(0),a}_z)(\partial_u A^{(2),a}_r), \tag{3.50}$$
$$T_{u\bar z} = \frac{2}{r^2}(\partial_u A^{(0),a}_{\bar z})(2\partial_{[\bar z} A^{(0),a}_{z]} + g f^{abc} A^{(0),b}_{\bar z} A^{(0),c}_z) + \frac{2}{r^2}(\partial_u A^{(0),a}_{\bar z})(\partial_u A^{(2),a}_r) \tag{3.51}$$

up to terms of order $\mathcal{O}(1/r^3)$.

We assume that we can work perturbatively with energies above the confinement scale $\Lambda > 0$ and can formally talk about asymptotic gluon states. We use the following parametrization and projection of the polarization vectors

$$\begin{gathered}\varepsilon^{+,\mu}(q) = \frac{1}{\sqrt{2}}(\bar z_q, 1, -i, -\bar z_q), \qquad \varepsilon^{-,\mu}(q) = \frac{1}{\sqrt{2}}(z_q, 1, i, -z_q), \\ (\partial_z x^\mu)\varepsilon^+_\mu(q) = 0 = (\partial_{\bar z} x^\mu)\varepsilon^-_\mu(q), \qquad (\partial_z x^\mu)\varepsilon^-_\mu(q) = -\frac{r}{\sqrt{2}} = (\partial_{\bar z} x^\mu)\varepsilon^+_\mu(q)\end{gathered} \tag{3.52}$$

and the canonical commutation relations

$$[a_\sigma(p), a^\dagger_{\sigma'}(q)] = \frac{4(2\pi)^3}{\omega_p}\delta_{\sigma\sigma'}\delta(\omega_p - \omega_q)\delta^2(z_{\hat{\boldsymbol p}} - z_{\hat{\boldsymbol q}}). \tag{3.53}$$

Starting with the gluon-ANEC operator at null infinity, we find

$$\mathcal{E}_{\text{gluon}}(\hat{\boldsymbol n}) = \int \mathrm{d}\Phi(p)\, \omega_p\, \delta^2(z_{\hat{\boldsymbol n}} - z_{\hat{\boldsymbol p}}) \sum_{\sigma=\pm1} :\left[a^{\dagger,a}_\sigma(\omega_p \hat{\boldsymbol n}) a^a_\sigma(\omega_p \hat{\boldsymbol n})\right]:, \tag{3.54}$$

where the saddle point estimates for $A_z^{(0),a}$ and $A_{\bar{z}}^{(0),a}$ are

$$A_z^{(0),a}(u,\hat{\boldsymbol{n}}) = \frac{-i}{(8\sqrt{2}\pi^2)}\int_0^{+\infty} \mathrm{d}\omega_p \left[a_-^{\dagger,a}(\omega_p\hat{\boldsymbol{n}})e^{i\frac{\omega_p u}{2}} - a_+^a(\omega_p\hat{\boldsymbol{n}})e^{-i\frac{\omega_p u}{2}}\right], \tag{3.55}$$

$$A_{\bar{z}}^{(0),a}(u,\hat{\boldsymbol{n}}) = \frac{-i}{(8\sqrt{2}\pi^2)}\int_0^{+\infty} \mathrm{d}\omega_p \left[a_+^{\dagger,a}(\omega_p\hat{\boldsymbol{n}})e^{i\frac{\omega_p u}{2}} - a_-^a(\omega_p\hat{\boldsymbol{n}})e^{-i\frac{\omega_p u}{2}}\right]. \tag{3.56}$$

Analogous to the scalar case, on-shell gluon states $|X\rangle = |p_1^{\sigma_1},...,p_n^{\sigma_n}\rangle$ are eigenstates of the gluon-ANEC at infinity

$$\mathcal{E}_{\text{gluon}}(\hat{\boldsymbol{n}})|X\rangle = \sum_{i=1}^{n}(\omega_i)\,\delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}_i})|X\rangle. \tag{3.57}$$

The gluon boost energy density operator $\mathcal{K}_{\text{gluon}}(\hat{\boldsymbol{n}})$ follows straightforwardly

$$\begin{aligned}\mathcal{K}_{\text{gluon}}(\hat{\boldsymbol{n}}) = \frac{(-i)}{4}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega_-}{(2\pi)^3}\delta^{(1)}(\omega_-)\int_{\max\{-\omega_-,\omega_-\}}^{+\infty}\mathrm{d}\omega_+\left((\omega_+)^2-(\omega_-)^2\right)\\ \times\sum_{\sigma=\pm1}:\left[a_\sigma^{a,\dagger}\left((\omega_++\omega_-)\hat{\boldsymbol{n}}\right)a_\sigma^a\left((\omega_+-\omega_-)\hat{\boldsymbol{n}}\right)\right]:.\end{aligned} \tag{3.58}$$

These operators are especially simple; the only fundamental difference between $\{\mathcal{E}_{\text{gluon}},\mathcal{K}_{\text{gluon}}\}$ and $\{\mathcal{E}_{\text{scalar}},\mathcal{K}_{\text{scalar}}\}$ is the sum over helicities.

On the other hand, the spin-1 angular momentum flux operators are complicated by boundary terms, which come from integrating by parts in u. Explicitly,

$$\begin{aligned}&\int \mathrm{d}u \lim_{r\to+\infty} r^2 T_{uz}\\ &\quad= \int \mathrm{d}u\Big\{4(\partial_u A_z^{(0),a})(\partial_z A_{\bar{z}}^{(0),a})\Big\} - 2A_z^{(0),a}(-\partial_u A_r^{(2),a} + \partial_z A_{\bar{z}}^{(0),a} + \partial_{\bar{z}} A_z^{(0),a})\Big|_{\mathcal{I}^+_+},\\ &\int \mathrm{d}u \lim_{r\to+\infty} r^2 T_{u\bar{z}}\\ &\quad= \int \mathrm{d}u\Big\{4(\partial_u A_{\bar{z}}^{(0),a})(\partial_{\bar{z}} A_z^{(0),a})\Big\} - 2A_{\bar{z}}^{(0),a}(-\partial_u A_r^{(2),a} + \partial_z A_{\bar{z}}^{(0),a} + \partial_{\bar{z}} A_z^{(0),a})\Big|_{\mathcal{I}^+_+},\end{aligned} \tag{3.59}$$

where we have used the leading u-equation of motion

$$\begin{aligned}&r^2\mathcal{D}_\mu F^\mu{}_u\\ &\quad= 2\partial_u\left[-\partial_u A_r^{(2),a} + \partial_z A_{\bar{z}}^{(0),a} + \partial_{\bar{z}} A_z^{(0),a}\right] + 2gf^{abc}\left[A_z^{(0),b}\partial_u A_{\bar{z}}^{(0),c} + A_{\bar{z}}^{(0),b}\partial_u A_z^{(0),c}\right] = 0.\end{aligned} \tag{3.60}$$

It is worth remarking that the non-abelian contribution appears through the boundary term in eq. (3.59); this is similar to the case when the light-sheet is placed in the bulk as shown in appendix C. For the first contribution to the angular momentum flux density, we use the identity

$$
\begin{aligned}
&4(\partial_u A_z^{(0),a})(\partial_z A_{\bar{z}}^{(0),a}) \\
&\quad = 2(\partial_u A_z^{(0),a})(\partial_z A_{\bar{z}}^{(0),a}) - 2(\partial_z\partial_u A_z^{(0),a})A_{\bar{z}}^{(0),a} + 2\partial_z\left[(\partial_u A_z^{(0),a})A_{\bar{z}}^{(0),a}\right],
\end{aligned}
\tag{3.61}
$$

just to isolate the contribution coming from the total derivative. Regarding the saddle point estimates for $\partial_z A_z^{(0),a}$ and $\partial_z A_{\bar{z}}^{(0),a}$, we have

$$
\begin{aligned}
\partial_z A_z^{(0),a}(u,\hat{\boldsymbol{n}}) &= \frac{-1}{(4\sqrt{2}\pi^2)}\int_0^{+\infty} \mathrm{d}\omega_p \left\{e^{i\frac{\omega_p u}{2}} L_{uz} a_-^{\dagger,a}(\omega_p, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}}) - e^{-i\frac{\omega_p u}{2}} L_{uz} a_+^a(\omega_p, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}})\right\}, \\
\partial_z A_{\bar{z}}^{(0),a}(u,\hat{\boldsymbol{n}}) &= \frac{-1}{(4\sqrt{2}\pi^2)}\int_0^{+\infty} \mathrm{d}\omega_p \left\{e^{i\frac{\omega_p u}{2}} L_{uz} a_+^{\dagger,a}(\omega_p, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}}) - e^{-i\frac{\omega_p u}{2}} L_{uz} a_-^a(\omega_p, z_{\hat{\boldsymbol{n}}}, \bar{z}_{\hat{\boldsymbol{n}}})\right\}.
\end{aligned}
\tag{3.62}
$$

This gives the following representation for the angular momentum flux density

- An orbital angular momentum contribution

$$
\mathcal{N}_{z,\text{gluon}}^{\text{orb}}(\hat{\boldsymbol{n}}) = i\int \mathrm{d}\Phi(p)\,\delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \sum_{\sigma=\pm1} :\left[a_\sigma^{\dagger,a}(\omega_p\hat{\boldsymbol{n}})\overleftrightarrow{\partial}_{z_{\hat{\boldsymbol{n}}}} a_\sigma^a(\omega_p\hat{\boldsymbol{n}})\right]: . \tag{3.63}
$$

- A spin type contribution

$$
\begin{aligned}
\mathcal{N}_{z,\text{gluon}}^{\text{spin}}(\hat{\boldsymbol{n}}) &= -i\int \mathrm{d}\Phi(p)\,\delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \sum_{\sigma=\pm1} \sigma :\partial_{z_{\hat{\boldsymbol{n}}}}\left[a_\sigma^{\dagger,a}(\omega_p\hat{\boldsymbol{n}}) a_\sigma^a(\omega_p\hat{\boldsymbol{n}})\right]: \\
&= i\int \mathrm{d}\Phi(p)\,\partial_{z_{\hat{\boldsymbol{n}}}}\delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \sum_{\sigma=\pm1} \sigma :\left[a_\sigma^{\dagger,a}(\omega_p\hat{\boldsymbol{n}}) a_\sigma^a(\omega_p\hat{\boldsymbol{n}})\right]: .
\end{aligned}
\tag{3.64}
$$

This piece is expected from the spin structure of the point particle stress tensor [213]. Indeed the covariant spin matrix reads [214] (with σ helicity)

$$
S_{\mu\nu} = \frac{\sigma}{p^0}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & p^3 & -p^2 \\ 0 & -p^3 & 0 & p^1 \\ 0 & p^2 & -p^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma(1-z\bar{z})}{1+z\bar{z}} & \frac{i\sigma(z-\bar{z})}{1+z\bar{z}} \\ 0 & -\frac{\sigma(1-z\bar{z})}{1+z\bar{z}} & 0 & \frac{\sigma(z+\bar{z})}{1+z\bar{z}} \\ 0 & -\frac{i\sigma(z-\bar{z})}{1+z\bar{z}} & -\frac{\sigma(z+\bar{z})}{1+z\bar{z}} & 0 \end{pmatrix}
\tag{3.65}
$$

and the component $S_{z\bar{z}} = \frac{1}{r^2}(\partial_z x^\mu)(\partial_{\bar{z}} x^\nu) S_{\mu\nu} = i\frac{\sigma}{2}$ enters in the total hard angular momentum operator as showed in appendix A of [215].

Please note that the splitting of angular momentum into an orbital and spin part is not gauge invariant, and it is done here only for convenience in analyzing different terms. The boundary terms in eq. (3.59) are associated with soft contributions.[9] Using the equations of motion, we can write

$$
-2(-\partial_u A_r^{(2)} + \partial_z A_{\bar{z}}^{(0)} + \partial_{\bar{z}} A_z^{(0)})\Big|_{\mathcal{I}_+^+} = 2\int_{-\infty}^{+\infty} \mathrm{d}u\,[A_z^{(0)}, \overleftrightarrow{\partial}_u A_{\bar{z}}^{(0)}]. \tag{3.66}
$$

[9] Similar boundary terms were also observed in [209], although He and Mitra considered the total angular momentum flux charge.

This makes it clear that for pure Maxwell theory (which is conformal) this term vanishes identically. Using the saddle point we obtain

$$\begin{aligned}-2(-\partial_u A_r^{(2),a}+&\partial_z A_{\bar{z}}^{(0),a}+\partial_{\bar{z}} A_z^{(0),a})\Big|_{\mathcal{I}^+_+}\\ &= -2igf^{abc}\int \mathrm{d}\Phi p \sum_{\sigma=\pm 1}\left[a_\sigma^{\dagger,b}(p)a_\sigma^c(p)\right]\delta^2(z_{\hat{\boldsymbol{n}}}-z_{\hat{\boldsymbol{p}}}),\end{aligned} \tag{3.67}$$

where in the last line we recognize the gluon number density operator

$$\rho^a_{\text{gluon}}(p) := -if^{abc}\sum_{\sigma=\pm 1}\left[a_\sigma^{\dagger,b}(p)a_\sigma^c(p)\right], \tag{3.68}$$

which contributes to the hard part of the non-abelian (large gauge) charge [1, 209]. Defining the boundary value of the field [209, 212]

$$C_z^a(z,\bar{z}) := A_z^a(u=+\infty,z,\bar{z}) \qquad C_{\bar{z}}^a(z,\bar{z}) := A_{\bar{z}}^a(u=+\infty,z,\bar{z}) \tag{3.69}$$

the total boundary contribution to the angular momentum flux becomes

$$\begin{aligned}-2A_z^{(0),a}(-\partial_u A_r^{(2),a}&+\partial_z A_{\bar{z}}^{(0),a}+\partial_{\bar{z}} A_z^{(0),a})\Big|_{\mathcal{I}^+_+}\\ &= 2gC_z^a(z,\bar{z})\int \mathrm{d}\Phi p\, \rho^a_{\text{gluon}}(p)\delta^2(z_{\hat{\boldsymbol{n}}}-z_{\hat{\boldsymbol{p}}}).\end{aligned} \tag{3.70}$$

This contribution is soft, since it contains the soft mode $C_z^a(z,\bar{z})$.[10] It is also contains the hard gluon charge, which might seem strange at first. We can gain some insight by turning on a matter source j^μ, which modifies the u-equation of motion and therefore the final density of angular momentum flux

$$\begin{aligned}\int \mathrm{d}u \lim_{r\to+\infty} r^2 T_{uz} &= \int \mathrm{d}u\Big\{4(\partial_u A_z^{(0),a})(\partial_z A_{\bar{z}}^{(0),a})\Big\} + \int \mathrm{d}u\, A_z^{(0),a} j_u^{a,(2)} + C_z^a q_H^a,\\ \int \mathrm{d}u \lim_{r\to+\infty} r^2 T_{u\bar{z}} &= \int \mathrm{d}u\Big\{4(\partial_u A_{\bar{z}}^{(0),a})(\partial_{\bar{z}} A_z^{(0),a})\Big\} + \int \mathrm{d}u\, A_{\bar{z}}^{(0),a} j_u^{a,(2)} + C_{\bar{z}}^a q_H^a,\end{aligned} \tag{3.71}$$

where

$$q_H(z,\bar{z}) := \int_{-\infty}^{+\infty}\mathrm{d}u\,\left[2[A_z^{(0)},\overleftrightarrow{\partial}_u A_{\bar{z}}^{(0)}]+j_u^{(2)}\right] \tag{3.72}$$

is the density of the hard color charge. These terms were first found by Ashtekar et al. [216, 217] in the abelian spin 1 case (coupled with a matter distribution) and they are related to mixing of the so-called radiative modes (i.e. our $A_z^{(0),a}$ and $A_{\bar{z}}^{(0),a}$) and of coulombic modes (i.e. the hard charge distribution, potential modes). A similar expression holds for $\mathcal{N}_{\bar{z},\text{gluon}}(\hat{\boldsymbol{n}})$ which can be obtained by exchanging $z\leftrightarrow\bar{z}$ and flipping all helicity terms $(+)\leftrightarrow(-)$.

[10] Please notice that as in scalar case, we could have changed the boundary conditions to set $C_z^a(z,\bar{z})=0$ in order to allow only hard modes.

3.5 The Spin-2 light-ray operators

It is well-known that the equivalence principle prevents a universal definition of a stress tensor for the gravitational field. One can try to define various nontensorial objects called "pseudotensors" which when linearized transform as Lorentz tensors. However, all such definitions are not gauge invariant (a notable example is the Landau-Lifshitz pseudotensor [218]). In this work, we will only consider linearized gravity, i.e. we split the metric tensor as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and assume $|h_{\mu\nu}(x)| \ll 1$. In the physically relevant situation of gravitational waves propagating in an asymptotically flat region, a suitable notion of a *gauge-invariant* stress tensor for long-wavelength modes can be defined using the so-called Isaacson averaging procedure [198, 199]. In our conventions, the effective stress tensor for gravitational waves is [219]

$$T^{\text{eff,GW}}_{\mu\nu} = \frac{\kappa^2}{32\pi G}\langle \partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\rangle, \tag{3.73}$$

where $\langle\cdot\rangle$ stands for the average over short-wavelength graviton modes.[11] It is possible to extend the Isaacson construction of an effective gravitational wave stress tensor to general classical theories of gravity [220]: for example, the leading order contribution at infinity is proportional to the Ricci mode for $f(R)$ theories [221], conversely the spectrum is unchanged for most higher derivative theories of gravity since higher derivative terms usually drop off [220]. It is worth stressing that this stress-tensor is defined canonically a la' Noether, and therefore it represents only the physical energy and the momentum flux gravitational wave contribution whereas the angular momentum requires a different analysis.

Asymptotically flat geometries have the following large r asymptotic expansion near $\mathcal{I}^+$ in the Bondi gauge [222, 223]

$$\begin{aligned} ds^2 &= dv^2 + 2dvdr - 2r^2\gamma_{\zeta\bar\zeta}d\zeta d\bar\zeta \\ &- \frac{2m_B}{r}dv^2 - rC_{\zeta\zeta}d\zeta^2 - rC_{\bar\zeta\bar\zeta}d\bar\zeta^2 - D^\zeta C_{\zeta\zeta}dvd\zeta - D^{\bar\zeta}C_{\bar\zeta\bar\zeta}dvd\bar\zeta \\ &- \frac{1}{r}\left[\frac{4}{3}N_\zeta - \frac{1}{4}\partial_\zeta(C_{\zeta\zeta}C^{\zeta\zeta})\right]dvd\zeta - \frac{1}{r}\left[\frac{4}{3}N_{\bar\zeta} - \frac{1}{4}\partial_{\bar\zeta}(C_{\bar\zeta\bar\zeta}C^{\bar\zeta\bar\zeta})\right]dvd\bar\zeta + \ldots \end{aligned} \tag{3.74}$$

where the leading contribution comes from Minkowski spacetime, and subleading contributions encode deviations from the flat background. The system of coordinates as

$$x^\mu = \left(v + r, r\frac{\zeta+\bar\zeta}{(1+\zeta\bar\zeta)}, -ir\frac{\zeta-\bar\zeta}{(1+\zeta\bar\zeta)}, r\frac{1-\zeta\bar\zeta}{(1+\zeta\bar\zeta)}\right) \tag{3.75}$$

where $v = t - r$ is the retarded time. In this formulation $m_B(v,\zeta,\bar\zeta)$ is the Bondi mass aspect, $C_{\alpha\beta}(v,\zeta,\bar\zeta)$ is the shear tensor and $(N_\zeta(v,\zeta,\bar\zeta), N_{\bar\zeta}(v,\zeta,\bar\zeta))$ is the Bondi angular momentum aspect. Here we use the conventions of Pasterski-Strominger-Zhiboedov [215], which differs from Barnich-Troessaert [224], Hawking-Perry-Strominger [225] or Flanagan-Nichols [226]. Since different conventions change the definition of the light-ray operators, this is of particular relevance here.[12]

Spaces that admit expansions of the form in eq. (3.74) are called Christodoulou-Klainerman (CK) spaces in the literature [228, 229]. Physically, the shear tensor encodes gravitational radiation analogously to Maxwell field strength[13] $F_{v\zeta} = \partial_v A_\zeta$.

[11] We stress that the gauge invariance of eq. (3.73) holds only under the average prescription [219].
[12] See also [227] where some of the relations between different conventions are spelled out in detail.
[13] In light-cone gauge, $A_v = 0$.

The Bondi news is defined to be [228]

$$N_{\zeta\zeta}(v,\zeta,\bar{\zeta}) = \partial_v C_{\zeta\zeta}(v,\zeta,\bar{\zeta}). \tag{3.76}$$

In these coordinates,

$$C_{\zeta\zeta}(v,\zeta,\bar{\zeta}) = -\kappa \lim_{r\to\infty} \frac{1}{r} h_{\zeta\zeta}(r,v,\zeta,\bar{\zeta}), \tag{3.77}$$

where $\kappa = \sqrt{32\pi G}$ [230]. The time-time component of the Einstein equations for eq. (3.74) gives the Bondi mass loss formula

$$\partial_v m_B = \frac{1}{4}\left[D_\zeta^2 N^{\zeta\zeta} + D_{\bar{\zeta}}^2 N^{\bar{\zeta}\bar{\zeta}} - N_{\zeta\zeta}N^{\zeta\zeta}\right] - 4\pi G \lim_{r\to\infty} r^2 T_{vv}^{\text{matter}}. \tag{3.78}$$

For CK spaces, the asymptotics of the Weyl curvature component Ψ_2^0 implies the following conditions [228]

$$\begin{aligned} m_B(v,\zeta,\bar{\zeta})\Big|_{\mathcal{I}_-^+} &= M_i^{\text{ADM}}(\zeta,\bar{\zeta})\,, \qquad m_B(v,\zeta,\bar{\zeta})\Big|_{\mathcal{I}_+^+} = M_f(\zeta,\bar{\zeta})\,, \\ C_{\zeta\zeta} &= -2D_\zeta^2 C(\zeta,\bar{\zeta}), \end{aligned} \tag{3.79}$$

where $C(\zeta,\bar{\zeta})$ labels inequivalent BMS vacua related to each other under action of the supertranslation mode. After integrating over v from $-\infty$ to $+\infty$, we get [230, 231]

$$\Delta m_B = \frac{1}{2} D_\zeta^2 \Delta C^{\zeta\zeta} - 4\pi G \int_{-\infty}^{+\infty} \mathrm{d}v \Big\{ \frac{1}{16\pi G} N_{\zeta\zeta}N^{\zeta\zeta} + \lim_{r\to\infty} r^2 T_{vv}^{\text{matter}} \Big\}, \tag{3.80}$$

where $\Delta C^{\zeta\zeta}(\zeta,\bar{\zeta}) = C^{\zeta\zeta}(+\infty,\zeta,\bar{\zeta}) - C^{\zeta\zeta}(-\infty,\zeta,\bar{\zeta})$ and $\Delta m_B = M_f(\zeta,\bar{\zeta})$ since $M_i^{\text{ADM}}(\zeta,\bar{\zeta}) = 0$ for flat Minkowski spacetime. Using supertranslation invariance, one can choose a Bondi frame [228] where $C^{\zeta\zeta}(-\infty,\zeta,\bar{\zeta}) = 0$ but in general the final vacuum will be non-trivial $C^{\zeta\zeta}(+\infty,\zeta,\bar{\zeta}) \neq 0$. Focusing on Bondi news squared term

$$\frac{1}{16\pi G} N_{\zeta\zeta}N^{\zeta\zeta} = \frac{\kappa^2}{16\pi G} \lim_{r\to\infty} \frac{1}{r^2} (\gamma^{\zeta\bar{\zeta}})^2 (\partial_v h_{\zeta\zeta}(x))(\partial_v h_{\bar{\zeta}\bar{\zeta}}(x)), \tag{3.81}$$

we see that it agrees with the (light-cone) time-time component of Isaacson effective gravitational wave stress tensor at infinity $T_{vv}^{\text{eff},GW}$ in these coordinates (albeit this is defined with the implicit averaging over short wavelength modes). The action of eq. (3.80) on an on-shell n-graviton state $|X\rangle = |p_1^{\sigma_1}...p_n^{\sigma_n}\rangle$ is

$$\tilde{\mathcal{E}}_{\text{Bondi}}(\hat{\boldsymbol{n}})|X\rangle = \int_{-\infty}^{+\infty} \mathrm{d}v\, \frac{1}{16\pi G} N_{\zeta\zeta}N^{\zeta\zeta}|X\rangle = \sum_{i=1}^{n} (E_{p_i})\delta^2(\Omega_{\hat{\boldsymbol{n}}} - \Omega_{\hat{\boldsymbol{p}}_i})|X\rangle, \tag{3.82}$$

where we used the saddle-point estimate for the free field mode expansion for the linearized graviton mode

$$h_{\mu\nu}(x) = \int \mathrm{d}\Phi(p) \sum_{\sigma=\pm 2} \left[\varepsilon_{\mu\nu}^{\sigma,*}(p) a_\sigma(p) e^{-ip\cdot x} + \varepsilon_{\mu\nu}^{\sigma}(p) a_\sigma^\dagger(p) e^{ip\cdot x}\right]. \tag{3.83}$$

The weight factor of the Bondi news squared term is given by the standard local factor E_p in retarded Bondi coordinates, which is expected because here $v = t - r$ is the standard light-cone time.

We can now come back to eq. (3.80) which can be solved in terms of $\Delta C_{\zeta\zeta}$ [231]

$$\Delta C_{\zeta\zeta} = \frac{4}{\pi}\int \mathrm{d}^2\zeta'\,\gamma_{\zeta'\bar{\zeta}'}\frac{\bar{\zeta}-\bar{\zeta}'}{\zeta-\zeta'}\frac{(1+\zeta'\bar{\zeta})^2}{(1+\zeta'\bar{\zeta}')(1+\zeta\bar{\zeta})^3}\left(4\pi G\,\mathcal{E}_{\text{shear-inclusive}}(\zeta',\bar{\zeta}')+\Delta M_B(\zeta',\bar{\zeta}')\right), \tag{3.84}$$

where we have defined a suitable extension of the ANEC operator to a shear-inclusive ANEC at infinity which includes both matter and gravity contributions

$$\tilde{\mathcal{E}}_{\text{shear-inclusive}}(\hat{\boldsymbol{n}}) := \int_{-\infty}^{+\infty}\mathrm{d}v\Big\{\frac{1}{16\pi G}(N_{\zeta\zeta}(v,\boldsymbol{n}))(\gamma^{\zeta\bar{\zeta}})^2(N_{\bar{\zeta}\bar{\zeta}}(v,\boldsymbol{n}))+\lim_{r\to\infty}r^2T^{\text{matter}}_{vv}(r,v,\boldsymbol{n})\Big\}, \tag{3.85}$$

whose action on an on-shell n-particle state of different massless species $\prod_{\alpha\in\text{species}}|X_\alpha\rangle$ is

$$\tilde{\mathcal{E}}_{\text{shear-inclusive}}(\hat{\boldsymbol{n}})\prod_{\alpha\in\text{species}}|X_\alpha\rangle = \sum_{i=1}^{n}E_i\delta^2(\Omega_{\hat{\boldsymbol{n}}}-\Omega_{\hat{\boldsymbol{p}}_i})\prod_{\alpha\in\text{species}}|X_\alpha\rangle. \tag{3.86}$$

As we will prove in section 4.4, expectation values of this operator will be positive-definite for a scattering process where we treat (perturbative) gravity as an EFT, similarly to what happens in the massless spin 0 and spin 1 case. Originally, the shear-inclusive ANEC operator was defined for complete achronal null geodesics (also called null lines) to cure the violations of the averaged null energy condition appearing for linearized graviton perturbations, see the appendix B for more details.

At the quantum level we can promote both sides of eq. (3.84) to operators and act on an on-shell state composed of massless particles reaching null infinity (i.e. radiation): a direct calculation shows that

$$\Delta C_{\zeta\zeta}\prod_{\alpha\in\text{species}}|X_\alpha\rangle = \left[16G\sum_{i=1}^{n}\frac{\bar{\zeta}-\bar{\zeta}_i}{\zeta-\zeta_i}\frac{(1+\zeta_i\bar{\zeta})^2E_i}{(1+\zeta_i\bar{\zeta}_i)(1+\zeta\bar{\zeta})^3}\right]\prod_{\alpha\in\text{species}}|X_\alpha\rangle, \tag{3.87}$$

where $\Delta M_B = 0$ in flat Minkowski picture at $v\to\pm\infty$. This is the leading memory effect due to the radiation flux generated in the scattering process [232], which can be directly related to Christodoulou memory effect [233]. Our results are consistent with the memory being given by the (transverse traceless part of) soft factor [231, 232]

$$M_{\zeta\zeta} = \frac{1}{r^2}(\partial_\zeta x^\mu)(\partial_\zeta x^\nu)\frac{\kappa}{2}\left[\sum_{i=1}^{n}\eta_i\frac{(p_\mu)_i(p_\nu)_i}{p_i\cdot n}\right]^{\text{TT}}, \tag{3.88}$$

as we will show in more details in section 4.6.

The angular momentum flux, requires the Einstein equations of the subleading terms in the Bondi gauge expansion [215]

$$\begin{aligned}\partial_v N_\zeta &= \frac{1}{4}\partial_\zeta\left[D_\zeta^2C^{\zeta\zeta}-D_{\bar{\zeta}}^2C^{\bar{\zeta}\bar{\zeta}}\right]+\partial_\zeta m_B\\ &\quad -8\pi G\left[\lim_{r\to+\infty}r^2T^{\text{matter}}_{v\zeta}-\frac{1}{32\pi G}D_\zeta(C_{\zeta\zeta}N^{\zeta\zeta})-\frac{1}{16\pi G}C_{\zeta\zeta}D_\zeta N^{\zeta\zeta}\right].\end{aligned} \tag{3.89}$$

Like the shear-inclusive ANEC at infinity, all angular momentum flux contributions that are quadratic in the fields combine together into a single expression for all massless particles

$$\tilde{\mathcal{N}}^{\rm PSZ}_{\zeta,{\rm Bondi}}(\hat{\boldsymbol{n}}) := \int_{-\infty}^{+\infty} {\rm d}v \Big\{ \lim_{r\to+\infty} r^2 T^{\rm matter}_{v\zeta} + \frac{1}{32\pi G}(\partial_\zeta C_{\zeta\zeta})(\gamma^{\zeta\bar{\zeta}})^2(\partial_v C_{\bar{\zeta}\bar{\zeta}}) - \frac{1}{32\pi G} C_{\zeta\zeta}(\gamma^{\zeta\bar{\zeta}})^2(\partial_\zeta\partial_v C_{\bar{\zeta}\bar{\zeta}}) - \frac{1}{16\pi G}\partial_\zeta(C_{\zeta\zeta}(\gamma^{\zeta\bar{\zeta}})^2\partial_v C_{\bar{\zeta}\bar{\zeta}})\Big\}, \tag{3.90}$$

where the gravity contributions are expressed only in terms of the radiative data. The saddle point estimate of the first two terms of such contributions gives the density of orbital angular momentum flux

$$\begin{aligned}\tilde{\mathcal{N}}^{\rm PSZ,orb}_{\zeta,{\rm Bondi}}(\hat{\boldsymbol{n}}) &= \frac{1}{32\pi G}\int_{-\infty}^{+\infty} {\rm d}v \Big\{(\partial_\zeta C_{\zeta\zeta})(\gamma^{\zeta\bar{\zeta}})^2(\partial_v C_{\bar{\zeta}\bar{\zeta}}) - C_{\zeta\zeta}(\gamma^{\zeta\bar{\zeta}})^2(\partial_\zeta\partial_v C_{\bar{\zeta}\bar{\zeta}})\Big\} \\ &= -\frac{1}{2}\int {\rm d}\Phi(p)\, \frac{\delta^2(\zeta_{\hat{\boldsymbol{n}}}-\zeta_{\hat{\boldsymbol{p}}})}{\gamma_{\zeta\bar{\zeta}}} \sum_{\sigma=\pm2} : \Big[a^\dagger_\sigma(E_p\hat{\boldsymbol{n}})\, i\overleftrightarrow{\partial}_{\zeta_{\hat{\boldsymbol{n}}}} a_\sigma(E_p\hat{\boldsymbol{n}})\Big] : .\end{aligned} \tag{3.91}$$

The remaining term gives the spin contribution

$$\begin{aligned}\tilde{\mathcal{N}}^{\rm PSZ,spin}_{\zeta,{\rm Bondi}}(\hat{\boldsymbol{n}}) &= -\frac{i}{2}\int {\rm d}\Phi(p)\left[\frac{\delta^2(\zeta_{\hat{\boldsymbol{n}}}-\zeta_{\hat{\boldsymbol{p}}})}{\gamma_{\zeta\bar{\zeta}}}\right]\sum_{\sigma=\pm2}\sigma : \partial_{\zeta_{\hat{\boldsymbol{n}}}}\Big[a^\dagger_\sigma(E_p\hat{\boldsymbol{n}})a_\sigma(E_p\hat{\boldsymbol{n}})\Big] : \\ &= \frac{i}{2}\int {\rm d}\Phi(p)\, \partial_{\zeta_{\hat{\boldsymbol{n}}}}\left[\frac{\delta^2(\zeta_{\hat{\boldsymbol{n}}}-\zeta_{\hat{\boldsymbol{p}}})}{\gamma_{\zeta\bar{\zeta}}}\right]\sum_{\sigma=\pm2}\sigma : \Big[a^\dagger_\sigma(E_p\hat{\boldsymbol{n}})a_\sigma(E_p\hat{\boldsymbol{n}})\Big] : .\end{aligned} \tag{3.92}$$

Other conventions like Hawking-Perry-Strominger (HPS) differ from eq. (3.90) in total derivative terms in ζ or $\bar{\zeta}$ of $C_{\zeta\zeta}N^{\zeta\zeta}$ and $C_{\bar{\zeta}\bar{\zeta}}N^{\bar{\zeta}\bar{\zeta}}$, which look like contact/spin terms under a saddle-point analysis similar to the one discuss before. While we are not attempting any kind of rigorous analysis of all these terms in different conventions here, our expectation is that the PSZ convention is the most natural[14] choice which gives the expected spin term of the graviton. This was explored further in [207], where the most general expression of the total angular momentum flux was analyzed and it was shown that eq. (3.90) corresponds to the simplest case $\alpha = \beta = 0$ in their notation.

To make contact with the system of coordinates used in the rest of the chapter, we re-write the corresponding expressions in flat null coordinates[15]

$$\begin{aligned}&\mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}) := \frac{1}{8\pi G}\int_{-\infty}^{+\infty} {\rm d}u\,(\partial_u C_{zz})(\partial_u C_{\bar{z}\bar{z}}), \quad \mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}}) := \frac{1}{8\pi G}\int_{-\infty}^{+\infty} {\rm d}u\, u(\partial_u C_{zz})(\partial_u C_{\bar{z}\bar{z}}), \\ &\mathcal{N}^{\rm PSZ}_{z,{\rm GR}}(\hat{\boldsymbol{n}}) := \frac{1}{16\pi G}\int_{-\infty}^{+\infty} {\rm d}u\, \left[(\partial_z C_{zz})(\partial_u C_{\bar{z}\bar{z}}) - C_{zz}(\partial_z\partial_u C_{\bar{z}\bar{z}}) - 2\partial_z(C_{zz}\partial_u C_{\bar{z}\bar{z}})\right],\end{aligned} \tag{3.93}$$

[14] Probably also because it makes direct contact with the Weyl tensor component Ψ_1^0.

[15] These expressions can be derived in other ways. For example, $\mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}})$ and $\mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}})$ can be deduced from the saddle point estimate of the Isaacson effective stress tensor in flat null coordinates.

whose saddle point estimate gives

$$
\begin{aligned}
\mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}) &= \int \mathrm{d}\Phi(p)\; \omega_p\, \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \sum_{\sigma=\pm 2} :\left[a^\dagger_\sigma(\omega_p \hat{\boldsymbol{n}}) a_\sigma(\omega_p \hat{\boldsymbol{n}})\right]:, \\
\mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}}) &= \frac{(-i)}{4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega_-}{(2\pi)^3} \delta^{(1)}(\omega_-) \int_{\max\{-\omega_-,\omega_-\}}^{+\infty} \mathrm{d}\omega_+ \left((\omega_+)^2 - (\omega_-)^2\right) \\
&\qquad \times \sum_{\sigma=\pm 2} :\left[a^\dagger_\sigma\left((\omega_+ + \omega_-)\hat{\boldsymbol{n}}\right) a_\sigma\left((\omega_+ - \omega_-)\hat{\boldsymbol{n}}\right)\right]:, \\
\mathcal{N}^{\rm PSZ,orb}_{z,\rm GR}(\hat{\boldsymbol{n}}) &= i \int \mathrm{d}\Phi(p)\, \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \sum_{\sigma=\pm 2} :\left[a^\dagger_\sigma(\omega_p \hat{\boldsymbol{n}}) \overleftrightarrow{\partial}_{z_{\hat{\boldsymbol{n}}}} a_\sigma(\omega_p \hat{\boldsymbol{n}})\right]:, \\
\mathcal{N}^{\rm PSZ,spin}_{z,\rm GR}(\hat{\boldsymbol{n}}) &= -i \int \mathrm{d}\Phi(p)\, \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \sum_{\sigma=\pm 2} \sigma : \partial_{z_{\hat{\boldsymbol{n}}}} \left[a^\dagger_\sigma(\omega_p \hat{\boldsymbol{n}}) a_\sigma(\omega_p \hat{\boldsymbol{n}})\right]: \\
&= i \int \mathrm{d}\Phi(p)\, \partial_{z_{\hat{\boldsymbol{n}}}} \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \sum_{\sigma=\pm 2} \sigma : \left[a^\dagger_\sigma(\omega_p \hat{\boldsymbol{n}}) a_\sigma(\omega_p \hat{\boldsymbol{n}})\right]: .
\end{aligned}
\tag{3.94}
$$

Note that the spin operator above differs from the spin-1 counterpart by a factor of 2 since the graviton has helicities $\sigma = \pm 2$.

3.6 On-shell detector algebra of hard light-ray operators

Light-ray operators constructed from the components of the stress tensor are naturally related to the global Poincaré charges by turning the line integral in the light-ray definition into an integral over all of space. These light-ray operators were recently studied in the context of unitary CFTs by Cordova and Shao [234] and shown, via symmetry arguments, to form a closed algebra. While it is not immediately clear that the same light-ray operators will form a closed algebra in a generic QFT, the underlying universality of the stress tensor algebra [235] hints at this possibility. The stress tensor algebra was also reconsidered more recently [236, 237] to derive a "universal" effective light-cone algebra for CFTs in $d > 2$. Moreover, there is an intriguing relation with the BMS algebra [222, 223] – and its extended version [10, 224, 238–240] – with light-ray operators [234] which is still worth exploring for light-ray placed at null infinity. We will then compute in our approach the associated algebra of light-ray operators in 3+1-dimensional QFTs for massless particles of integer spin and compare with [234].

In general, computing the commutator algebra of composite (non-local[16]) operators is quite subtle unless completely fixed by symmetries. For example, with the Schwinger action principle [241–243] we can compute the structure of all commutators of the (covariantly conserved) stress tensor but the commutators of stress tensor components with spatial-spatial indices; such commutators are model-dependent [235]. Since our family of light-ray operators contain either the time-time components or mixed time-spatial components of the stress tensor, it is conceivable that our light-ray operators satisfy a universal algebra [234, 236, 237]

$$
[L_1(\hat{\boldsymbol{n}}_1), L_2(\hat{\boldsymbol{n}}_2)] = \sum_{L_3} C_{L_1 L_2 L_3}(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2)\, L_3(\hat{\boldsymbol{n}}_2). \tag{3.95}
$$

[16] The fact that our operators are non-local does not guarantee the convergence of the OPE of such operators in the first place. This problem can be solved in CFT [155], but it is not clear how to generalize it to a general quantum field theory.

To make contact with physical detectors at infinity, we study the light ray algebra by taking the difference between the expectation value of Wightman two-point functions:

$$\langle [L(\hat{\boldsymbol{n}}_1), L'(\hat{\boldsymbol{n}}_2)]\rangle := \langle \psi_L | L(\hat{\boldsymbol{n}}_1) L'(\hat{\boldsymbol{n}}_2) | \psi_R\rangle \; - \langle \psi_L | L'(\hat{\boldsymbol{n}}_2) L(\hat{\boldsymbol{n}}_1) | \psi_R\rangle. \tag{3.96}$$

Without loss of generality, we can set the $|\psi\rangle$ above to be a single particle state. In the following calculations we will assume that canonical commutation relations are valid, motivated by the fact we will compute the algebra on a flat section of null infinity thanks to the choice of flat null coordinates.

3.6.1 Spin 0

The simplest commutator is $\mathcal{P}^\mu_{\text{scalar}}(\hat{\boldsymbol{n}}_1)$ and $\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)$:

$$\langle q | [\mathcal{P}^\mu_{\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)] | p\rangle = 0. \tag{3.97}$$

The vanishing of this commutator -at all order in perturbation theory- is physically interpreted as the statement that measurements of energy and momentum in two localized directions $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$ are compatible at the quantum mechanical level.

The commutator $[\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)]$ is slightly more non-trivial. Using

$$\begin{aligned}&\langle q | [\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)] | p\rangle \\ &\qquad = \left(\delta^2(z_{\hat{\boldsymbol{n}}_2} - z_{\hat{\boldsymbol{p}}})\omega_p - \delta^2(z_{\hat{\boldsymbol{n}}_2} - z_{\hat{\boldsymbol{q}}})\omega_q\right) \langle q | \mathcal{K}_{\text{scalar}}\left(\hat{\boldsymbol{n}}_1\right) | p\rangle,\end{aligned} \tag{3.98}$$

one obtains[17]

$$\begin{aligned}
&\langle q | [\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)] | p\rangle = \\
&= 8(2\pi)^3 i \underbrace{(\omega_q - \omega_p)\, \delta^{(1)}\left(\omega_q - \omega_p\right)}_{-\delta(\omega_q - \omega_p)} \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\delta^2(z_{\hat{\boldsymbol{n}}_2} - z_{\hat{\boldsymbol{p}}})\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{q}}}) \\
&= -8(2\pi)^3 i\delta\left(\omega_q - \omega_p\right) \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\delta^2(z_{\hat{\boldsymbol{n}}_2} - z_{\hat{\boldsymbol{p}}})\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{q}}}) \\
&= -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2}) \left[\left(\omega_p \delta^2(z_{\hat{\boldsymbol{n}}_2} - z_{\hat{\boldsymbol{p}}})\right) \left((2\pi)^3 4 \frac{\delta\left(\omega_q - \omega_p\right)}{\omega_p} \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{q}}})\right)\right] \\
&= -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2}) \langle q | \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2) | p\rangle,
\end{aligned} \tag{3.99}$$

which implies

$$[\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)] = -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2). \tag{3.100}$$

Moreover, it is straightforward to check that

$$\begin{aligned}&[\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)] = 0, \\ &[\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)] = 0.\end{aligned} \tag{3.101}$$

Moving onto the commutators involving the angular momentum flux density operator we need to introduce some smearing in the transverse directions $(z, \bar{z})$ to be able to

[17] The following steps can be made rigorous by smearing the states with appropriate smooth wave-functions, as explained earlier.

regularize the expressions, as in [157, 234]. With this technique we find that

$$\begin{aligned}
&\langle q|[\mathcal{N}_{z,\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)]|p\rangle \\
&= (\omega_p \delta^2(z_{\hat{p}} - z_{\hat{\boldsymbol{n}}_2}) - \omega_q \delta^2(z_{\hat{\boldsymbol{q}}} - z_{\hat{\boldsymbol{n}}_2}))\langle q|\mathcal{N}_{z,\text{scalar}}(\hat{\boldsymbol{n}}_1)|p\rangle \\
&= -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\langle q|\partial_z \mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)|p\rangle + 2i\partial_z \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\langle q|\mathcal{E}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)|p\rangle. \quad (3.102)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\langle q|[\mathcal{N}_{z,\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)]|p\rangle \\
&\quad = -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\langle q|\partial_z \mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)|p\rangle + 2i\partial_z \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\langle q|\mathcal{K}_{\text{scalar}}(\hat{\boldsymbol{n}}_2)|p\rangle. \quad (3.103)
\end{aligned}$$

The last commutator $[\mathcal{N}_{z,\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{N}_{\bar{z},\text{scalar}}(\hat{\boldsymbol{n}}_2)]$ gives

$$\begin{aligned}
&\langle q|[\mathcal{N}_{z,\text{scalar}}(\hat{\boldsymbol{n}}_1), \mathcal{N}_{\bar{z},\text{scalar}}(\hat{\boldsymbol{n}}_2)]|p\rangle \\
&\quad = +2i\partial_z \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\langle q|\mathcal{N}_{\bar{z},\text{scalar}}(\hat{\boldsymbol{n}}_2)|p\rangle - 2i\partial_{\bar{z}} \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\langle q|\mathcal{N}_{z,\text{scalar}}(\hat{\boldsymbol{n}}_2)|p\rangle \\
&\qquad - 2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\langle q|\partial_z \mathcal{N}_{\bar{z},\text{scalar}}(\hat{\boldsymbol{n}}_2)|p\rangle. \quad (3.104)
\end{aligned}$$

This concludes the calculation of the scalar light-ray algebra. It is worth stressing that in this case our boundary conditions at infinity naturally select only the hard modes.

3.6.2 Spin 1

The spin 1 abelian case (i.e. Maxwell theory) does not differ much from the spin 0 case except for the presence of the helicity term. The additional helicity term commutes with

$$\begin{aligned}
\mathcal{E}_{\text{photon}}(\hat{\boldsymbol{n}}) &= \int \mathrm{d}\Phi(p)\, \omega_p \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}}) \sum_{\sigma=\pm 1} : \left[a_\sigma^\dagger(\omega_p \hat{\boldsymbol{n}}) a_\sigma(\omega_p \hat{\boldsymbol{n}})\right] :, \\
\mathcal{K}_{\text{photon}}(\hat{\boldsymbol{n}}) &= \frac{(-i)}{4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega_-}{(2\pi)^3} \delta^{(1)}(\omega_-) \int_{\max\{-\omega_-, \omega_-\}}^{+\infty} \mathrm{d}\omega_+ \left((\omega_+)^2 - (\omega_-)^2\right) \\
&\qquad\qquad \times : \sum_{\sigma=\pm 1} \left[a_\sigma^\dagger\left((\omega_+ + \omega_-)\hat{\boldsymbol{n}}\right) a_\sigma\left((\omega_+ - \omega_-)\hat{\boldsymbol{n}}\right)\right] :, \qquad (3.105)
\end{aligned}$$

because it is proportional to the number operator which counts the difference between the number of helicity plus and helicity minus photons

$$\mathcal{N}^{\text{spin}}_{z,\text{photon}}(\hat{\boldsymbol{n}})|p^\sigma\rangle = i\partial_{z_{\hat{\boldsymbol{n}}}} \delta^2(z_{\hat{\boldsymbol{n}}} - z_{\hat{\boldsymbol{p}}})(\delta_{+,\sigma} - \delta_{-,\sigma})|p^\sigma\rangle. \qquad (3.106)$$

The commutation relation of $\mathcal{N}_z^{\text{orb}}$ with $\mathcal{N}_{\bar{z}}^{\text{orb}}$ is exactly the same as the scalar case

$$\begin{aligned}
&[\mathcal{N}^{\text{orb}}_{z,\text{photon}}(\hat{\boldsymbol{n}}_1), \mathcal{N}^{\text{orb}}_{\bar{z},\text{photon}}(\hat{\boldsymbol{n}}_2)] \\
&\quad = +2i\partial_z \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{N}^{\text{orb}}_{\bar{z},\text{photon}}(\hat{\boldsymbol{n}}_2) - 2i\partial_{\bar{z}} \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{N}^{\text{orb}}_{z,\text{photon}}(\hat{\boldsymbol{n}}_2) \\
&\qquad - 2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\partial_z \mathcal{N}^{\text{orb}}_{\bar{z},\text{photon}}(\hat{\boldsymbol{n}}_2). \qquad (3.107)
\end{aligned}$$

This is obvious from the fact that the $\mathcal{N}^{\mathrm{orb}}_{z,\mathrm{photon}}$ is simply the sum of two copies of $\mathcal{N}^{\mathrm{orb}}_{z,\mathrm{scalar}}$

$$\mathcal{N}^{\mathrm{orb}}_{z,\mathrm{photon}}(\hat{\boldsymbol{n}}) = i\int \mathrm{d}\Phi(p)\,\delta^2(z_{\hat{\boldsymbol{n}}}-z_{\hat{\boldsymbol{p}}})\sum_{\sigma=\pm1} :\left[a^\dagger_\sigma(\omega_p\hat{\boldsymbol{n}})\overleftrightarrow{\partial}_{z_{\hat{\boldsymbol{n}}}} a_\sigma(\omega_p\hat{\boldsymbol{n}})\right]:. \tag{3.108}$$

Moreover it is clear that $\mathcal{N}^{\mathrm{spin}}_{z,\mathrm{photon}}(\hat{\boldsymbol{n}}_1)$ commutes with $\mathcal{N}^{\mathrm{spin}}_{\bar{z},\mathrm{photon}}(\hat{\boldsymbol{n}}_2)$

$$\left[\mathcal{N}^{\mathrm{spin}}_{z,\mathrm{photon}}(\hat{\boldsymbol{n}}_1),\mathcal{N}^{\mathrm{spin}}_{\bar{z},\mathrm{photon}}(\hat{\boldsymbol{n}}_2)\right] = 0. \tag{3.109}$$

The remaining orbital-spin commutators take the form

$$\begin{aligned}\left[\mathcal{N}^{\mathrm{orb}}_{z,\mathrm{photon}}(\hat{\boldsymbol{n}}_1),\mathcal{N}^{\mathrm{spin}}_{\bar{z},\mathrm{photon}}(\hat{\boldsymbol{n}}_2)\right] &= +2i\partial_z\delta^2(z_{\hat{\boldsymbol{n}}_1}-z_{\hat{\boldsymbol{n}}_2})\mathcal{N}^{\mathrm{spin}}_{\bar{z},\mathrm{photon}}(\hat{\boldsymbol{n}}_2),\\ \left[\mathcal{N}^{\mathrm{spin}}_{z,\mathrm{photon}}(\hat{\boldsymbol{n}}_1),\mathcal{N}^{\mathrm{orb}}_{\bar{z},\mathrm{photon}}(\hat{\boldsymbol{n}}_2)\right] &= -2i\partial_{\bar{z}}\delta^2(z_{\hat{\boldsymbol{n}}_1}-z_{\hat{\boldsymbol{n}}_2})\mathcal{N}^{\mathrm{spin}}_{z,\mathrm{photon}}(\hat{\boldsymbol{n}}_2)\\ &\quad + 2i\delta^2(z_{\hat{\boldsymbol{n}}_1}-z_{\hat{\boldsymbol{n}}_2})\partial_{\bar{z}}\mathcal{N}^{\mathrm{spin}}_{z,\mathrm{photon}}(\hat{\boldsymbol{n}}_2).\end{aligned} \tag{3.110}$$

The above commutators are quite a bit more subtle than the others encountered so far and we do find an ambiguity in their derivation.[18] However, any problems with the orbital-spin commutator disappear in the integrated charges since $\mathcal{N}^{\mathrm{spin}}_{z/\bar{z}}$ are total derivatives of z or $\bar{z}$.

If we turn on matter contributions, we find an additional (hard) interaction term which is mixing Coulombic and radiative data

$$\int \mathrm{d}u\, A^{(0),a}_z j^{a,(2)}_u, \tag{3.111}$$

which clearly breaks the algebra because it does not conserve the particle number.

Regarding the soft non-abelian contribution which comes from the boundary term, we stress that as in the spin 0 case we are only interested in the hard contributions and therefore ignore the soft terms in our analysis, which requires a more delicate study of the soft non-abelian sector and its quantization as done by He and Mitra in [209]. We expect such terms to be related with the gluon soft theorem [212], and therefore to be relevant for the light-ray algebra of pure Yang-Mills in the soft sector: the non-commutativity of soft gluon limits for different helicities might give a non-trivial extension of Cordova-Shao algebra. We leave this very interesting problem for a future study.

3.6.3 Spin 2

Here similar comments to the ones made on the spin 1 abelian case apply, except for the fact that we have some freedom in the choice of the Bondi angular momentum aspect and therefore of the density of the angular momentum flux as discussed before.

[18] Since the $\mathcal{N}_{z/\bar{z}}$ do not contain integrals over the transverse directions, one must smear them with a test function. A more rigorous analysis would involve studying Wightman functions in position space with a careful prescription for sending the operator positions to the same light-sheet (see [244], for example) in addition to smearing with test functions.

The final algebra is

$$
\begin{aligned}
&[\mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}_2)] = 0, \\
&[\mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}}_1), \mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}}_2)] = 0, \\
&[\mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}_2)] = -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}_2), \\
&[\mathcal{N}^{\rm PSZ}_{z,{\rm GR}}(\hat{\boldsymbol{n}}_1), \mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}_2)] = -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\partial_z\mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}_2) + 2i\partial_z\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}}_2), \\
&[\mathcal{N}^{\rm PSZ}_{z,{\rm GR}}(\hat{\boldsymbol{n}}_1), \mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}}_2)] = -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\partial_z\mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}}_2) + 2i\partial_z\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{K}_{\rm GR}(\hat{\boldsymbol{n}}_2), \\
&[\mathcal{N}^{\rm PSZ}_{z,{\rm GR}}(\hat{\boldsymbol{n}}_1), \mathcal{N}^{\rm PSZ}_{\bar{z},{\rm GR}}(\hat{\boldsymbol{n}}_2)] = +2i\partial_z\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{N}^{\rm PSZ}_{\bar{z},{\rm GR}}(\hat{\boldsymbol{n}}_2) - 2i\partial_{\bar{z}}\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{N}^{\rm PSZ}_{z,{\rm GR}}(\hat{\boldsymbol{n}}_2) \\
&\qquad\qquad - 2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\partial_z\mathcal{N}^{\rm PSZ}_{\bar{z},{\rm GR}}(\hat{\boldsymbol{n}}_2).
\end{aligned} \tag{3.112}
$$

and therefore with our definitions the algebra of light-ray operators for the graviton case is consistent with complexified Cordova-Shao algebra. In particular, there are no contributions which are mixing coulombic and radiative data as expected [245]. Moreover, the convention adopted by Pasterski-Strominger-Zhiboedov [215] seems to be the most natural for the system of light-ray operators since it provides the standard helicity term which we would expect from a spin 2 point particle stress tensor.

3.6.4 Comparison with complexified Cordova-Shao algebra

It is straightforward to check that the commutators

$$
\begin{aligned}
&[\mathcal{E}(\hat{\boldsymbol{n}}_1), \mathcal{E}(\hat{\boldsymbol{n}}_2)] = 0, \\
&[\mathcal{K}(\hat{\boldsymbol{n}}_1), \mathcal{K}(\hat{\boldsymbol{n}}_2)] = 0, \\
&[\mathcal{K}(\hat{\boldsymbol{n}}_1), \mathcal{E}(\hat{\boldsymbol{n}}_2)] = -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{E}(\hat{\boldsymbol{n}}_2)
\end{aligned} \tag{3.113}
$$

agree exactly with [234] since $T_{uu} = T_{--}$.[19] However, in order to compare the other commutators, we need to compare how the components of the stress tensor at null infinity in our cooordinates differ from those in the standard light-cone components

$$T_{uz} = \frac{x^+}{2}\left(T_{-1} - iT_{-2}\right) + \left(x^1 - ix^2\right)T_{--}, \tag{3.114}$$

$$T_{u\bar{z}} = \frac{x^+}{2}\left(T_{-1} + iT_{-2}\right) + \left(x^1 + ix^2\right)T_{--}. \tag{3.115}$$

Therefore, we see that a full detailed comparison with the (complexified) Cordova Shao algebra for the angular momentum flux requires understanding the following operator

$$\lim_{x^+\to+\infty}(x^+)^3\left[\frac{1}{2}\left(T_{-1} \pm iT_{-2}\right) + \frac{x^1 \pm ix^2}{x^+}T_{--}\right], \tag{3.116}$$

[19] It is worth remembering here that $\delta^2(\Omega_{\hat{\boldsymbol{n}}_1} - \Omega_{\hat{\boldsymbol{n}}_2}) = 2\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})$.

where there is an explicit mixing with the original contribution of the complexified version of $\mathcal{N}_A$ at infinity. In general we have ($r = x^+$)

$$
\begin{aligned}
\lim_{r\to+\infty} r^2 T_{uu} &= \lim_{x^+\to+\infty} (x^+)^2 T_{--}, \\
\lim_{r\to+\infty} r^2 T_{uz} &= \lim_{x^+\to+\infty} (x^+)^3 \left[\frac{1}{2}\left(T_{-1} - iT_{-2}\right) + \frac{x^1 - ix^2}{x^+} T_{--}\right], \\
\lim_{r\to+\infty} r^2 T_{u\bar{z}} &= \lim_{x^+\to+\infty} (x^+)^3 \left[\frac{1}{2}\left(T_{-1} + iT_{-2}\right) + \frac{x^1 + ix^2}{x^+} T_{--}\right],
\end{aligned}
\tag{3.117}
$$

which makes it clear how the operators are mixing with each other. Nevertheless, the commutators

$$
\begin{aligned}
[\mathcal{N}_z(\hat{\boldsymbol{n}}_1), \mathcal{E}(\hat{\boldsymbol{n}}_2)] &= -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\partial_z \mathcal{E}(\hat{\boldsymbol{n}}_2) + 2i\partial_z \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{E}(\hat{\boldsymbol{n}}_2), \\
[\mathcal{N}_z(\hat{\boldsymbol{n}}_1), \mathcal{K}(\hat{\boldsymbol{n}}_2)] &= -2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\partial_z \mathcal{K}(\hat{\boldsymbol{n}}_2) + 2i\partial_z \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{K}(\hat{\boldsymbol{n}}_2)
\end{aligned}
\tag{3.118}
$$

match the complexified Cordova-Shao algebra. The only commutation relation which is new and differ from their result is $[\mathcal{N}_z(\hat{\boldsymbol{n}}_1), \mathcal{N}_{\bar{z}}(\hat{\boldsymbol{n}}_2)]$ [20], due to mixing in eq. (3.117).

[20] Their prediction is

$$
[\mathcal{N}_z(\hat{\boldsymbol{n}}_1), \mathcal{N}_{\bar{z}}(\hat{\boldsymbol{n}}_2)] = +2i\partial_z \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{N}_{\bar{z}}(\hat{\boldsymbol{n}}_2) + 2i\partial_{\bar{z}} \delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\mathcal{N}_z(\hat{\boldsymbol{n}}_2) - 2i\delta^2(z_{\hat{\boldsymbol{n}}_1} - z_{\hat{\boldsymbol{n}}_2})\partial_z \mathcal{N}_{\bar{z}}(\hat{\boldsymbol{n}}_2).
$$

Chapter 4

Classical radiative observables in the two-body problem

In this chapter, we will discuss several classical radiative observables arising from the scattering of waves and point particles. We will start by studying global observables – like the impulse – and we will derive the light deflection in the geometric optics limit for Thomson scattering. We will then consider the field strength for the outgoing radiation field in the two-body problem and we will impose the uncertainty principle to its expectation value to make contact with the classical prediction. An infinity of amplitude relations will follow from it, and we will discuss the consequences for classical radiative observables. At that point, we will focus on localized observables like the waveform and gravitational event shapes and we will perform some explicit calculation in scalar QED and gravity. A new general relation between energy event shapes and the amplitude of the waveform will then be derived. Finally, we will discuss the choice of the BMS frame at the amplitude level and the connection to disconnected amplitude contributions with the emission of zero-energy gravitons.

4.1 Global observables: impulse in wave scattering

Let us investigate the general structure of the impulse, $\langle \Delta p_1^\mu \rangle$, on a massive particle during a scattering event with a classical wave. Using the KMOC formalism as discussed in chapter 2, we get from the definition of the impulse

$$\begin{aligned}\langle \Delta p_1^\mu \rangle &= \langle \psi_w | S^\dagger \mathbb{P}_1^\mu S | \psi_w \rangle - \langle \psi_w | \mathbb{P}_1^\mu | \psi_w \rangle \\ &= \langle \psi_w | \, i[\mathbb{P}_1^\mu, T] \, | \psi_w \rangle + \langle \psi_w | \, T^\dagger [\mathbb{P}_1^\mu, T] \, | \psi_w \rangle \\ &= I^\mu_{w(1)} + I^\mu_{w(2)} \, ,\end{aligned}$$

with the initial state

$$|\psi_w\rangle = \int \mathrm{d}\Phi(p_1) \; \psi_A(p_1) \, e^{ib\cdot p_1/\hbar} |p_1 \, \alpha_2^\sigma\rangle \, . \tag{4.1}$$

We remark here that there is an equivalent formulation in terms of the background field,

$$\begin{aligned}\langle \Delta p_1^\mu \rangle &= \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_1')\, \psi_A(p_1)\psi_A^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \langle p_1' | \mathbb{C}^\dagger_{\alpha,(\sigma)} i[\mathbb{P}_1^\mu, T]\mathbb{C}_{\alpha,(\sigma)} | p_1 \rangle \\ &\quad + \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_1')\, \psi_A(p_1)\psi_A^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \langle p_1' | \mathbb{C}^\dagger_{\alpha,(\sigma)} T^\dagger [\mathbb{P}_1^\mu, T]\mathbb{C}_{\alpha,(\sigma)} | p_1 \rangle \\ &= \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_1')\, \psi_A(p_1)\psi_A^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \langle p_1' | i[\mathbb{P}_1^\mu, T(A_{\mathrm{cl}}^{(\sigma)})] | p_1 \rangle \\ &\quad + \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_1')\, \psi_A(p_1)\psi_A^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \langle p_1' | T^\dagger(A_{\mathrm{cl}}^{(\sigma)}) [\mathbb{P}_1^\mu, T(A_{\mathrm{cl}}^{(\sigma)})] | p_1 \rangle \,,\end{aligned} \tag{4.2}$$

where the scattering matrix computed from the background $A_{\mathrm{cl}}^{(\sigma)}$ is denoted by $T(A_{\mathrm{cl}}^{(\sigma)})$, and we have used the relation $\mathbb{C}^\dagger_{\alpha,(\sigma)}\mathbb{C}_{\alpha,(\sigma)} = 1$. While we will focus on the formulation in eq. (4.1), it is intriguing to notice the linear term of the impulse $I^\mu_{w(1)}$ is closely related to the two-point function of the massive scalar field in the coherent state background. As a consequence, we should expect a resummation of all higher-order results.

Returning to (4.1), we note that — just as in the scattering of two massive particles — only the first term contributes at leading order in the generic coupling g. This contribution arises at $\mathcal{O}(g^2)$; the second term only contributes starting at $\mathcal{O}(g^4)$. Let us focus on the $I^\mu_{w(1)}$ term, and write out the details of the wavefunction in eq. (2.86),

$$I^\mu_{w(1)} = \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_1')\; e^{-ib\cdot(p_1'-p_1)/\hbar} \psi_A(p_1)\psi_A^*(p_1') i(p_1'-p_1)^\mu \langle p_1'\, \alpha_2^\sigma | T | p_1\, \alpha_2^\sigma \rangle \,. \tag{4.3}$$

The matrix elements of coherent states are not of definite order in perturbation theory. In order to isolate the contributions at each order, one would ordinarily introduce a complete set of states of definite particle number on each side of the T matrix,

$$\begin{aligned} I^\mu_{w(1)} = \sum_{X,X'} \sum_{\tilde\sigma,\tilde\sigma'=\pm 1} \int & \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_1')\mathrm{d}\Phi(r_1)\mathrm{d}\Phi(r_1')\mathrm{d}\Phi(k_2)\mathrm{d}\Phi(k_2') \\ & \times e^{-ib\cdot(p_1'-p_1)/\hbar} \psi_A(p_1)\psi_A^*(p_1')\, i(p_1'-p_1)^\mu \\ & \times \langle p_1'\, \alpha_2^\sigma | r_1'\, k_2'^{\tilde\sigma'}\, X' \rangle \langle r_1'\, k_2'^{\tilde\sigma'}\, X' | T | r_1\, k_2^{\tilde\sigma}\, X \rangle \langle r_1\, k_2^{\tilde\sigma}\, X | p_1\, \alpha_2^\sigma \rangle \,. \end{aligned} \tag{4.4}$$

The sums over X and X' are over different numbers of messengers, including none, and include the phase-space integrals over their momenta. Charge conservation implies that each intermediate state must contain one net massive-particle number; we drop additional particle–antiparticle pairs as their effects will disappear in the classical limit, and we denote the massive-particle momenta by r_1 and r_1'. Moreover, in order to satisfy on-shell conditions of the T matrix element, each intermediate state must contain at least one messenger, whose momenta are denoted by k_2 and k_2'.

The leading order contribution to $I^\mu_{w(1)}$ is the simplest. One may be tempted to believe that it arises from terms with $X = X' = \emptyset$, but this is not quite right: that would omit disconnected parts of the S-matrix. In the situation at hand, a great many photons are present in the initial state; the dominant contribution to the interaction occurs when most photons pass directly from the initial to the final state. Thus rather than taking $X = X' = \emptyset$, we instead need to sum over additional messengers in the coherent states. These sums over non-interacting messengers, contributing disconnected S-matrix terms, are necessary to recover the correct normalization.

One can carry out these sums explicitly, but it is convenient instead to introduce an alternate representation for the T matrix in terms of creation and annihilation operators. As the incoming state $|\psi_w\rangle$ given in eq. (2.86) contains one massive particle and an arbitrary number of photons (or messengers more generally), we must consider terms with a pair of massive-particle annihilation and creation operators, and an arbitrary nonzero number of messenger annihilation and creation operators (not necessarily paired). That representation has the form,

$$T = \sum_{\tilde{\sigma},\tilde{\sigma}'=\pm 1} \int \mathrm{d}\Phi(\tilde{r}_1, \tilde{r}_1', \tilde{k}_2, \tilde{k}_2') \, \langle \tilde{r}_1', \tilde{k}_2'^{\tilde{\sigma}'} | T | \tilde{r}_1, \tilde{k}_2^{\tilde{\sigma}} \rangle \; a^\dagger_{\tilde{\sigma}'}(\tilde{k}_2') a^\dagger(\tilde{r}_1') \, a(\tilde{r}_1) a_{\tilde{\sigma}}(\tilde{k}_2) + \cdots \, , \tag{4.5}$$

where the ellipsis indicates higher order terms in the coupling g as well as amplitudes which do not contribute in the classical limit. We will summarily drop all these terms in the following, retaining only the explicit $\mathcal{O}(g^2)$ term. The measure here is a shorthand,

$$\mathrm{d}\Phi(\tilde{r}_1, \tilde{r}_1', \tilde{k}_2, \tilde{k}_2') = \mathrm{d}\Phi(\tilde{r}_1)\mathrm{d}\Phi(\tilde{r}_1')\mathrm{d}\Phi(\tilde{k}_2)\mathrm{d}\Phi(\tilde{k}_2') \, . \tag{4.6}$$

The advantage of the representation in eq. (4.5) is that the creation and annihilation operators act simply on coherent states, yielding factors of $\alpha(k_2)$ and $\alpha^*(k_2')$, and taking care of the normalization for us. Each term within this representation contains an ordinary (connected) amplitude with a definite number of external messengers.

The required matrix element for the integrand term in eq. (4.5) can be computed easily,

$$\begin{aligned} \langle p_1' \, \alpha_2^\sigma | T | p_1 \, \alpha_2^\sigma \rangle &= \langle \tilde{r}_1' \, \tilde{k}_2'^{\tilde{\sigma}'} | T | \tilde{r}_1 \, \tilde{k}_2^{\tilde{\sigma}} \rangle \; \langle p_1' \, \alpha_2^\sigma | a^\dagger_{\tilde{\sigma}'}(\tilde{k}_2') a^\dagger(\tilde{p}') \, a_{\tilde{\sigma}}(\tilde{k}_2) a(\tilde{p}) | p_1 \, \alpha_2^\sigma \rangle \\ &= \delta^3(\tilde{r}_1 - p_1) \, \delta^3(\tilde{r}_1' - p_1') \, \delta_{\tilde{\sigma},\sigma} \delta_{\tilde{\sigma}',\sigma} \alpha_2(\tilde{k}_2) \alpha_2^*(\tilde{k}_2') \, \langle \tilde{r}_1' \, \tilde{k}_2'^{\tilde{\sigma}'} | T | \tilde{r}_1 \, \tilde{k}_2^{\tilde{\sigma}} \rangle \, , \end{aligned} \tag{4.7}$$

where we neglected all the terms in the ellipsis of eq. (4.5). Notice that we encountered the matrix element $\langle \alpha_2^\sigma | \alpha_2^\sigma \rangle = 1$: this conveniently takes care of all the disconnected diagrams. The remaining matrix element introduces the desired scattering amplitude,

$$\langle \tilde{r}_1' \, \tilde{k}_2'^{\tilde{\sigma}'} | T | \tilde{r}_1 \, \tilde{k}_2^{\tilde{\sigma}} \rangle = \mathcal{A}_4(\tilde{r}_1, \tilde{k}_2^{\tilde{\sigma}} \to \tilde{r}_1', k_2'^{\tilde{\sigma}'}) \, \delta^4(\tilde{r}_1 + \tilde{k}_2 - \tilde{r}_1' - \tilde{k}_2') \, . \tag{4.8}$$

As usual, the superscripts on the messenger momenta denote the corresponding physical helicity. To write it in the usual amplitudes convention, $\mathcal{A}(0 \to p_1, p_2, \ldots)$, we must cross the momenta to the other side. This flips the helicity of incoming messengers.

Using the results of eq. (4.7) and eq. (4.8) in eq. (4.3) and carrying out the sums over $\tilde{\sigma}, \tilde{\sigma}'$, we obtain,

$$\begin{aligned} I^\mu_{w(1)} = \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_1')\mathrm{d}\Phi(k_2)\mathrm{d}\Phi(k_2') \; &\psi_A(p_1)\psi_A^*(p_1')\alpha_2(k_2)\alpha_2^*(k_2') \\ &\times e^{-ib\cdot(p_1'-p_1)/\hbar} \, i(p_1' - p_1)^\mu \\ &\times \mathcal{A}_4(p_1, k_2^\sigma \to p_1', k_2'^\sigma) \, \delta^4(p_1 + k_2 - p_1' - k_2') \, , \end{aligned} \tag{4.9}$$

where we have dropped the tildes on k_2 and k_2'.

If we make the usual change of variables to the momentum mismatches $q_{1,2}$,

$$\begin{aligned} q_1 &= p_1' - p_1\,, \\ q_2 &= k_2' - k_2\,; \end{aligned} \tag{4.10}$$

use the delta function to integrate over q_2; and drop the subscript on q_1, we find,

$$\begin{aligned} I^{\mu}_{w(1)} = \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(k_2)\hat{\mathrm{d}}^4 q\;\delta(2q\cdot p_1+q^2)\delta(2q\cdot k_2-q^2)\Theta(p_1^0+q^0)\Theta(k_2^0-q^0) \\ \times\,\psi_A(p_1)\psi_A^*(p_1+q)\,\alpha_2^*(k_2-q)\alpha_2(k_2) \\ \times\, e^{-ib\cdot q/\hbar}\,iq^{\mu}\,\mathcal{A}_4(p_1,k_2^{\sigma}\to p_1+q,(k_2-q)^{\sigma})\,. \end{aligned} \tag{4.11}$$

The analysis of the classical limit as far as the $\psi_A(p_1)\psi_A^*(p_1+q)$ factor is concerned is the same as in [166]. It requires us to take the wavenumber mismatch as our integration variable in lieu of the momentum mismatch. At leading order, we do not have to worry about terms singular in $\hbar$, so the evaluation as far as the massive particle is concerned will take,

$$\delta(2q\cdot p_1+q^2)\to\hbar^{-1}\delta(2\bar{q}\cdot p_1)\,, \tag{4.12}$$

$$\psi_A(p_1+q)\to\psi_A(p_1)\,. \tag{4.13}$$

Using the definition of fragments we gave in eq. (2.20) we find for the classical limit,

$$\begin{aligned} I^{\mu,\mathrm{cl}}_{w(1)} = g^2\left\langle\!\!\left\langle \int \mathrm{d}\Phi(\bar{k}_2)\hat{\mathrm{d}}^4\bar{q}\;\delta(2\bar{q}\cdot p_1)\delta(2\bar{q}\cdot\bar{k}_2-\bar{q}^2)\,\Theta(\bar{k}_2^0-\bar{q}^0)\,\bar{\alpha}_2^*(\bar{k}_2-\bar{q})\bar{\alpha}_2(\bar{k}_2)\right.\right. \\ \left.\left.\times\; e^{-ib\cdot\bar{q}}\,i\bar{q}^{\mu}\,\mathcal{A}^{(0)}_{4,0}(p_1,\hbar\bar{k}_2^{\sigma}\to p_1+\hbar\bar{q},\hbar(\bar{k}_2-\bar{q})^{\sigma})\right\rangle\!\!\right\rangle\,. \end{aligned} \tag{4.14}$$

As in [166], the double-angle brackets indicate an average over the wave function of the point-like particle. Classically, this is a function of the momentum p_1 with a very sharp peak at $p_1=m_Av_A$ where v_A is the classical (proper) velocity and m_A is the particle's mass.

We can now apply this general result in a variety of specific cases. We shall describe two examples in detail: Thomson scattering of a charge by a wave, with $b\simeq 0$, and gravitational scattering of light by a mass in the geometric-optics limit.

4.1.1 Impulse in Thomson scattering

Our first application is to Thomson scattering, of a particle of charge Qe and mass m, by a collimated beam of light. We take the light beam to have positive helicity, corresponding to the coherent state $|\alpha^+\rangle$. We need the four-point tree Compton amplitude in scalar QED,

$$\begin{aligned} \mathcal{A}^{(0)}_4(p_1,k_2^{\sigma}\to p_1',k_2'^{\sigma'}) &= 2Q^2\,\varepsilon^*_{\sigma}(k_2)\cdot\varepsilon_{\sigma'}(k_2') \\ &= 2Q^2\,\varepsilon_{-\sigma}(k_2)\cdot\varepsilon_{\sigma'}(k_2')\,, \end{aligned} \tag{4.15}$$

where we have chosen the gauge,

$$\varepsilon^{\sigma}(k)\cdot p_1 = 0\,, \tag{4.16}$$

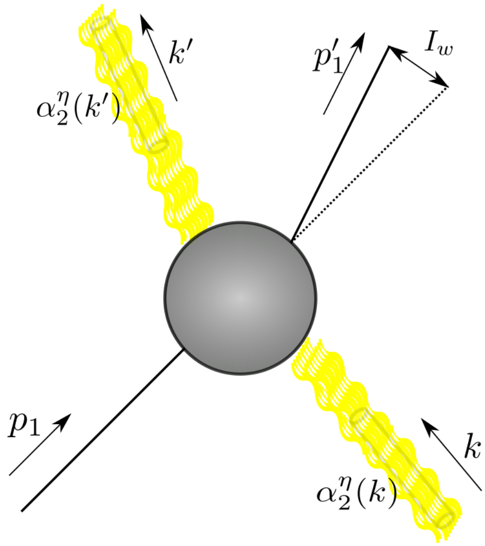


FIGURE 4.1: Impulse in scattering of a massive object off a coherent state background.

for both photons. Alternatively, in spinor variables, we have a gauge-invariant expression for the helicity amplitude, namely

$$\mathcal{A}_{4,0}^{(0)}(p_1, k_2^+ \to p_1', k_2'^+) = -\frac{Q^2}{2}\frac{\langle k_2|p_1|k_2']^2}{k_2\cdot p_1\, k_2'\cdot p_1}\,. \tag{4.17}$$

This form of the amplitude is manifestly gauge independent, but it depends explicitly on spinors $|k_2'\rangle$ and $|k_2]$ associated with photon momenta. As usual, in the classical limit we prefer to work with photon wavenumbers. We therefore introduce rescaled spinors,

$$\begin{aligned} |\bar{k}_2'\rangle &\equiv \hbar^{-1/2}\,|k_2'\rangle\,, \\ |\bar{k}_2] &\equiv \hbar^{-1/2}\,|k_2]\,, \end{aligned} \tag{4.18}$$

which are directly associated with the photon wavenumbers. The amplitude then has the expression,

$$\mathcal{A}_{4,0}^{(0)}(p_1, k_2^+ \to p_1', k_2'^+) = -\frac{Q^2}{2}\frac{\langle \bar{k}_2|p_1|\bar{k}_2']^2}{\bar{k}_2\cdot p_1\, \bar{k}_2'\cdot p_1}\,. \tag{4.19}$$

Choosing $b=0$, and for a more symmetric presentation, writing $k=k_2$ and $k'=k_2-q$, the impulse in eq. (4.14) takes the form,

$$\langle \Delta p^\mu\rangle = \frac{Q^2e^2}{2}\int \mathrm{d}\Phi(\bar{k})\mathrm{d}\Phi(\bar{k}')\,\delta(2p\cdot(\bar{k}-\bar{k}'))\,\bar{\alpha}^*(\bar{k}')\bar{\alpha}(\bar{k})\,i(\bar{k}'-\bar{k})^\mu\frac{\langle\bar{k}|p|\bar{k}']^2}{(\bar{k}\cdot p)^2}\,. \tag{4.20}$$

This expression may be compared with the classical electromagnetic result, obtained by iterating the classical Lorentz force twice. Thus we see in an explicit example that a vanishing impact parameter is perfectly acceptable in the classical scattering of waves off matter, in contrast to the situation for two massive particles scattering.

It is interesting that the Compton amplitude appears at tree level in the classical physics of wave scattering off massive particles. This amplitude is also relevant [246] for purely massive particle scattering, though at one loop order. While the amplitude is very simple for spinless particles, it is considerably more complicated [247] for particles with large spins. Currently we do not have a clear understanding of the appropriate Compton amplitude for the Kerr black hole, or of what principle we could use to determine it. This is an important area for further research. This work suggests one angle of attack: information about the classical part of the Compton amplitude could be extracted by a purely classical analysis of the impulse on a massive spinning object in scattering off a messenger wave. This is one topic under independent study in [181].

4.1.2 Light deflection in gravitational scattering

A second interesting application of the formulas derived in the previous section is to the gravitational deflection of light by a massive object. We may access this observable by computing the change in momentum of a narrow (small $\ell_\perp$) beam of light passing with non-zero impact parameter b past a massive point-like particle. At leading order, there is no radiation of momentum, so the change in momentum of the wave is simply the negative of the change in momentum of the massive point source: our starting point is once again eq. (4.14).

Before we discuss the details of the calculation, it is worth dwelling for a moment on our setup. Eddington's famous observations demonstrated that starlight is deflected by the sun in accordance with General Relativity. Near the sun, light emitted by a distant star is essentially a spherical wave, and so the incoming wave is extremely delocalized. In contrast, we have chosen to study a collimated, narrow beam of light. Nevertheless, the difference between our setup and Eddington's case is immaterial. We work in the situation where the wavelength $\lambda\!\!\!^{-}$ of the light is very small compared to the impact parameter: this is the domain of geometric optics, and also applies to Eddington's case. It is in the context of geometric optics that the bending is well-defined; the geometric bending does not depend on the details of the wave.

For our purposes the setup of a narrow beam in the far past is just a simpler place to start. The reason is that we can then determine the bending of light by computing the impulse on the beam: this impulse is directly the change in direction of the wave. By contrast the impulse on starlight due to the sun involves integrating over the whole incoming spherical wavefront: this is not related in a simple manner to the bending of light.

In the geometric-optics regime, we need the wavelength of the light $\lambda\!\!\!^{-}$ to be small. At the same time we must suppress all quantum effects, so we choose $\lambda\!\!\!^{-}$ to be large compared to the Compton wavelength ℓ_c of our point source. To keep our beam collimated, eq. (2.75) requires that $\ell_\perp \gg \lambda\!\!\!^{-}$. The requirement that our beam is narrow is $\ell_\perp \ll b$. Thus there is a series of inequalities:

$$\ell_c \ll \lambda\!\!\!^{-} \ll \ell_\perp \ll \ell_s \sim b\,. \tag{4.21}$$

Note that the scattering length ℓ_s is expected to be of order of the impact parameter in this case, as we are considering a t channel process. For simplicity, we consider a monochromatic beam with $\sigma_\parallel \to 0$. The final length scale to consider is the size ℓ_w of the point-particle's wave packet. As usual we require $\ell_c \ll \ell_w \ll \ell_s$. Once these conditions are met, there will be little overlap between the beam and the wave packet, so we do not anticipate that the values of the ratios $\lambda\!\!\!^{-}/\ell_w$ or $\ell_\perp/\ell_w$ will be important.

The impulse given in eq. (4.14) simplifies due to the constraints of eq. (4.21). Note that the quantity $|\bar{q}\cdot\bar{k}_2| \gg |\bar{q}^2|$ in the second delta function, as $\bar{k}_2 \sim 1/\lambda\!\!\!^{-}$ while $\bar{q} \sim 1/\ell_s$. The wavenumber $\bar{q}$ is then dominantly in the plane of scattering. In this plane, the coherent waveshape $\bar{\alpha}_2$ is of width $1/\ell_\perp$ so that we may approximate $\bar{\alpha}_2^*(\bar{k}_2 - \bar{q}) \simeq \bar{\alpha}_2^*(\bar{k}_2)$. For the same reason, the explicit theta function in the impulse simplifies: $\Theta(\bar{k}_2^0 - \bar{q}^0) = 1$. Taking into account the sign demanded by momentum balance, the impulse on the wave is,

$$\langle \Delta p_2^\mu \rangle = -g^2 \left\langle\!\!\left\langle \int \mathrm{d}\Phi(\bar{k}_2) \hat{\mathrm{d}}^4\bar{q}\; \delta(2\bar{q}\cdot p_1)\delta(2\bar{q}\cdot\bar{k}_2)\; |\bar{\alpha}_2(\bar{k}_2)|^2 \right.\right.$$
$$\left.\left. \times\; e^{-ib\cdot\bar{q}}\, i\bar{q}^\mu\, \mathcal{A}_{4,0}^{(0)}(p_1, \hbar\bar{k}_2^\sigma \to p_1 + \hbar\bar{q}, \hbar(\bar{k}_2 - \bar{q})^\sigma) \right\rangle\!\!\right\rangle . \tag{4.22}$$

The integral over $\bar{k}_2$ is now in a great many respects analogous to the integral over the massive particle wave function which is hidden in our double-angle brackets. In the geometric optics limit, $\bar{\alpha}_2(\bar{k}_2)$ is a steeply-peaked function of the wave number peaked at $\bar{k}_2 = \bar{k}_\odot$; in view of eq. (2.54), its normalization is related to the number of photons in the beam. The amplitude, meanwhile, is a smooth function in this region. The $\bar{k}_2$ integral then has the structure,

$$\int \mathrm{d}\Phi(\bar{k}_2)\, \delta(2\bar{q}\cdot\bar{k}_2)\, |\bar{\alpha}_2(\bar{k}_2)|^2\, f(\bar{k}_2) \;\simeq\; f(\bar{k}_\odot) \int \mathrm{d}\Phi(\bar{k}_2)\, \delta(2\bar{q}\cdot\bar{k}_2)\, |\bar{\alpha}_2(\bar{k}_2)|^2 \,, \tag{4.23}$$

where f is a slowly-varying function. We thus encounter the convolution of a delta function and the sharply-peaked $|\alpha_2(k)|^2$. The result of the convolution is a broadened delta function centered at $\bar{k}_2 = \bar{k}_\odot$. Neglecting the width (of order $\sigma_\perp$) of this function we have,

$$\int \mathrm{d}\Phi(\bar{k}_2)\, \delta(2\bar{q}\cdot\bar{k}_2)\, |\bar{\alpha}_2(\bar{k}_2)|^2\, f(\bar{k}_2) \;\simeq\; f(\bar{k}_\odot)\, N_\gamma \hbar\, \delta(2\bar{q}\cdot\bar{k}_\odot)\,. \tag{4.24}$$

Notice the appearance of the number of photons N_γ in the beam: this normalization constant emerges from the integral over $|\alpha_2(k)|^2$. The classical geometric optics approximation does not have access to this number of photons, and correspondingly it will cancel in our expression for the deflection angle below. Certain other physical quantities do involve this number of photons: for example, the total momentum of the beam is,

$$K_\odot^\mu = \int \mathrm{d}\Phi(\bar{k}) |\bar{\alpha}(\bar{k})|^2\, \bar{k}^\mu \simeq N_\gamma \hbar \bar{k}_\odot^\mu\,. \tag{4.25}$$

Returning to the impulse on the beam, use of eq. (4.24) leads to the expression,

$$\langle \Delta p_{\mathrm{geom}}^\mu \rangle = -N_\gamma \hbar\, g^2 \left\langle\!\!\left\langle \int \hat{\mathrm{d}}^4\bar{q}\; \delta(2\bar{q}\cdot p_1)\delta(2\bar{q}\cdot\bar{k}_\odot) \right.\right.$$
$$\left.\left. \times\; e^{-ib\cdot\bar{q}}\, i\bar{q}^\mu\, \mathcal{A}_{4,0}^{(0)}(p_1, \hbar\bar{k}_\odot^\sigma \to p_1 + \hbar\bar{q}, \hbar(\bar{k}_\odot - \bar{q})^\sigma) \right\rangle\!\!\right\rangle . \tag{4.26}$$

The subscript reminds us that the approximation is valid in the geometric-optics limit.

At leading order, we only need the four-point tree-level amplitude. As there are no contributions singular in $\hbar$ at this order, we can simply retain only the terms that survive in the classical limit:

$$\begin{aligned} \mathcal{A}_{4,0}^{(0)}(p_1, k_2^\sigma \to p_1', k_2'^\sigma) &= \frac{p_1\cdot k_2\, p_1\cdot k_2'}{q^2}\, \varepsilon_\sigma^*(k_2)\cdot\varepsilon^\sigma(k_2') + \cdots\,, \\ &= \frac{p_1\cdot \bar{k}_2\, p_1\cdot \bar{k}_2'}{\bar{q}^2}\, \varepsilon_\sigma^*(\bar{k}_2)\cdot\varepsilon^\sigma(\bar{k}_2') + \cdots\,, \end{aligned} \tag{4.27}$$

where we have chosen the gauge $p_1\cdot\varepsilon^\sigma(k) = 0$ for each polarization vector, and the ellipsis indicates terms which are suppressed by powers of $\hbar$.

This amplitude simplifies further in the geometric-optics limit. The inequalities in eq. (4.21) require in particular that the wave number $\bar{q} \sim 1/b \ll \bar{k}_2$. We may therefore replace the scalar product $p \cdot \bar{k}_2'$ with $p \cdot \bar{k}_2$ in eq. (4.27), up to terms which are neglected in the geometric-optics limit. At the same time, we may replace the polarization vector $\varepsilon^\sigma(\bar{k}_2')$ with $\varepsilon^\sigma(\bar{k}_2)$ to the same order of approximation. The amplitude is then simply,

$$\mathcal{A}^{(0)}_{4,0}(p_1, k_2^\sigma \to p_1', k_2'^\sigma) = -\frac{(p_1 \cdot \bar{k}_2)^2}{\bar{q}^2} + \cdots \tag{4.28}$$

We note that the geometric-optics limit of the amplitude for the scattering of a photon off a massive scalar is helicity-independent. Up to constant factors, it reduces to the amplitude between one massless and one massive scalar[1]. This is as expected from the equivalence principle: if the classical limit weren't universal, the impulse and hence the scattering angle would have helicity-dependent contributions.

In order to the evaluate the impulse, we insert the geometric-optics amplitude of eq. (4.28) into the expression in eq. (4.26)) for the impulse in the geometric-optics limit. We obtain,

$$\begin{aligned}\langle \Delta p^\mu_{\text{geom}} \rangle &= i\kappa^2\, N_\gamma \hbar\, (p_1 \cdot \bar{k}_\odot)^2 \int \hat{\mathrm{d}}^4\bar{q}\; \delta(2\bar{q}\cdot p_1)\delta(2\bar{q}\cdot \bar{k}_\odot)\; e^{-ib\cdot\bar{q}} \frac{\bar{q}^\mu}{\bar{q}^2} \\ &= i\kappa^2\, (p_1 \cdot K_\odot)^2 \int \hat{\mathrm{d}}^4\bar{q}\; \delta(2\bar{q}\cdot p_1)\delta(2\bar{q}\cdot K_\odot)\; e^{-ib\cdot\bar{q}} \frac{\bar{q}^\mu}{\bar{q}^2}\,.\end{aligned} \tag{4.29}$$

Here, we have replaced the general coupling g by the appropriate gravitational coupling κ, and the wavenumber $\bar{k}_\odot$ by the total beam momentum $K_\odot$. The second line of the last equation is strikingly similar to the impulse in a scattering process between two *massive* classical objects. Indeed, the integral remaining in eq. (4.29) is essentially the same as the integral appearing in the leading order impulse in [166]. It can easily be performed by taking the light beam in the z direction, $K_\odot^\mu = (E, 0, 0, E)$. The result is,

$$\langle \Delta p^\mu_{\text{geom}} \rangle = -\kappa^2\, \frac{p_1 \cdot K_\odot}{8\pi\, b^2}\, b^\mu\,. \tag{4.30}$$

The impact parameter b^μ is directed from the massive particle towards the wave, so the sign above indicates that the interaction is attractive.

The scattering angle θ is then determined geometrically in terms of the impulse: at leading order we have

$$\sin\theta = \frac{|b \cdot \Delta p|}{|\boldsymbol{b}|\, E}\,, \tag{4.31}$$

once we have fixed a frame. We have taken the absolute value to drop the sign of the angle, understanding that the bending is towards the scatterer. Working in the rest frame of the massive scalar, and using $\kappa^2 = 32\pi G$, we reproduce the well-known value for the gravitational bending of light,

$$\theta = \frac{4Gm_A}{|\boldsymbol{b}|} + \cdots\,. \tag{4.32}$$

As a final comment, it is satisfying that the impulse we have obtained in eq. (4.29) is essentially the same as the impulse on massive point particles as discussed in [166]. This occurred as the inequalities in eq. (4.21) greatly simplified the impulse. These inequalities themselves are very similar to the Goldilocks conditions in eq. (2.16) for classical point-like particles. The fact that the dynamics of massive particles is

[1] See the beautiful and pedagogical discussion in [248] for more details.

so similar to the behavior of waves in the geometric-optics regime was a celebrated aspect of nineteenth and early twentieth century physics, known as the Hamiltonian analogy. This analogy was highlighted by Schrödinger [249] and others as an important consideration in the early days of quantum mechanics.

4.2 Zero-variance principle for classical observables

In classical electrodynamics, a key role is played by the field strength $F_{\mu\nu}(x)$. This object is a complete gauge-invariant characterisation of the field; once it is known, quantities such as the energy-momentum radiated to infinity and the field angular momentum are easily determined. In a quantum description, the field becomes an operator $\mathbb{F}_{\mu\nu}(x)$. In a semiclassical situation, the expectation value of this operator on a state $|\psi_{\text{in}}\rangle$ should equal the classical field, up to negligible quantum corrections:

$$\langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)|\psi_{\text{in}}\rangle = F_{\mu\nu}(x) + \mathcal{O}(\hbar)\,. \tag{4.33}$$

Note that we have schematically indicated the presence of small, order $\hbar$, quantum corrections. More precisely, these corrections must be suppressed by dimensionless ratios involving Planck's constant; the precise ratios depend on the actual physical context.

Since in the quantum theory a single-valued field is replaced by the expectation value of an operator, we must address the quintessentially quantum mechanical issue of uncertainty. The uncertainty can be characterised by the variance

$$\langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\xi}(y)|\psi_{\text{in}}\rangle - \langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|\mathbb{F}_{\rho\xi}(y)|\psi_{\text{in}}\rangle\,. \tag{4.34}$$

In the domain of validity of the classical approximation, this variance must be negligible.

Precisely the same remarks hold in a quantum mechanical approach to GR. The curvature tensor $R_{\mu\nu\rho\xi}(x)$ in the classical theory is replaced by the expectation value of the curvature operator $\mathbb{R}_{\mu\nu\rho\xi}(x)$. The variance

$$\langle\psi_{\text{in}}|\mathbb{R}_{\mu\nu\rho\xi}(x)\mathbb{R}_{\alpha\beta\gamma\delta}(y)|\psi_{\text{in}}\rangle - \langle\psi_{\text{in}}|\mathbb{R}_{\mu\nu\rho\xi}(x)|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|\mathbb{R}_{\alpha\beta\gamma\delta}(y)|\psi_{\text{in}}\rangle \tag{4.35}$$

must be negligible.

This section is devoted to an investigation of this condition of negligible uncertainty. Working in the KMOC formalism at lowest order in perturbation theory, we will see that the expectations $\langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\xi}(y)|\psi_{\text{in}}\rangle$ and $\langle\psi_{\text{in}}|\mathbb{R}_{\mu\nu\rho\xi}(x)\mathbb{R}_{\alpha\beta\gamma\delta}(y)|\psi_{\text{in}}\rangle$ are determined by tree-level *six*-point amplitudes while $\langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)|\psi_{\text{in}}\rangle$ and $\langle\psi_{\text{in}}|\mathbb{R}_{\mu\nu\rho\xi}(x)|\psi_{\text{in}}\rangle$ are determined by *five*-point tree amplitudes. We must then face the question of how it can be that the variance is negligible.

4.2.1 Field strength expectations

We begin by reviewing the evaluation of single field-strength observables in KMOC. We will discuss the electromagnetic case in some detail. The gravitational case is completely analogous to the electromagnetic case, so we only quote key results.

Let us now consider the leading-order field strength in our situation. In the far past, we have

$$\begin{aligned}\langle\psi_{\rm in}|\mathbb{F}_{\mu\nu}(x)|\psi_{\rm in}\rangle &= \frac{1}{\sqrt{\hbar}}\sum_{\sigma=\pm1}\int \mathrm{d}\Phi(k)\left[\langle\psi_{\rm in}|-ia_\sigma(k)|\psi_{\rm in}\rangle\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\,e^{-i\bar{k}\cdot x}+\text{h.c.}\right]\\ &=0\,.\end{aligned} \tag{4.36}$$

The expectation value vanishes because there are no photons in the initial state: $a_\sigma(k)|\psi_{\rm in}\rangle = 0$. Classically, the interpretation is that the initial state contains no incoming radiation. Notice that the expectation value is not sensitive to the Coulomb fields of the incoming particles; instead, we are computing the asymptotic value of the field at infinity, namely the coefficient of the 1/distance piece of the field strength.

In the far future, the expectation value is

$$\begin{aligned}&\langle\psi_{\rm in}|S^\dagger\mathbb{F}_{\mu\nu}(x)S|\psi_{\rm in}\rangle\\ &\qquad=\frac{1}{\sqrt{\hbar}}\sum_{\sigma=\pm1}\int \mathrm{d}\Phi(k)\left[-i\langle\psi_{\rm in}|S^\dagger a_\sigma(k)S|\psi_{\rm in}\rangle\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\,e^{-i\bar{k}\cdot x}+\text{h.c.}\right].\end{aligned} \tag{4.37}$$

This no longer vanishes. We may evaluate it at lowest perturbative order by writing the S matrix in terms of the transition matrix T as $S=1+iT$. The matrix elements of T on momentum eigenstates are the scattering amplitudes, which we may organise (as usual in perturbation theory) in terms of the coupling e or κ depending on whether we are interested in electrodynamics or gravity. Generically we will denote the perturbative coupling as g. Thus we have

$$\begin{aligned}\langle\psi_{\rm in}|S^\dagger a_\sigma(k)S|\psi_{\rm in}\rangle &= i\langle\psi_{\rm in}|(a_\sigma(k)T-T^\dagger a_\sigma(k))|\psi_{\rm in}\rangle+\langle\psi_{\rm in}|T^\dagger a_\sigma(k)T|\psi_{\rm in}\rangle\\ &= i\langle\psi_{\rm in}|a_\sigma(k)T|\psi_{\rm in}\rangle+\langle\psi_{\rm in}|T^\dagger a_\sigma(k)T|\psi_{\rm in}\rangle\\ &\simeq i\langle\psi_{\rm in}|a_\sigma(k)T|\psi_{\rm in}\rangle\,.\end{aligned} \tag{4.38}$$

In the middle line above, we used the fact that $a_\sigma(k)|\psi_{\rm in}\rangle=0$; in the last line we neglected the term involving two T matrices which does not contribute at lowest order by counting powers of g.

Further expanding the state using eq. (2.19), and taking advantage of the shorthand notation of eq. (2.17) and eq. (2.18), we may write

$$\begin{aligned}\langle\psi_{\rm in}|S^\dagger\mathbb{F}_{\mu\nu}(x)S|\psi_{\rm in}\rangle = 2\Re\frac{1}{\sqrt{\hbar}}\sum_{\sigma=\pm1}\int &\mathrm{d}\Phi(p_1',p_2',p_1,p_2,k)\psi_b^*(p_1',p_2')\psi_b(p_1,p_2)\\ &\times\langle k^\sigma\,p_1'\,p_2'|T|p_1\,p_2\rangle\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\,e^{-i\bar{k}\cdot x}\,.\end{aligned} \tag{4.39}$$

The matrix element $\langle k^\sigma\,p_1'\,p_2'|T|p_1\,p_2\rangle$ is, at lowest order, a five-point tree amplitude so it is proportional to g^3. This is consistent with a classical analysis of the outgoing radiation field.

In GR, the equivalent expression is

$$\begin{aligned}\langle\psi_{\rm in}|S^\dagger\mathbb{R}_{\mu\nu\rho\xi}(x)S|\psi_{\rm in}\rangle = 2\Re\frac{-i}{\sqrt{\hbar}}\frac{\kappa}{2}\sum_{\sigma=\pm2}\int &\mathrm{d}\Phi(p_1',p_2',p_1,p_2,k)\psi_b^*(p_1',p_2')\psi_b(p_1,p_2)\\ &\times\langle k^\sigma,p_1',p_2'|T|p_1,p_2\rangle\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\,\bar{k}_{[\xi}\varepsilon^{*\sigma}_{\rho]}(k)\,e^{-i\bar{k}\cdot x}\,.\end{aligned} \tag{4.40}$$

It will be useful for us to simplify these expressions further, following again the

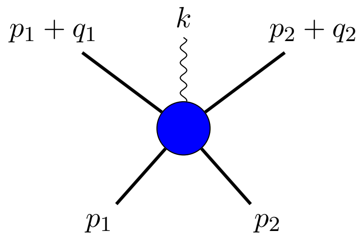


FIGURE 4.2: The kinematic configuration we choose for the five-point amplitude which determines the leading-order radiation field.

discussion of reference [166] at leading order. The matrix element $\langle k^\sigma p_1' p_2' | T | p_1 p_2 \rangle$ is the amplitude times a momentum-conserving delta function; our expectation value instructs us to integrate over all momenta in the amplitude. We may relabel these external momenta as shown in Fig. 4.2. The measure can then be written as

$$\mathrm{d}\Phi(p_1', p_2', p_1, p_2, k) = \mathrm{d}\Phi(p_1, p_2, k) \hat{\mathrm{d}}^4 q_1 \hat{\mathrm{d}}^4 q_2 \hat{\delta}(2p_1 \cdot q_1 + q_1^2) \hat{\delta}(2p_2 \cdot q_2 + q_2^2)\,. \tag{4.41}$$

In this form, the overall momentum-conserving delta function reads $\hat{\delta}^4(q_1 + q_2 + k)$. Now, in the classical regime the photon momentum is of order $\hbar$, as are the momentum mismatches q_1 and q_2. The q_i^2 terms in the delta functions above are therefore small shifts compared to the width (or order $1/\ell_w$) of the wavefunctions in the expectation values. As the observable has the structure of a convolution of the sharply-peaked wavefunctions multiplied by delta functions and otherwise smooth functions, we may neglect the q_i^2 shifts[2]. Similarly, the wavefunctions themselves are

$$\psi_b^*(p_1 + q_1, p_2 + q_2) \psi_b(p_1, p_2) = \psi^*(p_1 + q_1, p_2 + q_2) \psi(p_1, p_2) e^{-iq_1 \cdot b/\hbar}\,. \tag{4.42}$$

We may neglect the q_i shifts in the wavefunctions $\psi_i^*(p_i + q_i)$ because the momenta q_i are negligible compared to the width of the wavefunctions. Thus the field strength becomes

$$\begin{aligned}
\langle \psi_{\text{in}} | S^\dagger \mathbb{F}_{\mu\nu}(x) S | \psi_{\text{in}} \rangle = 2\Re \frac{1}{\sqrt{\hbar}} \sum_{\sigma = \pm 1} \int & \mathrm{d}\Phi(p_1, p_2, k) |\psi(p_1, p_2)|^2 \, \hat{\mathrm{d}}^4 q_1 \hat{\mathrm{d}}^4 q_2 \\
& \times \hat{\delta}_{\sigma_w}(2p_1 \cdot q_1) \hat{\delta}_{\sigma_w}(2p_2 \cdot q_2) \, \bar{k}_{[\mu} \varepsilon^{*\sigma}_{\nu]}(k) \, e^{-i(\bar{k} \cdot x + \bar{q}_1 \cdot b)} \\
& \times \mathcal{A}_5^{(0)}(p_1, p_2 \to p_1 + q_1, p_2 + q_2, k^\sigma) \, \hat{\delta}^4(k + q_1 + q_2)\,.
\end{aligned} \tag{4.43}$$

The notation $\hat{\delta}_{\sigma_w}(x)$ indicates that the delta functions have been "broadened" so that their width is of order $\sigma_w \sim \hbar/\ell_w$. In our point-particle treatment, we neglect this scale in the rest of this work. The amplitude $\mathcal{A}_5^{(0)}(p_1, p_2 \to p_1 + q_1, p_2 + q_2, k^\sigma)$ is a five-point, tree amplitude.

The integral in eq. (4.43) still depends on the wavefunctions of the particles. But now the role of the wavefunctions is transparent: they are steeply-peaked functions of the momenta p_1 and p_2 which allows us to simply replace these variables of integration with the incoming classical momenta $m_A v_A$ and $m_B v_B$. We thus write the field

[2] Usually care must be taken with these delta functions at higher order. Here we work at leading order, so the situation is simple.

expectation value as

$$\langle\psi_{\text{in}}|S^\dagger\mathbb{F}_{\mu\nu}(x)S|\psi_{\text{in}}\rangle = 2\Re\hbar^{7/2}\sum_{\sigma=\pm1}\left\langle\!\!\left\langle\int \mathrm{d}\Phi(\bar{k})\,\hat{\mathrm{d}}^4\bar{q}_1\hat{\mathrm{d}}^4\bar{q}_2\,\hat{\delta}(2p_1\cdot\bar{q}_1)\hat{\delta}(2p_2\cdot\bar{q}_2)\right.\right.$$
$$\left.\left.\times\,\mathcal{A}_5^{(0)}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma)\,\hat{\delta}^4(\bar{k}+\bar{q}_1+\bar{q}_2)\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}\,e^{-i(\bar{k}\cdot x+\bar{q}_1\cdot b)}\right\rangle\!\!\right\rangle. \tag{4.44}$$

The double-angle brackets are shorthand notation that instructs us to evaluate the momenta at their classical values, and remind us to take care of q^2 shifts in the delta functions. Recalling that k, q_1 and q_2 are all of order $\hbar$, we have scaled out all the $\hbar$ dependence except that of the scattering amplitude itself. Similarly, in gravity, one finds

$$\langle\psi_{\text{in}}|S^\dagger\mathbb{R}_{\mu\nu\rho\xi}(x)S|\psi_{\text{in}}\rangle = -2\Re\hbar^{7/2}\frac{i\kappa}{2}\sum_{\sigma=\pm2}\left\langle\!\!\left\langle\int \mathrm{d}\Phi(\bar{k})\,\hat{\mathrm{d}}^4\bar{q}_1\hat{\mathrm{d}}^4\bar{q}_2\,\hat{\delta}(2p_1\cdot\bar{q}_1)\hat{\delta}(2p_2\cdot\bar{q}_2)\right.\right.$$
$$\left.\left.\times\,\mathcal{M}_5^{(0)}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma)\hat{\delta}^4(\bar{k}+\bar{q}_1+\bar{q}_2)\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\,\bar{k}_{[\xi}\varepsilon^{*\sigma}_{\rho]}(k)\,e^{-i(\bar{k}\cdot x+\bar{q}_1\cdot b)}\right\rangle\!\!\right\rangle. \tag{4.45}$$

For these expressions to make sense classically, it better be that the overall $\hbar$ dependence of the amplitudes cancels that of the observable. Indeed we may write

$$\begin{aligned}\mathcal{A}_5^{(0)}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma) &= \hbar^{-7/2}\mathcal{A}_{5,0}^{(0)}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma)+\mathcal{O}(\hbar)\\ \mathcal{M}_5^{(0)}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma) &= \hbar^{-7/2}\mathcal{M}_{5,0}^{(0)}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma)+\mathcal{O}(\hbar)\,,\end{aligned} \tag{4.46}$$

where the quantities $\mathcal{A}_{5,0}^{(0)}$ and $\mathcal{M}_{5,0}^{(0)}$ are independent of $\hbar$, as was noticed in [166]. We will return to this structure below.

The physical interpretation of these expectation values is that they compute the radiative part of the field at large distances. To see this explicitly, the $\bar{k}$ integral needs to be performed taking advantage of the large distance between the point of measurement x and the particles. The integration will be discussed in detail in section 4.3. The question of central interest to us in this section, however, is to compute the uncertainty in the field strength; to do so, we turn to computing the expectation of two field strengths.

4.2.2 Expectation of two field strengths

It will be quite straightforward for us to compute expectations of products of operators using precisely the methods of the previous subsection. In electrodynamics, we need to compute

$$\begin{aligned}&\langle\psi_{\text{in}}|S^\dagger\mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\delta}(y)S|\psi_{\text{in}}\rangle\\ &= -\frac{1}{\hbar}\sum_{\sigma,\sigma'=\pm1}\int \mathrm{d}\Phi(k',k)\langle\psi_{\text{in}}|S^\dagger\left[-ia_\sigma(k)\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\,e^{-i\bar{k}\cdot x}+\text{h.c.}\right]\\ &\qquad\times\left[-ia_{\sigma'}(k')\,\bar{k}'_{[\rho}\varepsilon^{*\sigma'}_{\delta]}(k')\,e^{-i\bar{k}'\cdot y}+\text{h.c.}\right]S|\psi_{\text{in}}\rangle\,.\end{aligned} \tag{4.47}$$

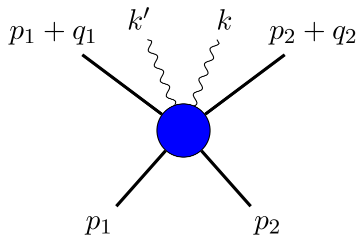


FIGURE 4.3: The kinematic configuration we choose for the six-point amplitude appearing at leading-order expectation of a pair of field strength operators.

Working at lowest order, and taking advantage of the fact that $a_\sigma(k)|\psi_{\text{in}}\rangle = 0$, the expectation simplifies to

$$\begin{aligned}&\langle\psi_{\text{in}}|S^\dagger \mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\delta}(y)S|\psi_{\text{in}}\rangle \\ &= -\frac{2}{\hbar}\Re\sum_{\sigma,\sigma'=\pm1}\int \mathrm{d}\Phi(k',k)\langle\psi_{\text{in}}|a_\sigma(k)a_{\sigma'}(k')iT|\psi_{\text{in}}\rangle \bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\bar{k}'_{[\rho}\varepsilon^{*\sigma'}_{\delta]}(k')\, e^{-i(\bar{k}\cdot x+\bar{k}'\cdot y)}\,,\end{aligned} \tag{4.48}$$

up to a purely quantum single-photon effect [4]. Expanding the wavefunctions, we encounter the matrix element $\langle p_1' p_2'|a_\sigma(k)a_{\sigma'}(k')T|p_1p_2\rangle$: a six-point tree amplitude. The classical limit is determined precisely as in the previous section with the result

$$\begin{aligned}\langle\psi_{\text{in}}|S^\dagger \mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\xi}(y)S|\psi_{\text{in}}\rangle = -2\hbar^5\Re\sum_{\sigma,\sigma'=\pm1}\Bigg\langle\!\!\Bigg\langle\int \mathrm{d}\Phi(\bar{k}',\bar{k})\,\hat{\mathrm{d}}^4\bar{q}_1\hat{\mathrm{d}}^4\bar{q}_2\,\hat{\delta}(2p_1\cdot\bar{q}_1)\hat{\delta}(2p_2\cdot\bar{q}_2) \\ \times\, i\mathcal{A}_6^{(0)}\,\hat{\delta}^4(\bar{k}+\bar{k}'+\bar{q}_1+\bar{q}_2)\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\bar{k}'_{[\rho}\varepsilon^{*\sigma'}_{\xi]}(k')\,e^{-i(\bar{k}\cdot x+\bar{k}'\cdot y+\bar{q}_1\cdot b)}\Bigg\rangle\!\!\Bigg\rangle\,.\end{aligned} \tag{4.49}$$

The amplitude is shown in Fig. 4.3.

Similarly, in gravity, we find

$$\begin{aligned}&\langle\psi_{\text{in}}|S^\dagger \mathbb{R}_{\mu\nu\rho\xi}(x)\mathbb{R}_{\alpha\beta\gamma\delta}(y)S|\psi_{\text{in}}\rangle \\ &= -2\hbar^5\Re\left(-i\frac{\kappa}{2}\right)^2\sum_{\sigma,\sigma'=\pm2}\Bigg\langle\!\!\Bigg\langle\int \mathrm{d}\Phi(\bar{k}',\bar{k})\,\hat{\mathrm{d}}^4\bar{q}_1\hat{\mathrm{d}}^4\bar{q}_2\,\hat{\delta}(2p_1\cdot\bar{q}_1)\hat{\delta}(2p_2\cdot\bar{q}_2)\,\mathcal{M}_6^{(0)} \\ &\times\,\hat{\delta}^4(\bar{k}+\bar{k}'+\bar{q}_1+\bar{q}_2)\,\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]}(k)\bar{k}_{[\rho}\varepsilon^{*\sigma}_{\xi]}(k)\,\bar{k}'_{[\alpha}\varepsilon^{*\sigma'}_{\beta]}(k')\bar{k}'_{[\gamma}\varepsilon^{*\sigma'}_{\delta]}(k')\,e^{-i(\bar{k}\cdot x+\bar{k}'\cdot y+\bar{q}_1\cdot b)}\Bigg\rangle\!\!\Bigg\rangle\,.\end{aligned} \tag{4.50}$$

In both cases, the expectation of two field strengths is given to leading order in g by a tree-level six-point amplitude.

4.2.3 Negligible variance?

We have now seen explicitly that the expectation of a single field strength is determined by a five-point amplitude, while the expectation of two field strengths is a six-point tree amplitude at lowest order in the coupling g. But for the uncertainty in the field strength to be negligible, we need the variance to be negligible. How can this happen?

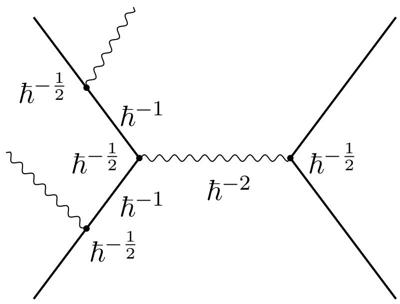


FIGURE 4.4: A sample Feynman diagram contributing to the six-point tree amplitude, indicating powers of $\hbar$ assigned by naive power counting to the propagators and vertices.

Let us count powers of the coupling in the electromagnetic variance. The product of two field-strength expectations is

$$\langle\psi_{\rm in}|\mathbb{F}_{\mu\nu}(x)|\psi_{\rm in}\rangle\langle\psi_{\rm in}|\mathbb{F}_{\rho\xi}(y)|\psi_{\rm in}\rangle \sim (\mathcal{A}_5^{(0)})^2 \sim (g^3)^2\,. \tag{4.51}$$

while the expectation of two field strengths is

$$\langle\psi_{\rm in}|\mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\delta}(y)|\psi_{\rm in}\rangle \sim \mathcal{A}_6^{(0)} \sim g^4\,. \tag{4.52}$$

Thus the situation seems to very bad: the variance is dominated by the expectation of two field-strength operators! For the classical limit to emerge as expected, we need the six-point tree amplitude to be suppressed somehow.

One possibility is that it is suppressed by powers of $\hbar$, but naively that is not the case. Consider, for example, the 6-point Feynman diagram shown in Fig. 4.4. We can count the powers of $\hbar$ associated to the diagram as follows. Each vertex contributes a factor $\hbar^{-\frac{1}{2}}$; this is because the dimensionless coupling is $e/\sqrt{\hbar}$ or $\kappa/\sqrt{\hbar}$. Each messenger propagator contributes a factor $\hbar^{-2}$ because messenger momenta are of order $\hbar$. Meanwhile each massive propagator contributes a factor $\hbar^{-1}$; this arises since the momenta flowing through these propagators are a sum of a massive on-shell momentum p and a messenger momentum k, so that the propagator denominator is $(p+k)^2-m^2 \simeq 2\hbar\, p\cdot\bar{k}$. We conclude that the six-point tree amplitude contains terms of order $\hbar^{-6}$. Referring back to eq. (4.49) or eq. (4.50) for the expectation of two field strengths, we see that the observables contribute a total of $\hbar^{+5}$. Based on this counting, the observable seems to scale as $\hbar^{-1}$, which would be a serious obstruction to the emergence of a classical limit. Evidently there is more to understand here: the powers of $\hbar$ don't make sense.

It is a familiar story that power counting Feynman diagrams can be misleading: upon combining diagrams to evaluate an amplitude, there can be cancellations. In fact this already happens in the case of the five-point tree; there, naive power counting suggests that the amplitude scales as $\hbar^{-9/2}$ but in fact the leading term in the amplitude is of order $\hbar^{-7/2}$ [95][3]. The question, then, of the fate of the six-point tree amplitude in the expectation value of two field strengths becomes a question of the overall $\hbar$ scaling of six-point tree amplitudes in QED and gravity. We will shortly demonstrate explicitly that the QED amplitude in fact scale as $\hbar^{-4}$; the gravitational

[3] In reference [95], the analysis of the five-point tree amplitude was performed in gravity using the large mass expansion, which is equivalent to expanding in small $\hbar$.

case will be discussed in section 6.2.5. Two powers of $\hbar$ cancel; consequently the contribution of the six-point tree to the variance is entirely at the quantum level.

It is amusing that at next-to-leading order in the perturbative coupling g, namely order g^6, the expectation value of two field strengths is sensitive to one-loop six-point amplitudes and to products of two tree five-point amplitudes[4]. These products of tree-level five-point amplitudes can be viewed as the cut of a six-point one-loop amplitude. It is easy to check that these scale as $\hbar^{-5}$, so in this sense they are *enhanced* relative to the six-point tree amplitude. This is as desired for negligible uncertainty:

$$\begin{aligned}&\langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|\mathbb{F}_{\rho\delta}(y)|\psi_{\text{in}}\rangle \sim (\mathcal{A}_5^{(0)})^2 \sim (g^3)^2\,;\\&\langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\xi}(y)|\psi_{\text{in}}\rangle \sim \mathcal{A}_6^{(1)} \sim (\mathcal{A}_5^{(0)})^2 \sim (g^3)^2\,.\end{aligned} \tag{4.53}$$

4.2.4 Mixed variances

Our discussion so far reveals that scattering amplitudes, viewed as Laurent series in $\hbar$, obey certain properties which permit the emergence of a classical limit through negligible uncertainty. This Laurent expansion can also be viewed as an expansion in small momentum transfers divided by the centre-of-mass energy $\sqrt{s}$. In fact, the emergence of the classical limit imposes an *infinite* set of these relationships, which we will call "transfer relations" on scattering amplitudes. In this subsection we will describe the origin of these relations, and explicitly demonstrate a non-trivial example at one loop and five points.

To see where these relationships are coming from, recall that the double field-strength expectation (4.49) depends on a six-point amplitude. We have seen that the dominant term is actually the six-point one-loop amplitude, occurring at next-to-leading order in the expansion in g. At this order an additional term contributes to the double field-strength expectation; this term is the product of two five-point amplitudes. Now, negligible uncertainty demands that the complete double field-strength expectation must be the product of two single field-strength expectations. At leading order in the coupling, and leading non-trivial order in $\hbar$, we conclude that there must exist a relationship between the leading-in-$\hbar$ six-point one-loop amplitude and the product of two five-point trees.

Further examples of relationships between amplitudes can be obtained by considering expectations of three (or more) field strengths, leading to relationships between seven- (or higher-) point loop amplitudes and products of three (or more) five-point amplitudes.

Yet more relationships occur by considering expectations of products of operators including field strengths and momenta. For example, consider the variance

$$V_{\mu\nu\rho} \equiv \langle\psi_{\text{in}}|S^\dagger\mathbb{F}_{\nu\rho}(x)S\mathbb{P}_\mu|\psi_{\text{in}}\rangle - \langle\psi_{\text{in}}|S^\dagger\mathbb{F}_{\nu\rho}(x)S|\psi\rangle_{\text{in}}\langle\psi_{\text{in}}|\mathbb{P}_\mu|\psi_{\text{in}}\rangle\,. \tag{4.54}$$

This is the variance in a measurement of the initial momentum and the future field strength; it must be negligible in the classical regime. In a quantum-first approach, however, this variance will not vanish. Indeed it need not be real:

$$V^*_{\mu\nu\rho} = \langle\psi_{\text{in}}|\mathbb{P}_\mu S^\dagger\mathbb{F}_{\nu\rho}(x)S|\psi_{\text{in}}\rangle - \langle\psi_{\text{in}}|S^\dagger\mathbb{F}_{\nu\rho}(x)S|\psi\rangle_{\text{in}}\langle\psi_{\text{in}}|\mathbb{P}_\mu|\psi_{\text{in}}\rangle \neq V_{\mu\nu\rho}. \tag{4.55}$$

[4] We learned from G. Veneziano that the use of a cut 6-point function to compute observables which are quadratic in the fields (such as number or energy densities in inclusive cross-sections) goes back to Mueller's generalized optical theorem from the seventies (see e.g. [250] and references therein).

We can derive an interesting constraint on the five-point one-loop amplitude by demanding that imaginary part of this variance vanishes in the classical approximation. We therefore define

$$\begin{aligned}\mathcal{O}_{\mu\nu\rho} &= i(V^*_{\mu\nu\rho} - V_{\mu\nu\rho}) \\ &= i\langle\psi_{\text{in}}|\mathbb{P}_\mu S^\dagger \mathbb{F}_{\nu\rho}(x) S - S^\dagger \mathbb{F}_{\nu\rho}(x) S \mathbb{P}_\mu|\psi_{\text{in}}\rangle\,.\end{aligned} \tag{4.56}$$

Expanding the states as usual, we easily find

$$\begin{aligned}\mathcal{O}_{\mu\nu\rho} = \int \mathrm{d}\Phi(p_1', p_2', p_1, p_2)\, \psi_b^*(p_1', p_2')\psi_b(p_1, p_2) i(p_{1\mu}' - p_{1\mu}) \\ \times \langle p_1' p_2'| i(\mathbb{F}_{\nu\rho}(x) T - T^\dagger \mathbb{F}_{\nu\rho}(x)) + T^\dagger \mathbb{F}_{\nu\rho}(x) T|p_1 p_2\rangle\,.\end{aligned} \tag{4.57}$$

The factor $i(p_{1\mu}' - p_{1\mu})$ is important here: working at leading perturbative order, this factor is of order $\hbar$. It is also worth noting that we may write the expectation of the field strength itself as

$$\begin{aligned}\langle\psi_{\text{in}}|\mathbb{F}_{\nu\rho}|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi(p_1', p_2', p_1, p_2)\, \psi_b^*(p_1', p_2')\psi_b(p_1, p_2) \\ \times \langle p_1' p_2'| i(\mathbb{F}_{\nu\rho}(x) T - T^\dagger \mathbb{F}_{\nu\rho}(x)) + T^\dagger \mathbb{F}_{\nu\rho}(x) T|p_1 p_2\rangle\,.\end{aligned} \tag{4.58}$$

Thus the $i(p_{1\mu}' - p_{1\mu}) \sim \hbar$ factor in the variance is the key distinction between the variance, which vanishes classically, and the field strength which of course should not vanish classically. As we have already seen that the field strength is related to five-point amplitudes, it is now clear that the condition of vanishing $\mathcal{O}_{\mu\nu\rho}$ will become a condition on five-point amplitudes.

It is useful to break the variance $\mathcal{O}_{\mu\nu\rho}$ up into two structures:

$$\begin{aligned}\mathcal{O}^{(1)}_{\mu\nu\rho} = \int \mathrm{d}\Phi(p_1', p_2', p_1, p_2)\, \psi_b^*(p_1', p_2')\psi_b(p_1, p_2) i(p_{1\mu}' - p_{1\mu}) \\ \times \langle p_1' p_2'| i(\mathbb{F}_{\nu\rho}(x) T - T^\dagger \mathbb{F}_{\nu\rho}(x))|p_1 p_2\rangle\,,\end{aligned} \tag{4.59}$$

and

$$\begin{aligned}\mathcal{O}^{(2)}_{\mu\nu\rho} = \int \mathrm{d}\Phi(p_1', p_2', p_1, p_2)\, \psi_b^*(p_1', p_2')\psi_b(p_1, p_2) i(p_{1\mu}' - p_{1\mu}) \\ \times \langle p_1' p_2'| T^\dagger \mathbb{F}_{\nu\rho}(x) T|p_1 p_2\rangle\,.\end{aligned} \tag{4.60}$$

Both of these objects are real, which is convenient in terms of keeping the expressions simple.

We may simplify these structures using the explicit expression for the field strength given in eq. (2.40). For $\mathcal{O}^{(1)}$ we find

$$\begin{aligned}\mathcal{O}^{(1)}_{\mu\nu\rho} = 2\Re \frac{1}{\sqrt{\hbar}} \sum_{\sigma=\pm 1} \int \mathrm{d}\Phi(p_1', p_2', p_1, p_2, k) \psi_b^*(p_1', p_2')\psi_b(p_1, p_2) \times \\ \times\, i(p_{1\mu}' - p_{1\mu}) \langle k^\sigma p_1' p_2'|T|p_1 p_2\rangle\, \bar{k}_{[\nu} \varepsilon^{*\sigma}_{\rho]}(k)\, e^{-i\bar{k}\cdot x}\,,\end{aligned} \tag{4.61}$$

which should be compared to eq. (4.39). Again we see that the crucial new ingredient is a factor $i(p'_{1\mu} - p_{1\mu})$. In the classical regime, we may write this term as

$$\mathcal{O}^{(1)}_{\mu\nu\rho} = 2\Re\hbar^{9/2} \sum_{\sigma=\pm1} \left\langle\!\!\left\langle \int \mathrm{d}\Phi(\bar{k})\, \hat{\mathrm{d}}^4\bar{q}_1 \hat{\mathrm{d}}^4\bar{q}_2\, \hat{\delta}(2p_1\cdot\bar{q}_1)\hat{\delta}(2p_2\cdot\bar{q}_2) \right.\right.$$
$$\left.\left. \times\, i\bar{q}_\mu \mathcal{A}_5(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma)\, \hat{\delta}^4(\bar{k}+\bar{q}_1+\bar{q}_2)\, \bar{k}_{[\nu}\varepsilon^{*\sigma}_{\rho]}(k)\, e^{-i(\bar{k}\cdot x+\bar{q}_1\cdot b)} \right\rangle\!\!\right\rangle . \tag{4.62}$$

Referring back once more to eq. (4.44), the additional $\hbar$ suppression is now manifest.

Following eq. (2.20), at five-points we may write

$$\begin{aligned} \mathcal{A}^{(0)}_5(i\to f) &= \hbar^{-7/2}\left(\mathcal{A}^{(0)}_{5,0}(i\to f) + \hbar\mathcal{A}^{(0)}_{5,1}(i\to f) + \cdots\right), \\ \mathcal{A}^{(1)}_5(i\to f) &= \hbar^{-9/2}\left(\mathcal{A}^{(1)}_{5,0}(i\to f) + \hbar\mathcal{A}^{(1)}_{5,1}(i\to f) + \cdots\right). \end{aligned} \tag{4.63}$$

where we have scaled out the dominant (inverse) power of $\hbar$. Now at classical order ($\hbar^0$) the tree level amplitude $\mathcal{A}^{(0)}_5$ does not appear in $\mathcal{O}^{(1)}_{\mu\nu\rho}$ on account of the explicit factor $\hbar^{9/2}$ in eq. (4.62). The leading in g, non-trivial, classical contribution arises from the fragment $\mathcal{A}^{(1)}_{5,0}$. We conclude then that

$$\mathcal{O}^{(1)}_{\mu\nu\rho} = 2\Re \sum_{\sigma=\pm1} \left\langle\!\!\left\langle \int \mathrm{d}\Phi(\bar{k})\, \hat{\mathrm{d}}^4\bar{q}_1 \hat{\mathrm{d}}^4\bar{q}_2\, \hat{\delta}(2p_1\cdot\bar{q}_1)\hat{\delta}(2p_2\cdot\bar{q}_2)\, \times \right.\right.$$
$$\left.\left. \times\, i\bar{q}_\mu \mathcal{A}^{(1)}_{5,0}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma)\, \hat{\delta}^4(\bar{k}+\bar{q}_1+\bar{q}_2)\, \bar{k}_{[\nu}\varepsilon^{*\sigma}_{\rho]}(k)\, e^{-i(\bar{k}\cdot x+\bar{q}_1\cdot b)} \right\rangle\!\!\right\rangle . \tag{4.64}$$

The relevant fragmentary amplitude is the leading-in-$\hbar$ five-point one-loop amplitude, sometimes known as the "superclassical" part of the one-loop amplitude. Of course in this context this fragment of the amplitude is contributing precisely at classical order.

Now the full $\mathcal{O}_{\mu\nu\rho}$ should vanish at classical order. Since $\mathcal{O}^{(1)}_{\mu\nu\rho} \neq 0$, it must be that the second structure $\mathcal{O}^{(2)}_{\mu\nu\rho}$ cancels the contribution of eq. (4.64). We find that

$$\mathcal{O}^{(2)}_{\mu\nu\rho} = 2\Re \sum_{\sigma=\pm1} \left\langle\!\!\left\langle \int \mathrm{d}\Phi(\bar{k})\, \hat{\mathrm{d}}^4\bar{q}_1 \hat{\mathrm{d}}^4\bar{q}_2 \hat{\mathrm{d}}^4\bar{w}_1 \hat{\mathrm{d}}^4\bar{w}_2\, \hat{\delta}(2p_1\cdot\bar{q}_1)\hat{\delta}(2p_2\cdot\bar{q}_2)\hat{\delta}(2p_1\cdot\bar{w}_1)\hat{\delta}(2p_2\cdot\bar{w}_2) \right.\right.$$
$$\times\, \bar{q}_\mu\, \hat{\delta}^4(\bar{k}+\bar{q}_1+\bar{q}_2)\hat{\delta}^4(\bar{q}_1+\bar{q}_2-\bar{w}_1-\bar{w}_2)\, \bar{k}_{[\nu}\varepsilon^{*\sigma}_{\rho]}(k)\, e^{-i(\bar{k}\cdot x+\bar{q}_1\cdot b)}$$
$$\left.\left. \times\, \mathcal{A}^{(0)}_{5,0}(p_1,p_2\to p_1+w_1,p_2+w_2,k^\sigma)\mathcal{A}^{(0)}_{4,0}(p_1+w_1,p_2+w_2\to p_1+q_1,p_2+q_2) \right\rangle\!\!\right\rangle . \tag{4.65}$$

Comparing eq. (4.64) and eq. (4.65), the condition for vanishing $\mathcal{O}$ is

$$\begin{aligned} &i\mathcal{A}^{(1)}_{5,0}(p_1p_2\to p_1+q_1,p_2+q_2,k^\sigma) \\ &\quad= -\int \hat{\mathrm{d}}^4\bar{w}_1\hat{\mathrm{d}}^4\bar{w}_2\, \hat{\delta}(2p_1\cdot\bar{w}_1)\hat{\delta}(2p_2\cdot\bar{w}_2)\hat{\delta}^4(\bar{q}_1+\bar{q}_2-\bar{w}_1-\bar{w}_2) \\ &\qquad \times\, \mathcal{A}^{(0)}_{5,0}(p_1,p_2\to p_1+w_1,p_2+w_2,k^\sigma)\mathcal{A}^{(0)}_{4,0}(p_1+w_1,p_2+w_2\to p_1+q_1,p_2+q_2). \end{aligned} \tag{4.66}$$

Thus the dominant part of the five-point one-loop amplitude is given by the tree five-point and tree four-point amplitudes; as we checked explicitly in appendix G.

Clearly this explicit example is one among an infinite set of relationships. Variances involving one field strength operator and two momenta will lead to relationships among two-loop five-point amplitudes and the product of one five-point tree and two four-point trees. We can continue, in principle, as far as we wish generating similar relations. These negligible uncertainty relations generalise the well-known relations between multiloop four-point amplitudes required for eikonal exponentiation. Indeed consideration of expectations such as

$$\langle\psi_{\rm in}|S^\dagger\mathbb{P}_{\mu_1}\mathbb{P}_{\mu_2}\cdots\mathbb{P}_{\mu_n}S|\psi_{\rm in}\rangle \simeq \langle\psi_{\rm in}|S^\dagger\mathbb{P}_{\mu_1}|\psi_{\rm in}\rangle\langle\psi_{\rm in}|S^\dagger\mathbb{P}_{\mu_2}|\psi_{\rm in}\rangle\cdots\langle\psi_{\rm in}|S^\dagger\mathbb{P}_{\mu_n}|\psi_{\rm in}\rangle\,, \tag{4.67}$$

shows that there must be a relationship between the $n-1$ loop four-point amplitude and the product of n tree amplitudes.

Thus we find a remarkable abundance of relationships between multiloop, multileg amplitudes, considered as Laurent series in $\hbar$, forced on us by the *absence* of uncertainty in the classical regime. These are a direct consequences of the radiative generalisation of the eikonal exponentiation we discussed in section 7.1.3.

As well as finding explicit relations between different fragmentary amplitudes, we can use similar ideas to determine the $\hbar$ scaling associated with fragments in the transfer expansion. The kinds of multiple cancellations of $\hbar$ powers we saw at six points must continue to occur at higher points. The reason again follows from considering expectations of products of more than two field-strength operators.

The arguments are based simply on counting powers of coupling and $\hbar$. We know that for the single expectation, at leading order, we have

$$\langle\mathbb{F}_{\mu\nu}\rangle \sim g^3. \tag{4.68}$$

This means that we must also have $\langle\mathbb{F}^n\rangle \sim (g^3)^n$. Now we perform the KMOC analysis of $\langle\mathbb{F}^n\rangle$. Following the steps of the calculation earlier in section 4.2.2 we find, schematically, that

$$\langle\mathbb{F}^n\rangle \sim \hbar^{3n/2+2}\int\mathcal{A}_{4+n}. \tag{4.69}$$

These relations allow us to deduce two things. Firstly the relevant fragment of the complete amplitude $\mathcal{A}_{4+n}$ must scale as $\hbar^{-3n/2-2}$. Secondly this fragment must have $3n$ powers of the coupling g — this corresponds to having $n-1$ loops. From this we can also infer the scaling of all other loop and tree amplitudes, in the classical limit, since each loop contributes an extra factor of $\hbar^{-1}$. In particular the tree scaling will be

$$\mathcal{A}_{4+n,0} \sim \hbar^{-n/2-3}. \tag{4.70}$$

This is consistent with the scaling we computed above for six points ($n=2$), and we have also checked explicitly at seven points. It is interesting to see how these two very simple power counting arguments have completely constrained the $\hbar$ scaling of all $2\to 2+n$ amplitudes.

4.3 Localized observables I: waveform and Newman-Penrose scalars

In the section 4.1, we built on [166] to analyze what we may call *global* observables, requiring an array of detectors covering the celestial sphere at infinity in order to measure the quantity. This is most manifest for the total radiated momentum, defined by eq. (3.33) of [166],

$$R^\mu \equiv \langle k^\mu\rangle = \langle\psi_{\rm in}|\,S^\dagger\mathbb{K}^\mu S\,|\psi_{\rm in}\rangle = \langle\psi_{\rm in}|\,T^\dagger\mathbb{K}^\mu T\,|\psi_{\rm in}\rangle\,. \tag{4.71}$$

Even in electromagnetic scattering, achieving 4π coverage would make this a challenging measurement. In the gravitational context, where we would be looking to detect emission from scattering of distant black holes, such a measurement would be hopelessly impractical. Instead, we turn to what we may call *local* observables, which can be measured with a localized detector, albeit still sitting somewhere on the celestial sphere, say at x. The paradigm for such a measurement is that of the waveform $W(t, \hat{\boldsymbol{n}}; x)$ of radiation emitted during a scattering event in direction $\hat{\boldsymbol{n}}$ from an event at the coordinate origin. (That is, we adopt the convention that $-\hat{\boldsymbol{n}}$ points back from the observer towards the scattering event.) We will focus on electromagnetic radiation here, but much of the formalism will carry over to the gravitational case. Let us keep in mind that we will be interested in several detectors, all nearby x, though with separations that are completely negligible compared to the distance from the origin.

Local observables have a general structure which, as we will see, is determined by some source (the scattering event) and the propagation of messengers over very large distances. In fact it is convenient to break up our discussion of these observables along these lines. Here we will discuss this overall structure in more detail, with a focus on the crucial aspect of propagation. In the following sections, we will then extract general expressions for local observables from quantum field theory, and connect to the Newman-Penrose formalism. Then we will examine global observables in cases where a classical wave scatters off a massive particle before turning to the physically important case where two massive particles scatter and radiate.

It will be easier to discuss and manipulate the Fourier transform of the waveform with respect to time. We will refer to this as the spectral waveform $f(\omega, \hat{\boldsymbol{n}}; x)$:

$$f(\omega, \hat{\boldsymbol{n}}; x) = \int_{-\infty}^{+\infty} \mathrm{d}t\, W(t, \hat{\boldsymbol{n}}; x)\, e^{i\omega t}\,. \tag{4.72}$$

Given a result for the spectral waveform, we can of course recover the time-dependent waveform via an inverse Fourier transform. Because we are interested in radiation produced by long-range forces, the idealized waveforms for the scattering processes we will consider stretch infinitely far back and forward in time. The idealization is implicit in the infinite limits for the integral in eq. (4.72). In an actual measurement, however, the waveform would be below the noise floor of the detector for all times before a 'signal start time' preceding the moment of closest approach, and likewise for all times after a 'signal end time' following that moment. We can then take the theoretical waveforms to be approximations to actual ones cut off at the start and end times. Label the interval between the two by Δt_s.

Let us imagine that the point of closest approach during the scattering event is at the coordinate origin, $(t, \mathbf{x}) = (0, \mathbf{0})$. When a massless wave scatters off a point particle, the wave may overlap the particle; we take a suitable event of maximum overlap as the origin. We can treat the scattering as occurring in a box of temporal length Δt_s, and of spatial size Δx_s. Radiation is emitted inside the box during the scattering event, and then spreads out. We will take an (idealized) measurement of the radiation in some direction $\hat{\boldsymbol{n}}$, at a much later time and at a point very far away in that direction. The details of the scattering — the particles' interaction and spins — will determine the radiation emitted inside the box. Modifying those details could radically change the emission. Those details, however, will have no effect on the propagation of the radiation out to the distant measuring apparatus. Only the spin of the radiated field can have any effect. We thus expect the form of the result to be a Green's function convoluted with a source. More precisely, given that we have only outgoing radiation, we expect a retarded Green's function G_{ret}. We can then expand the Green's function in the large-distance limit to obtain the connection between the observable and the emitted radiation inside the box.

The details of the scattering inside the box around $(0, \mathbf{0})$ define a current for our radiation. In a real-world context, we are interested in electromagnetic or gravitational radiation, but we can equally well treat the case of (massless) scalar radiation as well. The details of the scattering inside the box give rise to a wavenumber-space field-strength current, $\widetilde{J}_{\boldsymbol{\mu}}(\bar{k})$, where the notation $\boldsymbol{\mu}$ denotes a number of indices appropriate to the radiated messenger: none for a scalar, two for a photon, and four for a graviton,

$$\begin{aligned} \widetilde{J}(\bar{k}) :& \qquad \text{scalar}\,, \\ \widetilde{J}_{\mu\nu}(\bar{k}) :& \qquad \text{electromagnetism}\,, \\ \widetilde{J}_{\mu\nu\rho\sigma}(\bar{k}) :& \qquad \text{gravity}\,. \end{aligned} \tag{4.73}$$

In a slight abuse of language, we will refer to these quantities simply as currents. They will satisfy appropriate conservation conditions. We will later obtain an expression for such a current in terms of scattering amplitudes.

Given this current, the usual position-space current can of course be obtained by taking a Fourier transform,

$$J_{\boldsymbol{\mu}}(x) = \int \hat{\mathrm{d}}^4\bar{k}\; \widetilde{J}_{\boldsymbol{\mu}}(\bar{k})\, e^{-i\bar{k}\cdot x}\,. \tag{4.74}$$

Clearly we can also write $\widetilde{J}_{\boldsymbol{\mu}}(\bar{k})$ in terms of $J_{\boldsymbol{\mu}}(x)$ via an inverse transform,

$$\widetilde{J}_{\boldsymbol{\mu}}(\bar{k}) = \int \mathrm{d}^4 x\; J_{\boldsymbol{\mu}}(x)\, e^{i\bar{k}\cdot x}\,. \tag{4.75}$$

Both of these forms of the current will be helpful for us below.

We obtain an x-dependent radiation observable in the general form,

$$R_{\boldsymbol{\mu}}(x) = i\int \mathrm{d}\Phi(\bar{k})\; \left[\widetilde{J}_{\boldsymbol{\mu}}(\bar{k})\, e^{-i\bar{k}\cdot x} - \widetilde{J}^*_{\boldsymbol{\mu}}(\bar{k})\, e^{+i\bar{k}\cdot x}\right], \tag{4.76}$$

that is, as an integral of the source $\widetilde{J}_{\boldsymbol{\mu}}(\bar{k})$ over the on-shell massless phase space for the radiated messenger. Examples will include expectations of hermitian operators, such as the field-strength operator in electromagnetism, or the Riemann tensor in gravity.

The hermiticity properties of our radiation observables is manifest in eq. (4.76). But notice that the observables are defined as integrals over positive frequencies $\bar{k}^0 \geq 0$. Yet in writing the innocuous-seeming Fourier transform in eq. (4.74), we have assumed knowledge of the current for both positive *and* negative frequency. So we must fill a gap: what do we mean by the current for negative frequency? In fact, the reality condition provides the necessary information. Our currents are real in position space, and we may note that,

$$J_{\boldsymbol{\mu}}(x) = \int \hat{\mathrm{d}}^4\bar{k}\; \theta(\bar{k}^0)\left[\widetilde{J}_{\boldsymbol{\mu}}(\bar{k})\, e^{-i\bar{k}\cdot x} + \widetilde{J}_{\boldsymbol{\mu}}(-\bar{k}) e^{i\bar{k}\cdot x}\right]. \tag{4.77}$$

The reality condition then leads to the relation,

$$\widetilde{J}_{\boldsymbol{\mu}}(-\bar{k}) = \widetilde{J}^*_{\boldsymbol{\mu}}(\bar{k})\,. \tag{4.78}$$

We use this relation to define the current for negative frequency.

A key simplification arises because the source event, occurring in our box, is sourced in a comparatively localized region compared to the very large propagation distance of the outgoing radiation. To access this simplification, we follow a well-trodden path by rewriting our radiation observables as integrals over the spatial extent of the source. Thus, we express the observable of eq. (4.76) in terms of the spatial current $J_{\boldsymbol{\mu}}(x)$, yielding

$$R_{\boldsymbol{\mu}}(x) = i \int \mathrm{d}\Phi(\bar{k})\, \mathrm{d}^4 y\; J_{\boldsymbol{\mu}}(y) \big[e^{-i\bar{k}\cdot(x-y)} - e^{+i\bar{k}\cdot(x-y)} \big]\,. \tag{4.79}$$

Next, we interchange orders of integration. Judicious forethought reveals the combination of phase space integrals to be a difference of retarded and advanced Green's functions,

$$R_{\boldsymbol{\mu}}(x) = \int \mathrm{d}^4 y\; J_{\boldsymbol{\mu}}(y) \big[G_{\mathrm{ret}}(x-y) - G_{\mathrm{adv}}(x-y) \big]\,. \tag{4.80}$$

In the far future, where the observer measures the wavetrain emitted from the scattering event, G_{adv} will vanish. Put in an explicit form for G_{ret}, and switch back to the wavenumber-space current in order to make the complete dependence of the integrand on x and y manifest. The result is,

$$\begin{aligned} R_{\boldsymbol{\mu}}(x) &= \int \hat{\mathrm{d}}\omega \hat{d}^3\bar{\boldsymbol{k}}\, d^4 y\, \widetilde{J}_{\boldsymbol{\mu}}(\bar{k})\, e^{-i\bar{k}\cdot y} \frac{\delta(x^0 - y^0 - |\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\ &= \int \hat{\mathrm{d}}\omega \hat{d}^3\bar{\boldsymbol{k}}\, d^3\boldsymbol{y}\; \widetilde{J}_{\boldsymbol{\mu}}(\bar{k})\, \frac{e^{-i\omega x^0}\, e^{+i\omega|\boldsymbol{x}-\boldsymbol{y}|}\, e^{+i\bar{\boldsymbol{k}}\cdot\boldsymbol{y}}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,. \end{aligned} \tag{4.81}$$

Notice that the integral is now over *all* wavenumbers. We have split the four-dimensional momentum integration into integrals over spatial and frequency components for later convenience.

From the earlier discussion, we know that $J_{\boldsymbol{\mu}}(y)$ is concentrated around $y \simeq 0$, whereas x is far away ($x \gg y$). Accordingly we can expand the integrand there, using,

$$\begin{aligned} |\boldsymbol{x}-\boldsymbol{y}| &\sim \big[\boldsymbol{x}^2 - 2\boldsymbol{x}\cdot\boldsymbol{y}\big]^{1/2} \\ &\sim |\boldsymbol{x}|\Big(1 - \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{y}}{|\boldsymbol{x}|}\Big)\,. \end{aligned} \tag{4.82}$$

We must be careful in performing this expansion: while it is sufficient to retain the leading term in the denominator, we must retain formally subleading terms that contribute to nontrivial phases. Even in those exponents, we can of course still drop terms beyond the subleading, as they give rise to no nontrivial phases.

Substituting the expansion in eq. (4.82) into eq. (4.81), we obtain,

$$R_{\boldsymbol{\mu}}(x) = \int \hat{\mathrm{d}}\omega \hat{d}^3\bar{\boldsymbol{k}}\, d^3\boldsymbol{y}\; \widetilde{J}_{\boldsymbol{\mu}}(\bar{k})\, \frac{e^{-i\omega x^0}\, e^{+i\omega|\boldsymbol{x}|} e^{-i\omega\hat{\boldsymbol{n}}\cdot\boldsymbol{y}}\, e^{+i\bar{\boldsymbol{k}}\cdot\boldsymbol{y}}}{4\pi|\boldsymbol{x}|}\,; \tag{4.83}$$

performing in turn the $\boldsymbol{y}$ and $\boldsymbol{k}$ integrals, we finally obtain,

$$\begin{aligned} R_{\boldsymbol{\mu}}(x) &= \frac{(2\pi)^3}{4\pi|\boldsymbol{x}|} \int \hat{\mathrm{d}}\omega \hat{\mathrm{d}}^3\bar{\boldsymbol{k}}\; \widetilde{J}_{\boldsymbol{\mu}}(\bar{k})\, e^{-i\omega x^0}\, e^{+i\omega|\boldsymbol{x}|}\, \delta^3(\bar{\boldsymbol{k}} - \omega\hat{\boldsymbol{n}}) \\ &= \frac{1}{4\pi|\boldsymbol{x}|} \int \hat{\mathrm{d}}\omega\; \widetilde{J}_{\boldsymbol{\mu}}(\omega, \omega\hat{\boldsymbol{n}})\, e^{-i\omega(x^0 - |\boldsymbol{x}|)}\,. \end{aligned} \tag{4.84}$$

We can thus identify the waveform with the coefficient of the leading-power term $|\boldsymbol{x}|^{-1}$,

$$W_{\boldsymbol{\mu}}(t, \hat{\boldsymbol{n}}; x) = \frac{1}{4\pi} \int \hat{\mathrm{d}}\omega \; \widetilde{J}_{\boldsymbol{\mu}}(\omega, \omega\hat{\boldsymbol{n}})\, e^{-i\omega(x^0 - |\boldsymbol{x}|)} \quad . \tag{4.85}$$

In this equation, t represents the observer's clock time. We could take it to be x^0, or $x^0 - |\boldsymbol{x}|$, or some other convenient time. We must nonetheless retain the separate dependence on x^0 and $|\boldsymbol{x}|$, because these quantities will differ between the cluster of nearby observers in which we are interested. That is, the absolute phase of the waveform at any given observer's location is not measurable and is therefore irrelevant, but the relative phases between nearby observers are measurable.

Choosing $t = x^0 - |\boldsymbol{x}|$, the corresponding spectral waveform is then simply,

$$f_{\boldsymbol{\mu}}(\omega, \hat{\boldsymbol{n}}) = \frac{1}{4\pi} \widetilde{J}_{\boldsymbol{\mu}}(\omega, \omega\hat{\boldsymbol{n}})\,. \tag{4.86}$$

More precisely, eq. (4.86) is the waveform for positive frequencies. For negative frequencies, the waveform follows from eq. (4.78),

$$f_{\boldsymbol{\mu}}(\omega, \hat{\boldsymbol{n}}) = \frac{1}{4\pi} \widetilde{J}^*_{\boldsymbol{\mu}}(-\omega, -\omega\hat{\boldsymbol{n}})\,; \tag{4.87}$$

notice that $-\omega$ is now positive. In both cases, once we know the current $\widetilde{J}_{\boldsymbol{\mu}}(\bar{k})$, we can immediately write down the spectral waveform.

As we have seen, the waveform is directly related to the current $\widetilde{J}_{\boldsymbol{\mu}}(\bar{k})$ generated by the scattering event. We must choose a specific local radiation observable to determine this current using its definition, eq. (4.76). In this section we will study examples in both electrodynamics and gravity.

Let us begin with a simple case: the field-strength tensor in electrodynamics. We choose an observer at x, in the far future of the event, equipped to measure the expectation value of the electric and magnetic field at the point x. The observable is therefore,

$$F_{\mu\nu} = \langle \psi_{\text{in}} | S^\dagger \mathbb{F}_{\mu\nu}(x) S | \psi_{\text{in}} \rangle\,, \tag{4.88}$$

where (as usual) $|\psi_{\text{in}}\rangle$ is the incoming state in the far past. In principle, we can also use $|\psi_w\rangle$ as our incoming state to study the scattered radiation field in a Thomson scattering process, but in the following we will restrict ourselves to the two-body problem.

Inserting the expression for the field-strength tensor of eq. (2.40) into this expectation value, and converting to integrals over wavenumbers, we learn that,

$$\begin{aligned}\langle F^{\text{out}}_{\mu\nu}(x)\rangle = -i\hbar^{3/2} \sum_{\sigma=\pm1} \int \mathrm{d}\Phi(\bar{k}) \, \big[&\langle \psi_{\text{in}} | S^\dagger a_\sigma(k) S | \psi_{\text{in}} \rangle \, \bar{k}^{\mu}_{[\mu} \varepsilon^{*;\sigma}_{\nu]}(\bar{k})\, e^{-i\bar{k}\cdot x} \\ &- \langle \psi_{\text{in}} | S^\dagger a^\dagger_\sigma(k) S | \psi_{\text{in}} \rangle \, \bar{k}^{\mu}_{[\mu} \varepsilon^{\sigma}_{\nu]}(\bar{k})\, e^{+i\bar{k}\cdot x} \big]\,,\end{aligned} \tag{4.89}$$

where we have again dropped the 'in' subscript, leaving it implicit in the rest of our discussion.

We now see the virtue of our definition of the general class of radiation observables in eq. (4.76). Evidently the expectation value $\langle F^{\text{out}}_{\mu\nu}(x)\rangle$ is of precisely this form, and we can read off the current $\widetilde{J}_{\mu\nu}(\bar{k})$ as

$$\widetilde{J}_{\mu\nu}(\bar{k}) = -\hbar^{3/2} \sum_{\sigma=\pm 1} \langle \psi_{\text{in}} | S^\dagger a_\sigma(k) S | \psi_{\text{in}} \rangle \, \bar{k}_{[\mu} \varepsilon^{*\sigma}_{\nu]}(\bar{k}) \,. \tag{4.90}$$

The discussion of the previous section therefore applies, and we see from eq. (4.86) that the corresponding spectral waveform is,

$$f_{\mu\nu}(\omega, \hat{\boldsymbol{n}}) = -\frac{1}{4\pi} \hbar^{3/2} \sum_{\sigma=\pm 1} \langle \psi_{\text{in}} | S^\dagger a_\sigma(k) S | \psi_{\text{in}} \rangle \, \bar{k}^\mu_{[\mu} \varepsilon^{*\sigma}_{\nu]}(\bar{k}) \Big|_{\bar{k}=(\omega, \omega \hat{\boldsymbol{n}})} \,, \tag{4.91}$$

for positive frequency ($\omega > 0$). For negative frequency ($\omega < 0$) the waveform is,

$$f_{\mu\nu}(\omega, \hat{\boldsymbol{n}}) = -\frac{1}{4\pi} \hbar^{3/2} \sum_{\sigma=\pm 1} \langle \psi_{\text{in}} | S^\dagger a^\dagger_\sigma(k) S | \psi_{\text{in}} \rangle \, \bar{k}^\mu_{[\mu} \varepsilon^{\sigma}_{\nu]}(\bar{k}) \Big|_{\bar{k}=-(\omega, \omega \hat{\boldsymbol{n}})} \,. \tag{4.92}$$

This result holds to all orders in perturbation theory.

It is straightforward to extend this result to gravity. We work in Einstein gravity, and assume that the spacetime is asymptotically Minkowskian. In this case our observer at x is very far from the source of gravitational waves, and is equipped to measure the expectation value of the local spacetime curvature $\langle R^{\text{out}}_{\mu\nu\rho\sigma}(x) \rangle$. The corresponding spectral waveform is nothing but the double copy of eq. (4.91),

$$f_{\mu\nu\rho\xi}(\omega, \hat{\boldsymbol{n}}) = \frac{i\kappa}{4\pi} \hbar^{3/2} \sum_{\sigma=\pm 2} \langle \psi_{\text{in}} | S^\dagger a_\sigma(k) S | \psi_{\text{in}} \rangle \, \bar{k}^\mu_{[\mu} \varepsilon^{*,\sigma}_{\nu]}(\bar{k}) \, \bar{k}^\mu_{[\rho} \varepsilon^{*,\sigma}_{\xi]}(\bar{k}) \Big|_{\bar{k}=(\omega, \omega \hat{\boldsymbol{n}})} \,, \tag{4.93}$$

for $\omega > 0$. In this equation, the operator $a_\sigma(k)$ annihilates perturbative gravitational states. We have included a factor $\kappa/2$ so that the Riemann tensor has the conventional normalization. Noting that the metric perturbation falls off as inverse distance, it follows that non-linear terms in the Riemann tensor produce corrections which fall off faster than inverse distance. Consequently, we have neglected them. Notice that all possible traces of eq. (4.93) vanish, consistent with the fact that the Riemann tensor in vacuum equals the Weyl tensor. The waveform for negative frequency is,

$$f_{\mu\nu\rho\xi}(\omega, \hat{\boldsymbol{n}}) = -\frac{i\kappa}{4\pi} \hbar^{3/2} \sum_{\sigma=\pm 2} \langle \psi_{\text{in}} | S^\dagger a^\dagger_\sigma(k) S | \psi_{\text{in}} \rangle \, \bar{k}^\mu_{[\mu} \varepsilon^{\sigma}_{\nu]}(\bar{k}) \, \bar{k}^\mu_{[\rho} \varepsilon^{\sigma}_{\xi]}(\bar{k}) \Big|_{\bar{k}=-(\omega, \omega \hat{\boldsymbol{n}})} \,. \tag{4.94}$$

The Lorentz indices on these observables reflects the tensor structure of electrodynamics and gravity. In both cases, however, there are only two possible polarizations of the outgoing radiation. It is helpful to project the waveform onto one of these polarizations. Classically, a convenient way to do so is to use the Newman–Penrose (NP) [251] formalism, which is intimately connected to the spinor-helicity method of scattering amplitudes [106, 181, 252]. We can adopt the same idea in the present context. For us, a simple route to the NP formalism is to pick a complex basis of vectors which is aligned with our setup. We choose the vectors[5]

$$L^\mu = \bar{k}^\mu / \omega = (1, \hat{\boldsymbol{n}})^\mu, \quad N^\mu = \zeta^\mu, \quad M^\mu = \varepsilon_+{}^\mu, \quad M^{*\mu} = \varepsilon_-{}^\mu \,. \tag{4.95}$$

The null vector ζ^μ is simply a gauge choice, satisfying $\zeta \cdot \varepsilon_\pm(k) = 0$ and $L \cdot N = L \cdot \zeta = 1$. Furthermore note that $M \cdot M^* = -1$. The scaling of the NP vector L ensures that it

[5] We use capital letters to denote the elements of our NP basis rather than the more traditional lower case symbols in order to distinguish the vectors from loop momenta, masses, etc.

does not depend on frequency ω, and is dimensionless. Indeed the polarization vectors $\varepsilon_\pm(k)$ do not depend on the scaling of $\bar{k}$ so they are also independent of frequency. These vectors therefore make sense as a spacetime basis, not merely as a basis in Fourier space.

It is easy to check that the only non-zero components of $f_{\mu\nu}$ in the NP basis are $f_{\mu\nu}M^{*\mu}N^\nu$ and $f_{\mu\nu}M^\mu N^\nu$. These are the leading radiative NP scalar, traditionally [253] denoted Φ_2^0, and its conjugate. We can write these NP scalars as Fourier transforms:

$$\Phi_2^0(t,\hat{\boldsymbol{n}}) = \int \hat{\mathrm{d}}\omega\, e^{-i\omega t}\,\tilde{\Phi}_2^0(\omega,\hat{\boldsymbol{n}})\,. \tag{4.96}$$

Notice that we commuted the NP basis vectors through the frequency integration sign. This is permissible as the basis vectors are independent of frequency. For positive frequency ω, we find,

$$\tilde{\Phi}_2^0(\omega,\hat{\boldsymbol{n}}) = -\frac{\omega}{4\pi}\hbar^{3/2}\langle\psi_{\mathrm{in}}|S^\dagger a_-(k)S|\psi_{\mathrm{in}}\rangle\Big|_{\bar{k}=(\omega,\omega\hat{\boldsymbol{n}})}\,, \tag{4.97}$$

while for negative frequency, the corresponding expression reads,

$$\tilde{\Phi}_2^0(\omega,\hat{\boldsymbol{n}}) = +\frac{\omega}{4\pi}\hbar^{3/2}\langle\psi_{\mathrm{in}}|S^\dagger a_+^\dagger(k)S|\psi_{\mathrm{in}}\rangle\Big|_{\bar{k}=-(\omega,\omega\hat{\boldsymbol{n}})}\,. \tag{4.98}$$

Combining these results, we find that the time-domain NP scalar is,

$$\begin{aligned}\Phi_2^0(t,\hat{\boldsymbol{n}}) = -\frac{\hbar^{3/2}}{4\pi}\int \hat{\mathrm{d}}\omega\,\Theta(\omega)\,\omega\big[&e^{-i\omega t}\langle\psi_{\mathrm{in}}|S^\dagger a_-(k)S|\psi_{\mathrm{in}}\rangle \\ &+ e^{+i\omega t}\langle\psi_{\mathrm{in}}|S^\dagger a_+^\dagger(-k)S|\psi_{\mathrm{in}}\rangle\big]\Big|_{\bar{k}=(\omega,\omega\hat{\boldsymbol{n}})}\,.\end{aligned} \tag{4.99}$$

In gravity, the corresponding radiative NP scalar is defined by

$$\Psi_4(x) = -N_\mu M_\nu^* N_\rho M_\xi^*\langle W^{\mu\nu\rho\xi}(x)\rangle\,, \tag{4.100}$$

where $W^{\mu\nu\rho\xi}(x)$ is the Weyl tensor, equal to the Riemann tensor in our case. Expanded at large distances, the leading term in the NP scalar is Ψ_4^0:

$$\Psi_4(x) = \frac{1}{|\boldsymbol{x}|}\Psi_4^0 + \cdots\,. \tag{4.101}$$

This object is directly relevant to gravitational waveforms [24, 25, 254]. We find that the spectral version of the NP scalar is,

$$\tilde{\Psi}_4^0(\omega,\hat{\boldsymbol{n}}) = -i\frac{\kappa\,\omega^2}{8\pi}\hbar^{3/2}\langle\psi_{\mathrm{in}}|S^\dagger a_{--}(k)S|\psi_{\mathrm{in}}\rangle\Big|_{\bar{k}=(\omega,\omega\hat{\boldsymbol{n}})}\,, \tag{4.102}$$

for positive ω. Let us emphasize once again that these results hold to all orders of perturbation theory.

NP scalars are particularly well-suited for comparison with helicity amplitudes in quantum field theory. However, they may be slightly less familiar than the more elementary field strengths; field strengths also have the virtue of being hermitian quantities. Therefore, we will also study the expectation of the radiative field-strength tensor in perturbation theory. This entails rewriting the scattering matrix in terms of the transition matrix T, $S = 1 + iT$,

$$\begin{aligned}\langle F^{\text{out}}_{\mu\nu}(x)\rangle &= \langle\psi_{\text{in}}|(1 - iT^\dagger)\mathbb{F}_{\mu\nu}(x)(1 + iT)|\psi_{\text{in}}\rangle \\ &= \langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)|\psi_{\text{in}}\rangle + 2\Re i\langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)T|\psi_{\text{in}}\rangle + \langle\psi_{\text{in}}|T^\dagger\mathbb{F}_{\mu\nu}(x)T|\psi_{\text{in}}\rangle\,.\end{aligned} \tag{4.103}$$

As we will see in section 4.6, there can be disconnected contributions to such radiative observables which start at order $\mathcal{O}(g)$ as shown in [178]. Nevertheless, those are related to a gauge choice at null infinity (i.e. to the choice BMS frame) and they come from degenerate amplitude contributions which have support on the zero-energy kinematics for the messenger. In the following, we will avoid this subtlety and we will focus only on the connected contribution. The first term in eq. (4.103) is the expectation value of the field strength due to any incoming radiation which may be present in $|\psi_{\text{in}}\rangle$; the following term is linear in amplitudes, and thus of $\mathcal{O}(g^3)$ (or higher); the last term is quadratic in amplitudes (or equivalently, linear in a cut amplitude), and contains terms of $\mathcal{O}(g^5)$ and higher. Please see Fig. (4.5) for a pictorial representation of these different contributions.

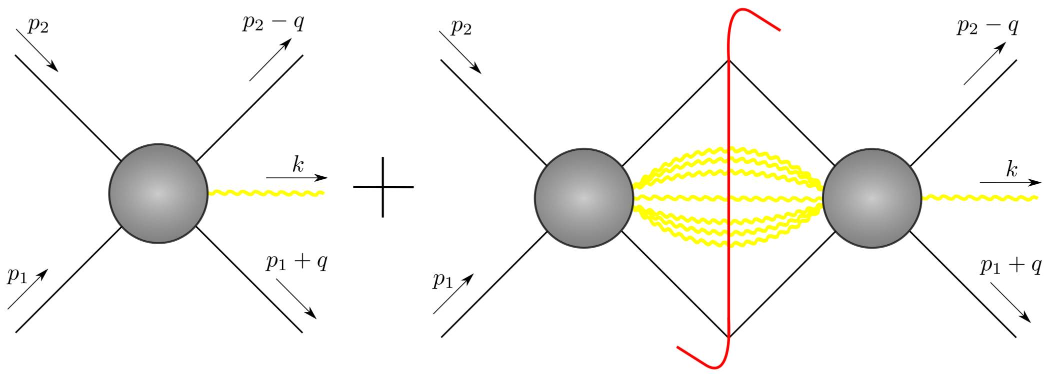


FIGURE 4.5: The waveform receives a linear and a quadratic contributions from scattering amplitudes, as a consequence of unitarity.

Using unitarity, we can rewrite eq. (4.103),

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle = \langle\psi_{\text{in}}|\mathbb{F}_{\mu\nu}(x)|\psi_{\text{in}}\rangle + i\langle\psi_{\text{in}}|[\mathbb{F}_{\mu\nu}(x), T]|\psi_{\text{in}}\rangle + \langle\psi_{\text{in}}|T^\dagger[\mathbb{F}_{\mu\nu}(x), T]|\psi_{\text{in}}\rangle\,. \tag{4.104}$$

The commutator in the second term of this expression is reminiscent of the form of the impulse Δp (although in case of the field strength, the first term above need not vanish). This second form of the field strength can be both instructive and useful, but it has a slight disadvantage that reality properties are somewhat obscured compared to eq. (4.103). When taking the classical limit, we are interested in the leading term in the large-distance expansion as well; for such radiation observables, we will understand the $\left\langle\!\!\left\langle \cdots \right\rangle\!\!\right\rangle$ notation to impose that expansion as well.

4.3.1 Emission waveform in scalar QED

We turn now to photon emission in the scattering of two charged point particles. At leading order in perturbation theory, only the second term in eq. (4.103) contributes. The connected contribution will be of order $\mathcal{O}(g^3)$, whereas the second term will be of $\mathcal{O}(g^5)$. If we now substitute eq. (2.40), along with eq. (2.9) for the initial-state wavefunction for the scattering particles into the first term of eq. (4.103), we obtain,

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_1 &= \frac{2}{\hbar^{3/2}}\Re\sum_{\sigma=\pm1}\int \mathrm{d}\Phi(p_1,p_2,p_1',p_2',k)e^{-ib\cdot(p_1'-p_1)/\hbar}\psi(p_1,p_2)\psi^*(p_1',p_2')\\ &\qquad\qquad\times k^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'|a_\sigma(k)\,T|p_1\,p_2\rangle\\ &= \frac{2}{\hbar^{3/2}}\Re\sum_{\sigma=\pm1}\int \mathrm{d}\Phi(p_1,p_2,p_1',p_2',k)e^{-ib\cdot(p_1'-p_1)/\hbar}\psi(p_1,p_2)\psi^*(p_1',p_2')\\ &\qquad\qquad\times k^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'\,k^\sigma|T|p_1\,p_2\rangle\,.\end{aligned}\tag{4.105}$$

We can identify the matrix element as a five-point amplitude,

$$\langle p_1'\,p_2'\,k^\sigma|T|p_1\,p_2\rangle = \mathcal{A}_5(p_1,p_2\to p_1',p_2',k^\sigma)\delta^4(p_1+p_2-p_1'-p_2'-k)\,.\tag{4.106}$$

At leading order, we replace the amplitude by its leading order contribution, given by a tree-level expression. To compute the required waveform, we must identify the expectation of $F^{\mu\nu}(x)$ as the spatial current $J_{\boldsymbol{\mu}}(x)$ in eq. (4.74)-(4.75), and via eq. (4.75), in eq. (4.85).

Beyond leading order, the expectation of $F^{\mu\nu}(x)$ will receive higher-order contributions to the amplitudes in eq. (4.106), alongside contributions from the last term in eq. (4.104),

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_2 = -\frac{i}{\hbar^{3/2}}&\sum_{\sigma=\pm1}\int \mathrm{d}\Phi(p_1,p_2,p_1',p_2',k)e^{-ib\cdot(p_1'-p_1)/\hbar}\psi(p_1,p_2)\psi^*(p_1',p_2')\\ &\times\Big[k^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'|T^\dagger a_\sigma(k)\,T|p_1\,p_2\rangle\\ &\qquad - k^{[\mu}\varepsilon^{(\sigma)\nu]}(k)e^{+ik\cdot x/\hbar}\langle p_1'\,p_2'|T^\dagger a_\sigma^\dagger(k)T|p_1\,p_2\rangle\Big]\end{aligned}\tag{4.107}$$

Insert a complete set of states to the right of each $T^\dagger$,

$$\langle\psi_{\text{in}}|T^\dagger\mathbb{F}^{\mu\nu}T|\psi_{\text{in}}\rangle = \sum_X\int \mathrm{d}\Phi(r_1)\mathrm{d}\Phi(r_2)\;\langle\psi_{\text{in}}|T^\dagger|r_1\,r_2\,X\rangle\langle r_1\,r_2\,X|\mathbb{F}^{\mu\nu}T|\psi_{\text{in}}\rangle\,,\tag{4.108}$$

where the sum over X is over all states, including no additional particles, and includes an implicit integral over momenta of any particles in X and a sum over any other quantum numbers. As in [166], we assume that each of the incoming massive particles carries a separately conserved global charge, so that each intermediate state has one net particle of each type. We can ignore additional particle-antiparticle pairs of the massive particles, as these contributions will disappear in the classical limit. As there are no messengers in the initial state, and hence no coherent states, there is no need to sum over arbitrary numbers of messengers. Accordingly, we do not need to switch to a coherent-friendly representation in eq. (4.5) of the T matrix. We obtain,

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_2 = &-\frac{i}{\hbar^{3/2}}\sum_X\sum_{\sigma=\pm1}\int d\Phi(r_1,r_2,p_1,p_2,p_1',p_2',k)e^{-ib\cdot(p_1'-p_1)/\hbar}\psi(p_1,p_2)\psi^*(p_1',p_2')\\ &\times\left[k^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'|T^\dagger|r_1\,r_2\,X\rangle\langle r_1\,r_2\,X|a_\sigma(k)\,T|p_1\,p_2\rangle\right.\\ &\left.-\,k^{[\mu}\varepsilon^{(\sigma)\nu]}(k)e^{+ik\cdot x/\hbar}\langle p_1'\,p_2'|T^\dagger|r_1\,r_2\,X\rangle\langle r_1\,r_2\,X|a_\sigma^\dagger(k)T|p_1\,p_2\rangle\right]\\ = &-\frac{i}{\hbar^{3/2}}\sum_X\sum_{\sigma=\pm1}\int d\Phi(r_1,r_2,p_1,p_2,p_1',p_2',k)e^{-ib\cdot(p_1'-p_1)/\hbar}\psi(p_1,p_2)\psi^*(p_1',p_2')\\ &\times\left[k^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'|T^\dagger|r_1\,r_2\,X\rangle\langle r_1\,r_2\,k^\sigma\,X|T|p_1\,p_2\rangle\right.\\ &\left.-\,k^{[\mu}\varepsilon^{(\sigma)\nu]}(k)e^{+ik\cdot x/\hbar}\langle p_1'\,p_2'|T^\dagger|r_1\,r_2\,k^\sigma\,X\rangle\langle r_1\,r_2\,X|T|p_1\,p_2\rangle\right].\end{aligned}\tag{4.109}$$

In the second term within brackets, the creation operator requires a photon in the intermediate state, and eliminates it from the bra. We then relabeled X to exclude it. Note as well that at next-to-next-leading order and beyond, we necessarily require amplitudes with three incoming particles. These can just as easily be obtained by crossing. The term in eq. (4.109) has the interpretation of a cut of an amplitude, just as for the second term in the impulse in [166], as seen in eqs. (3.26–3.31) therein.

4.3.2 The detected wave at leading order

The leading-order connected contribution to the waveform will arise at $\mathcal{O}(g^3)$, as described in the previous section. We apply the approach of [166] to eq. (4.105). Similarly to that reference, we define the momentum mismatches,

$$\begin{aligned}q_1 &= p_1' - p_1\,,\\ q_2 &= p_2' - p_2\,;\end{aligned}\tag{4.110}$$

and trade the integrals over the p_i' for integrals over the q_i,

$$\begin{aligned}&\langle F^{\mu\nu}(x)\rangle_1 =\\ &\frac{2}{\hbar^{3/2}}\Re\sum_{\sigma=\pm1}\int d\Phi(p_1)d\Phi(p_2)\hat{d}^4q_1\hat{d}^4q_2 d\Phi(k)\,\delta(2p_1\cdot q_1+q_1^2)\delta(2p_2\cdot q_2+q_2^2)\\ &\qquad\times e^{-ib\cdot q_1/\hbar}\Theta(p_1^0+q_1^0)\Theta(p_2^0+q_2^0)\psi(p_1)\psi^*(p_1+q_1)\psi(p_2)\psi^*(p_2+q_2)\\ &\qquad\times k^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)e^{-ik\cdot x/\hbar}\mathcal{A}_5(p_1,p_2\to p_1+q_1,p_2+q_2,k^\sigma)\delta^4(q_1+q_2+k)\,.\end{aligned}\tag{4.111}$$

We can take the classical limit, and change to the required wavenumber variables for the q_i and k,

$$\begin{aligned}&\langle F^{\mu\nu}(x)\rangle_{1,\text{cl}} =\\ &\frac{1}{2}g^3\left\langle\!\!\left\langle\hbar^2\Re\sum_{\sigma=\pm1}\int d\Phi(\bar{k})\bar{k}^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)e^{-i\bar{k}\cdot x}\right.\right.\\ &\left.\left.\times\prod_{i=1,2}\int\hat{d}^4\bar{q}_i\,\delta(p_i\cdot\bar{q}_i)\,e^{-ib\cdot\bar{q}_1}\delta^4(\bar{q}_1+\bar{q}_2+\bar{k})\mathcal{A}_{5,0}^{(0)}(p_1,p_2\to p_1+\hbar\bar{q}_1,p_2+\hbar\bar{q}_2,\hbar\bar{k}^\sigma)\right\rangle\!\!\right\rangle.\end{aligned}\tag{4.112}$$

We have also extracted powers of $\hbar$ from the coupling, and dropped the $\hbar$-suppressed terms inside the on-shell delta functions as well as the positive-energy theta functions.

We recognize the inner integral in the second term as the radiation kernel defined in eq. (4.42) of [166] (after changing variables there $p_i \to p_i - \hbar\bar{w}_i$ and $\bar{w}_i \to -\bar{q}_i$),

$$\begin{aligned}\mathcal{R}^{(0)}(\bar{k}^\sigma; b) \equiv \hbar^2 \prod_{i=1,2} \int \hat{\mathrm{d}}^4\bar{q}_i\, \delta(p_i\cdot\bar{q}_i)\, e^{-ib\cdot\bar{q}_1}\delta^4(\bar{q}_1+\bar{q}_2+\bar{k}) \\ \times \mathcal{A}^{(0)}_{5,0}(p_1,p_2\to p_1+\hbar\bar{q}_1, p_2+\hbar\bar{q}_2, \hbar\bar{k}^\sigma)\,.\end{aligned} \tag{4.113}$$

We have made the impact parameter an explicit argument here. At leading order, we can then write,

$$\langle F^{\mu\nu}(x)\rangle_{1,\mathrm{cl}} = \frac{1}{2}g^3 \left\langle\!\!\left\langle \Re \sum_{\sigma=\pm1}\int \mathrm{d}\Phi(\bar{k})\bar{k}^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)e^{-i\bar{k}\cdot x}\mathcal{R}^{(0)}(\bar{k}^\sigma;b)\right\rangle\!\!\right\rangle. \tag{4.114}$$

The spectral waveform is then,

$$\begin{aligned} f_{\mu\nu}(\omega,\hat{\boldsymbol{n}}) = -\frac{ig^3}{16\pi}\sum_{\sigma=\pm1}\Big[\Theta(\omega)\bar{k}^{[\mu}\varepsilon^{(\sigma)\nu]*}(k)\mathcal{R}^{(0)}(\bar{k}^\sigma;b)\big|_{\bar{k}=\omega(1,\hat{\boldsymbol{n}})} \\ - \Theta(-\omega)\bar{k}^{[\mu}\varepsilon^{(\sigma)\nu]}(k)\mathcal{R}^{(0)*}(\bar{k}^\sigma;b)\big|_{\bar{k}=-\omega(1,\hat{\boldsymbol{n}})}\Big]\end{aligned} \tag{4.115}$$

The corresponding result for the Fourier-space NP scalar is,

$$\tilde{\Phi}^0_2(\omega,\hat{\boldsymbol{n}}) = -\frac{ig^3\omega}{16\pi}\left\langle\!\!\left\langle \Theta(\omega)\mathcal{R}^{(0)}(\omega(1,\hat{\boldsymbol{n}})^-;b) + \Theta(-\omega)\mathcal{R}^{(0)*}(-\omega(1,\hat{\boldsymbol{n}})^+;b)\right\rangle\!\!\right\rangle. \tag{4.116}$$

Equivalently, we may write,

$$\begin{aligned}\Phi^0_2(t,\hat{\boldsymbol{n}}) = -\frac{ig^3}{16\pi}\left\langle\!\!\left\langle\int \hat{\mathrm{d}}\omega\,\Theta(\omega)\,\omega\big[\,e^{-i\omega\cdot t}\mathcal{R}^{(0)}(\omega(1,\hat{\boldsymbol{n}})^-;b)\right. \\ \left. -\ e^{+i\omega\cdot t}\mathcal{R}^{(0)*}(\omega(1,\hat{\boldsymbol{n}})^+;b)\big]\right\rangle\!\!\right\rangle.\end{aligned} \tag{4.117}$$

As the leading order radiation kernel $\mathcal{R}^{(0)}$ is given by a five-point amplitude, the waveform as a function of frequency ω, is simply the five-point amplitude up to the additional factor of ω. The explicit form of eq. (4.113) for electromagnetic scattering is given in eq. (5.46) of [166]. We evaluate them to obtain,

$$\begin{aligned}\mathcal{R}^{(0)}(\bar{k};b) = &\frac{Q_1^2Q_2\,e^{ib\cdot\bar{k}}}{m_A\,v_A\cdot\bar{k}}\big[v_B\cdot\bar{k}\,v_A\cdot\varepsilon - v_A\cdot\bar{k}\,v_B\cdot\varepsilon\big] \\ &\times\frac{1}{2\pi\sqrt{\gamma^2-1}}K_0\big(\sqrt{-b^2}\,v_A\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\ &+\frac{Q_1^2Q_2\gamma\,e^{ib\cdot\bar{k}}}{m_A\,v_A\cdot\bar{k}}\big[v_A\cdot\bar{k}\,\tilde{b}\cdot\varepsilon - \tilde{b}\cdot\bar{k}\,v_A\cdot\varepsilon\big] \\ &\times\frac{i}{2\pi\,(\gamma^2-1)}K_1\big(\sqrt{-b^2}\,v_A\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\ &+\big(1\leftrightarrow 2\text{ modulo phases}\big)\,.\end{aligned} \tag{4.118}$$

A side calculation shows that (with ζ a null reference momentum),

$$
\begin{aligned}
v_B \cdot \bar{k}\, v_A \cdot \varepsilon - v_A \cdot \bar{k}\, v_B \cdot \varepsilon = &\\
&\frac{1}{\sqrt{2}\,\langle\zeta\, \bar{k}\rangle}\left[\langle\bar{k}|\, v_B\,|\bar{k}]\,\langle\zeta|\, v_A\,|\bar{k}] - \langle\bar{k}|\, v_A\,|\bar{k}]\,\langle\zeta|\, v_B\,|\bar{k}]\right] \\
&= \frac{1}{\sqrt{2}}\,[\bar{k}|\, v_B\, v_A\,|\bar{k}]
\end{aligned}
\tag{4.119}
$$

for positive-helicity emission, and

$$
\frac{1}{\sqrt{2}}\,\langle\bar{k}|\, v_B\, v_A\,|\bar{k}\rangle \tag{4.120}
$$

for negative-helicity emission.

Then,

$$
\begin{aligned}
\mathcal{R}^{(0)}(\bar{k}^+; b) = &\frac{Q_1^2 Q_2\, e^{ib\cdot\bar{k}}}{2\sqrt{2}\pi m_A\, v_A \cdot \bar{k}\,\sqrt{\gamma^2-1}} \\
&\times \Bigg\{[\bar{k}|\, v_B\, v_A\,|\bar{k}]\;K_0\big(\sqrt{-b^2}\, v_A\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\
&\quad + \frac{i\,[\bar{k}|\, b\, v_A\,|\bar{k}]}{\sqrt{\gamma^2-1}\,\sqrt{-b^2}} K_1\big(\sqrt{-b^2}\, v_A\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\Bigg\} \\
&+ \frac{Q_1 Q_2^2}{2\sqrt{2}\pi m_B\, v_B\cdot\bar{k}\,\sqrt{\gamma^2-1}} \\
&\times \Bigg\{[\bar{k}|\, v_A\, v_B\,|\bar{k}]\;K_0\big(\sqrt{-b^2}\, v_B\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\
&\quad + \frac{i\,[\bar{k}|\, b\, v_B\,|\bar{k}]}{\sqrt{\gamma^2-1}\,\sqrt{-b^2}} K_1\big(\sqrt{-b^2}\, v_B\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\Bigg\}\,.
\end{aligned}
\tag{4.121}
$$

There is a similar result for the other photon helicity.

Using the integrals,

$$
\begin{aligned}
\int_0^\infty \mathrm{d}\omega\;\omega e^{-i\omega(t+a_0)} K_0(\omega a_1) = &\frac{1}{a_1^2+(a_0+t)^2} - \frac{(t+a_0)}{[a_1^2+(a_0+t)^2]^{3/2}}\,\mathrm{arcsinh}\Big(\frac{1}{a_1}(t+a_0)\Big) \\
&- \frac{i\pi}{2}\frac{(t+a_0)}{[a_1^2+(a_0+t)^2]^{3/2}}\,, \\
\int_0^\infty \mathrm{d}\omega\;\omega e^{-i\omega(t+a_0)} K_1(\omega a_1) = &\frac{\pi a_1}{2[a_1^2+(a_0+t)^2]^{3/2}} - i\frac{(a_0+t)}{a_1[a_1^2+(a_0+t)^2]} \\
&- i\frac{a_1}{[a_1^2+(a_0+t)^2]^{3/2}}\,\mathrm{arcsinh}\Big(\frac{1}{a_1}(t+a_0)\Big)\,;
\end{aligned}
\tag{4.122}
$$

and defining,

$$
\begin{aligned}
v_{j,\hat{\boldsymbol{n}}} &\equiv v_j\cdot\bar{k}/\omega = v_j\cdot(1,\hat{\boldsymbol{n}})\,, \\
\rho_1(t) &\equiv -b^2 v_{A,\hat{\boldsymbol{n}}}^2 + (\gamma^2-1)(t+\mathbf{b}\cdot\hat{\boldsymbol{n}})^2\,, \\
\rho_2(t) &\equiv -b^2 v_{B,\hat{\boldsymbol{n}}}^2 + (\gamma^2-1)t^2\,,
\end{aligned}
\tag{4.123}
$$

along with,

$$
\begin{aligned}
\Xi^{\zeta}_{ia}(t,\hat{\boldsymbol{n}};\mathbf{v}) &= \frac{\sqrt{\gamma^2-1}}{\rho_1(t)} - \zeta\frac{(\gamma^2-1)(t+\mathbf{v}\cdot\hat{\boldsymbol{n}})}{\rho_1^{3/2}(t)}\,\mathrm{arcsinh}\left(\frac{\sqrt{\gamma^2-1}}{\sqrt{-b^2}v_{A,\hat{\boldsymbol{n}}}}(t+\mathbf{v}\cdot\hat{\boldsymbol{n}})\right)\\
&\quad - \frac{i\pi}{2}\frac{(\gamma^2-1)(t+\mathbf{v}\cdot\hat{\boldsymbol{n}})}{\rho_1^{3/2}(t)},\\
\Xi_{ib}(t,\hat{\boldsymbol{n}};\mathbf{v}) &= \frac{\pi v_{A,\hat{\boldsymbol{n}}}}{\rho_1^{3/2}(t)} + i\frac{\sqrt{\gamma^2-1}(t+\mathbf{v}\cdot\hat{\boldsymbol{n}})}{b^2 v_{A,\hat{\boldsymbol{n}}}\rho_1(t)}\\
&\quad - i\frac{v_{A,\hat{\boldsymbol{n}}}}{\rho_1^{3/2}(t)}\,\mathrm{arcsinh}\left(\frac{\sqrt{\gamma^2-1}}{\sqrt{-b^2}v_{A,\hat{\boldsymbol{n}}}}(t+\mathbf{v}\cdot\hat{\boldsymbol{n}})\right),
\end{aligned}
\tag{4.124}
$$

we can write,

$$
\begin{aligned}
&\Phi^0_2(t,\hat{\boldsymbol{n}}) =\\
&\quad -\frac{ig^3Q_1^2Q_2}{(4\pi)^3\sqrt{2}m_A\,v_{A,\hat{\boldsymbol{n}}}}\Big[\langle\hat{n}|\,v_B\,v_A\,|\hat{n}\rangle\;\Xi^{+}_{1a}(t,\hat{\boldsymbol{n}};\mathbf{b}) - [\hat{n}|\,v_B\,v_A\,|\hat{n}]\;\Xi^{-}_{1a}(t,\hat{\boldsymbol{n}};\mathbf{b})\\
&\qquad\qquad + i\big(\langle\hat{n}|\,b\,v_A\,|\hat{n}\rangle - [\hat{n}|\,b\,v_A\,|\hat{n}]\big)\,\Xi_{1b}(t,\hat{\boldsymbol{n}};\mathbf{b})\Big]\\
&\quad -\frac{ig^3Q_1Q_2^2}{(4\pi)^3\sqrt{2}m_B\,v_{B,\hat{\boldsymbol{n}}}}\Big[\langle\hat{n}|\,v_A\,v_B\,|\hat{n}\rangle\;\Xi^{+}_{2a}(t,\hat{\boldsymbol{n}};\mathbf{0}) - [\hat{n}|\,v_A\,v_B\,|\hat{n}]\;\Xi^{-}_{2a}(t,\hat{\boldsymbol{n}};\mathbf{0})\\
&\qquad\qquad + i\big(\langle\hat{n}|\,b\,v_B\,|\hat{n}\rangle - [\hat{n}|\,b\,v_B\,|\hat{n}]\big)\,\Xi_{2b}(t,\hat{\boldsymbol{n}};\mathbf{0})\Big].
\end{aligned}
\tag{4.125}
$$

Here, $|\hat{n}\rangle$ and $|\hat{n}]$ are spinors built out of the null vector $(1,\hat{\boldsymbol{n}})$.

4.4 Localized observables II: Gravitational energy event shapes

In analogy to the standard event shapes considered in QCD jet physics, we explore here gravitational energy event shapes, which are of current interest to the gravitational waves community. Our light-ray operators probe the underlying structure of gravitational radiation as registered by a detector placed at null infinity in the direction $\hat{\boldsymbol{n}}$. Of particular interest is the gravitational radiation emitted by the scattering of compact objects. The relevant amplitude for such processes is of the form

$$
\begin{aligned}
\langle p_3p_4\{k_j^{\sigma_j}\}_{j=1,..,M}|T|p_1p_2\rangle &= \mathcal{M}_{4+M}(p_1,p_2\to p_3,p_4,\{k_j^{\sigma_j}\}_{j=1,..,M})\\
&\quad\times\delta^4\left(p_1+p_2-p_3-p_4-\sum_{j=1}^{M}k_j\right),
\end{aligned}
\tag{4.126}
$$

As a first approximation, we can focus on the soft kinematic region of the emitted gravitons in order to take advantage of the simple form of graviton soft factors and of the fact the quantum state representing the radiation is known as we will discuss later in section 5.5. To be more precise, the detectors are assumed to have a lower energy resolution λ and the emitted gravitons to have a total energy E_{wave} with

$$
\lambda \ll E_{\text{wave}} \ll \Lambda,
\tag{4.127}
$$

where Λ is a suitable UV cutoff which can be thought as the total energy of the process [255, 256]. The classical limit requires the number of gravitons emitted to be large $M \to +\infty$, as dictated by the coherent state structure [167]: this implies an infinite resummation of all graviton contributions.

The 1-point gravitational energy event shape is simply the on-shell expectation value of the gravitational ANEC at infinity

$$\langle \psi_{\text{out}} | \tilde{\mathcal{E}}(\hat{\boldsymbol{n}}) | \psi_{\text{out}} \rangle = \langle \psi_{\text{in}} | S^\dagger \tilde{\mathcal{E}}(\hat{\boldsymbol{n}}) S | \psi_{\text{in}} \rangle. \tag{4.128}$$

Inserting the completeness relation of on-shell states $\sum_X |X\rangle\langle X|$ yields

$$\langle \psi_{\text{out}} | \tilde{\mathcal{E}}(\hat{\boldsymbol{n}}) | \psi_{\text{out}} \rangle = \sum_X \langle \psi_{\text{in}} | S^\dagger \tilde{\mathcal{E}}(\hat{\boldsymbol{n}}) | X \rangle \langle X | S | \psi_{\text{in}} \rangle = \sum_X w_{\tilde{\mathcal{E}}}(\hat{\boldsymbol{n}}) |\langle X | S | \psi_{\text{in}} \rangle|^2, \tag{4.129}$$

which is always positive definite.

The graviton emission amplitude for M soft gravitons of momenta k_j and helicity σ_j is then[6]

$$\mathcal{M}_{4+M}(p_1, p_2 \to p_3, p_4, \{k_j^{\sigma_j}\}_{j=1,..,M}) = \mathcal{M}_4(p_1, p_2 \to p_3, p_4) \prod_{l=1}^{M} \left[\frac{\kappa}{2} \sum_{i=1}^{4} \eta_i \frac{\varepsilon^{\sigma_l}_{\mu\nu}(k_l) p_i^\mu p_i^\nu}{p_i \cdot k_l} \right], \tag{4.130}$$

where we have used the leading soft graviton factor [255, 257] and $\eta_i = +1$ (resp. $\eta_i = -1$) if the particle is outgoing (resp. ingoing). One can go further in the soft expansion by taking into account the sub- (and subsub-) leading contributions, see [168, 170, 258–260] for further details.

We will be interested in the classical limit of this quantity, and the proper way to do so would be to use the in-in KMOC formalism [166]. In such approach, the limit is taken with a careful analysis of the wavepackets of the external massive fields which localize the particles on their classical trajectory as $\hbar \to 0$. Here for simplicity we will work in the approximation $\omega \ll b^{-1}$, where ω is the typical graviton frequency of the emitted wave and b is the impact parameter of the two incoming particles: this will allow us to get a simple universal b-independent result which will be useful to analyze the properties of gravitational energy event shapes. In order to make transparent the power counting for the amplitudes involved, we will consider the tree level classical contribution for $\mathcal{M}_4(p_1, p_2 \to p_3, p_4)$ so that $|\mathcal{M}_4(p_1, p_2 \to p_3, p_4)|^2$ will always be of order G^2 in this section.

The leading contribution will be given by the single graviton emission amplitude,

$$\begin{aligned} \langle \psi_{\text{out}} | \tilde{\mathcal{E}}_{\text{GR}}(\hat{\boldsymbol{n}}) | \psi_{\text{out}} \rangle \Big|_{\mathcal{O}(G^3)} &= \int \mathrm{d}\Phi(p_3) \int \mathrm{d}\Phi(p_4) \sum_{\sigma_1 = \pm 2} \int \mathrm{d}\Phi(k_1) \\ &\times \left[(E_{k_1}) \delta^2(\Omega_{\hat{\boldsymbol{k}}_1} - \Omega_{\hat{\boldsymbol{n}}}) \right] (2\pi)^4 \delta^4 \left(\sum_{i=1}^{4} \eta_i p_i + k_1 \right) |\mathcal{M}_5(p_1, p_2 \to p_3, p_4, k_1^{\sigma_1})|^2. \end{aligned} \tag{4.131}$$

[6] At leading order there is no distinction between consecutive double soft limits and simultaneous double soft limits, so we don't need to assume any hierarchy in the energy of the soft gravitons.

In the soft regime for the graviton we can isolate the graviton phase space integration,

$$
\begin{aligned}
&\sum_{\sigma_1=\pm 2}\int \mathrm{d}\Phi(k_1)(2\pi)^4\delta^4\left(\sum_{i=1}^{4}\eta_i p_i + k_1\right)\left[(E_{k_1})\delta^2(\Omega_{\hat{\boldsymbol{k}}_1}-\Omega_{\hat{\boldsymbol{n}}})\right]|\mathcal{M}_5(p_1,p_2\to p_3,p_4,k_1^{\sigma_1})|^2 \\
&\simeq (2\pi)^4\delta^4\left(\sum_{i=1}^{4}\eta_i p_i\right)\sum_{\sigma_1=\pm 2}\int \mathrm{d}\Phi(k_1)\left[(E_{k_1})\delta^2(\Omega_{\hat{\boldsymbol{k}}_1}-\Omega_{\hat{\boldsymbol{n}}})\right]|\mathcal{M}_5(p_1,p_2\to p_3,p_4,k_1^{\sigma_1})|^2.
\end{aligned}
\tag{4.132}
$$

To make further progress we need to use the polarization sum identity for the graviton polarization vectors which reads

$$
\begin{aligned}
\sum_{\sigma=\pm 2}\varepsilon^{\sigma}_{\mu\nu}(k)\varepsilon^{\sigma,*}_{\alpha\beta}(k) &= \frac{1}{2}\left[P_{\mu\alpha}P_{\nu\beta}+P_{\mu\beta}P_{\nu\alpha}-P_{\mu\nu}P_{\alpha\beta}\right] \\
P_{\mu\nu} &= \eta_{\mu\nu}-\frac{q_\mu p_\nu + q_\nu p_\mu}{p\cdot q},
\end{aligned}
\tag{4.133}
$$

where q^μ is a suitable reference momentum. The gauge dependent terms will cancel from our calculation using momentum conservation [255], so we do not need to worry about spurious contributions. Since the amplitude factorizes, we can actually perform the graviton phase space integration in a universal form

$$
\begin{aligned}
&\sum_{\sigma_1=\pm 2}\frac{1}{2(2\pi)^3}\int_{\lambda}^{E_{\mathrm{wave}}}\mathrm{d}E_{k_1}\int \mathrm{d}\Omega_{\hat{\boldsymbol{k}}_1}\,\delta^2(\Omega_{\hat{\boldsymbol{k}}_1}-\Omega_{\hat{\boldsymbol{n}}})(E_{k_1})^2\left|\sum_{i=1}^{4}\eta_i\frac{\kappa}{2}\frac{\varepsilon^{\sigma_1}_{\mu\nu}(k_1)p_i^\mu p_i^\nu}{p_i\cdot k_1}\right|^2 \\
&= G\frac{E_{\mathrm{wave}}}{4\pi^2}\left(\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{n}})(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{n}})}\right).
\end{aligned}
\tag{4.134}
$$

Working in the center of mass frame[7] for the two massive particles we have

$$
\begin{aligned}
\langle\psi_{\mathrm{out}}|\tilde{\mathcal{E}}_{\mathrm{GR}}(\hat{\boldsymbol{n}})|\psi_{\mathrm{out}}\rangle\Big|_{\mathcal{O}(G^3)} &= \int \mathrm{d}\Omega_{\boldsymbol{p}_{\mathrm{CM}}}\frac{|\boldsymbol{p}_{\mathrm{CM}}|}{16\pi^2\sqrt{s}}|\mathcal{M}_4(p_1,p_2\to p_3,p_4)|^2 \\
&\quad\times G\frac{E_{\mathrm{wave}}}{4\pi^2}\left(\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{n}})(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{n}})}\right),
\end{aligned}
\tag{4.135}
$$

which is infrared finite at the leading order G^3. It is worth noticing that eq. (4.135) is only the leading order contribution, and we still need to sum over all possible graviton emissions.

[7]Here $s=(p_1+p_2)^2$ is the center of mass energy.

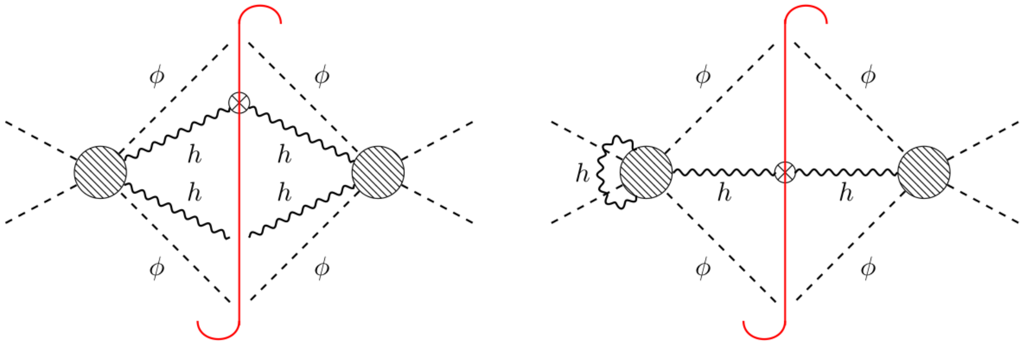


FIGURE 4.6: The picture shows how the cancellation of infrared divergences between real and virtual graviton emissions works for the gravitational energy event shape

Before looking at the classical limit, it is worth looking at how infrared divergences will cancel for the gravitational energy event shape in general. At order G^4 we have,

$$
\begin{aligned}
&\langle\psi_{\text{out}}|\tilde{\mathcal{E}}_{\text{GR}}(\hat{\boldsymbol{n}})|\psi_{\text{out}}\rangle\Big|_{\mathcal{O}(G^4)} \\
&= \int \mathrm{d}\Phi(p_3) \int \mathrm{d}\Phi(p_4) \sum_{\sigma_1=\pm 2} \int \mathrm{d}\Phi(k_1)\, (2\pi)^4 \delta^4\left(\sum_{i=1}^{4} \eta_i p_i + k_1\right) \\
&\qquad \times \left[(E_{k_1})\delta^2(\Omega_{\hat{\boldsymbol{k}}_1} - \Omega_{\hat{\boldsymbol{n}}})\right] |\mathcal{M}_5(p_1, p_2 \to p_3, p_4, k_1^{\sigma_1})|^2 \\
&+ \int \mathrm{d}\Phi(p_3) \int \mathrm{d}\Phi(p_4)\, \frac{1}{2} \sum_{\sigma_1,\sigma_2=\pm 2} \int \mathrm{d}\Phi(k_1) \int \mathrm{d}\Phi(k_2)\, (2\pi)^4 \delta^4\left(\sum_{i=1}^{4} \eta_i p_i + \sum_{l=1}^{2} k_l\right) \\
&\qquad \times \left[\sum_{l=1}^{2} (E_{k_l})\delta^2(\Omega_{\hat{\boldsymbol{k}}_l} - \Omega_{\hat{\boldsymbol{n}}})\right] |\mathcal{M}_5(p_1, p_2 \to p_3, p_4, k_1^{\sigma_1}, k_2^{\sigma_2})|^2.
\end{aligned}
\tag{4.136}
$$

The first contribution includes loop contributions related to virtual gravitons

$$
\begin{aligned}
&|\mathcal{M}_5(p_1, p_2 \to p_3, p_4, k_1^{\sigma_1})|^2\Big|_{\mathcal{O}(G^4)} = |\mathcal{M}_5(p_1, p_2 \to p_3, p_4, k_1^{\sigma_1})|^2\Big|_{\mathcal{O}(G^3)} \\
&\times \left[-\frac{G}{4\pi^2} \log\left(\frac{\Lambda}{\lambda}\right) \int \mathrm{d}\Omega_{\hat{\boldsymbol{k}}} \left(\sum_{i,j=1}^{4} \eta_i \eta_j \frac{2(p_i \cdot p_j)^2 - m_i^2 m_j^2}{(E_{p_i} - \boldsymbol{p}_i \cdot \hat{\boldsymbol{k}})(E_{p_j} - \boldsymbol{p}_j \cdot \hat{\boldsymbol{k}})}\right)\right],
\end{aligned}
\tag{4.137}
$$

where we have regularized the integral with the ultraviolet cutoff Λ [255]. The second term instead has a divergent contribution which comes from the emission of one real graviton, since integrating over the 2-gravitons phase space gives

$$\begin{aligned}
&\frac{1}{2}\sum_{\sigma_1,\sigma_2=\pm 2}\int_\lambda^{E_{\rm wave}}\frac{\mathrm{d}E_{k_1}(E_{k_1})}{2(2\pi)^3}\int_\lambda^{E_{\rm wave}}\frac{\mathrm{d}E_{k_2}(E_{k_2})}{2(2\pi)^3}\int \mathrm{d}\Omega_{\hat{\boldsymbol{k}}_1}\int \mathrm{d}\Omega_{\hat{\boldsymbol{k}}_2}\,\Theta(E_{\rm wave}-E_{k_1}-E_{k_2})\\
&\qquad\qquad\times\left[\sum_{l=1}^{2}(E_{k_l})\delta^2(\Omega_{\hat{\boldsymbol{k}}_l}-\Omega_{\hat{\boldsymbol{n}}})\right]\prod_{l=1}^{2}\left|\sum_{i=1}^{4}\eta_i\frac{\kappa}{2}\frac{\varepsilon^{\sigma_l}_{\mu\nu}(k_l)p_i^\mu p_i^\nu}{p_i\cdot k_l}\right|^2\\
&= G\frac{E_{\rm wave}}{4\pi^2}\left(\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{n}})(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{n}})}\right)\\
&\qquad\qquad\times\log\left(\frac{E_{\rm wave}}{\lambda}\right)\frac{G}{4\pi^2}\left[\int \mathrm{d}\Omega_{\hat{\boldsymbol{k}}}\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{k}})(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{k}})}\right],
\end{aligned}\tag{4.138}$$

where we have used the fact that

$$\int_\lambda^{E_{\rm wave}}\mathrm{d}E_{k_1}\int_\lambda^{E_{\rm wave}}\mathrm{d}E_{k_2}\,\frac{1}{E_{k_2}}\Theta(E_{\rm wave}-E_{k_1}-E_{k_2})\overset{E_{k_1},E_{k_2}\ll E_{\rm wave}}{\simeq}E_{\rm wave}\log\left(\frac{E_{\rm wave}}{\lambda}\right),\tag{4.139}$$

which is valid as long as we can neglect the single graviton energies (suppressed by $\hbar$) compared to the energy of the wave $E_{k_1}, E_{k_2} \ll E_{\rm wave}$, as noticed in [256].

One can then perform the regularized sum in eq. (4.136)

$$\begin{aligned}
\langle\psi_{\rm out}|\tilde{\mathcal{E}}_{\rm GR}(\hat{\boldsymbol{n}})|\psi_{\rm out}\rangle\Big|_{\mathcal{O}(G^4)} &= \int \mathrm{d}\Omega_{\boldsymbol{p}_{\rm CM}}\frac{|\boldsymbol{p}_{\rm CM}|}{16\pi^2\sqrt{s}}|\mathcal{M}_4(p_1,p_2\to p_3,p_4)|^2\\
&\times G\frac{E_{\rm wave}}{4\pi^2}\left(\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{n}})(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{n}})}\right)\\
&\times\log\left(\frac{E_{\rm wave}}{\Lambda}\right)\frac{G}{4\pi^2}\left[\int \mathrm{d}\Omega_{\hat{\boldsymbol{k}}}\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{k}})(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{k}})}\right],
\end{aligned}\tag{4.140}$$

which is infrared finite as expected. One can then repeat the same argument for the G^{3+n}-th order contribution

$$\begin{aligned}
\langle\psi_{\rm out}|\mathcal{E}_{\rm GR}(\hat{\boldsymbol{n}})|\psi_{\rm out}\rangle\Big|_{\mathcal{O}(G^{3+n})} &= \int \mathrm{d}\Omega_{\boldsymbol{p}_{\rm CM}}\frac{|\boldsymbol{p}_{\rm CM}|}{16\pi^2\sqrt{s}}|\mathcal{M}_4(p_1,p_2\to p_3,p_4)|^2\\
&\times G\frac{E_{\rm wave}}{4\pi^2}\left(\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{n}})(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{n}})}\right)\\
&\times\sum_{M'=0}^{n}\frac{1}{M'!}\log\left(\frac{E_{\rm wave}}{\Lambda}\right)^{M'}\left[\frac{G}{4\pi^2}\int \mathrm{d}\Omega_{\hat{\boldsymbol{k}}}\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{k}})(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{k}})}\right]^{M'},
\end{aligned}\tag{4.141}$$

where $M' = M - 1$.

It is worth noticing that this situation is in contrast with QCD case where the gluon energy event shape is infrared divergent and one has to sum also over quarks energy event shapes to get a well-defined (i.e. an infrared finite) answer [141]. The cancellation of infrared divergences in this setting is similar to what was found for the graviton emission cross section by Donoghue [261].

The energy-energy correlator is defined as,

$$\langle\psi_{\rm out}|\tilde{\mathcal{E}}(\hat{\boldsymbol{n}}_1)\tilde{\mathcal{E}}'(\hat{\boldsymbol{n}}_2)|\psi_{\rm out}\rangle = \langle\psi_{\rm in}|S^\dagger\tilde{\mathcal{E}}(\hat{\boldsymbol{n}}_1)\tilde{\mathcal{E}}'(\hat{\boldsymbol{n}}_2)S|\psi_{\rm in}\rangle. \tag{4.142}$$

By inserting a completeness relation of on-shell final states $\sum_X |X\rangle\langle X|$ between the two gravitational ANEC operators at infinity we get

$$\begin{aligned}\langle\psi_{\rm out}|\tilde{\mathcal{E}}(\hat{\boldsymbol{n}}_1)\tilde{\mathcal{E}}'(\hat{\boldsymbol{n}}_2)|\psi_{\rm out}\rangle &= \sum_X\langle\psi_{\rm in}|S^\dagger\tilde{\mathcal{E}}(\hat{\boldsymbol{n}}_1)|X\rangle\langle X|\tilde{\mathcal{E}}'(\hat{\boldsymbol{n}}_2)S|\psi_{\rm in}\rangle\\ &= \sum_X w_{\tilde{\mathcal{E}}}(\hat{\boldsymbol{n}}_1)w'_{\tilde{\mathcal{E}}}(\hat{\boldsymbol{n}}_2)|\langle X|S|\psi_{\rm in}\rangle|^2.\end{aligned} \tag{4.143}$$

In details

$$\begin{aligned}&\langle\psi_{\rm out}|\tilde{\mathcal{E}}_{\rm GR}(\hat{\boldsymbol{n}}_1)\tilde{\mathcal{E}}'_{\rm GR}(\hat{\boldsymbol{n}}_2)|\psi_{\rm out}\rangle\Big|_{\mathcal{O}(G^4)}\\ &= \int {\rm d}\Phi(p_3)\int {\rm d}\Phi(p_4)\frac{1}{2}\sum_{\sigma_1,\sigma_2=\pm2}\int {\rm d}\Phi(k_1)\int {\rm d}\Phi(k_2)\\ &\times\Big[(E_{k_1})(E_{k_1})\delta^2(\Omega_{\hat{\boldsymbol{k}}_1}-\Omega_{\hat{\boldsymbol{n}}_1})\delta^2(\Omega_{\hat{\boldsymbol{k}}_1}-\Omega_{\hat{\boldsymbol{n}}_2})+(E_{k_2})(E_{k_2})\delta^2(\Omega_{\hat{\boldsymbol{k}}_2}-\Omega_{\hat{\boldsymbol{n}}_1})\delta^2(\Omega_{\hat{\boldsymbol{k}}_2}-\Omega_{\hat{\boldsymbol{n}}_2})\\ &\quad+2(E_{k_1})(E_{k_2})\delta^2(\Omega_{\hat{\boldsymbol{k}}_1}-\Omega_{\hat{\boldsymbol{n}}_1})\delta^2(\Omega_{\hat{\boldsymbol{k}}_2}-\Omega_{\hat{\boldsymbol{n}}_2})\Big]|\langle k_1^{\sigma_1}k_2^{\sigma_2}p_3p_4|S|p_1p_2\rangle|^2.\end{aligned} \tag{4.144}$$

Contrary to the 1-pt energy event shape, here we get also contact terms which correspond to the case where the two detectors are aligned along the same direction. These terms are usually included in QCD energy event shapes and they are required in order to remove collinear divergences [142], but classical gravity does not have such type of collinear divergences [262]. We will thus restrict our attention to the generic case $\hat{\boldsymbol{n}}_1 \neq \hat{\boldsymbol{n}}_2$:

$$\begin{aligned}&\langle\psi_{\rm out}|\tilde{\mathcal{E}}_{\rm GR}(\hat{\boldsymbol{n}}_1)\tilde{\mathcal{E}}'_{\rm GR}(\hat{\boldsymbol{n}}_2)|\psi_{\rm out}\rangle\Big|_{\mathcal{O}(G^4)} = \int {\rm d}\Phi(p_3)\int {\rm d}\Phi(p_4)\,\frac{1}{2}\sum_{\sigma_1,\sigma_2=\pm2}\int {\rm d}\Phi(k_1)\int {\rm d}\Phi(k_2)\\ &\times(2E_{k_1}E_{k_2}\delta^2(\Omega_{\hat{\boldsymbol{k}}_1}-\Omega_{\hat{\boldsymbol{n}}_1})\delta^2(\Omega_{\hat{\boldsymbol{k}}_2}-\Omega_{\hat{\boldsymbol{n}}_2}))|\langle k_1^{\sigma_1}k_2^{\sigma_2}p_3p_4|S|p_1p_2\rangle|^2.\end{aligned} \tag{4.145}$$

At this point we can repeat the integration as we did before to get

$$\begin{aligned}\langle\psi_{\rm out}|\tilde{\mathcal{E}}_{\rm GR}(\hat{\boldsymbol{n}}_1)\tilde{\mathcal{E}}'_{\rm GR}(\hat{\boldsymbol{n}}_2)|\psi_{\rm out}\rangle\Big|_{\mathcal{O}(G^4)} &= \int {\rm d}\Omega_{\boldsymbol{p}_{\rm CM}}\frac{|\boldsymbol{p}_{\rm CM}|}{16\pi^2\sqrt{s}}|\mathcal{M}_4(p_1,p_2\to p_3,p_4)|^2\\ &\times\prod_{l=1}^{2}\left[G\frac{E_{\rm wave}}{4\pi^2}\left(\sum_{i,j=1}^{4}\eta_i\eta_j\frac{2(p_i\cdot p_j)^2-m_i^2m_j^2}{(E_{p_i}-\boldsymbol{p}_i\cdot\hat{\boldsymbol{n}}_l)(E_{p_j}-\boldsymbol{p}_j\cdot\hat{\boldsymbol{n}}_l)}\right)\right].\end{aligned} \tag{4.146}$$

As we can see, despite being a natural quantum observable the 2-point gravitational energy event shape factorizes, i.e. there is no degree of correlation between the two emissions. This is due to the soft limit we are considering and ultimately related to the coherent state structure of the radiation [263]. It would be very interesting to compute the non-trivial (connected) 2-point energy event shape in a quantum theory of gravity from the six point amplitude with 4 massive scalars and 2 gravitons, which should provide the leading infrared finite contribution to this observable.

So far we have discussed the quantum picture, where energy event shape correlators are a non-trivial infrared-safe prediction. For our problem in the classical soft limit, as we will see, only the 1-point event shape will be non-trivial because of factorization properties. In our case we can take the sum over all contributions $M = 1, .., +\infty$, which implies to sum over $M' = 0, .., +\infty$ in our expression for the gravitational wave event shape in eq. (4.141):

$$\langle \psi_{\text{out}} | \tilde{\mathcal{E}}_{\text{GR}}(\hat{\boldsymbol{n}}) | \psi_{\text{out}} \rangle \Big|_{\text{class}} = G \frac{E_{\text{wave}}}{4\pi^2} \int \mathrm{d}\Omega_{\boldsymbol{p}_{\text{CM}}} \frac{|\boldsymbol{p}_{\text{CM}}|}{16\pi^2 \sqrt{s}} |\mathcal{M}_4(p_1, p_2 \to p_3, p_4)|^2$$
$$\times \left(\sum_{i,j=1}^{4} \eta_i \eta_j \frac{2(p_i \cdot p_j)^2 - m_i^2 m_j^2}{(E_{p_i} - \boldsymbol{p}_i \cdot \hat{\boldsymbol{n}})(E_{p_j} - \boldsymbol{p}_j \cdot \hat{\boldsymbol{n}})} \right) \left(\frac{E_{\text{wave}}}{\Lambda} \right)^{\frac{G}{2\pi} \sum_{i,j} \eta_i \eta_j \frac{m_i m_j (1+\beta_{ij}^2)}{\beta_{ij}(1-\beta_{ij}^2)^{\frac{1}{2}}} \log\left(\frac{1+\beta_{ij}}{1-\beta_{ij}} \right)}, \tag{4.147}$$

where $\beta_{ij} = \sqrt{1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}}$. This is consistent with the graviton production rate in Weinberg [255] under our hypotheses. Once normalized by the total cross section we have

$$\left\langle \tilde{\mathcal{E}}_{\text{GR}}(\hat{\boldsymbol{n}}) \right\rangle \Big|_{\text{class}} = \frac{E_{\text{wave}} G}{4\pi^2} \left(\sum_{i,j=1}^{4} \eta_i \eta_j \frac{2(p_i \cdot p_j)^2 - m_i^2 m_j^2}{(E_{p_i} - \boldsymbol{p}_i \cdot \hat{\boldsymbol{n}})(E_{p_j} - \boldsymbol{p}_j \cdot \hat{\boldsymbol{n}})} \right)$$
$$\times \left(\frac{E_{\text{wave}}}{\Lambda} \right)^{\frac{G}{2\pi} \sum_{i,j} \eta_i \eta_j \frac{m_i m_j (1+\beta_{ij}^2)}{\beta_{ij}(1-\beta_{ij}^2)^{\frac{1}{2}}} \log\left(\frac{1+\beta_{ij}}{1-\beta_{ij}} \right)} \tag{4.148}$$

and if we try to compute the gravitational energy-energy correlator at the classical level

$$\left\langle \tilde{\mathcal{E}}_{\text{GR}}(\hat{\boldsymbol{n}}_1) \tilde{\mathcal{E}}'_{\text{GR}}(\hat{\boldsymbol{n}}_2) \right\rangle \Big|_{\text{class}} = \left\langle \tilde{\mathcal{E}}_{\text{GR}}(\hat{\boldsymbol{n}}_1) \right\rangle \Big|_{\text{class}} \left\langle \tilde{\mathcal{E}}'_{\text{GR}}(\hat{\boldsymbol{n}}_2) \right\rangle \Big|_{\text{class}}, \tag{4.149}$$

which is a consequence of the classical (soft) factorization. This can be shown explicitly using the coherent state in [264], and it is a simple consequence of the uncertainty principle discussed in section 4.2.

4.4.1 IR safety and invariance under BMS soft supertranslations

It is possible to do an interesting consistency check about infrared safety of the gravitational energy event shapes by looking at how soft supertranslation symmetry affects such matrix elements. In the rest of the section we have used an infrared cutoff and we have observed the cancellation of infrared divergences as in the Bloch-Nordsieck mechanism, but it would be nice to understand if there is a theoretical reason for these event shapes to be infrared finite.

It is well known that the action of the soft supertranslation mode corresponds to an insertion of a soft graviton at the level of S-matrix elements [228, 230]. In particular the generator of such symmetry reads

$$T_{\text{soft}}(f) = \frac{1}{16\pi G} \int \mathrm{d}^2\zeta \left[\partial_\zeta D^{\bar{\zeta}} C_{\bar{\zeta}\bar{\zeta}} + \partial_{\bar{\zeta}} D^{\zeta} C_{\zeta\zeta} \right] f(\zeta, \bar{\zeta})$$
$$= \lim_{E \to 0} \frac{E}{4\pi\kappa} \int \mathrm{d}^2\zeta \left[(a_+(E\hat{\boldsymbol{n}}) + a_-^\dagger(E\hat{\boldsymbol{n}})) D_{\bar{\zeta}}^2 f + \text{h.c.} \right], \tag{4.150}$$

where $f(\zeta,\bar{\zeta})$ is an arbitrary function on the celestial sphere. Its action on graviton creation operators is given by

$$[T_{\text{soft}}(f), a_+(k)] = \frac{8\pi^2}{\kappa}\frac{\delta(E_k)}{\gamma_{\zeta_k\bar{\zeta}_k}}D_\zeta^2 f \qquad [T_{\text{soft}}(f), a_-(k)] = \frac{8\pi^2}{\kappa}\frac{\delta(E_k)}{\gamma_{\zeta_k\bar{\zeta}_k}}D_{\bar{\zeta}}^2 f, \tag{4.151}$$

and the one with creation operators is fixed because $T_{\text{soft}}(f)$ is hermitian. The latter implies that

$$[T_{\text{soft}}(f), a_\sigma^\dagger(k)]^\dagger = -[T_{\text{soft}}(f), a_\sigma(k)] \qquad \forall\, \sigma = \pm 2. \tag{4.152}$$

We now consider a gravitational event shape of the form

$$\langle\psi_{\text{out}}|\tilde{\mathcal{P}}_{\mu,\text{GR}}(\hat{\boldsymbol{n}})|\psi_{\text{out}}\rangle = \sum_X \langle\psi_{\text{in}}|S^\dagger\tilde{\mathcal{P}}_{\mu,\text{GR}}(\hat{\boldsymbol{n}})|X\rangle\langle X|S|\psi_{\text{in}}\rangle \tag{4.153}$$

and we want to check whether this definition is invariant or not under BMS soft supertranslations, i.e. under the addition of a soft graviton. If it is, this provides a strong evidence that eq. (4.153) is actually infrared finite, i.e. insensitive to soft (graviton) physics. A short calculation shows that

$$[T_{\text{soft}}(f), \tilde{\mathcal{P}}_{\mu,\text{GR}}(\hat{\boldsymbol{n}})] = \frac{n_\mu}{2\pi\kappa\gamma_{\zeta_{\hat{\boldsymbol{n}}}\bar{\zeta}_{\hat{\boldsymbol{n}}}}}\lim_{E_p\to 0}(E_p)^2\left[D_\zeta^2 f\, a_+^\dagger(E_p\hat{\boldsymbol{n}}) + D_{\bar{\zeta}}^2 f\, a_-^\dagger(E_p\hat{\boldsymbol{n}}) - \text{h.c.}\right] \tag{4.154}$$

and since the addition of a soft graviton gives the usual Weinberg soft factor with a pole in E_p, the action in eq. (4.154) annihilates every S-matrix element (once we insert the completeness relation) [230].

It would be interesting to understand in more details if the invariance under the soft modes of large gauge symmetries, BMS soft supertranslations and general asymptotic symmetries of massless particles always guarantees the IR-finiteness (in the soft sense[8]) of a matrix element in a QFT. This could be helpful to prove in general whether for other on-shell observables, for example like the ones defined in [166], infrared divergences are always going to cancel.

4.5 Amplitude of the waveform and energy event shapes

In section 4.3, we presented the general form for the waveform observable. We worked out the leading-order form in two-particle scattering in Sect. 4.3.1, and computed the explicit form for electromagnetic scattering. The appearance of the radiation kernel suggests a connection to the radiated momentum previously computed in [166], and more specifically with the event shape analogue discussed in section 4.4. Let us elucidate that connection in this section.

In eq. (3.33) of [166], we find an expression for time-averaged radiated momentum,

$$R^\mu \equiv \langle k^\mu\rangle = \langle\psi_{\text{in}}|\,S^\dagger\mathbb{K}^\mu S\,|\psi_{\text{in}}\rangle = \langle\psi_{\text{in}}|\,T^\dagger\mathbb{K}^\mu T\,|\psi_{\text{in}}\rangle\,. \tag{4.155}$$

This quantity is also integrated over the entire celestial sphere; we need a more differential observable. Furthermore, this expression is related to the energy emitted, rather than the amplitude of the emitted wave.

[8]Here we are not considering collinear divergences, which play a huge role in QCD.

We can use Mellin transforms to extract a more restricted observable, passing through the spectral waveform to relate the emitted power to the amplitude. Write the expectation of the observable $\langle (k^0)^{z-1}\rangle$,

$$R(z) \equiv \langle (k^0)^{z-1}\rangle = \langle \psi_{\text{in}}|\, T^\dagger (\mathbb{K}^0)^{z-1} T\, |\psi_{\text{in}}\rangle\,. \tag{4.156}$$

The inverse Mellin transform is related to the unpolarized energy density function,

$$f_\epsilon(E) = -iE \int_{c-i\infty}^{c+i\infty} \mathrm{d}z\; E^{-z} R(z)\,, \tag{4.157}$$

where the integral is taken along a line parallel to the imaginary axis, with $c \in (0,1)$ (or a deformation of that contour that doesn't cross any poles or branch points)[9]. The total energy is given by the integral,

$$E_{\text{tot}} = \int_0^\infty \mathrm{d}E\, f_\epsilon(E)\,. \tag{4.158}$$

Using the form in eq. (3.38) of [166], we can write,

$$R(z) = \sum_X \int \mathrm{d}\Phi(k)\mathrm{d}\Phi(r_1)\mathrm{d}\Phi(r_2)\; (k_X^0)^{z-1} \sum_{\sigma=\pm 1} |\hat{\mathcal{R}}(k^\sigma, r_X)|^2\,, \tag{4.159}$$

for the expression in the quantum theory. In this equation, $\hat{\mathcal{R}}$ represents the quantum radiation kernel, given by the integral over wavefunctions inside the absolute square in eq. (3.38). The quantum radiation kernel is expressed directly in terms of a scattering amplitude.

In the classical limit, the density function is more naturally a function of frequency rather than of energy,

$$f_{\epsilon,\text{cl}}(\omega) = -i\,\omega \int_{c-i\infty}^{c+i\infty} \mathrm{d}z\; \omega^{-z} R_{\text{cl}}(z)\,, \tag{4.160}$$

so that $R_{\text{cl}}(z) = \hbar^{-z-1} R(z)$. We can use eqs. (4.40–4.41) of [166] to write,

$$R_{\text{cl}}(z) = \sum_X \hbar^{-z-1} \left\langle\!\!\left\langle \int \mathrm{d}\Phi(k)\, (k_X^0)^{z-1} \sum_{\sigma=\pm 1} |\mathcal{R}(k^\sigma, r_X)|^2 \right\rangle\!\!\right\rangle. \tag{4.161}$$

The radiation kernel here is expressed in terms of the appropriate limit of a quantum scattering amplitude.

We next need to restrict the measured radiation from the entire celestial sphere to a narrow cone in a given direction. We take the limit of the cone, and measure only the radiation in a given direction from the scattering event. We implicitly assume that the measurement distance is much larger than the impact parameter, so that there is a unique and well-defined direction. It's not clear exactly what a formal expression for the operator would be, but what we want is,

$$\mathbb{K}^\mu\, \delta^{(2)}(\hat{\boldsymbol{k}} - \hat{\boldsymbol{n}})\,, \tag{4.162}$$

[9] With our conventions, the expected power of $(2\pi)^{-1}$ is in the forward rather than the inverse Mellin transform.

for radiation in the $\hat{\mathbf{n}}$ direction. This operator is to be understood as inserting,

$$\sum_{i\in\text{messengers}} k_i^\mu \, \delta^{(2)}(\hat{\mathbf{k}}_i - \hat{\boldsymbol{n}}) \,, \tag{4.163}$$

into a sum over states or equivalently the phase-space integral. Focusing on the energy component, this can be understood as a light ray operator [3, 144–147] given by,

$$\tilde{\mathcal{E}}(\hat{\mathbf{n}}) = \int_{-\infty}^{+\infty} \mathrm{d}v \lim_{r\to\infty} r^2 T_{vv}\,(v, r, \hat{\mathbf{n}}) \tag{4.164}$$

where v denotes the light-cone time $v = t - r$ and $T_{vv}\,(v, r, \hat{\mathbf{n}})$ is the (light-cone) time-time component of the stress-energy tensor (in gravity, this will be replaced by the Bondi news squared operator [3]). By applying the saddle point approximation for the fields in the energy momentum tensor, the plane wave expansion will localize to the point on the sphere in the direction of propagation. Schematically we will have (see refs. [10, 230] for further details)

$$e^{ix\cdot k/\hbar} = e^{i\omega v + i\omega r\left(1 - \hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\right)} \overset{r\to\infty}{\sim} \frac{1}{i\omega r} e^{i\omega v} \delta^{(2)}\left(\hat{\mathbf{n}} - \hat{\mathbf{k}}\right) \tag{4.165}$$

where $\omega = \bar{k}^0$. Then one finds,

$$\tilde{\mathcal{E}}(\hat{\mathbf{n}}) = \sum_{\sigma=\pm 1} \int \mathrm{d}\Phi(k)\, k^0\, \delta^{(2)}\left(\hat{\mathbf{n}} - \hat{\mathbf{k}}\right) \left[a_\sigma^\dagger(k) a_\sigma(k)\right] \tag{4.166}$$

where the action on on-shell particle states is equivalent to the time component of eq. (4.163). The analogous Mellin kernel for $(\mathbb{K}^0)^{z-1}$ is presumably,

$$(\mathbb{K}^0)^{z-1}\, \delta^{(2)}(\hat{\mathbb{K}} - \hat{\boldsymbol{n}}) \,, \tag{4.167}$$

which is to be understood as inserting,

$$\sum_{\substack{i\in \text{distinct} \\ \text{messengers}}} \Bigg(\sum_{\substack{j\parallel i \\ j\in\text{messengers}}} k_j^0 \Bigg)^{z-1} \delta^{(2)}(\hat{\mathbf{k}}_i - \hat{\boldsymbol{n}}) \,, \tag{4.168}$$

into a sum over states or the phase-space integral. The sum over distinct messengers is a sum over messengers which are not collinear; the sum over the collinear messengers is taken in the inner sum. The inner sum includes i itself.

This form is motivated by a subtlety about overlapping directions: if $\hat{\mathbf{k}}_j = \hat{\mathbf{k}}_l$ with the remaining directions distinct we want,

$$\sum_{\substack{i\in\text{messengers} \\ i\neq j,l}} (k_i^0)^{z-1}\, \delta^{(2)}(\hat{\mathbf{k}}_i - \hat{\boldsymbol{n}}) + (k_j^0 + k_l^0)^{z-1}\, \delta^{(2)}(\hat{\mathbf{k}}_j - \hat{\boldsymbol{n}}) \,, \tag{4.169}$$

which is what eq. (4.168) is designed to give. At leading order this subtlety is irrelevant.

The analog to eq. (4.159) is,

$$
\begin{aligned}
R(z,\hat{\boldsymbol{n}}) = \sum_{\substack{i\in\text{ distinct}\\ \text{messengers}}} \sum_X \int \mathrm{d}\Phi(k_i)\mathrm{d}\Phi(r_1)\mathrm{d}\Phi(r_2) & \Big(\sum_{\substack{j\parallel i\\ j\in\text{messengers}}} k_j^0\Big)^{z-1} \\
& \times \delta^{(2)}(\hat{\mathbf{k}}_i-\hat{\boldsymbol{n}}) \sum_{\sigma=\pm1} |\hat{\mathcal{R}}(k_i^\sigma,r_X)|^2\,,
\end{aligned}
\tag{4.170}
$$

and to eq. (4.161),

$$
R_{\rm cl}(z,\hat{\boldsymbol{n}}) = \sum_{\substack{i\in\text{ distinct}\\ \text{messengers}}} \hbar^{-z-1}\Big\langle\!\!\Big\langle \int \mathrm{d}\Phi(k_i)\Big(\sum_{\substack{j\parallel i\\ j\in\text{messengers}}} k_j^0\Big)^{z-1}\delta^{(2)}(\hat{\mathbf{k}}_i-\hat{\boldsymbol{n}})\sum_{\sigma=\pm1}|\mathcal{R}(k_i^\sigma,r_X)|^2\Big\rangle\!\!\Big\rangle\,.
\tag{4.171}
$$

At leading order, eq. (4.171) simplifies to just,

$$
R^{(0)}_{\rm cl}(z,\hat{\boldsymbol{n}}) = g^6\Big\langle\!\!\Big\langle \int \mathrm{d}\Phi(\bar{k})\,(\bar{k}^0)^{z-1}\,\delta^{(2)}(\hat{\mathbf{k}}-\hat{\boldsymbol{n}})\sum_{\sigma=\pm1}\Big|\mathcal{R}^{(0)}(\bar{k}^\sigma;b)\Big|^2\Big\rangle\!\!\Big\rangle\,.
\tag{4.172}
$$

The corresponding result for the spectral density in the $\hat{\boldsymbol{n}}$ direction is,

$$
f_{\epsilon,\rm cl}(\omega,\hat{\boldsymbol{n}}) = g^6\omega\Big\langle\!\!\Big\langle \int \mathrm{d}\Phi(\bar{k})\,\frac{\delta(\ln\bar{k}^0-\ln\omega)}{\bar{k}^0}\,\delta^{(2)}(\hat{\mathbf{k}}-\hat{\boldsymbol{n}})\sum_{\sigma=\pm1}\Big|\mathcal{R}^{(0)}(\bar{k}^\sigma;b)\Big|^2\Big\rangle\!\!\Big\rangle\,.
\tag{4.173}
$$

Writing out,

$$
\begin{aligned}
\mathrm{d}\Phi(\bar{k}) &= \frac{d^3\bar{\mathbf{k}}}{2(2\pi)^3\,|\bar{\mathbf{k}}|} \\
&= \frac{|\bar{\mathbf{k}}|d|\bar{\mathbf{k}}|\,d\Omega_{\bar{\mathbf{k}}}}{2(2\pi)^3}\,,
\end{aligned}
\tag{4.174}
$$

we can perform the integrals in eq. (4.173) to obtain,

$$
f_{\epsilon,\rm cl}(\omega,\hat{\boldsymbol{n}}) = \frac{g^6\omega^2}{8\pi^2}\sum_{\sigma=\pm1}\Big\langle\!\!\Big\langle\Big|\mathcal{R}^{(0)}(\omega(1,\hat{\boldsymbol{n}})^\sigma;b)\Big|^2\Big\rangle\!\!\Big\rangle\,.
\tag{4.175}
$$

We can now compare this with the amplitude of each component of the waveform, expanded at the leading order order in the coupling: for $|f_{\mu\nu}M^{*\mu}N^\nu|$ and $|f_{\mu\nu}M^\mu N^\nu|$ we have, respectively

$$
\begin{aligned}
|f_{\mu\nu}(\omega(1,\hat{\boldsymbol{n}}))M^{*\mu}N^\nu| &= \frac{\omega}{16\pi}g^3\Big|\Big\langle\!\!\Big\langle\,\mathcal{R}^{(0)}(\omega(1,\hat{\boldsymbol{n}})^-;b)\Big\rangle\!\!\Big\rangle\Big| \\
|f_{\mu\nu}(\omega(1,\hat{\boldsymbol{n}}))M^{\mu}N^\nu| &= \frac{\omega}{16\pi}g^3\Big|\Big\langle\!\!\Big\langle\,\mathcal{R}^{(0)}(\omega(1,\hat{\boldsymbol{n}})^+;b)\Big\rangle\!\!\Big\rangle\Big|
\end{aligned}
\tag{4.176}
$$

At leading order, we can also write

$$
\Big\langle\!\!\Big\langle\Big|\mathcal{R}^{(0)}(\omega(1,\hat{\boldsymbol{n}})^\sigma;b)\Big|^2\Big\rangle\!\!\Big\rangle = \Big|\Big\langle\!\!\Big\langle\,\mathcal{R}^{(0)}(\omega(1,\hat{\boldsymbol{n}})^\sigma;b)\Big\rangle\!\!\Big\rangle\Big|^2
\tag{4.177}
$$

and therefore we can express the spectral density of emission from eq. (4.175) in terms of the amplitudes of the two helicity components of the waveform,

$$f_{\epsilon,\mathrm{cl}}(\omega,\hat{\boldsymbol{n}}) = 32\big[|f_{\mu\nu}(\omega(1,\hat{\boldsymbol{n}}))M^{*\mu}N^{\nu}|^2 + |f_{\mu\nu}(\omega(1,\hat{\boldsymbol{n}}))M^{\mu}N^{\nu}|^2\big]\,. \tag{4.178}$$

This relation is the avatar of the relation between the energy of the wave and the squared amplitude of the wave, the only difference being that here we are measuring the momentum emitted in a given direction at a large distance r from the source. The emitted radiation observable provides information about the magnitude of the observed messenger wave, but not about its phase. The direct derivation in previous sections adds that information. A recently proposed generalization of a standard event shape is sensitive to amplitude phases [265]. It would be interesting to explore a possible connection to the waveform.

4.6 BMS frame from amplitudes and static contributions to the waveform

One interesting aspect of studying the asymptotic structure at null infinity is that it impose constraints on the structure of the waveform and other radiative observables. In particular, the displacement memory effect must be included consistently in the generation of the waveform catalogue as stressed in [266]. In section 4.3, we have argued that that the waveform receives contributions at leading order from the five-point tree amplitude with the emission of one messenger, and we have studied the connected piece to compute the time-dependent contribution to the waveform. Here, we consider the disconnected piece in the gravitational setup.

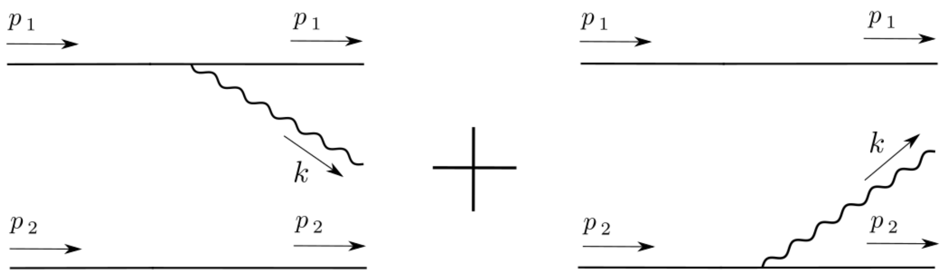


FIGURE 4.7: The picture shows the static contribution of zero-energy graviton emissions to the waveform at the lowest order.

This disconnected piece, pictured in Fig. 4.7, involves the product of the three-point amplitude and one on-shell delta function:

$$\langle k_1^{\sigma_1} p_1' p_2' | T | p_1 p_2\rangle = \langle k_1^{\sigma_1} p_1' | T | p_1\rangle \delta_\Phi(p_2,p_2') + \langle k_1^{\sigma_1} p_2' | T | p_2\rangle \delta_\Phi(p_1,p_1'). \tag{4.179}$$

The on-shell kinematics implies that the energy of the graviton must be exactly zero: we can therefore apply exactly Weinberg soft theorem

$$\begin{aligned}\langle k_1^{\sigma_1} p_1' | T | p_1\rangle &= \lim_{E_{k_1}\to 0} \frac{\kappa}{2}\varepsilon^{\sigma_1}_{\mu\nu}(k_1) p_1^\mu p_1^\nu \left(\frac{1}{k_1\cdot p_1 - i\epsilon} - \frac{1}{k_1\cdot p_1 + i\epsilon}\right)\delta_\Phi(p_1,p_1') \\ &= \pi i \kappa \varepsilon^{\sigma_1}_{\mu\nu}(k_1) p_1^\mu p_1^\nu \delta(k_1\cdot p_1)\delta_\Phi(p_1,p_1'). \end{aligned}\tag{4.180}$$

Therefore, the disconnected contribution to the five-point tree amplitude is of order $\mathcal{O}(\kappa)$ and it is highly degenerate because it is supported on zero-energy kinematics for the external graviton. We can easily get the classical contribution,

$$\begin{aligned}\langle k_1^{\sigma_1} p_1' p_2' | T | p_1 p_2 \rangle &= \pi i \kappa \varepsilon^{\sigma_1}_{\mu\nu}(k_1) \left[p_1^\mu p_1^\nu \delta(k_1 \cdot p_1) + p_2^\mu p_2^\nu \delta(k_1 \cdot p_2) \right] \delta_\Phi(p_1, p_1') \delta_\Phi(p_2, p_2') \\ &\overset{\hbar \to 0}{\sim} \pi i \frac{\kappa}{\hbar^{3/2}} \varepsilon^{\sigma_1}_{\mu\nu}(\bar{k}_1) \left[m_A v_A^\mu v_A^\nu \delta(\bar{k}_1 \cdot v_A) + m_B v_B^\mu v_B^\nu \delta(\bar{k}_1 \cdot v_B) \right] \\ &\qquad \times \delta_\Phi(p_1, p_1') \delta_\Phi(p_2, p_2'). \end{aligned} \tag{4.181}$$

If we compute the expectation value of the graviton field in the classical limit

$$\begin{aligned}\langle \psi_{\text{in}} | S^\dagger h_{\mu\nu}(x) S | \psi_{\text{in}} \rangle &= 2\Re \frac{1}{\sqrt{\hbar}} \sum_{\sigma_1 = \pm 2} \int \mathrm{d}\Phi(p_1', p_2', p_1, p_2, k_1) \psi_b^*(p_1', p_2') \psi_b(p_1, p_2) \\ &\qquad \times \langle k_1^{\sigma_1} p_1' p_2' | T | p_1 p_2 \rangle \, \varepsilon^{*\sigma_1}_{\mu\nu}(\bar{k}_1) \, e^{-i \bar{k}_1 \cdot x} \\ &= 2\Re \frac{1}{\sqrt{\hbar}} \sum_{\sigma_1 = \pm 2} \int \mathrm{d}\Phi(p_1, p_2, k_1) |\psi_A(p_1)|^2 |\psi_B(p_2)|^2 \, \varepsilon^{\sigma_1}_{\alpha\beta}(\bar{k}_1) \varepsilon^{*\sigma_1}_{\mu\nu}(\bar{k}_1) \\ &\qquad \times \pi i \frac{\kappa}{\hbar^{3/2}} \left[m_A v_A^\alpha v_A^\beta \delta(\bar{k}_1 \cdot v_A) e^{i \bar{k}_1 \cdot b} + m_B v_B^\alpha v_B^\beta \delta(\bar{k}_1 \cdot v_B) \right] \, e^{-i \bar{k}_1 \cdot x} \\ &= 2\Re \int \mathrm{d}\Phi(\bar{k}_1) \sum_{\sigma_1 = \pm 2} \left[\varepsilon^{\sigma_1}_{\alpha\beta}(\bar{k}_1) \varepsilon^{*\sigma_1}_{\mu\nu}(\bar{k}_1) \right] \, e^{-i \bar{k}_1 \cdot x} \\ &\qquad \times i \pi \kappa \left[m_A v_A^\alpha v_A^\beta \delta(\bar{k}_1 \cdot v_A) e^{i \bar{k}_1 \cdot b} + m_B v_B^\alpha v_B^\beta \delta(\bar{k}_1 \cdot v_B) \right] \\ &= 2 \, \Re \int \mathrm{d}\Phi(\bar{k}_1) \, i \sum_{j = A,B} \left[\frac{\kappa}{2} m_j P_{\mu\nu\alpha\beta} v_j^\alpha v_j^\beta (2\pi) \delta(\bar{k}_1 \cdot v_j) e^{i \bar{k}_1 \cdot b_j} \right] \, e^{-i \bar{k}_1 \cdot x}, \end{aligned}$$

where $b_j = b$ (resp. $b_j = 0$) for $j = A$ (resp. $j = B$) and

$$P_{\mu\nu\alpha\beta} := \frac{1}{2} \left[P_{\mu\alpha} P_{\nu\beta} + P_{\mu\beta} P_{\nu\alpha} - P_{\mu\nu} P_{\alpha\beta} \right], \tag{4.182}$$

as defined earlier in eq. (4.133). This matches exactly with the time-independent contribution to the waveform discussed in [131, 178], once we consider the leading $1/r$ piece of the field at large distances

$$h_{\mu\nu}(t, r, \hat{\boldsymbol{n}}) \sim \frac{1}{4\pi r} \int \frac{\mathrm{d}\omega_1}{2\pi} \, e^{-i\omega_1 (t - r)} \sum_{j = A,B} \frac{\kappa}{2} m_j P_{\mu\nu\alpha\beta} v_j^\alpha v_j^\beta (2\pi) \delta(\omega_1 (1, \hat{\boldsymbol{n}}) \cdot v_j) e^{i \bar{k}_1 \cdot b_j}. \tag{4.183}$$

Our result in eq. (4.183) depends only on zero-energy physics, and it might seem to be unphysical. But this is not the case, and the story is much more subtle. This contribution is related to the choice of the BMS frame at null infinity at the amplitude level and therefore it is a related to a gauge choice. Indeed, the vacuum in the full non-linear general relativity is labelled by the value of the soft BMS supertranslation charge [228], which is directly related to the contribution of zero-energy gravitons as discussed in section 4.4.1. Indeed, these terms do not contribute to the total energy emitted in gravitational waves but they will affect other observables like the total emitted angular momentum [131, 178], due to a known classical problem in GR known as BMS supertranslation ambiguity [267, 268]. Therefore, a byproduct of this analysis is that amplitudes with zero-energy gravitons give a new perspective on the choice of the BMS frame for radiative observables, which would be definitely worth investigating in the future.

Chapter 5

Coherent states from the S-matrix

Coherent states have a long history in applications to semi-classical physics [193, 269] as well as more recent interest [4, 82, 106, 128, 270]. In the very low-energy regime, a single coherent state provides the exact quantum state of radiation and taking the limit of the large number of quanta reproduce known classical limit results for the energy flux [255, 271–273]. A lot of attention has been devoted so far to the soft expansion where coherent states arise naturally from classical currents [106, 167, 170, 256, 263, 274], but the dynamics of how such states are generated by the full two-body classical dynamics is much less clear.

In this chapter, we will try to address this question from different perspectives. We will start by looking at the soft dynamics from the worldline perspective, where at the leading order it is possible to show analytically the emergence of the coherent state from the path integral. Then we will use the uncertainty principle introduced ealier in chapter 4 to impose the zero-variance classical requirement to the most generic density matrix in QFT written in the P-representation, both in the radiative and in the spin sector, and we will see how coherent states arise from this perspective. An alternative and more direct insight into the problem is offered by studying the properties of the particle distribution, like the mean, the variance and the factorial moments: as we will see, the deviation from coherence is an infrared safe concept which can be studied in perturbation theory. Finally, we will also establish a connection with asymptotic symmetries and to the definition of an infrared finite S-matrix in four dimensions.

5.1 Coherent states from the soft dynamics

In this section, we would like to provide a derivation of the emergence of coherent states from the soft dynamics in scalar QED in the classical scattering process of two charged point particles. While [119] considered the conservative case, here we extend the discussion to real radiation. This can be done using established methods which have been used in relatively recent literature on the eikonal approximation [275, 276]. We start from an incoming state of the form

$$|\psi_{\rm in}\rangle = \int \mathrm{d}\Phi(p_1,p_2) \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\tilde{\psi}(x_1,x_2)\,e^{i(p_1\cdot x_1+p_2\cdot x_2)/\hbar}e^{i(b\cdot p_1)/\hbar}\,|p_1p_2\rangle\,, \qquad (5.1)$$

and our aim is to show that this should evolve over time to a state

$$
\begin{aligned}
|\psi_{\text{out}}\rangle = &\int \mathrm{d}\Phi(p_1', p_2') \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}(x_1, x_2)\, e^{i(p_1'\cdot x_1 + p_2'\cdot x_2)/\hbar} \\
&\times \int \hat{\mathrm{d}}^4 q\, \mathrm{d}^4 x\, e^{iq\cdot x/\hbar} e^{ib\cdot p_1'/\hbar} e^{-ib\cdot q/\hbar} e^{iq\cdot(x_2 - x_1)/\hbar} e^{i\chi(x_\perp; s)/\hbar} \\
&\times \exp\left[\sum_{\sigma=\pm1} \int \mathrm{d}\Phi(k) \beta^{(\sigma)}(k, x_1, x_2) a_\sigma^\dagger(k)\right] |p_1', p_2'\rangle\,,
\end{aligned}
\tag{5.2}
$$

where we use β rather than α to refer to the coherent state parameter, as we will ultimately calculate this only in a particular limit, namely the forward approximation in which they follow classical straight-line trajectories. In such limit we are justified to effectively obtain

$$
\begin{aligned}
|\psi_{\text{out}}\rangle = &\int \mathrm{d}\Phi(p_1', p_2') \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}(x_1, x_2)\, e^{i(p_1'\cdot x_1 + p_2'\cdot x_2)/\hbar} \\
&\times e^{ib\cdot p_1'/\hbar} e^{i\chi(x_\perp - b_\perp; s)/\hbar} \exp\left[\sum_{\sigma=\pm1} \int \mathrm{d}\Phi(k) \beta^{(\sigma)}(k, x_1, x_2) a_\sigma^\dagger(k)\right] |p_1', p_2'\rangle\,.
\end{aligned}
\tag{5.3}
$$

The situation being considered here is similar to the analysis of [277], which considered particles emerging from an amplitude interacting with soft radiation (see reference [276] for the gravitational case). This analysis used path integral methods — also known as the Schwinger proper time formalism[1] — to write the propagators for the outgoing particles in terms of explicit sums over their spacetime trajectories. This provided a very physical picture for describing soft radiation, allowing the authors to generalise beyond the leading soft approximation, and to show that certain sets of corrections exponentiate in perturbation theory. However, only virtual radiation was included, and thus we must extend such methods to the real radiative case being considered here. We restrict our discussion to the case of QED for simplicity.

Let us first recall a useful result from the Schwinger formalism (reviewed here in appendix A), namely that the propagator for a particle in a background gauge field, produced at position x_i and ending up with final momentum p_f, can be written as

$$
D_F\,(x_i, p_f) = \int_0^\infty \mathrm{d}T e^{-T\epsilon/\hbar}\, \langle p_f| e^{-i(\hat{H}T)/\hbar} |x_i\rangle, \tag{5.4}
$$

where

$$
\langle p_f| e^{-i\hat{H}T/\hbar} |x_i\rangle = \int_{x(0)=x_i}^{p(T)=p_f} \mathcal{D}p\mathcal{D}x \exp\left[\frac{i}{\hbar} p(T)\cdot x(T) + \frac{i}{\hbar}\int_0^T \mathrm{d}t(-p\cdot\dot{x} - H(p,x))\right] \tag{5.5}
$$

is a double path integral in position and momentum space, subject to the above boundary conditions, and[2]

$$
\hat{H} = -(\hat{p} - e\hat{A})^2 + m^2 = -\hat{p}^2 + e\hat{p}\cdot\hat{A} + e\hat{A}\cdot\hat{p} - e^2\hat{A}^2 + m^2 \tag{5.6}
$$

is the Hamiltonian, where e is the coupling constant and $\hat{A}^\mu$ is the gauge field interpreted as an operator in the quantum mechanical Hilbert space. As discussed in

[1] A useful pedagogical review of Schwinger proper time methods may be found in ref. [278].

[2] As usual in the context of Schwinger proper time [278] we choose a convenient normalisation for "time" t and the associated Hamiltonian H. In particular the dimensions of H are mass^2.

appendix A, we should Weyl-order the Hamiltonian so that all momentum operators are to the left. Care must be taken with the term $\hat{A} \cdot \hat{p}$, where we note that we will be operating on momentum states to the left:

$$\langle p|\hat{A} \cdot \hat{p} = \hat{p}_\mu(\hat{A}^\mu)\langle p| + \langle p|\hat{p} \cdot \hat{A}\,. \tag{5.7}$$

Here $\hat{p}_\mu\,(A^\mu)$ represents the action of $\hat{p}_\mu$ on $\hat{A}^\mu$. Using the fact that $\hat{p} = -i\partial_\mu$ in position space, we can get rid of this term by using the Lorenz gauge $\partial_\mu A^\mu = 0$, so that our Hamiltonian becomes

$$\hat{H} = -\hat{p}^2 + 2e\hat{p} \cdot \hat{A} - e^2\hat{A}^2 + m^2. \tag{5.8}$$

Plugging this into eq. (5.4) and using eq. (5.5), we obtain a path integral representation for the propagator of a scalar particle in the presence of a gauge field:

$$\begin{aligned} D_F\,(x_i, p_f; A) = &\int_0^\infty \mathrm{d}T e^{-T\epsilon/\hbar} \int_{x(0)=x_i}^{p(T)=p_f} \mathcal{D}p\mathcal{D}x \\ &\times \exp\left[\frac{i}{\hbar}p(T)\cdot x(T) + \frac{i}{\hbar}\int_0^T \mathrm{d}t\left(-p\cdot\dot{x} + p^2 - 2ep\cdot A + e^2A^2 - m^2\right)\right]. \end{aligned} \tag{5.9}$$

It is of course impossible to carry out this path integral in general; this would amount to exactly solving for a quantum particle moving in an arbitrary electromagnetic field! But we can evaluate it approximately in many different cases. Relevant for our purposes is if a particle has position x_i at $t = 0$, and follows an approximate straight-line trajectory, given by

$$x_c^\mu = x_i^\mu + p_f^\mu t, \quad 0 \le t \le T \tag{5.10}$$

where p_f is the final momentum introduced above. In the sum over trajectories in eq. (5.9), we can then redefine

$$x(t) \to x_c(t) + x(t), \quad p(t) \to p_f + p(t)\,, \tag{5.11}$$

where now $x(t)$ and $p(t)$ are small fluctuations. Substituting this into eq. (5.9), one finds

$$\begin{aligned} D_F(x_i, p_f; A) = &\int_0^\infty \mathrm{d}T e^{-T\epsilon/\hbar} e^{\frac{i}{\hbar}p_f\cdot x_i + \frac{i}{\hbar}T\left(p_f^2 - m^2\right)} \\ \times &\int_{x(0)=0}^{p(T)=0} \mathcal{D}p\mathcal{D}x \exp\left[\frac{i}{\hbar}\int_0^T \mathrm{d}t\left(p\cdot(p_f - \dot{x}) + p^2 - 2ep\cdot A - 2ep_f\cdot A + e^2A^2\right)\right]. \end{aligned} \tag{5.12}$$

This still looks rather formidable, but it will simplify considerably in what follows.

Let us now apply this to the problem of a pair of propagating particles, interacting via photon exchange, in the conservative case. We will consider two different scalar fields ϕ_1 and ϕ_2, so that the scattering particles can in principle be different. We may then consider the 4-point Green's function:

$$\begin{aligned} G_4(\phi_1(x_1), \phi_1(x_2), &\phi_2(x_3), \phi_2(x_4)) \\ &= \int \mathcal{D}A_\mu \mathcal{D}\phi_1 \mathcal{D}\phi_2\, \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_2(x_4) e^{\frac{i}{\hbar}S(\phi_1,\phi_2,A_\mu)}, \end{aligned} \tag{5.13}$$

which will ultimately be related to S-matrix elements for the scattering particles, and we may separate the total classical action $S(\phi_1, \phi_2, A_\mu)$ into three distinct terms:

$$S(\phi_1, \phi_2, A_\mu) = S_A(A_\mu) + S_1(\phi_1, A_\mu) + S_2(\phi_2, A_\mu). \tag{5.14}$$

The first term on the right-hand side consists of terms involving only the gauge field A_μ, and $S_i\,(\phi_i, A_\mu)$ contains terms involving the individual scalar field ϕ_i only, or its coupling to the gauge field. Substituting eq. (5.14) into eq. (5.13), this may be rewritten as

$$\begin{aligned} G_4(\phi_1(x_1), \phi_1(x_2), &\phi_2(x_3), \phi_2(x_4)) \\ &= \int \mathcal{D}A_\mu G_2\,(\phi_1(x_1)\phi_1(x_2); A)\, G_2\,(\phi_2(x_3)\phi_2(x_4); A)\, e^{\frac{i}{\hbar}S_A(A_\mu)}\,, \end{aligned} \tag{5.15}$$

where

$$G_2\,(\phi_i(x)\phi_i(y); A) = \int \mathcal{D}\phi_i\, \phi_i(x)\phi_i(y)\, e^{\frac{i}{\hbar}S_i(\phi_i, A)}\,, \tag{5.16}$$

is the two-point function for the field ϕ_i in the presence of a background gauge field. This is simply the propagator, and indeed is almost exactly the object that we worked out in eq. (5.12). The only difference is that eq. (5.12) has the scalar particle moving from a state of given initial position and final momentum, while in eq. (5.16) both initial and final positions are specified. The form of eq. (5.12) is convenient for our problem given that we wish to consider particles that are separated by an definite distance Δx. Let us therefore consider a pair of particles "produced" at positions z_i, so that $z_1 = x_1$ and $z_2 = x_3$. Then we can use translational invariance to set

$$z_1^\mu = \Delta x^\mu, \quad z_2^\mu = 0\,. \tag{5.17}$$

Each particle i propagates out to infinity, ending up with a final momentum p_i', and so we are interested in the Green's function

$$G_4\left(\phi_1(z_1), \phi_1(p_1'), \phi_2(z_2), \phi_2(p_2')\right) = \int \mathcal{D}A_\mu\, D_F\left(z_1, p_1'; A\right) D_F\left(z_2, p_2'; A\right) e^{\frac{i}{\hbar}S_A(A_\mu)}\,, \tag{5.18}$$

where we used the notation for the propagators in eq. (5.12). To turn this into a transition matrix element, we need to truncate the free propagators for the final state particles according to the LSZ prescription. In other words, each of the full scalar field propagators will be modified

$$\begin{aligned} -i\left(p_i'^2 - m_i^2\right) D_F\left(z_i, p_i'; A\right) &= -i\left(p_i'^2 - m_i^2\right) \int_0^\infty \mathrm{d}T e^{-T\epsilon/\hbar} e^{\frac{i}{\hbar}p_i'\cdot z_i + \frac{i}{\hbar}T\left(p_i'^2 - m_i^2\right)} f_i(0,T) \\ &= -e^{\frac{i}{\hbar}p_i'\cdot z_i} \int_0^\infty \mathrm{d}T e^{-T\epsilon/\hbar} f_i(0,T) \frac{d}{dT} e^{\frac{i}{\hbar}T\left(p_i'^2 - m_i^2\right)}. \end{aligned} \tag{5.19}$$

Again we used eq. (5.12), and defined

$$f_i(0,T) = \int_{x_i(0)=0}^{p(T)=0} \mathcal{D}p\mathcal{D}x \exp\left[\frac{i}{\hbar}\int_0^T \mathrm{d}t\left(p\cdot\left(p_i' - \dot{x}\right) + p^2 - 2ep\cdot A - 2ep_i'\cdot A + e^2A^2\right)\right]. \tag{5.20}$$

Integrating by parts and enforcing the on shell constraint $p_i'^2 \to m_i^2$ gives

$$-i\left(p_i'^2 - m_i^2\right) D_F\left(z_i, p_i'; A\right) \to e^{i(p_i'\cdot z_i)/\hbar} f(0,\infty)\,. \tag{5.21}$$

Combining this with eq. (5.18) and using the initial positions of eq. (5.17), we find that the partially truncated Green's function associated with our chosen process is

$$G_4\left(p_1', p_2'; b\right)\Big|_{3,4} = \int \mathcal{D}A_\mu\, e^{i(\Delta x\cdot p_1')/\hbar} f_1(0,\infty) f_2(0,\infty) e^{\frac{i}{\hbar}S_A(A_\mu)}, \tag{5.22}$$

where the notation on the left-hand side denotes those particles for which the LSZ reduction has been carried out. After shifting $p \to p + eA$ in the integration over p, eq. (5.20) implies

$$f_i(0,\infty) = \int_{x(0)=0}^{p(T)=0} \mathcal{D}p\mathcal{D}x \exp\left\{\frac{i}{\hbar}\int_0^\infty \mathrm{d}t\left[\left(p - \frac{\dot{x}}{2} + \frac{p_i'}{2}\right)^2 - \frac{\dot{x}^2}{4} + \frac{p_i'\cdot\dot{x}}{2} - \frac{m_i^2}{4} - e\dot{x}\cdot A - ep_i'\cdot A\right]\right\}. \tag{5.23}$$

The p integral is Gaussian and can be absorbed into the overall normalisation of the path integral of eq. (5.20) (as can the term in m_i^2). One then has

$$f_i(0,\infty) = \int \mathcal{D}x \exp\left\{\frac{i}{\hbar}\int_0^\infty \mathrm{d}t\left[-\frac{\dot{x}^2}{4} + \frac{1}{2}p_i'\cdot\dot{x} - e\dot{x}\cdot A - ep_i'\cdot A\right]\right\}. \tag{5.24}$$

Eq. (5.22) has a nice interpretation: to represent the (half-truncated) Green's function for scalar particles interacting via a gauge field, one can describe the passage of each particle by a factor representing the sum over possible trajectories, weighted by an "action" containing the interaction of the particle with the gauge field. It is possible to carry out the path integral in eq. (5.24) perturbatively, which corresponds to summing over the various wobbles that the trajectory can have. These wobbles are caused by interactions with the gauge field, as one expects: each wobble corresponds to a recoil against an emitted photon. Then the path integral over the gauge field in eq. (5.22) sums over all possible photon emissions between the scalar particle lines.

Above, we have only carried out the LSZ reduction for the outgoing particles 3 and 4. We must also carry out the reduction for particles 1 and 2. That this is not as simple as in eq. (5.19) ultimately stems from the fact that the lower limit of the Schwinger proper time integral in eq. (5.4) is zero, rather than minus infinity. It is possible to transform proper time coordinates so that the LSZ reduction for incoming particles can be carried out in the above formulae, as argued recently in ref. [124]. An alternative approach was presented, some time ago, in ref. [279]. Here we will take a more pedestrian approach, and simply consider that the above argument has provided only half of the full four-point Green's function, as shown in Fig. 5.1(a). That is, the particles at the origin and Δx^μ are off-shell and propagate out to form the final states with momenta $\{p_i'\}$. We can then easily fill in the remaining half of the scattering process, as shown in Fig. 5.1(b), by appending the integrand of eq. (5.22) with two additional f-factors for the incoming particles:

$$\begin{aligned} S\left(\{p_i\},\{p_i'\};\Delta x\right) &= \int \mathcal{D}A_\mu\, e^{i(\Delta x\cdot p_1')/\hbar} f_1(-\infty,0) f_1(0,\infty) f_2(-\infty,0) f_2(0,\infty) e^{\frac{i}{\hbar}S_A(A_\mu)} \\ &= \int \mathcal{D}A_\mu\, e^{i(\Delta x\cdot p_1')/\hbar} f_1(-\infty,\infty) f_2(-\infty,\infty) e^{\frac{i}{\hbar}S_A(A_\mu)}, \end{aligned} \tag{5.25}$$

where in the second line we have used the definition of the f-factors to combine them into a single factor associated with each incoming particle. On the left-hand side, we have acknowledged that the LSZ reduction has now been carried out for the incoming particles, so that eq. (5.25) constitutes an S-matrix element. As such, it includes the

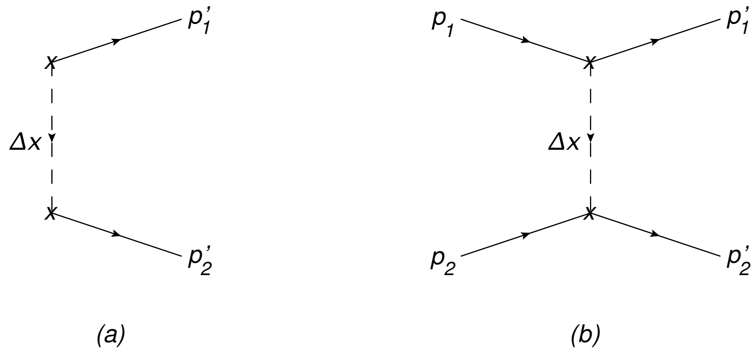


FIGURE 5.1: (a) The half-truncated 4-point Green's function considered in eq. (5.22), in which off-shell particles separated by a distance Δx^μ propagate out to form outgoing states; (b) the complete scattering process.

case of trivial scattering (which results if $A_\mu \to 0$). If we instead want the T-matrix element, we would replace eq. (5.25) with

$$T\left(\{p_i\},\{p_i'\};\Delta x\right) = e^{i(\Delta x\cdot p_1')/\hbar}\left[\int \mathcal{D}A_\mu\, f_1(-\infty,\infty)f_2(-\infty,\infty)e^{\frac{i}{\hbar}S_A(A_\mu)} - 1\right], \tag{5.26}$$

To make sense of these expressions, let us consider the leading contribution to the path integral for each incoming particle. This corresponds to the classical approximation in which the particles follow classical straight-line trajectories, such that the fluctuations $x^\mu(t) = \dot{x}^\mu = 0$. Then, one may simplify

$$f_i(-\infty,\infty) \to \exp\left[-\frac{i}{\hbar}e\int_{-\infty}^{\infty} \mathrm{d}t\, p_i'\cdot A\right] = \Phi_i\,. \tag{5.27}$$

The right-hand side is simply a Wilson line operator

$$\Phi_i = \exp\left[-\frac{i}{\hbar}e\int_{-\infty}^{\infty}\mathrm{d}x_i'^\mu A_\mu\right], \tag{5.28}$$

evaluated along the path

$$x_i'^\mu = z_i^\mu + p_i'^\mu t, \quad -\infty < t < \infty. \tag{5.29}$$

We have thus found that if our particles start at time $t = 0$ separated by an impact parameter b^μ, they evolve according to the operator

$$S\left(p_1', p_2'; \Delta x\right) = \int \mathcal{D}A_\mu e^{i(\Delta x\cdot p_1')/\hbar}\Phi_1\Phi_2 e^{\frac{i}{\hbar}S_A(A_\mu)}\,. \tag{5.30}$$

This amounts to saying that the amplitude is described by a vacuum expectation value of Wilson line operators, which is by no means a new observation: the description of the Regge (high energy) limit of $2\to 2$ scattering in terms of Wilson lines was given in QCD [280], and later generalised to gravity in [281, 282] (see also refs. [283, 284]). The QED case, however, is particularly simple. If we neglect pair production of the scalar fields, the path-integral in eq. (5.24) can be performed exactly. It may look more familiar if we write the Wilson line coupling in terms of a current:

$$-\frac{i}{\hbar}e\int \mathrm{d}x^\mu A_\mu \equiv -\frac{i}{\hbar}\int \mathrm{d}^4x J_i^\mu A_\mu, \quad J_i^\mu = ev_i^\mu \delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}(t))\,, \tag{5.31}$$

where the delta function localises the current onto the particle's worldline described by $\boldsymbol{z}(t)$. Then one may define an overall current

$$J = J_1 + J_2 \tag{5.32}$$

so that the path integral of eq. (5.30) assumes the familiar form

$$S\left(p_1', p_2'; \Delta x\right) = \int \mathcal{D}A_\mu e^{i(\Delta x\cdot p_1')/\hbar} e^{\frac{i}{\hbar}[S_A(A_\mu)-A_\mu J^\mu]}\,. \tag{5.33}$$

Given $S_A\left(A_\mu\right)$ is quadratic in QED, one finds

$$S\left(p_1', p_2'; \Delta x\right) = e^{i(\Delta x\cdot p_1')/\hbar}\exp\left[\frac{i}{\hbar}\int \mathrm{d}^4x\int \mathrm{d}^4y J^\mu(x) D_{\mu\nu}(x-y)J^\nu(y)\right]\,, \tag{5.34}$$

where $D_{\mu\nu}(x-y)$ is the photon propagator. This can be interpreted as follows. The exponent consists of all possible one-photon exchanges between the two Wilson lines (including those diagrams in which the photon may be emitted and absorbed by the same particle). This one-loop contribution then exponentiates, as we know it must! We did not have to force this property: it comes out simply from the formalism we are using. As was shown in e.g. [281], the one-loop VEV of Wilson lines generates the eikonal phase χ experienced by two interacting particles in precisely the situation we are examining. Note that this eikonal phase is only dependent on the transverse distance which we can identify with $x_\perp - b_\perp$. In general we expect it to be related to the 4-pt amplitude via eq.(7.6). Thus, we can write eq. (5.30) simply as[3]

$$S\left(p_1', p_2'; \Delta x\right) = e^{i(\Delta x\cdot p_1')/\hbar}e^{i\chi(x_\perp - b_\perp, s)/\hbar}. \tag{5.35}$$

In turn, this leads to a final state

$$\int \mathrm{d}\Phi(p_1', p_2')\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\;\tilde{\psi}(x_1,x_2)\,e^{i(p_1'\cdot x_1 + p_2'\cdot x_2)/\hbar}e^{i(\Delta x\cdot p_1')/\hbar}e^{i\chi(x_\perp - b_\perp, s)/\hbar}\left|p_1'p_2'\right\rangle\,. \tag{5.36}$$

At the leading order we're considering, there is no distinction between the impact parameter b^μ and Δx^μ and therefore we can equivalently write eq. (5.36) as

$$\int \mathrm{d}\Phi(p_1', p_2')\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\;\tilde{\psi}(x_1,x_2)\,e^{i(p_1'\cdot x_1 + p_2'\cdot x_2)/\hbar}e^{i(b\cdot p_1')/\hbar}e^{i\chi(x_\perp - b_\perp, s)/\hbar}\left|p_1'p_2'\right\rangle\,, \tag{5.37}$$

which agrees with eq. (7.7) within our approximation. In general, we need to relate $e^{i\chi(x_\perp - b_\perp, s)/\hbar}$ with the four-point amplitude via eq. (7.1) and this would imply that the relation between b^μ and Δx^μ is more subtle (see eq. (7.14)).

This has all been in the conservative regime with no radiation, and now we would like to extend this to the radiative case. To this end, we need to go back and include the contribution of the gauge field insertion in the Green's function correlator. In principle, we should allow any number (n) of photons and therefore we will generalise

[3] We stress that eq. (5.35) applies to the S-matrix. Substituting this into eq. (5.26) recovers the result that the scattering amplitude –related to the T-matrix – is given by $e^{i\chi} - 1$.

eq. (5.13) to the $(n+4)$-pt correlator

$$G_{n+4}(\phi_1(x_1), \phi_1(x_2), \phi_2(x_3), \phi_2(x_4), \{A_{\sigma_j}(x_j)\}_{j=5,\dots,n+4}) \\ = \int \mathcal{D}A_\mu \, \mathcal{D}\phi_1 \, \mathcal{D}\phi_2 \, \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_2(x_4)A_{\sigma_5}(x_5)\dots A_{\sigma_{n+4}}(x_{n+4})e^{\frac{i}{\hbar}S(\phi_1,\phi_2,A_\mu)}. \tag{5.38}$$

We may then carry out similar steps to above to find the S-matrix element for the radiative corrections to the process of Fig. 5.1(b). The result is (c.f. eq. (5.25))

$$S\left(\{p_i\},\{p_i'\},\{k_i\};\Delta x\right) = \prod_{j=5}^{n+4}\left[\int \mathrm{d}^4x_j\, e^{-i(k_j\cdot x_j)/\hbar}\Box_{x_j}\right] e^{i(\Delta x\cdot p_1')/\hbar}\int \mathcal{D}A_\mu\, e^{\frac{i}{\hbar}S_A(A_\mu)} \\ \times \prod_{j=5}^{n+4}\left[\left(\varepsilon^{\sigma_j}(k_j)\cdot A_{\sigma_j}(x_j)\right)\right] f_1(-\infty,\infty)f_2(-\infty,\infty), \tag{5.39}$$

where we have included the LSZ reduction for the outgoing photons, with momenta $\{k_i\}$ and polarisation vectors $\{\varepsilon(k_i)\}$. Let us clarify this expression by again taking the leading classical behaviour, such that the two massive particles are following straight-line trajectories. Then we have, using eq. (5.27),

$$S\left(\{p_i\},\{p_i'\},\{k_i\};\Delta x\right) \\ = \prod_{j=5}^{n+4}\left[\int \mathrm{d}^4x_j\, e^{-i(k_j\cdot x_j)/\hbar}\Box_{x_j}\right]\int \mathcal{D}A_\mu\, e^{i(\Delta x\cdot p_1')/\hbar}\Phi_1\Phi_2\prod_{j=5}^{n+4}\left[\left(\varepsilon^{\sigma_j}(k_j)\cdot A_{\sigma_j}(x_j)\right)\right] e^{\frac{i}{\hbar}S_A(A_\mu)}. \tag{5.40}$$

Since the path integral is Gaussian, it is easy to perform it analytically and to take the LSZ reduction to get

$$S\left(\{p_i\},\{p_i'\},\{k_i\};\Delta x\right) \\ = \frac{e^{i(\Delta x\cdot p_1')/\hbar}}{n!}\prod_{j=5}^{n+4}\left[\varepsilon^{\sigma_j}(k_j)\cdot\tilde{J}(k_j)\right]\exp\left[i\int \mathrm{d}^4x\int \mathrm{d}^4y J^\mu(x)D_{\mu\nu}(x-y)J^\nu(y)\right] \\ = \frac{e^{i(\Delta x\cdot p_1')/\hbar}}{n!}\prod_{j=5}^{n+4}\left[\varepsilon^{\sigma_j}(k_j)\cdot\tilde{J}(k_j)\right]e^{i\chi(x_\perp-b_\perp,s)/\hbar},$$

where we have defined the Fourier transform of the current

$$\tilde{J}_\mu(k_j) = \int \mathrm{d}^4x_j\, e^{-i(k_j\cdot x_j)/\hbar}J_\mu(x_j), \tag{5.41}$$

and recognised the eikonal phase from eq. 5.34 and eq. (5.35). We see that the photon distribution is exactly Poissonian in this approximation, which is the hallmark of a coherent state. To see this in more detail, note that we can construct the full final state by summing over infinitely many photon emissions. Including then the measure for the incoming wavepackets etc., in our leading soft approximation we obtain eq. (5.3) as desired.

It is worth noticing at this point that a similar coherent state structure arise in scattering amplitudes as a consequence of infrared divergences: indeed, this is just a consequence of Weinberg soft theorems [255]. The fact that the scalar QED amplitude factorizes at leading order in the soft expansion for the external photons is a classical phenomenon: indeed, it can be viewed as the soft photon emission from a classical hard current given by the massive particles following straight-line trajectories [285]. In this setup, there is a classical factorization for observables as discussed in section 4.4 which now we understand as a simple consequence of the uncertainty principle. We will touch more this points in section 5.5.

Going beyond the straight-line approximation, we cannot solve the path integral over particle trajectories exactly, and thus need to rely on perturbation theory. But still, it is possible to see that the (perturbatively defined) classical currents associated to massive particle trajectories are emitting on-shell photons according to a Poissonian distribution. Therefore, coherent states naturally appear in this setup. This can be proved by splitting the path integral in eq. (5.40) for the gauge field between potential and radiation modes[4]

$$\int \mathcal{D}A_\mu = \int_{\text{pot}} \mathcal{D}A_\mu^{\text{pot}} \int_{\text{rad}} \mathcal{D}A_\mu^{\text{rad}}. \tag{5.42}$$

For radiation modes we can use the on-shell plane-wave expansion and eq. (5.39) becomes

$$\begin{aligned} S\left(\{p_i\},\{p_i'\},\{k_i\};\Delta x\right) &= \prod_{j=5}^{n+4}\left[\int \mathrm{d}^4 x_j\, e^{-i(k_j\cdot x_j)/\hbar}\Box_{x_j}\right]\int_{\text{pot}} \mathcal{D}A_\mu^{\text{pot}}\int_{\text{rad}}\mathcal{D}A_\mu^{\text{rad}} \\ \times \left(\int \mathcal{D}x_1\, e^{W_1^{\text{pot}}+W_1^{\text{rad}}}\right)&\left(\int \mathcal{D}x_2\, e^{W_2^{\text{pot}}+W_2^{\text{rad}}}\right)\prod_{j=5}^{n+4}\left[\left(\varepsilon^{\sigma_j}(k_j)\cdot A_{\sigma_j}(x_j)\right)\right] e^{\frac{i}{\hbar}S_A(A_\mu)}, \end{aligned} \tag{5.43}$$

where we have written

$$f_i(-\infty,\infty) = \int \mathcal{D}x_i e^{W_i^{\text{pot}}+W_i^{\text{rad}}}, \tag{5.44}$$

and defined

$$\begin{aligned} W_i^{\text{pot}} &= \frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t\left(-\frac{1}{4}\dot{x}^2+\frac{1}{2}p_i'\cdot\dot{x}-e\dot{x}\cdot A^{\text{pot}}-ep_i'\cdot A^{\text{pot}}\right), \\ W_i^{\text{rad}} &= -\frac{i}{\hbar^{\frac{3}{2}}}e\int_{-\infty}^{+\infty}\mathrm{d}t(\dot{x}+p_i')_\mu\sum_{\sigma=\pm1}\int \mathrm{d}\Phi(k)\left[a_\sigma(k)\varepsilon_\sigma^{\mu *}(k)e^{-\frac{i}{\hbar}k\cdot x}+\text{h.c.}\right]. \end{aligned} \tag{5.45}$$

If we restore the power counting in $\hbar$ for the coupling $e \to e/\sqrt{\hbar}$ and we define a (perturbative) *trajectory-dependent* classical current

$$J_i^\mu(t,\boldsymbol{x}) := ie(\dot{x}+p_i')^\mu\,, \tag{5.46}$$

we can write

$$W_i^{\text{rad}} = -\int_{-\infty}^{+\infty}\mathrm{d}t\sum_{\sigma=\pm1}\int \mathrm{d}\Phi(\bar{k})\left[a_\sigma(k)(J_i(t,\boldsymbol{x})\cdot\varepsilon_\sigma^*(\bar{k}))e^{-i\bar{k}\cdot x}-\text{h.c.}\right]. \tag{5.47}$$

At this point we can express the current in frequency modes $J_i^\mu(\omega,x)$,

$$J_i^\mu(t,\boldsymbol{x}) = \int\frac{\mathrm{d}\bar{\omega}}{2\pi}\,e^{-i\bar{\omega}t}\tilde{J}_i^\mu(\bar{\omega},\boldsymbol{x})\,, \tag{5.48}$$

[4]Here, loosely speaking, we call radiation modes the ones corresponding to $k_0 \sim |\boldsymbol{k}|$ and potential modes the ones with $k_0 \ll |\boldsymbol{k}|$ [35].

which gives

$$W_i^{\mathrm{rad}} = \sum_{\sigma=\pm1} \int \mathrm{d}\Phi(\bar{k}) \, \left(\tilde{J}_i^*(-\omega_k, \boldsymbol{x}) \cdot \varepsilon_\sigma(\bar{k}) a_\sigma^\dagger(k) - \mathrm{h.c.}\right) . \tag{5.49}$$

This is indeed a coherent state, in the form of a displacement operator. We wish to remind the reader that at this stage we have not yet expanded the trajectory in perturbation theory, since eq. (5.43) is an exact result. Effectively, at every order in perturbation theory, there will be an effective on-shell current which is the classical "source" of coherent radiation for each trajectory. What remains to be established, perturbatively, is how these classical currents interact to produce the final conjectured coherent state.

5.2 Coherent states from the uncertainty principle

We would like to understand the final particle distribution for the photons emitted in a scattering process in the classical limit. In particular, we want to show here that for classical radiation the factorization property of expectation values of physical observables is directly connected to coherence. To simplify our discussion, we work in a spacetime box of finite dimension V with periodic boundary conditions so that there will be a finite number $|V^D|$ of allowed momenta $\{k_i\}_{i\in V^D}$ in the dual momentum lattice V^D. At the end of the discussion, we will take $V \to +\infty$. In this framework, we can construct a coherent state for each quantum-mechanical harmonic oscillator with momentum $\{k_i\}$

$$|\alpha^\sigma_{k_i}\rangle = \mathcal{N}_\alpha \exp\left(\alpha^\sigma(k_i) a_\sigma^\dagger(k_i)\right) |0\rangle, \qquad \forall\, k_i \in V^D . \tag{5.50}$$

It is known that we can write every classical radiation density matrix as a probability distribution in the coherent state space, as proved by Glauber and Sudarshan [286–289]

$$\hat{\rho}_{\mathrm{out}} = \sum_{\sigma=\pm1} \int \prod_{l_i \in V^D} \mathrm{d}^2\alpha^\sigma_{l_i} \, \mathcal{P}^\sigma(\alpha) |\alpha^\sigma_{l_i}\rangle\langle\alpha^\sigma_{l_i}| \qquad \mathrm{d}^2\alpha^\sigma_l := (\mathrm{d}\Re\alpha^\sigma_l \, \mathrm{d}\Im\alpha^\sigma_l)/\pi \tag{5.51}$$

where $\mathcal{P}^\sigma(\alpha)$ is a separable function of $\alpha^\sigma_{l_i}$ with $i \in V^D$

$$\mathcal{P}^\sigma(\alpha) := \prod_{l_i \in V^D} \mathcal{P}^\sigma(\alpha(l_i)). \tag{5.52}$$

For the classical case $\mathcal{P}^\sigma(\alpha) \geq 0$, and this is what allows to talk about probability distribution in the standard mathematical sense. Let us stress here that in the notation $\sum_{l_i\in V^D}$ we include not only the summation over the dual lattice vectors but also the appropriate finite-volume on-shell phase-space normalization.

What is the implication of the exact classical factorization on the final radiation density matrix? Based on the previous discussion of negligible uncertainty we expect that the expectation value $\langle\,\cdot\,\rangle_{\rho_{\mathrm{out}}}$ in the density matrix of eq. (5.51) gives

$$\langle \mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\delta}(y)\rangle_{\rho_{\mathrm{out}}} \overset{\hbar\to0}{=} \langle \mathbb{F}_{\mu\nu}(x)\rangle_{\rho_{\mathrm{out}}} \langle \mathbb{F}_{\rho\delta}(y)\rangle_{\rho_{\mathrm{out}}} . \tag{5.53}$$

We use the on-shell mode expansion for the field strength operator

$$\mathbb{F}_{\mu\nu}(x) = \frac{-i}{\sqrt{\hbar}} \sum_{\sigma=\pm 1} \sum_{k\in V^D} \left[a_\sigma(k) \bar{k}_{[\mu} \varepsilon^{*\sigma}_{\nu]} e^{-i\bar{k}\cdot x} - \text{h.c.} \right], \tag{5.54}$$

and by taking advantage of the completeness relation in the Hilbert space of photons[5]

$$\sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \sum_{l_1,\ldots,l_n\in V^D} |l_1^{\sigma_1}...l_n^{\sigma_n}\rangle\langle l_1^{\sigma_1}...l_n^{\sigma_n}| = 1\,, \tag{5.55}$$

we have,

$$\begin{aligned}
\mathrm{Tr}_{\rho_\text{out}}(\mathbb{F}_{\mu\nu}(x)) &= \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \sum_{l_1,\ldots,l_n\in V^D} \langle l_1^{\sigma_1}...l_n^{\sigma_n}|\mathbb{F}_{\mu\nu}(x)\rho_\text{out}|l_1^{\sigma_1}...l_n^{\sigma_n}\rangle \\
&= \sum_{n,m=0}^{+\infty} \frac{1}{n!m!} \sum_{\sigma_1,\sigma_1',\ldots,\sigma_n,\sigma_m'=\pm 1} \sum_{l_1,l_1',\ldots,l_n,l_m'\in V^D} \\
&\quad \times \langle l_1^{\sigma_1}...l_n^{\sigma_n}|\mathbb{F}_{\mu\nu}(x)|(l_1')^{\sigma_1}...(l_m')^{\sigma_m'}\rangle\langle (l_1')^{\sigma_1}...(l_m')^{\sigma_m'}|\rho_\text{out}|l_1^{\sigma_1}...l_n^{\sigma_n}\rangle \\
&= -i\hbar^{\frac{3}{2}} \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \sum_{\bar{l}_1,\ldots,\bar{l}_n\in \bar{V}^D} \\
&\quad \times \sum_{\bar{k}\in\bar{V}^D} \left[\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]} \langle l_1^{\sigma_1}...l_n^{\sigma_n} k^\sigma|\rho_\text{out}|l_1^{\sigma_1}...l_n^{\sigma_n}\rangle e^{-i\bar{k}\cdot x} - \text{h.c.}\right].
\end{aligned} \tag{5.56}$$

where in the last line we have restored also the $\hbar$ scaling implicit in the finite volume phase-space normalization. Further manipulations show that

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \sum_{\bar{l}_1,\ldots,\bar{l}_n\in \bar{V}^D} \langle l_1^{\sigma_1}...l_n^{\sigma_n} k^\sigma|\rho_\text{out}|l_1^{\sigma_1}...l_n^{\sigma_n}\rangle \\
&\quad = \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \sum_{\bar{l}_1,\ldots,\bar{l}_n\in \bar{V}^D} \prod_{i=1}^{n} \left[\mathcal{N}^2_{\alpha_{l_i}} \int \mathrm{d}^2\alpha^\sigma_{l_i} \, |\alpha^{\sigma_i}(l_i)|^2 \, \mathcal{P}^{\sigma_i}(\alpha(l_i)) \right] \alpha^\sigma(k) \\
&\quad = \mathrm{Tr}_{\rho_\text{out}}(1)\alpha^\sigma(k) \\
&\quad = \alpha^\sigma(k)
\end{aligned} \tag{5.57}$$

where we have used the fact that the density matrix is normalized $\mathrm{Tr}_{\rho_\text{out}}(1) = 1$. If we demand the expectation value of $\mathrm{Tr}_{\rho_\text{out}}(\mathbb{F}_{\mu\nu}(x))$ to be classical, restoring powers of $\hbar$ requires the waveshape to scale as determined in section 2.2

$$\alpha^\sigma(k) \to \hbar^{-\frac{3}{2}} \bar{\alpha}^\sigma(k)\,. \tag{5.58}$$

Using this condition and the classical scaling of the normalization of the final state, we get

$$\mathrm{Tr}_{\rho_\text{out}}(\mathbb{F}_{\mu\nu}(x)) = -i \sum_{\bar{k}\in\bar{V}^D} \sum_{\sigma=\pm 1} \int \mathrm{d}^2\bar{\alpha}^\sigma_k \, \mathcal{P}^\sigma(\bar{\alpha}_k) \left[\bar{k}_{[\mu}\varepsilon^{*\sigma}_{\nu]} \bar{\alpha}^{\sigma,*}(k) e^{-i\bar{k}\cdot x} - \text{h.c.}\right]. \tag{5.59}$$

[5]For $n = 0$, the first term in the sum has to be interpreted as $|0\rangle\langle 0|$. This will be implicitly assumed in the rest of the argument.

Using a similar argument, for the expectation value of the product we obtain

$$\begin{aligned}\mathrm{Tr}_{\rho_{\text{out}}}(\mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\delta}(y)) &= \sum_{n=0}^{+\infty}\frac{1}{n!}\sum_{\sigma_1,\ldots,\sigma_n=\pm1}\sum_{\bar{l}_1,\ldots,\bar{l}_n\in\bar{V}^D}\langle l_1^{\sigma_1}\ldots l_n^{\sigma_n}|\mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\delta}(y)\rho_{\text{out}}|l_1^{\sigma_1}\ldots l_n^{\sigma_n}\rangle\\ &= -\sum_{\bar{k}_1,\bar{k}_2\in\bar{V}^D}\sum_{\sigma_1,\sigma_2=\pm1}\int \mathrm{d}^2\bar{\alpha}_{k_1}^{\sigma_1}\,\mathrm{d}^2\bar{\alpha}_{k_2}^{\sigma_2}\mathcal{P}^{\sigma_1,\sigma_2}(\bar{\alpha}_{k_1},\bar{\alpha}_{k_2})\\ &\quad\times\Big[\bar{k}_{1,[\mu}\varepsilon_{\nu]}^{\sigma_1*}\bar{k}_{2,[\rho}\varepsilon_{\delta]}^{\sigma_2*}\bar{\alpha}^{\sigma_1,*}(k_1)\bar{\alpha}^{\sigma_2,*}(k_2)e^{-i\bar{k}_1\cdot x-i\bar{k}_2\cdot y}\\ &\qquad+\bar{k}_{1,[\mu}\varepsilon_{\nu]}^{\sigma_1}\bar{k}_{2,[\rho}\varepsilon_{\delta]}^{\sigma_2*}\bar{\alpha}^{\sigma_1}(k_1)\bar{\alpha}^{\sigma_2,*}(k_2)e^{i\bar{k}_1\cdot x-i\bar{k}_2\cdot y}+\text{h.c.}\Big]\,,\end{aligned} \tag{5.60}$$

up to the commutator term

$$4\hbar\partial_{[\mu}\eta_{\nu][\delta}\partial_{\rho]}\int \mathrm{d}\Phi(\bar{k})\,e^{-i\bar{k}\cdot(x-y)} = \frac{\hbar}{\pi^2}\partial_{[\mu}\eta_{\nu][\delta}\partial_{\rho]}\frac{1}{(\mathbf{x}-\mathbf{y})^2-(x^0-y^0-i\epsilon)^2}\,. \tag{5.61}$$

which can be neglected in the $\hbar\to 0$ limit [4, 187].

Therefore we are effectively asking whether the product of the averages is equal to the average of the product over a distribution $\mathcal{P}^\sigma(\alpha)$ for all $k\in V^D$, i.e. we are asking the distribution to have *zero variance* in the coherent state space. But distributions of zero variance are degenerate because it means that the random variable $\alpha^\sigma(k)$[6] is almost surely constant for each $k\in V^D$ (see page 173 in [290]). Therefore the distribution has support in a lower-dimensional space, and since we can apply this argument independently both for the real and for the imaginary part of $\alpha^\sigma(k)$ we have

$$\mathcal{P}^\sigma(\alpha) = \prod_{k\in V^D}\sum_{j=1}^{+\infty}c_j^\sigma\delta^2(\alpha^\sigma(k)-\alpha_j^\sigma(k))\,. \tag{5.62}$$

What this is essentially saying is that we get a (possibly infinite) sum of discrete distributions with a constant value $\alpha_j^\sigma(k)$. But then, making use of a crucial result due to Hillery [291], we get

$$\mathcal{P}^\sigma(\alpha) = \prod_{k\in V^D}\delta^2(\alpha^\sigma(k)-\alpha_\star^\sigma(k))\,, \tag{5.63}$$

which means that we can describe the final state only with one coherent state for each helicity and for each momentum k in the dual momentum lattice. At this point we can take the large volume limit and what this calculation implies is that we can describe the final state with a single coherent state

$$|\alpha^\sigma\rangle = \mathcal{N}_\alpha\exp\left(\int \mathrm{d}\Phi(k)\alpha^\sigma(k)a_\sigma^\dagger(k)\right)|0\rangle,\qquad \mathcal{N}_\alpha = \exp\left(-\frac{1}{2}\int \mathrm{d}\Phi(k)|\alpha^\sigma(k)|^2\right)\,, \tag{5.64}$$

which takes naturally into the account the infinite-dimensional superposition of harmonic oscillators of momentum k. The mininum uncertainty principle in this context

[6]Technically this is a random variable $\alpha^\sigma(k)$ for each value of the momentum in the dual lattice, i.e. there is an harmonic oscillator for each quanta of radiation. But since they are all independent, we can promote this statement to the full expression.

has been dubbed as the "complete coherence condition" in the literature[7], a term coined by Glauber [286, 287].

5.3 Spin coherent states from the uncertainty principle

It is possible to prove, similarly to what we have done for the radiation case in section 5.2, that the generic spin quantum state for the massive particles is necessarily a coherent spin state once we impose the uncertainty principle. We can represent the most generic quantum spin state of the initial and the final massive particles in our two-body scattering problem in terms of a density matrix. Such construction has been developed in the quantum optics literature [294], and for our setup it corresponds to dressing the external incoming and outgoing massive momentum states with the following spin density matrix

$$\hat{\rho}^S_{\text{in}} := \int \prod_{i=1}^{2} \mathrm{d}^2\alpha_i^S \mathcal{P}^S_{\text{in}}(\alpha_1^S, \alpha_2^S)|\alpha_1^S\rangle|\alpha_2^S\rangle\langle\alpha_2^S|\langle\alpha_1^S|,$$
$$\hat{\rho}^S_{\text{out}} := \int \prod_{i=1}^{2} \mathrm{d}^2\alpha_i^S \mathcal{P}^S_{\text{out}}(\alpha_1^S, \alpha_2^S)|\alpha_1^S\rangle|\alpha_2^S\rangle\langle\alpha_2^S|\langle\alpha_1^S|,$$

where in the classical limit the P-representation is separable [294]

$$\mathcal{P}^S_{\text{in}}(\alpha_1^S, \alpha_2^S) = \mathcal{P}^S_{\text{in}}(\alpha_1^S)\mathcal{P}^S_{\text{in}}(\alpha_2^S),$$
$$\mathcal{P}^S_{\text{out}}(\alpha_1^S, \alpha_2^S) = \mathcal{P}^S_{\text{out}}(\alpha_1^S)\mathcal{P}^S_{\text{out}}(\alpha_2^S).$$

We now impose the uncertainty principle for the generic expectation value of the spin operator,

$$\mathrm{Tr}_{\rho^S_{\text{in}}}(\boldsymbol{S}_i\boldsymbol{S}_j) \overset{\hbar\to 0}{=} \mathrm{Tr}_{\rho^S_{\text{in}}}(\boldsymbol{S}_i)\mathrm{Tr}_{\rho^S_{\text{in}}}(\boldsymbol{S}_j) \qquad \text{for } i = 1, 2\,,$$
$$\mathrm{Tr}_{\rho^S_{\text{out}}}(\boldsymbol{S}_i\boldsymbol{S}_j) \overset{\hbar\to 0}{=} \mathrm{Tr}_{\rho^S_{\text{out}}}(\boldsymbol{S}_i)\mathrm{Tr}_{\rho^S_{\text{out}}}(\boldsymbol{S}_j) \qquad \text{for } i = 1, 2\,.$$

Following some simple steps, it is easy to see that this is equivalent to demand the zero-variance property for both $\mathcal{P}^S_{\text{in}}(\alpha_i^S)$ and $\mathcal{P}^S_{\text{out}}(\alpha_i^S)$ $(i = 1, 2)$. Therefore, we conclude that

$$\mathcal{P}^S_{\text{in}}(\alpha_1^S, \alpha_2^S) = \delta(\alpha_1^S - \alpha^S_{1,*,\text{in}})\delta(\alpha_2^S - \alpha^S_{2,*,\text{in}}),$$
$$\mathcal{P}^S_{\text{out}}(\alpha_1^S, \alpha_2^S) = \delta(\alpha_1^S - \alpha^S_{1,*,\text{out}})\delta(\alpha_2^S - \alpha^S_{2,*,\text{out}}).$$

This implies that spin coherent states are suitable to represent the quantum-mechanical state of classical spin particles, and more in general an exponentiation of the spin degrees of freedom is required in the classical limit. This is also consistent with the fact that we should expect to not have any entanglement in the spin sector [295].

5.4 Coherent states from the particle distribution

Here we would like to study the particle statistics distribution of the gravitons emitted in the scattering of a pair of massive point particles of mass m_A and m_B in general relativity, using methods of perturbative QFT. In particular, we relate the expectation value of the graviton number operator to a sum of unitarity cuts involving scattering amplitudes with external gravitons.

[7] In the quantum optics literature the normal-ordered correlator of the electric field at different spatial locations can have various degrees of coherence [292, 293].

Let $\bar{P}_n$ be the probability of emitting n gravitons in the scattering of a pair of massive particles as described above. Unitarity implies that $\sum_{n=0}^{\infty} \bar{P}_n = 1$. In quantum field theory, this statement is equivalent to a completeness relation in the Hilbert space,

$$|0\rangle\langle 0| + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_n=\pm} \int \prod_{i=1}^{n} d\Phi(k_i) |k_1^{\sigma_1}...k_n^{\sigma_n}\rangle\langle k_1^{\sigma_1}...k_n^{\sigma_n}| = 1, \tag{5.65}$$

where $|k_1^{\sigma_1}...k_n^{\sigma_n}\rangle\langle k_1^{\sigma_1}...k_n^{\sigma_n}|$ is the n-graviton particle projector on states with definite momenta $k_1, ..., k_n$ and helicities $\sigma_1, ..., \sigma_n$, whose values are indicated by the single $+$ and $-$ signs.

We denote the scattering matrix operator by S, the momenta of the incoming (resp. outgoing) massive scalar particles by p_1, p_2 (resp. p_3, p_4), and the outgoing graviton momenta by $\{k_i\}_{i=1,\dots,n}$. It is clear that the probability $\bar{P}_n$ is given by taking the expectation value of the n-graviton particle projector,

$$\bar{P}_n = \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_n=\pm} \int d\Phi(p_3) d\Phi(p_4) \int \prod_{i=1}^{n} d\Phi(k_i) |\langle k_1^{\sigma_1}...k_n^{\sigma_n} p_3 p_4 |S| p_1 p_2\rangle|^2. \tag{5.66}$$

As it is written, (5.66) is formally divergent as it is known from the study of infrared divergences in quantum field theory (see [296]) because of the contribution of zero-energy gravitons. We will therefore work with a finite-resolution detector $\lambda > 0$, which implies that we will study only the probabilities of gravitons emitted with an energy $E_k > \lambda$. Correspondingly, we will replace

$$\bar{P}_n \to \bar{P}_n^{\lambda}, \qquad \int \prod_{i=1}^{n} d\Phi(k_i) \to \int_{\lambda} \prod_{i=1}^{n} d\Phi(k_i), \tag{5.67}$$

As we will see later, we will not be interested in the single probability but in a particular infrared-safe combination of probabilities. Therefore λ will be used only as an intermediate regulator, and in the end we will send $\lambda \to 0$.

We would like to scatter classically two massive point particles with classical momenta $m_A v_A$ and $m_B v_B$ with an impact parameter b^{μ}. Since the main purpose of this paper is to take the classical limit from a quantum field theory calculation, we use the KMOC formalism [166] and take instead as our incoming state

$$|\psi_{\text{in}}\rangle := \int d\Phi(p_1) d\Phi(p_2) e^{ib\cdot p_1/\hbar} \psi_A(p_1) \psi_B(p_2) |p_1 p_2\rangle, \tag{5.68}$$

where

$$d\Phi(p_1) := \frac{1}{(2\pi)^3} d^4 p_1 \delta(p_1^2 - m_A^2)\theta(p_1^0), \qquad d\Phi(p_2) := \frac{1}{(2\pi)^3} d^4 p_2 \delta(p_2^2 - m_B^2)\theta(p_2^0). \tag{5.69}$$

The wavefunctions $\psi_A(p_1), \psi_B(p_2)$ are defined as

$$\psi_A(p_1) := \mathcal{N} m_A^{-1} \exp\left[-\frac{p_1 \cdot v_A}{\hbar \ell_{c,A}/\ell_{w,A}^2}\right], \qquad \psi_B(p_2) := \mathcal{N} m_B^{-1} \exp\left[-\frac{p_2 \cdot v_B}{\hbar \ell_{c,B}/\ell_{w,B}^2}\right], \tag{5.70}$$

where $\mathcal{N}$ is a normalization factor, $\ell_{c,j} = \hbar/m_j$ is the Compton wavelength and $\ell_{w,j}$

is related to the intrinsic spread of the wavefunction for the j-th massive particle ($j = A, B$). We will also require the "Goldilocks conditions"

$$\ell_{c,j} \ll \ell_{w,j} \ll b \qquad \text{for} \quad j = A, B, \tag{5.71}$$

which ensure that wavefunctions such as those in (5.70) effectively localize the massive particles on their classical trajectories as $\hbar \to 0$. We expand the S-matrix in terms of the scattering matrix T,

$$S = 1 + iT. \tag{5.72}$$

For the expectation value of the graviton projector operator, only the amplitudes with at least one graviton emitted are going to contribute. We can read off from (5.66) the probability of emitting n gravitons with energies $E_{k_i} > \lambda$,

$$P_n^\lambda = \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm} \int d\Phi(r_1) d\Phi(r_2) \int_\lambda \prod_{i=1}^n d\Phi(k_i) \langle \psi_{\text{in}} | T^\dagger | r_1 r_2 k_1^{\sigma_1} \ldots k_n^{\sigma_n} \rangle \langle r_1 r_2 k_1^{\sigma_1} \ldots k_n^{\sigma_n} | T | \psi_{\text{in}} \rangle. \tag{5.73}$$

We introduce now the momentum transfers [166],

$$q_j := p_j' - p_j, \qquad w_j := r_j - p_j, \tag{5.74}$$

with which (5.73) can be written as

$$\begin{aligned} P_n^\lambda = \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm} & \int d\Phi(p_1) d\Phi(p_2) \int_\lambda \prod_{i=1}^n d\Phi(k_i) \int \frac{d^4 q}{(2\pi)^4} \int \prod_{j=1,2} d^4 w_j \\ & \times \delta\left(2p_1 \cdot q + q^2\right) \delta\left(2p_2 \cdot q - q^2\right) \Theta\left(p_1^0 + q^0\right) \Theta\left(p_2^0 - q^0\right) e^{-ib\cdot q/\hbar} \delta^{(4)}\left(w_1 + w_2 + \sum_{i=1}^n k_i\right) \\ & \times \psi_A(p_1) \psi_A^*(p_1 + q) \psi_B(p_2) \psi_B^*(p_2 - q) \prod_{j=1,2} \left[\delta\left(2p_j \cdot w_j + w_j^2\right) \Theta\left(p_j^0 + w_j^0\right)\right] \\ & \times \mathcal{A}_{n+4}\left(p_1, p_2 \to p_1 + w_1, p_2 + w_2, k_1^{\sigma_1}, \ldots, k_n^{\sigma_n}\right) \\ & \qquad \times \mathcal{A}_{n+4}^*\left(p_1 + q, p_2 - q \to p_1 + w_1, p_2 + w_2, k_1^{\sigma_1}, \ldots, k_n^{\sigma_n}\right). \end{aligned} \tag{5.75}$$

where $q = q_1 = -q_2$. We now conveniently define a set of symmetrized variables for the external momenta [139]

$$p_A^\mu := p_1^\mu + \frac{q^\mu}{2}, \qquad p_B^\mu := p_2^\mu - \frac{q^\mu}{2}, \tag{5.76}$$

which has the nice property of enforcing exactly the condition $p_A \cdot q = p_B \cdot q = 0$. In terms of these new variables [128], we have

$$\begin{aligned} P_n^\lambda = \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm} \Bigg\langle\!\!\Bigg\langle \int_\lambda \prod_{i=1}^n d\Phi(k_i) \int \frac{d^4 q}{(2\pi)^4} \delta\left(2p_A\cdot q\right)\delta\left(2p_B\cdot q\right)\Theta\left(p_A^0+\frac{q^0}{2}\right)\Theta\left(p_B^0-\frac{q^0}{2}\right) \\ \times \int d^4 w_1 d^4 w_2 e^{-ib\cdot q/\hbar}\delta^{(4)}\left(w_1+w_2+\sum_{i=1}^n k_i\right) \\ \times\,\delta\left(2p_A\cdot w_1+w_1^2-q\cdot w_1\right)\Theta\left(p_A^0+w_1^0-\frac{q^0}{2}\right)\delta\left(2p_B\cdot w_2+w_2^2+q\cdot w_2\right)\Theta\left(p_B^0+w_2^0+\frac{q^0}{2}\right) \\ \times\,\mathcal{A}_{n+4}\left(p_A-\frac{q}{2},p_B+\frac{q}{2}\to p_A+w_1-\frac{q}{2},p_B+w_2+\frac{q}{2},k_1^{\sigma_1},\ldots,k_n^{\sigma_n}\right) \\ \times\,\mathcal{A}^*_{n+4}\left(p_A+\frac{q}{2},p_B-\frac{q}{2}\to p_A+w_1-\frac{q}{2},p_B+w_2+\frac{q}{2},k_1^{\sigma_1},\ldots,k_n^{\sigma_n}\right)\Bigg\rangle\!\!\Bigg\rangle, \end{aligned} \tag{5.77}$$

where we use the double bracket notation $\langle\!\langle\,\cdot\,\rangle\!\rangle$ introduced in [166], which contains the implicit phase space integral over p_A, p_B and the appropriate wavefunctions

$$\Big\langle\!\!\Big\langle f\left(p_A,p_B,\ldots\right)\Big\rangle\!\!\Big\rangle \equiv \int d\Phi(p_A)d\Phi(p_B)|\psi_A(p_A)|^2|\psi_B(p_B)|^2 f\left(p_A,p_B,\ldots\right), \tag{5.78}$$

where

$$\begin{aligned} d\Phi(p_A) &:= \frac{1}{(2\pi)^3} d^4 p_A \delta\left(p_A^2-m_A^2+q^2/4\right)\theta\left(p_A^0-q^0/2\right), \\ d\Phi(p_B) &:= \frac{1}{(2\pi)^3} d^4 p_B \delta\left(p_B^2-m_B^2+q^2/4\right)\theta\left(p_B^0+q^0/2\right). \end{aligned} \tag{5.79}$$

Note that eq.(5.75) is expressed in terms of unitarity cuts involving n gravitons and the two massive particles in the intermediate state. The same result can be obtained by applying the LSZ reduction with the appropriate KMOC wavefunctions from the in-in formalism, as shown in chapter 2.

In classical physics, we are interested in knowing whether the final graviton particle distribution is exactly Poissonian or super-Poissonian (the most general case). We refer the reader to appendix D for a brief review of the two cases. Poissonian statistics are known to be equivalent to having a single coherent state representing the quantum state for the classical radiation field. Here we give a short argument [4] for why we expect a single coherent state, based on the fact that the we expect the incoming state to be a pure state in the classical limit and on the unitarity of the S-matrix. The work of Glauber in 1963 [286, 287] shows that every quantum state of radiation (i.e. every density matrix) can be written as a superposition of coherent states,

$$\hat\rho_{k,\mathrm{out}} = \sum_{\sigma=\pm}\int d^2\alpha_k^\sigma\; \mathcal{P}^\sigma_{\mathrm{out}}(\alpha_k)|\alpha_k^\sigma\rangle\langle\alpha_k^\sigma|, \qquad d^2\alpha_k^\sigma := \frac{d\Re(\alpha_k^\sigma)d\Im(\alpha_k^\sigma)}{\pi}, \tag{5.80}$$

where $\mathcal{P}^\sigma_{\mathrm{out}}(\alpha_k)$ is a well-defined probability density ($\mathcal{P}^\sigma_{\mathrm{out}}(\alpha_k)\geq 0$) in the coherent state space in the classical limit, and $|\alpha_k^\sigma\rangle$ represents a coherent state of a graviton excitation ("harmonic oscillator") of momentum k and definite helicity σ, which we can write generically as

$$|\alpha_k^\sigma\rangle := \exp\left[\alpha_k a_\sigma^\dagger(k) - \alpha_k^* a_\sigma(k)\right]|0\rangle, \tag{5.81}$$

where $a_\sigma^\dagger(k)$ and $a_\sigma(k)$ are the creation and annihilation operators of a graviton of helicity σ. This representation is known as the Glauber-Sudarshan P-representation [286, 288], and it is widely used in the quantum optics literature. In quantum field theory, we need to consider an infinite superposition of harmonic oscillators for all momenta $k \in \mathbb{R}^{1,3}$, and therefore we will promote eq.(5.80) to[8]

$$\hat{\rho}_{\text{radiation,out}} = \sum_{\sigma=\pm} \int \mathcal{D}^2\alpha^\sigma\, \mathcal{P}^\sigma_{\text{out}}(\alpha)|\alpha^\sigma\rangle\langle\alpha^\sigma|, \tag{5.82}$$

where now

$$|\alpha^\sigma\rangle = \exp\left[\int d\Phi(k)(\alpha(k)a_\sigma^\dagger(k) - \alpha^*(k)a_\sigma(k))\right]|0\rangle. \tag{5.83}$$

Since we are dealing with scattering boundary conditions and our incoming KMOC state $|\psi_{\text{in}}\rangle$ is a pure state, the unitarity of the S-matrix $SS^\dagger = 1$ implies that $|\psi_{\text{in}}\rangle$ is mapped to outgoing pure states. Therefore, in particular, the outgoing radiation state must be a superposition of pure states,

$$\mathcal{P}^\sigma_{\text{out}}(\alpha) = \sum_{j=1}^{\infty} c^\sigma_{j,\text{out}}\delta^2(\alpha^\sigma - \alpha^\sigma_j), \tag{5.84}$$

But thanks to a crucial theorem of Hillery [291], we know that every such superposition of pure states is trivial in the classical limit $\hbar \to 0$,

$$\mathcal{P}^\sigma_{\text{out},\star}(\alpha) = \delta^2(\alpha^\sigma - \alpha^\sigma_\star). \tag{5.85}$$

We therefore expect, on general grounds, to be able to describe the final radiation state for a scattering process involving point particles with a single coherent state.

From the pure amplitude perspective, the same question is hard to answer unless we work strictly in the soft approximation [3, 256, 264]. But in general, we can address this question perturbatively by studying the mean, the variance and the factorial moments of the particle distribution. A similar approach has been taken by F. Gelis and R. Venugopalan [297–299] in the standard in-in formalism, which we try to specialize here from a fully on-shell perspective and in the classical limit.

The graviton number operator is defined as

$$\hat{N} = \sum_{\sigma=\pm} \int d\Phi(k)\, a_\sigma^\dagger(k)a_\sigma(k). \tag{5.86}$$

Having defined

$$|\psi_{\text{out}}\rangle := S|\psi_{\text{in}}\rangle, \tag{5.87}$$

the expectation value of the number operator in the final state gives the mean of the

[8] See also [6] for a more rigorous approach by taking the large volume limit of a finite spacetime box, where momenta are quantized and we only need to consider a finite superposition of harmonic oscillators. We thank Donal O'Connell for emphasizing this point.

distribution, which can be expressed in terms of unitarity cuts in a similar fashion to the derivation of eq. (5.75), as depicted in Fig. 5.2. The mean of the distribution is defined as

$$\begin{aligned}\mu^\lambda_{\text{out}} &:= \langle\psi_{\text{out}}|\hat{N}|\psi_{\text{out}}\rangle \\ &= \int d\Phi(r_1)d\Phi(r_2)\sum_{n_X}\int_\lambda d\Phi(X) n_X\langle\psi_{\text{in}}|T^\dagger|r_1r_2X\rangle\langle r_1r_2X|T|\psi_{\text{in}}\rangle \\ &= \sum_{n=0}^{\infty} n\, P^\lambda_n, \end{aligned} \tag{5.88}$$

where $|r_1r_2X\rangle$ denotes the state with n_X gravitons and two massive particles of momenta r_1 and r_2, and $\int_\lambda d\Phi(X)$ stands for the phase space integration for the gravitons.

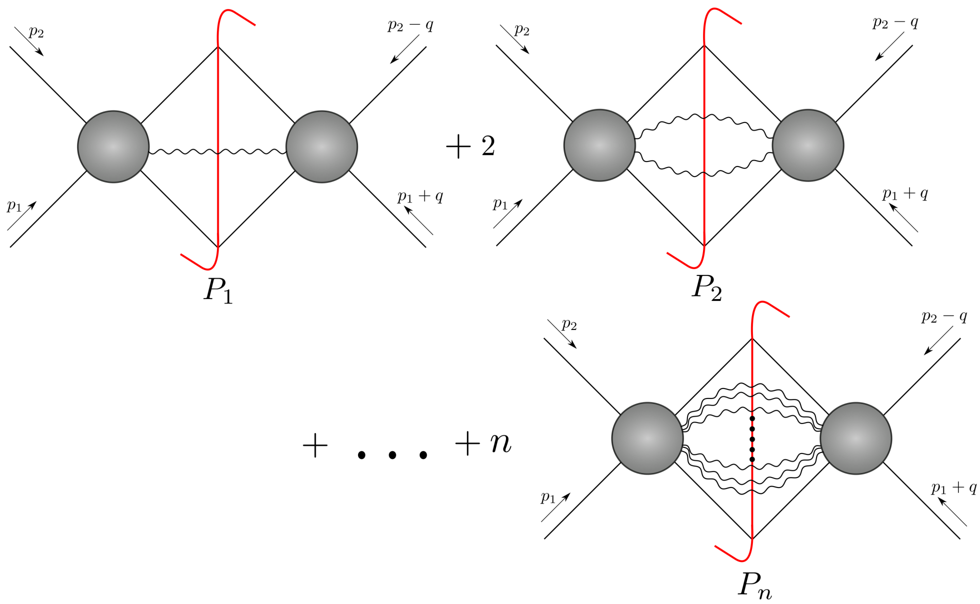


FIGURE 5.2: Diagrammatic representation of the on-shell amplitude contribution to the graviton number operator expectation value.

We define the variance of the distribution as

$$\Sigma^\lambda_{\text{out}} := \langle\psi_{\text{out}}|(\hat{N})^2|\psi_{\text{out}}\rangle - \Big(\langle\psi_{\text{out}}|\hat{N}|\psi_{\text{out}}\rangle\Big)^2 = \sum_{n=0}^{\infty} n^2\, P^\lambda_n - \left(\sum_{n=0}^{\infty} n\, P^\lambda_n\right)^2. \tag{5.89}$$

If the variance is equal to the mean, i.e. if

$$\Sigma^\lambda_{\text{out}} \stackrel{?}{=} \mu^\lambda_{\text{out}}, \tag{5.90}$$

then the distribution is consistent with a Poissonian distribution. This means that the deviation from the Poissonian distribution,

$$\begin{aligned}\Delta_{\text{out}} :&= \Sigma^\lambda_{\text{out}} - \mu^\lambda_{\text{out}} \\ &= \sum_{n=0}^{\infty}(n^2-n)\, P^\lambda_n - \left(\sum_{n=0}^{\infty} n\, P^\lambda_n\right)^2, \end{aligned} \tag{5.91}$$

characterizes the deviation from the coherent state description.

We claim here that the difference between the mean μ^λ and the variance Σ^λ is an infrared-safe quantity in perturbative quantum gravity. While the probability of emission of n gravitons is generally ill-defined because of infrared divergences, there is a non-trivial cancellation which happens for Δ_{out}. Indeed, the contribution of zero-energy gravitons to the final state, which give rise to the infrared divergent contributions, is known to be exactly represented by a coherent state. This can be proved either from a Faddeev-Kulish approach [3, 264] or from a path integral perspective [6, 275, 276]. Let us denote the mean and the variance of this coherent state for zero-energy gravitons by $\mu_{\text{out}}^{E_k\sim 0}$ and $\Sigma_{\text{out}}^{E_k\sim 0}$ respectively. In appendix D, we show that for such coherent state of soft gravitons we have[9]

$$\begin{aligned}\Sigma_{\text{out}}^{E_k\sim 0} &= \mu_{\text{out}}^{E_k\sim 0} = \sum_{\sigma=\pm}\int_{E_k\sim 0} d\Phi(k)|\alpha^\sigma_{E_k\sim 0}(k)|^2,\\ \Delta_{\text{out}}^{E_k\sim 0} &= \Sigma_{\text{out}}^{E_k\sim 0} - \mu_{\text{out}}^{E_k\sim 0} = 0.\end{aligned} \tag{5.92}$$

This is the reason why the cutoff λ was removed in eq. (5.91).[10]

We can easily check by induction on the number of loops and legs that [11]

$$P_n^\lambda = \sum_{L_1,L_2=0}^{\infty} G^{2+n+L_1+L_2} P_n^{(L_1,L_2)}, \tag{5.93}$$

where we have explicitly extracted the scaling in G of the product of an L_1-loop amplitude with an L_2-loop amplitude with n gravitons. The lowest order contribution to Δ_{out} is of order $\mathcal{O}(G^4)$, which corresponds to

$$\Delta_{\text{out}}\Big|_{\mathcal{O}(G^4)} = 2G^4 P_2^{(0,0)}. \tag{5.94}$$

This leading term is the unitarity cut involving the 6-pt tree amplitude $\mathcal{A}_6^{(0)}(\phi_A\phi_B \to \phi_A\phi_B h_1 h_2)$ and its conjugate $\mathcal{A}_6^{(0)*}(\phi_A\phi_B \to \phi_A\phi_B h_1 h_2)$. It is important also to understand the higher order terms in Δ_{out}, since they will give non-trivial amplitude relations if we assume coherence at all orders. From the definition eq. (5.91), we have

$$\boxed{\begin{aligned}\Delta_{\text{out}} = &\sum_{L_1,L_2=0}^{\infty}\sum_{n=2}^{\infty} G^{2+n+L_1+L_2}(n^2-n)P_n^{(L_1,L_2)}\\ &- \sum_{L_1,L_2,L_1',L_2'=0}^{\infty}\sum_{n,m=1}^{\infty} nm\, G^{4+n+m+L_1+L_2+L_1'+L_2'} P_n^{(L_1,L_2)} P_m^{(L_1',L_2')}.\end{aligned}} \tag{5.95}$$

[9] It is not necessary to specify $\alpha^\sigma_{E_k\sim 0}(k)$ for the argument to work. The interested reader can find additional details in [264].

[10] This argument does not apply directly to non-abelian theories because of the presence of collinear divergences, which for perturbative gravity are known to cancel exactly [262]. It would be interesting to develop this idea further, along the lines of [270, 300].

[11] To avoid cluttering the notation, we keep the λ dependence implicit in $P_n^{(L_1,L_2)} \equiv P_n^{(L_1,L_2),\lambda}$.

Let us examine the first several terms appearing explicitly in the expansion of eq. (5.95),

$$\begin{aligned}\Delta_{\rm out} &= 2G^4 P_2^{(0,0)} + 6G^5 P_3^{(0,0)} + 12G^6 P_4^{(0,0)} + 20G^7 P_5^{(0,0)} \\ &+ G^5(2P_2^{(1,0)} + 2P_2^{(0,1)}) + G^6(2P_2^{(0,2)} + 2P_2^{(2,0)} + 6P_3^{(1,0)} + 6P_3^{(0,1)}) \\ &+ G^7(2P_2^{(3,0)} + 2P_2^{(0,3)} + 6P_3^{(2,0)} + 6P_3^{(0,2)} + 6P_3^{(1,1)} + 12P_4^{(1,0)} + 12P_4^{(0,1)} - 4P_1^{(0,0)}P_2^{(0,0)}) \\ &+ \Big[G^6(2P_2^{(1,1)} - (P_1^{(0,0)})^2) + G^7(2P_2^{(1,2)} + 2P_2^{(2,1)} - 2P_1^{(0,1)}P_1^{(0,0)} - 2P_1^{(1,0)}P_1^{(0,0)})\Big],\end{aligned} \tag{5.96}$$

where we have organized each different line according to the expected behavior of the terms in the classical limit. We expect that the first three lines of eq. (5.96) are related to "quantum" contributions and are therefore irrelevant in the classical limit. The last line of eq. (5.96), instead, contains a combination of unitarity cuts which will give non-trivial quadratic relations between "classical" loop amplitudes with a higher number of emitted gravitons of the form $P_n^{(L_1,L_2)}$ with $n \geq 2$ and $L_1 + L_2 \geq 1$, and 5-point amplitude contributions involving $P_1^{(L_1,L_2)}$. We will discuss this interpretation in more detail in section 6.3, where we will also emphasize the relevance of the 5-pt amplitude for the calculation of classical radiative observables.

It is important to consider also higher moments of the statistical distribution for the graviton number production. We can define a generating functional

$$F(x) = \sum_{n=0}^{\infty} P_n^\lambda e^{nx}, \tag{5.97}$$

from which all higher moments can be derived,

$$\langle \psi_{\rm out}|\hat{N}^m|\psi_{\rm out}\rangle = \sum_{n=0}^{\infty} n^m\, P_n^\lambda = \frac{d^m F(x)}{dx^m}\Big|_{x=0}. \tag{5.98}$$

Therefore, the knowledge of all graviton emission probabilities P_n^λ is enough to completely determine the distribution of the particles above the energy cutoff. In practice, we can rely on perturbation theory and therefore computing the first few moments is enough to accurately determine the particle distribution. We can also defined connected moments (or "cumulants"), like the variance and its higher order generalizations. Having defined a generating functional

$$G(x) := \log(F(x)), \tag{5.99}$$

for a Poissonian distribution we would expect, given a certain waveshape $\alpha^\sigma(k)$, that

$$\Sigma^{(m),\lambda}_{\rm Poisson} := \frac{d^m G^{\rm Poisson}(x)}{dx^m}\Big|_{x=0} \sim \sum_{\sigma=\pm}\int_\lambda d\Phi(k)|\alpha^\sigma(k)|^2 \qquad \text{for all } m>0. \tag{5.100}$$

because all the cumulants should be equal. In particular, the variance is a special case for $m = 2$, i.e. $\Sigma^{(2),\lambda} = \Sigma^\lambda$.

For our purposes it is more convenient to consider factorial moments $\Gamma^{(m)}$, which correspond to a linear combination of the connected moments discussed above. We define the factorial moments

$$\Gamma_{\text{out}}^{(m),\lambda} := \langle\psi_{\text{out}}|\prod_{j=1}^{m}(\hat{N}-j+1)|\psi_{\text{out}}\rangle$$
$$= \langle\psi_{\text{out}}|\hat{N}(\hat{N}-1)\dots(\hat{N}-m+1)|\psi_{\text{out}}\rangle\ . \tag{5.101}$$

For a Poissonian distribution it is possible to prove that

$$\Gamma^{(m)}_{\text{out,Poisson}} = (\mu^{\lambda}_{\text{out,Poisson}})^m \tag{5.102}$$

and therefore we can also consider in perturbation theory other infrared-safe combinations of probabilities like

$$\Delta^{(m)}_{\text{out}} := \Gamma^{(m),\lambda}_{\text{out}} - (\mu^{\lambda}_{\text{out}})^m, \tag{5.103}$$

where for $m=2$ one can check that we recover the difference between the mean and the variance in eq. (5.95). By expanding eq. (5.103) we get immediately

$$\boxed{\begin{aligned}\Delta^{(m)}_{\text{out}} =&\sum_{n=0}^{\infty}\sum_{L_1,L_2=0}^{\infty} G^{2+n+L_1+L_2}\frac{n!}{(n-m)!}P_n^{(L_1,L_2)}\\ &-\sum_{n_1,\dots,n_m=1}^{\infty}\ \sum_{L_1^{(1)},\dots,L_1^{(m)}=0}^{\infty}\ \sum_{L_2^{(1)},\dots,L_2^{(m)}=0}^{\infty} G^{2m+\sum_{k=1}^{m}\left[n_k+L_1^{(k)}+L_2^{(k)}\right]}\prod_{j=1}^{m}\left[n_j P_{n_j}^{\left(L_1^{(j)},L_2^{(j)}\right)}\right].\end{aligned}} \tag{5.104}$$

It is interesting to consider the first terms in this expansion of $\Delta^{(3)}_{\text{out}}$,

$$\begin{aligned}\Delta^{(3)}_{\text{out}} =&6G^5P_3^{(0,0)}+24G^6P_4^{(0,0)}+60G^7P_5^{(0,0)}\\ &+G^6(6P_3^{(1,0)}+6P_3^{(0,1)})+G^7(6P_3^{(0,2)}+6P_3^{(2,0)}+6P_3^{(1,1)}+24P_4^{(1,0)}+24P_4^{(0,1)})\,,\end{aligned} \tag{5.105}$$

and of $\Delta^{(4)}_{\text{out}}$,

$$\begin{aligned}\Delta^{(4)}_{\text{out}} =&24G^6P_4^{(0,0)}+120G^7P_5^{(0,0)}\\ &+G^7(24P_4^{(1,0)}+24P_4^{(0,1)})\,,\end{aligned} \tag{5.106}$$

where we have organized the terms similarly to what was done in eq. (5.96). We will explore the deep consequences of assuming coherence at all orders, i.e. $\Delta^{(m)}_{\text{out}}=0$, in section 6.3. In [6], it is shown how coherence properties are linked to the factorization of radiative observables in the KMOC formalism.[12] In classical physics, we expect only the 1-point function to play a role for any observable of interest. Such an observable is essentially uniquely determined by the classical equations of motion and the retarded boundary conditions at $t\to-\infty$: all two-point and higher-point functions then have to factorize as $\hbar\to 0$. There the following relation was established,

Poissonian distribution in Fock space	$\Longleftrightarrow$	Zero-variance property in the Glauber-Sudarshan coherent state basis

[12]Similar statements about the classical factorization have been made in [3] for infrared divergences and in [4] for the classical expansion.

which implies that Poissonian distributions in the number operator basis correspond to a degenerate distribution ($\propto \delta^2(\alpha^\sigma - \alpha^\sigma_\star)$) in the Glauber-Sudarshan space.

5.5 Coherent states from asymptotic symmetries and IR-finite S-matrix

There is an interesting connection between asymptotic symmetries [10], coherent states and the definition of an infrared finite S-matrix for perturbative quantum gravity [301, 302]. It is well known that while physical observables computed from it are always infrared finite, the S-matrix is ill-defined in four dimensions due to presence of long-range gravity interactions.

Weinberg showed that in gravity infrared divergences exponentiate as a consequence of soft theorems [255], but the real revolution was to understand that there is a general symmetry principle behind it. Indeed, soon after the discovery of the BMS symmetry the gravitational scattering by Strominger [228], it was realized that Weinberg soft theorem can be understood from the Ward identity of the BMS supertranslation charge [230]: schematically, if we define

$$\begin{aligned} N(z,\bar{z}) &= \gamma^{\zeta\bar{\zeta}} \int_{-\infty}^{\infty} \mathrm{d}v\, N_{\zeta\zeta} \\ &= -\frac{\kappa}{8\pi} \lim_{E\to 0} E \left[a_+(E\hat{n}) + a_-^\dagger (E\hat{n}) \right] \end{aligned} \tag{5.107}$$

then

$$\langle \mathrm{out}|(N(z,\bar{z})S - SN(z,\bar{z}))|\mathrm{in}\rangle = -\frac{\kappa^2}{8\pi} \sum_{i=1}^{4} \eta_i \frac{p_i^\mu p_i^\nu \varepsilon^+_{\mu\nu}(\hat{n})}{p_i\cdot \hat{n}} \langle \mathrm{out}|S|\mathrm{in}\rangle, \tag{5.108}$$

where $|\mathrm{out}\rangle := S|\mathrm{in}\rangle$ and for simplicity we can take $|\mathrm{in}\rangle = |p_1 p_2\rangle$, consistently with the notation in section 4.4 for the two-body problem. Clearly, a similar equation holds for the other graviton helicity by using the conjugate memory operator $\bar{N}(z,\bar{z}) = \gamma^{z\bar{z}} \int_{-\infty}^{\infty} \mathrm{d}v\, N_{\bar{\zeta}\bar{\zeta}}$.

We will follow [302] for the next discussion. The interesting part about the memory operator eq. (5.107) is that we can easily find the eigenstate which diagonalizes the action of $N(z,\bar{z})$: not surprisingly, it takes the form of a coherent state

$$|N_{\mathrm{in/out}}\rangle = \exp\left\{ \sum_{\sigma=\pm 2} \int \mathrm{d}\Phi(k) N^{\mu\nu}_{\mathrm{in/out}}(k) \left[\varepsilon^\sigma_{\mu\nu}(k) a^\dagger_\sigma(k) - \varepsilon^{*\sigma}_{\mu\nu}(k) a_\sigma(k) \right] \right\} |0\rangle, \tag{5.109}$$

where the waveshape is required to have a pole in the energy

$$\lim_{E\to 0} E\, a_+(E\hat{n})\, |N_{\mathrm{in/out}}\rangle = \lim_{E\to 0} E\, N^{\mu\nu}_{\mathrm{in/out}}(E\hat{n})\, \varepsilon^+_{\mu\nu}(\hat{n}) |N_{\mathrm{in/out}}\rangle. \tag{5.110}$$

consistently with the leading Weinberg soft theorem. At this point we can rewrite the Ward identity between eigenstates of the memory operator as

$$(N_{\mathrm{out}} - N_{\mathrm{in}})\langle \mathrm{out}|S|\mathrm{in}\rangle = \Omega^{\mathrm{soft}} \langle \mathrm{out}|S|\mathrm{in}\rangle, \tag{5.111}$$

where the soft factor is

$$\Omega^{\text{soft}} = -\frac{\kappa^2}{8\pi} \sum_i \eta_i \frac{p_i^\mu p_i^\nu}{p_i \cdot \hat{n}} \varepsilon^+_{\mu\nu}(E\hat{n}) \tag{5.112}$$

and the eigenvalue of the memory operator is given explicitly by

$$N_{\text{in/out}} = -\frac{\kappa}{4\pi} \left(\lim_{E\to 0} E\, N^{\mu\nu}_{\text{in/out}}(E\hat{n}) \epsilon^+_{\mu\nu}(\hat{n}) \right) \to N^{\mu\nu}_{\text{out}} - N^{\mu\nu}_{\text{in}} = \frac{\kappa}{2} \sum_{i=1}^{4} \eta_i \frac{p_i^\mu p_i^\nu}{p_i \cdot k}. \tag{5.113}$$

Therefore, we see that the states are labelled by the value of the soft supertranslation charge. Without loss of generality, we can assume that the incoming state is a Fock state $|p_1 p_2\rangle$ with $N_{\text{in}} = 0$. Then, for consistency, the final state has to include a coherent state of the form

$$|N_{\text{out}}\rangle = \exp\left\{ \frac{\kappa}{2} \sum_{i=1}^{4} \eta_i \sum_{\sigma=\pm 2} \int \mathrm{d}\Phi(k) \left(\frac{p_i^\mu p_i^\nu}{p_i \cdot k} \right) \left[\varepsilon^\sigma_{\mu\nu}(k) a^\dagger_\sigma(k) - \varepsilon^{*\sigma}_{\mu\nu}(k) a_\sigma(k) \right] \right\} |0\rangle. \tag{5.114}$$

The appearance of this coherent state in the gravitational S-matrix can be shown in other ways, which make manifest that this is classical result. Indeed, this is generated by the stress tensor of free massive particles moving in a straight-line trajectory as can be shown in the Faddeev-Kulish approach [301]. To make this precise, one can study the interaction Hamiltonian for our theory of scalars minimally coupled with gravity in the linearized gravity approximation and in asymptotic limit $|t| \to +\infty$ [264]

$$\begin{aligned} H(t) &= H_0 + V^{\text{asy}}(t) = H_0 - \int \mathrm{d}^3x\, h^{\mu\nu}(t, \boldsymbol{x}) T^{\text{asy}}_{\mu\nu}(t, \boldsymbol{x}), \\ T^{\text{asy}}_{\mu\nu} &= \sum_{i=1,2} \frac{\kappa}{2} \int \mathrm{d}\Phi(p) \frac{p^\mu p^\nu}{E_p} \rho_i(\boldsymbol{p}) \delta^3\left(\boldsymbol{x} - t\frac{\boldsymbol{p}}{E_p} \right), \end{aligned} \tag{5.115}$$

where H_0 is the free Hamiltonian, $\rho_i(\boldsymbol{p}) = a_i^\dagger(\boldsymbol{p}) a_i(\boldsymbol{p})$ is the number operator for the i-th scalar field and $(t, \boldsymbol{x})$ are the coordinates of the asymptotic trajectory of the scalar particles. This result is is valid only at first order in the perturbation $h_{\mu\nu}$, but it tells something important about the asymptotic dynamics: solving the asymptotic Schrödinger equation for the potential $V^{\text{asy}}(t)$ gives an evolution operator which generates the coherent state of gravitons given by eq. (5.114).[13] This gives an insight about how the quantum scattering theory generates a classical gravitational wave, composed of infinitely many gravitons, at least in the soft kinematic regime.

In this specific context, the coherent state in eq. (5.114) is referred as Faddeev-Kulish state [285]. It turns out that, if we dress the external hard massive scalar particles with the appropriate coherent states of gravitons in a such a way that the the BMS supertranslation Ward identity is obeyed, then the new dressed S-matrix in perturbative quantum gravity is infrared finite as proved in [301]. In particular, for

[13] There is also an additional term which is the analogue of the Coulomb phase contribution in QED, which plays an important role at higher orders in the soft expansion but not at leading order.

our two-body problem this means that we should consider

$$\langle\text{out}|S|\text{in}\rangle$$
$$= \langle p_3 p_4| \exp\Big\{ -\frac{\kappa}{2}\sum_{i=1}^{4}\eta_i \int \mathrm{d}\Phi(k)\frac{p_i^\mu p_i^\nu}{p_i\cdot k}\Big[\varepsilon^{\sigma}_{\mu\nu}(k)a^\dagger_\sigma(k) - \varepsilon^{*\sigma}_{\mu\nu}(k)a_\sigma(k)\Big]\Big\} S|p_1p_2\rangle. \tag{5.116}$$

Finally, it is worth mentioning that while this procedure can be easily generalized to scalar QED this is not the case for non-abelian theories like quantum chromodynamics (QCD). Indeed, a crucial ingredient in our discussion was the fact that IR divergences are only coming from the soft region: in general, collinear divergences are also relevant. The coherent states required for the dressing of the S-matrix in gravity are generated by classical sources given by the free massive particles moving on straight-line trajectories, but collinear divergences do not arise from such simple asymptotic dynamics. At leading logarithmic order, [14] one can still follow the coherent state approach of Catani, Ciafaloni and Marchesini [307–311] which uses energy ordering in each interaction to systematically organise the divergences due to soft gluons in QCD, and indeed a relation with asymptotic symmetries has been recently established [1]. But despite the attempts to study collinear divergences fully within the coherent operator approach [312, 313], it is likely that a more complicated structure should arise at higher orders and might also involve other particles distributions than the Poissonian one discussed in section 5.4. It would be very interesting to explore this more in the future.

[14] See [303] for an introduction to the topic, [304] for a recent review on the problem and [305, 306] for an alternative approach using old-fashioned perturbation theory.

Chapter 6

Tree amplitudes in classical gravitational scattering

In this chapter, we will compute several tree-level amplitudes for the quantum gravitational scattering with radiation both by using Feynman diagrams and BCFW recursion relations. Then. we will take the classical limit and we will prove that the leading deviation from coherence in the final state, which corresponds to the six-point tree amplitude $\mathcal{M}_6^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^{\sigma_1}, 6^{\sigma_2})$, is quantum suppressed. Just for this section, we will use the mostly plus convention for the spacetime signature.

6.1 Tree amplitudes from Feynman diagrams

In this section, we extend the parametrization of the pure Lagrangian used by Cheung and Remmen [314] to the case of real scalar fields minimally coupled with gravity. This will make use of an auxiliary field, the connection, whose job is to effectively resum higher order graviton pure contact vertices in the same spirit as the first order Palatini formulation developed by Deser [315, 316]. We can then compute in a straightforward way all the tree level amplitudes we need for this work.

Let us consider the Lagrangian of two real scalars minimally coupled with gravity in $D = 4$ dimensions,

$$\begin{aligned} S &:= S_{GR} + S_{matter}, \\ S_{GR} &:= \frac{1}{16\pi G}\int \mathrm{d}^4x \left[\partial_a\sigma_{ce}\partial_b\sigma^{de}\left(\frac{1}{4}\sigma^{ab}\delta^c_d - \frac{1}{2}\sigma^{cd}\delta^a_d\right) + \frac{1}{2}\sigma^{ab}\omega_a\omega_b\right], \\ S_{matter} &:= -\sum_{j=A,B}\int \mathrm{d}^4x \left[\frac{1}{2}\sigma^{ab}\partial_a\phi_j\partial_b\phi_j + \frac{1}{2}\sqrt{-\det(\sigma^{-1})}m_j^2\phi_j^2\right], \end{aligned} \tag{6.1}$$

where we have used the following conventions:

$$\begin{aligned} &\sigma_{ab} := \frac{1}{\sqrt{-g}}g_{ab}, \quad \sigma^{ab} = \sqrt{-g}g^{ab}, \quad \det(g) = \det(\sigma^{-1}), \\ &\omega_a := \partial_a \log\sqrt{-g} = \frac{1}{2}\sigma_{bc}\partial_a\sigma^{bc}. \end{aligned} \tag{6.2}$$

We introduce the auxiliary field A^a_{bc}, which allows us to rewrite the pure gravity Lagrangian as

$$S_{GR} = \frac{1}{16\pi G}\int \mathrm{d}^4x \left[-\left(A^a_{bc}A^b_{ad} - \frac{1}{3}A^a_{ac}A^b_{bd}\right)\sigma^{cd} + A^a_{bc}\partial_a\sigma^{bc}\right]. \tag{6.3}$$

Before setting up the perturbation theory in the new variables, it is useful to unmix the graviton and auxiliary field by doing the shift

$$A^a_{bc} \to A^a_{bc} - \frac{1}{2}\left(\partial_b h^a_c + \partial_c h^a_b - \partial^a h_{bc} + \frac{1}{2}\eta_{bc}\partial^a h^d_d\right) \tag{6.4}$$

and adding the gauge fixing term

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2}\partial_a h^{ac}\partial^b h_{bc} = -\frac{1}{2}\eta_{cd}\partial_a\left(\sqrt{-g}g^{ac}\right)\partial_b\left(\sqrt{-g}g^{bd}\right). \tag{6.5}$$

Using

$$\sigma^{ab} = \eta^{ab} - \kappa h^{ab} \tag{6.6}$$

with $\kappa = \sqrt{32\pi G}$, and the expansion

$$\begin{aligned}
\sqrt{-\det(\sigma^{-1})} &= \exp\left[\frac{1}{2}\operatorname{Tr}\log\left(1 - \kappa\eta h^{-1}\right)\right] \\
&= 1 - \frac{\kappa}{2}\operatorname{Tr}\left(\eta h^{-1}\right) - \frac{\kappa^2}{4}\operatorname{Tr}\left(\eta h^{-1}\right)^2 - \frac{\kappa^3}{8}\operatorname{Tr}\left(\eta h^{-1}\right)^3 + \frac{\kappa^2}{8}\operatorname{Tr}^2\left(\eta h^{-1}\right) \\
&\quad + \frac{\kappa^3}{6}\operatorname{Tr}\left(\eta h^{-1}\right)\operatorname{Tr}\left(\eta h^{-1}\right)^2 + \mathcal{O}(h^3) \\
&= 1 - \frac{\kappa}{2}h^a_a - \frac{\kappa^2}{4}h^{ab}h_{ab} - \frac{\kappa^3}{8}h^{bc}h_{ab}h^a_c + \frac{\kappa^2}{8}(h^a_a)^2 + \frac{\kappa^3}{6}h^a_a\left(h_{bc}h^{bc}\right) + \mathcal{O}(h^3),
\end{aligned} \tag{6.7}$$

we get explicitly up to $\mathcal{O}(h^3)$ a Lagrangian of the form

$$\begin{aligned}
\mathcal{L} = \mathcal{L}_{\mathrm{GR}} + \mathcal{L}_{\mathrm{matter}} + \mathcal{L}_{\mathrm{GF}} &= \mathcal{L}_{hh} + \mathcal{L}_{AA} + \mathcal{L}_{hhh} \\
&\quad + \mathcal{L}_{hhA} + \mathcal{L}_{hAA} + \mathcal{L}_{\phi\phi} + \mathcal{L}_{h\phi\phi} + \mathcal{L}_{hh\phi\phi} + \mathcal{L}_{hhh\phi\phi}.
\end{aligned} \tag{6.8}$$

The quadratic terms in the Lagrangian are given by

$$\begin{aligned}
\mathcal{L}_{hh} &:= \frac{1}{2}\left(h_{ab}\Box h^{ab} - \frac{1}{2}h^e_e\Box h^f_f\right), \\
\mathcal{L}_{AA} &:= -2\left(A^a_{bc}A^b_{ad} - \frac{1}{3}A^a_{ac}A^b_{bd}\right)\eta^{cd}, \\
\mathcal{L}_{\phi\phi} &:= -\sum_{j=A,B}\left[\frac{1}{2}\partial^a\phi_j\partial_a\phi_j + m_j^2\phi_j^2\right],
\end{aligned} \tag{6.9}$$

and the interaction terms are

$$\begin{aligned}
\mathcal{L}_{hhh} &:= \kappa\frac{1}{2}h^{ab}\left[\partial_a h_{cd}\partial_b h^{cd} + 2\partial_{[c}h_{d]b}\partial^d h^c_a + \frac{1}{2}\left(2\partial_c h_{ab}\partial^c h^e_e - \partial_a h^e_e\partial_b h^f_f\right)\right], \\
\mathcal{L}_{hhA} &:= 2\kappa h^{ab}\left[A^c_{ad}\left(\partial^d h_{bc} - \partial_{(b}h^d_{c)}\right) - \frac{1}{2}\left(\eta_{ad}A^d_{bc}\partial^c h^e_e - A^c_{ca}\partial_b h^e_e\right)\right], \\
\mathcal{L}_{hAA} &:= 2\kappa h^{ab}\left(A^c_{ad}A^d_{bc} - \frac{1}{3}A^c_{ac}A^d_{bd}\right), \\
\mathcal{L}_{h\phi\phi} &:= \frac{\kappa}{2}\sum_{j=A,B}\left[h^{ab}\partial_a\phi_j\partial_b\phi_j + \frac{1}{2}h^a_a m_j^2\phi_j^2\right],
\end{aligned}$$

$$\mathcal{L}_{hh\phi\phi} := \frac{\kappa^2}{8} \sum_{j=A,B} \left[h^{ab} h_{ab} - \frac{1}{2} (h^a_a)^2 \right] m_j^2 \phi_j^2,$$
$$\mathcal{L}_{hhh\phi\phi} := \frac{\kappa^3}{16} \sum_{j=A,B} \left[\frac{4}{3} h^{bc} h_{ab} h^a_c - h^a_a \left(h_{bc} h^{bc} \right) \right] m_j^2 \phi_j^2. \tag{6.10}$$

In the massless limit $m_A, m_B \to 0$, the interaction terms become purely trivalent. In that case, it is possible to set up the standard Berends-Giele recursion relations. But even with the mass terms, the final expressions are more compact than in the standard perturbative expansion of gravity: the gravity pure self-interactions are nicely resummed by the auxiliary field, which makes it possible to avoid the cumbersome expressions for higher point vertices (at least at tree level, where ghosts are absent). The Feynman rules for the propagators are then

$$(\Delta^{hh})_{abcd}(p) = -\frac{i}{2p^2} \left(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc} - \eta_{ab}\eta_{cd} \right),$$
$$(\Delta^{AA})^{ad}_{bcef}(p) = -\frac{i}{4} \left[\frac{1}{2} \delta^d_{(b}\eta_{c)(e}\delta^a_{f)} + \eta^{ad} \left(\frac{1}{2}\eta_{bc}\eta_{ef} - \frac{1}{2}\eta_{b(e}\eta_{f)c} \right) \right],$$
$$(\Delta^{\phi_j\phi_j})(p) = -\frac{i}{p^2 + m_j^2} \qquad \text{for } j = A, B, \tag{6.11}$$

and the rules for the interaction vertices are

$$\begin{aligned}
\langle h_{ab} h_{cd} h_{ef} \rangle (p_1, p_2, p_3) = i\frac{\kappa}{2} \Bigg\{ & \left[\frac{1}{2} \left(\eta_{a(c}\eta_{d)(e}\eta_{f)b} + \eta_{b(c}\eta_{d)(e}\eta_{f)a} \right) (p_1 \cdot p_2) \right. \\
& - \frac{1}{2} \left(\eta_{ab}\eta_{c(e}\eta_{f)d} + \eta_{cd}\eta_{a(e}\eta_{f)b} \right) (p_1 \cdot p_2) \\
& \left. + \left(\frac{1}{2}\eta_{ab}\eta_{cd} - \frac{1}{2}\eta_{a(c}\eta_{d)b} \right) p_{1(e}p_{2f)} - \frac{1}{2} p_{2(a}\eta_{b)(e}\eta_{f)(d}p_{1c)} \right] \\
& + \begin{bmatrix} p_2 \leftrightarrow p_3 \\ cd \leftrightarrow ef \end{bmatrix} + \begin{bmatrix} p_1 \leftrightarrow p_3 \\ ab \leftrightarrow ef \end{bmatrix} \Bigg\}, \\
\left\langle h_{ab} A^c_{de} A^f_{gh} \right\rangle (p_1, p_2, p_3) = i\frac{\kappa}{2} & \left(\delta^c_{(g}\eta_{h)(a}\eta_{b)(d}\delta^f_{e)} - \frac{1}{3}\delta^f_{(g}\eta_{h)(a}\eta_{b)(d}\delta^c_{e)} \right), \\
\left\langle h_{ab} h_{cd} A^e_{fg} \right\rangle (p_1, p_2, p_3) = \frac{\kappa}{2} \Bigg\{ & \left[\frac{1}{2} \delta^e_{(a} \left(\eta_{b)(f}\eta_{g)(c}p_{1d)} - \eta_{b)(c}\eta_{d)(f}p_{1g)} \right) \right. \\
& \left. + \frac{1}{2}\eta_{ab} \left(p_{1(f}\eta_{g)(c}\delta^e_{d)} - p_{1(c}\eta_{d)(f}\delta^e_{g)} \right) \right] + \begin{bmatrix} p_1 \leftrightarrow p_2 \\ ab \leftrightarrow cd \end{bmatrix} \Bigg\} \\
& - \frac{\kappa}{4} p_3^e \left(\eta_{f(a}\eta_{b)(c}\eta_{d)g} + \eta_{g(a}\eta_{b)(c}\eta_{d)f} \right), \\
\langle h_{ab} \phi_j \phi_j \rangle (p_1, p_2, p_3) = -i\frac{\kappa}{2} & \left(p_{2(a}p_{3b)} - m_j^2 \eta_{ab} \right) \qquad \text{for } j = A, B, \\
\langle h_{ab} h_{cd} \phi_j \phi_j \rangle (p_1, p_2, p_3, p_4) = i\frac{\kappa^2}{4} & m_j^2 \left(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc} - \eta_{ab}\eta_{cd} \right) \qquad \text{for } j = A, B, \\
\langle h_{ab} h_{cd} h_{ef} \phi_j \phi_j \rangle (p_1, p_2, p_3, p_4, p_5) = i\frac{\kappa^3}{8} & m_j^2 \left(\eta_{fa}\eta_{b(c}\eta_{d)e} + \eta_{d(a}\eta_{e)c}\eta_{fb} + \eta_{a(d}\eta_{e)b}\eta_{fc} + \eta_{e(a}\eta_{b)c}\eta_{fd} \right. \\
& \left. - [\eta_{a(c}\eta_{d)b}\eta_{ef} + \eta_{a(e}\eta_{f)b}\eta_{cd} + \eta_{ab}\eta_{e(c}\eta_{d)f}] \right) \qquad \text{for } j = A, B,
\end{aligned} \tag{6.12}$$

where all momenta are chosen to be ingoing. At this point one can implement these

Feynman rules in the xAct package [317], which we use extensively in the following calculations.

For the purposes of simplifying computations, we adopt the following conventions for the momenta of our amplitude:

$$\begin{aligned}\mathcal{M}^{(0)}_{n+4}(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B,5^{\sigma_1},\dots,(n+4)^{\sigma_n}) &\equiv \mathcal{M}^{(0)}_{n+4}\left(p_1,p_2\to -p_3,-p_4,-p_5^{\sigma_1},\dots,-p_{n+4}^{\sigma_n}\right)\\ &\equiv \mathcal{M}^{(0)}_{n+4}\left(p_1,p_2\to -p_3,-p_4,-k_1^{\sigma_1},\dots,-k_n^{\sigma_n}\right),\end{aligned} \tag{6.13}$$

and we define the momentum invariants $s_{ij}=-(p_i+p_j)^2$, with Mandelstam invariants defined as $s=s_{12}$ and $t=s_{13}$ in the particular case of four-point kinematics.

6.1.1 Four-point and five-point tree amplitude

We have only one diagram in the 4-pt case, given in Fig. 6.1. The Feynman rules give the well-known result[1]

$$\mathcal{A}^{(0)}_4(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B) = -\frac{i\kappa^2}{2t}\left(\frac{1}{2}t\left(-m_A^2-m_B^2+s\right)+\frac{1}{2}\left(-m_A^2-m_B^2+s\right)^2-m_A^2m_B^2\right). \tag{6.14}$$

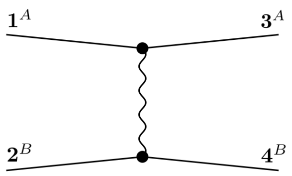


FIGURE 6.1: The only Feynman diagram contributing to $\mathcal{A}^{(0)}_4(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B)$.

For the 5-pt amplitude, we have explicitly computed the 7 diagrams pictured in Fig. 6.2. Notice that the first 6 diagrams are in one-to-one correspondence with the analogous calculation in scalar QED [95], while the last one is related to the graviton self-interaction.

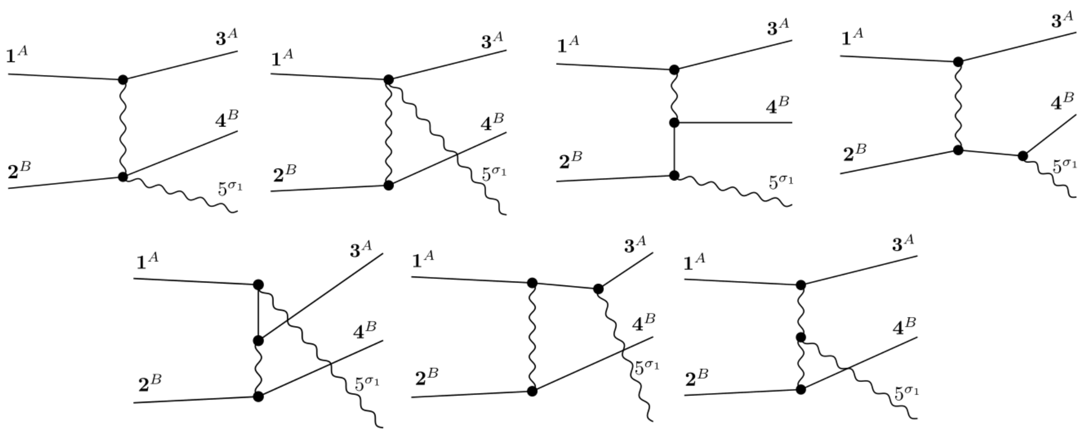


FIGURE 6.2: The Feynman diagrams contributing to $\mathcal{A}^{(0)}_5(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B,5^{\sigma_1})$.

[1]See for example eq. (3.1) of [318], with $D=4$ and $\kappa=\sqrt{2}\kappa_4$.

6.1.2 Six-point tree amplitude

We have computed the 68 diagrams in Fig. 6.3 for the 6-point tree amplitude. In order from the top left of the picture in Fig. 6.3, the first 42 of these diagrams can be compared with the analogous calculation in scalar QED done in [6], which in particular involve the 3-point and the 4-point vertices with one matter line and one or two gravitons. The remaining 26 diagrams are classified into the following three types:

- 21 diagrams involving the graviton self-interaction;
- 3 diagrams with the auxiliary field propagator;
- 2 diagrams with a 5-point contact vertex with 3 gravitons and one matter line.

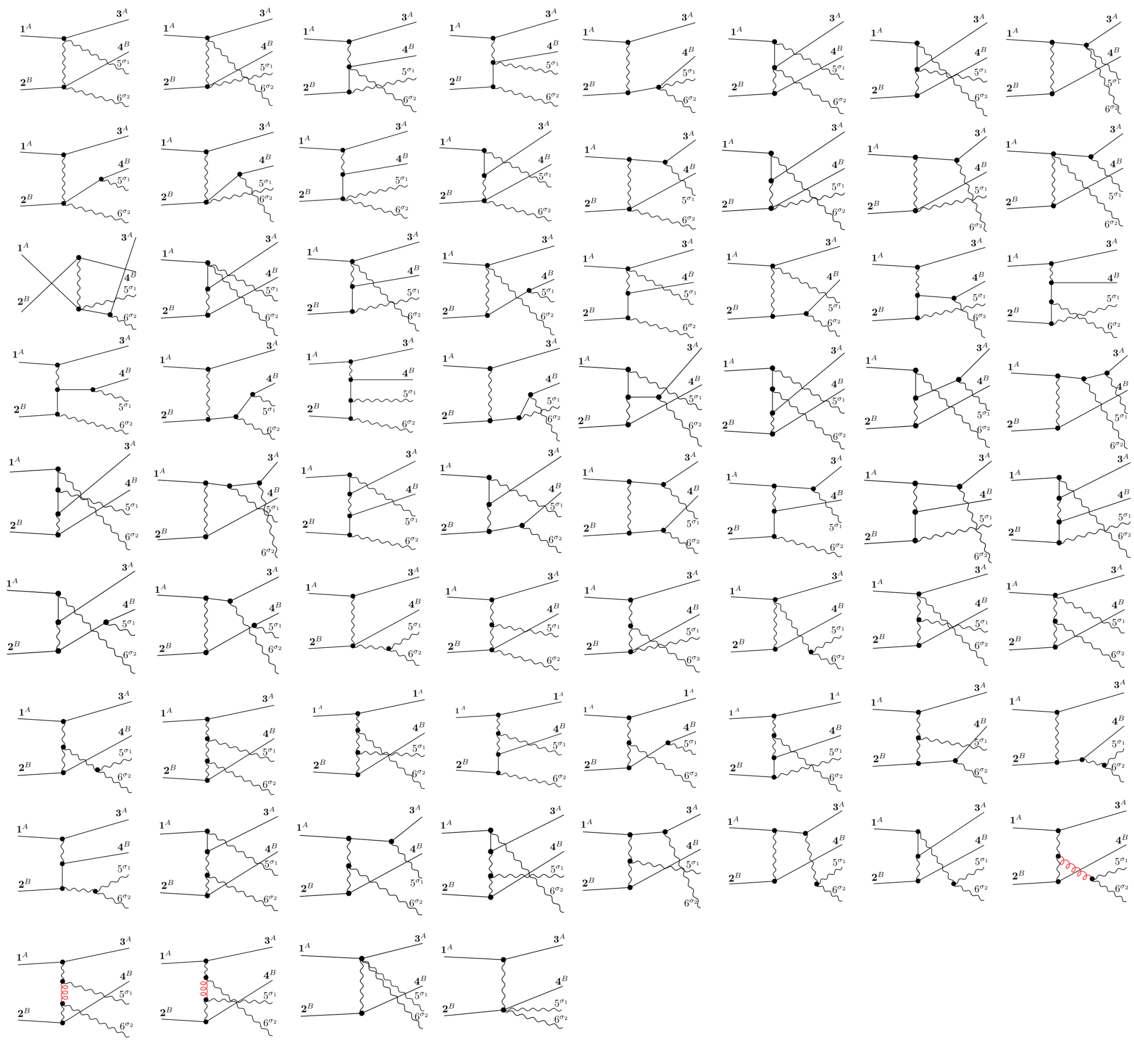


FIGURE 6.3: The Feynman diagrams contributing to $\mathcal{A}_6^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^{\sigma_1}, 6^{\sigma_2})$. We have highlighted in red the contribution of the auxiliary field, which is crucial to obtain the correct result.

The calculation of these tree level amplitudes agrees exactly with an independent on-shell BCFW calculation presented in the next section.

6.2 Tree amplitudes from on-shell recursion relations

In this section,we compute the necessary tree-level amplitudes for the theory defined in eq. (6.1) by using an on-shell diagrammar[2] to recursively construct all the amplitudes in the theory. A diagrammar requires basic amplitudes to serve as the atoms of the computation, and the on-shell recursive framework of BCFW [321]. In massless theories there are straightforward arguments to construct three-point amplitudes from little-group scaling [322, 323]. The simplicity comes from the on-shellness of the momenta, which is maintained throughout the computation and simplifies the expressions needed as input.

We begin with a brief review of BCFW recursion, in preparation for the new shift that we will introduce to compute the 5-point tree amplitude $\mathcal{A}_5^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^{\sigma_1})$ and to set the stage for its application to the 6-point tree amplitude $\mathcal{A}_6^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^{\sigma_1}, 6^{\sigma_2})$.

6.2.1 Review of BCFW

The basic mechanism of BCFW recursion is understood through elementary complex analysis. The derivation begins by introducing a complex variable z and considering a linear shift in (a subset of) the momenta p_i in the (yet-to-be-determined) n-point tree-level amplitude:

$$\mathcal{A}_n^{(0)}(\{p_i\}) \to \mathcal{A}_n^{(0)}(\{\hat{p}_i\}) , \tag{6.15}$$

where the shifted momenta are defined as

$$\hat{p_i} = p_i + z r_i . \tag{6.16}$$

The choice of r_i corresponds to a *choice of shift*.

As tree amplitudes are rational functions, we can consider $\mathcal{A}_n^{(0)}(\{\hat{p}_i\})$ as a meromorphic function of z which we denote as $\mathcal{A}_n^{(0)}(z)$. We then evaluate the contour integral

$$\oint_{\gamma_\infty} dz \frac{\mathcal{A}_n^{(0)}(z)}{z} = \mathcal{A}_n^{(0)}(0) + \sum_I \operatorname*{Res}_{z=z_I} \left[\frac{\mathcal{A}_n^{(0)}(z)}{z} \right] . \tag{6.17}$$

where the z_I are the poles in the complex plane, and the integration contour $\gamma_\infty := \lim_{R\to\infty} \gamma_R$, where γ_R is a circular contour around the origin with radius R.

The choice of the vectors r_i will to some extent determine the large-z behavior, but importantly must also satisfy [324]:

- For all i, j, we have $r_i \cdot r_j = 0$, which ensures linearity of deformed inverse propagators in z;
- On-shellness of the shifted momenta: $\hat{p}_i^2 = -m_i^2$, which implies $r_i \cdot p_i = 0$;
- Conservation of momentum is maintained on the shift, i.e. $\sum_i r_i = 0$.

With an appropriate choice of shift, and for generic kinematics, and the non-trivial residues on the right-hand side are thus encoded by the kinematic poles of the amplitude. In particular, the first condition implies that the poles in $\mathcal{A}_n^{(0)}(z)$ are simple

[2]This terminology is taken from [319, 320].

poles. The residues are defined by the product of lower-point on-shell amplitudes in the same theory and the scalar propagator,

$$\sum_I \operatorname*{Res}_{z=z_I}\left[\frac{\mathcal{A}_n^{(0)}(z)}{z}\right] = -\sum_I \sum_{\sigma=\pm} \mathcal{A}_L^{(0)}\big(\{\hat{p}_L\},\hat{P}_I^{\sigma}\big)\,\frac{-i}{P_I^2+m_I^2}\,\mathcal{A}_R^{(0)}\big(-\hat{P}_I^{-\sigma},\{\hat{p}_R\}\big)\,, \tag{6.18}$$

where L and R stand for the "left" and "right" amplitude in the factorization, and

$$P_I = \sum_R p_R = -\sum_L p_L. \tag{6.19}$$

The momentum channels which contribute a residue are those which contain at least one shifted external momentum in both $\{\hat{p}_L\}$ and $\{\hat{p}_R\}$, and the poles corresponding to each channel are the solutions of the linear equations

$$\hat{P}_I^2 = P_I^2 + z\sum_{i\in R} 2r_i\cdot P_I\,. \tag{6.20}$$

Note also that each pole contributes only a single residue, so partitioning into $\{p_L\}$ and $\{p_R\}$ should take into account global momentum conservation to avoid overcounting.

A "good" shift on $\mathcal{A}^{(0)}$ is defined as any shift for which the left-hand side of eq. (6.17) vanishes, behavior which corresponds to the vanishing of the residue at infinity, also known as the "boundary term",

$$\oint_{\gamma_\infty} dz\frac{\mathcal{A}_n^{(0)}(z)}{z} = \lim_{z\to\infty}\left[\mathcal{A}_n^{(0)}(z)\right] = 0\,. \tag{6.21}$$

For amplitudes in massless theories, it is understood what constitutes a good shift for various helicity configurations in various theories [322, 325–328]. Then, by combining eq. (6.18) with eq. (6.17), we get the recursive formula

$$\mathcal{A}_n^{(0)}(0) = \mathcal{A}_n^{(0)}(\{p\}) = \sum_I\sum_{\sigma=\pm}\mathcal{A}_L^{(0)}\big(\{\hat{p}_L\},\hat{P}_I^{\sigma}\big)\,\frac{-i}{P_I^2}\,\mathcal{A}_R^{(0)}\big(-\hat{P}_I^{-\sigma},\{\hat{p}_R\}\big). \tag{6.22}$$

Later in this section we introduce a new kind of shift which is applicable to massive legs as well. In particular, it will be only the first item, the on-shellness of the momenta, that needs modification to accommodate this case.

In the following section we apply BCF shifts [329] exclusively to massless legs: they are labelled as $[i,j\rangle$ and they modify the external legs as follows:

$$\begin{aligned}\hat{p}_i^{a\dot{b}} &= |\hat{i}]^a\langle i|^{\dot{b}} = \big(|i]+z|j]\big)^a\langle i|^{\dot{b}},\\ \hat{p}_j^{a\dot{b}} &= |j]^a\langle\hat{j}|^{\dot{b}} = |j]^a\big(\langle j|-z\langle i|\big)^{\dot{b}},\end{aligned} \tag{6.23}$$

which implies that

$$r_i^{a\dot{b}} = -r_j^{a\dot{b}} = |j]^a\langle i|^{\dot{b}}. \tag{6.24}$$

We now proceed to apply these shifts to graviton-scalar amplitudes.

We use the spinor-helicity formalism throughout, adopting the shorthand of [324] whereby Feynman-slashed four-momentum is replaced by the momentum labels with products denoted by simply concatenating momentum labels

$$\ldots ij|X] \equiv \ldots \not{p}_i \not{p}_j |X] \,. \tag{6.25}$$

Differences of momenta are similarly denoted, whilst sums of momenta are combined into an upper-case P:

$$(i-j)|X] \equiv (\not{p}_i - \not{p}_j)|X] \quad , \quad P_{ij} = p_i + p_j \,. \tag{6.26}$$

6.2.2 Building blocks of the amplitude diagrammar

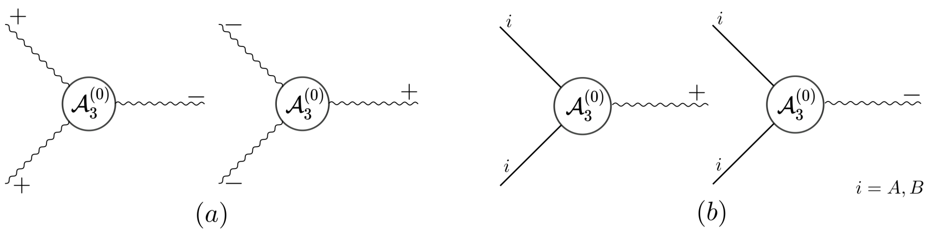


FIGURE 6.4: Generic on-shell diagrams that are the atoms of the diagrammar, corresponding to the amplitudes defined in eq. (6.27) (resp. eq. (6.28) for (a) (resp. (b)).

We begin by looking at amplitudes with a single flavor of massive scalar, which we pick as flavor A without loss of generality. To construct these amplitudes we require pure gravity amplitudes as well as minimally coupled graviton-scalar amplitudes. The diagrams for the three-point amplitudes needed are depicted in Fig. 6.4.

The massless three-point graviton amplitudes are

$$\mathcal{A}_3^{(0)}(1^+, 2^+, 3^-) = i\kappa \frac{[12]^6}{[23]^2[31]^2}, \qquad \mathcal{A}_3^{(0)}(1^-, 2^-, 3^+) = i\kappa \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}, \tag{6.27}$$

and the massive-scalar amplitudes are [323, 330]

$$\mathcal{A}_3^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^+) = i\kappa \frac{[3|2|\chi\rangle^2}{\langle 3\chi\rangle^2}, \qquad \mathcal{A}_3^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^-) = i\kappa \frac{\langle 3|2|\chi]^2}{[3\chi]^2}. \tag{6.28}$$

where we have introduced a reference spinor χ. Although it may appear as though the amplitudes in eq. (6.28) depend on the choice of χ, this is not the case, as long as the denominators do not vanish.

Using the amplitudes in eq. (6.27) and eq. (6.28) we can apply BCFW recursion to construct four-point amplitudes. Up to helicity conjugation and permutation (crossing) invariance, there are two independent configurations:

$$\mathcal{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^+, 4^+), \qquad \mathcal{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^-, 4^+). \tag{6.29}$$

We can apply a $[3,4\rangle$ shift to construct both,[3]

$$\hat{p}_3^{a\dot{b}} = |3\rangle^a [\hat{3}|^{\dot{b}} = |3\rangle^a ([3| + z[4|)^{\dot{b}} \,, \tag{6.30}$$

$$\hat{p}_4^{a\dot{b}} = |\hat{4}\rangle^a [4|^{\dot{b}} = (|4\rangle - z|3\rangle)^a [4|^{\dot{b}}. \tag{6.31}$$

[3]For simplicity, we will suppress the $a, \dot{b}$ indices from here on.

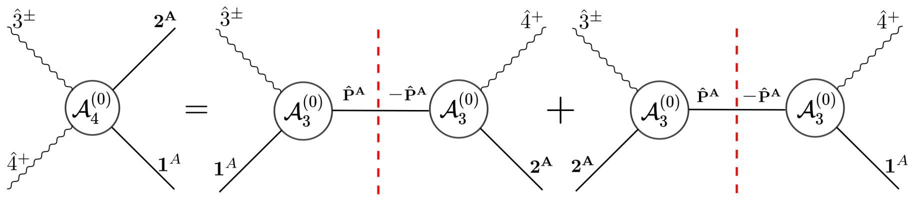


FIGURE 6.5: The sum of factorizations of $\mathcal{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^{\pm}, 4^{+})$ which is needed to reproduce the amplitude via BCFW on-shell techniques.

In massless theories the validity of such a shift follows directly from the scaling of the two-point propagator and polarization tensors [326, 328], and this analysis appears to hold for the massive case too, as it results in the correct amplitudes (see for example [331, 332]).

The momentum shift involves a subset of the physical poles of the theory,

$$\hat{s}_{31} - m_A^2 = 0 \Rightarrow z = z_{31} \equiv -\frac{[3|1|3\rangle}{[4|1|3\rangle}\,, \tag{6.32}$$

$$\hat{s}_{41} - m_A^2 = 0 \Rightarrow z = z_{41} \equiv -\frac{[4|1|4\rangle}{[4|1|3\rangle}\,, \tag{6.33}$$

and thus the amplitudes can be reproduced by the diagrams in Fig. 6.5.

At four points, some simple algebra reproduces a compact form of the amplitude from the factorizations

$$\begin{aligned}\mathcal{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^+, 4^+) &= i\kappa^2 \frac{[\hat{3}|1|\chi\rangle^2}{\langle 3\chi\rangle^2} \frac{-1}{s_{31} - m_A^2} \frac{[4|2|\chi\rangle^2}{\langle \hat{4}\chi\rangle^2} + i\kappa^2 \frac{[\hat{3}|2|\chi\rangle^2}{\langle 3\chi\rangle^2} \frac{-1}{s_{32} - m_A^2} \frac{[4|1|\chi\rangle^2}{\langle \hat{4}\chi\rangle^2} \\ &= -i\kappa^2 \frac{m_A^4 [34]^2}{(s_{31} - m_A^2)\langle 43\rangle^2} + (1 \leftrightarrow 2). \end{aligned} \tag{6.34}$$

The spurious double pole cancels upon summation with the symmetric term, and the technique also gives the correct result for the mixed-helicity configuration,[4]

$$\mathcal{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^+, 4^+) = -i\kappa^2 \frac{m_A^4 [34]^3}{\langle 34\rangle (s_{31} - m_A^2)(s_{32} - m_A^2)}, \tag{6.35}$$

$$\mathcal{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^-, 4^+) = i\kappa^2 \frac{[4|1|3\rangle^4}{s_{34}(s_{31} - m_A^2)(s_{32} - m_A^2)}. \tag{6.36}$$

Finally, we consider the four-point two-flavor amplitude computed from a single Feynman diagram and given in eq. (6.14), that is equivalent to

$$\mathcal{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B) = \frac{i\kappa^2}{2 s_{13}} \left(2(p_1 \cdot p_2)(p_1 \cdot p_4) + m_A^2 m_B^2\right). \tag{6.37}$$

There are well-established on-shell constraints on the classical contribution of this amplitude to eikonal scattering; it consists of a single residue in the form of a product of three-point amplitudes subject to a shift prescription which defines the residue in s_{13} [333]. The full QFT amplitude requires further information to fully reproduce eq. (6.37). Because of the simplicity of the Feynman diagram calculation, we treat it

[4]In particular, the $3 \leftrightarrow 4$ symmetry is restored, and the correct s_{34} factorization is reproduced.

as a fundamental amplitude in our diagrammar, and it joins the basic building blocks in eq. (6.27) and eq. (6.28).

6.2.3 The equal-mass shift

The results discussed in section 6.2.2 relied upon the presence of massless particles in the processes in question, but here we are interested in the amplitude with two massive particles with different flavors and just a single massless graviton, as depicted in Fig. 6.6. This raises the question of whether we can construct this amplitude with any kind of shift. In fact this is possible, but first we need to consider what actually makes on-shell recursion effective.

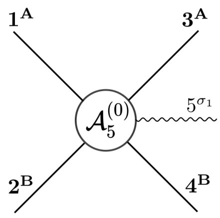


FIGURE 6.6: The five-point tree amplitude we would like to compute with BCFW-like techniques.

The principal advantage of the BCFW method is that it allows us to construct higher-point amplitudes from on-shell *expressions*. When we are dealing with massless theories/particles, this also implies that the on-shell *condition* for a particle is also satisfied: $\hat{p}_i^2 = 0$. These two statements are not completely equivalent when considering theories with equally-massive particles (particles 1 and 3): an on-shell expression need not be in terms of momenta and masses which satisfy the on-shell conditions

$$\hat{p}_1^2 = \hat{p}_3^2 = -m_A^2 \tag{6.38}$$

but can be loosened such that the mass is shifted, but by the same value for both particles:

$$\hat{p}_1^2 = \hat{p}_3^2 = -\hat{m}_A^2. \tag{6.39}$$

The mass m_A is thus treated like a kinematic variable rather than an invariant defining "on-shellness". Crucially, the equal-mass expressions now used in the recursion *remain equal-mass expressions*, and the diagrammar can be used to build amplitudes in the theory just like the massless case.

This approach still requires at least one massless external particle, which we label particle 5 and assume to have positive helicity, without loss of generality. The three-line shift that satisfies the requirements of on-shell recursion is

$$\begin{aligned}
\hat{p}_5 &= |\hat{5}\rangle[\hat{5}| = (|5\rangle + z\,(1-3)|5])[5|\,, \\
\hat{p}_1 &= p_1 + z\,3|5][5|\,, \\
\hat{p}_3 &= p_3 - z\,1|5][5|,
\end{aligned} \tag{6.40}$$

where one can easily verify that

$$\hat{p}_1^2 = p_1^2 - z[5|13|5] = -m_A^2 + z[5|31|5] = \hat{p}_3^2. \tag{6.41}$$

Thus the condition in eq. (6.39) is satisfied, and equal-mass amplitudes can be used in the recursion. Similarly to the BCFW shift, shifting the anti-holomorphic spinor $|5]$ produces a boundary term in $\mathcal{A}_5^{(0)}(z)$, i.e. it is a "bad" shift. From comparison with the extended-Cheung-Remmen Feynman diagram computation of section 6.1, we confirm that the holomorphic shift is a good shift for the five-point tree amplitude.

6.2.4 Five-point tree amplitude

We now apply the equal-mass shift to the tree-level amplitude with two flavors of pairs of minimally-coupled massive particles and one graviton.

The equal-mass shift we use is

$$\begin{aligned} \mathcal{A}_5^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^+) \to \mathcal{A}_5^{(0)}(\hat{\mathbf{1}}^A, \mathbf{2}^B, \hat{\mathbf{3}}^A, \mathbf{4}^B, \hat{5}^+)\ , \\ |\hat{5}\rangle[\hat{5}| = (|5\rangle + z(1-3)|5])\,[5|\ , \\ \hat{p}_1 = p_1 + z\ 3|5][5|\ , \\ \hat{p}_3 = p_3 - z\ 1|5][5|\ , \\ \hat{p}_1 - \hat{p}_3 = p_1 - p_3 + z\ P_{13}|5][5|. \end{aligned} \tag{6.42}$$

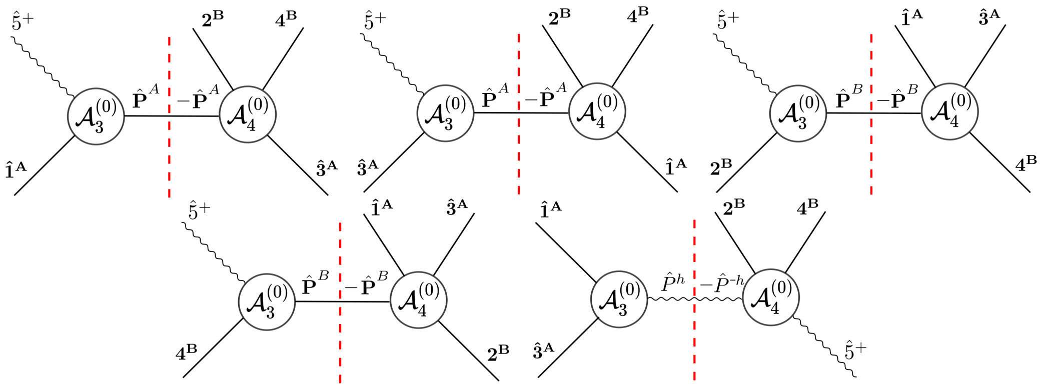


FIGURE 6.7: Factorizations of the five-point tree amplitude $\mathcal{A}_5^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^{\sigma_1})$ on the equal mass shift defined in the expressions in eq. (6.42).

The factorization on the equal-mass poles are depicted in Fig. 6.7, and the shift yields a total of five terms,

$$\begin{aligned} \mathcal{A}_5^{(0)}|_{51} &= \mathcal{A}_3^{(0)}\left(\hat{\mathbf{1}}^A, \hat{\mathbf{P}}^A, \hat{5}^+\right) \frac{i}{s_{51} - m_A^2} \mathcal{A}_4^{(0)}(-\hat{\mathbf{P}}^A, \mathbf{2}^B, \hat{\mathbf{3}}^A, \mathbf{4}^B), \\ \mathcal{A}_5^{(0)}|_{53} &= \mathcal{A}_3^{(0)}\left(\hat{\mathbf{3}}^A, \hat{\mathbf{P}}^A, \hat{5}^+\right) \frac{i}{s_{53} - m_A^2} \mathcal{A}_4^{(0)}(-\hat{\mathbf{P}}^A, \mathbf{2}^B, \hat{\mathbf{1}}^A, \mathbf{4}^B), \\ \mathcal{A}_5^{(0)}|_{52} &= \mathcal{A}_3^{(0)}\left(\mathbf{2}^B, \hat{\mathbf{P}}^B, \hat{5}^+\right) \frac{i}{s_{52} - m_B^2} \mathcal{A}_4^{(0)}(-\hat{\mathbf{P}}^B, \hat{\mathbf{3}}^A, \mathbf{4}^B, \hat{\mathbf{1}}^A), \\ \mathcal{A}_5^{(0)}|_{54} &= \mathcal{A}_3^{(0)}\left(\mathbf{4}^B, \hat{\mathbf{P}}^B, \hat{5}^+\right) \frac{i}{s_{54} - m_B^2} \mathcal{A}_4^{(0)}(-\hat{\mathbf{P}}^B, \hat{\mathbf{3}}^A, \mathbf{2}^B, \hat{\mathbf{1}}^A), \\ \mathcal{A}_5^{(0)}|_{13} &= \sum_{\sigma=\pm} \mathcal{A}_3^{(0)}\left(\hat{\mathbf{1}}^A, \hat{\mathbf{3}}^A, \hat{P}^\sigma\right) \frac{i}{s_{13}} \mathcal{A}_4^{(0)}(\mathbf{2}^B, \mathbf{4}^B, \hat{5}^+, -\hat{P}^{-\sigma}), \end{aligned} \tag{6.43}$$

where the factorizations correspond to residues at the following poles:

$$\begin{aligned} z_{51} &= \frac{[5|1|5\rangle}{[5|13|5]}\,, & z_{53} &= \frac{[5|3|5\rangle}{[5|13|5]}\,, \\ z_{52} &= -\frac{[5|2|5\rangle}{[5|2(1-3)|5]}\,, & z_{54} &= -\frac{[5|4|5\rangle}{[5|4(1-3)|5]}\,, \\ z_{13} &= \frac{s_{13}}{[5|P_{24}(1-3)|5]}\,. \end{aligned} \tag{6.44}$$

It is convenient to organize the calculation in terms of the variables[5]

$$K_A := p_1 - p_3, \qquad K_B := p_2 - p_4, \tag{6.45}$$

which are antisymmetric under the exchange of the corresponding pair of momenta. Each residue in eq. (6.43) yields an expression containing spurious poles, which are not present in the full amplitude. For example the P_{52} factorization gives

$$\begin{aligned} \mathcal{A}_5^{(0)}\big|_{52} =& i\kappa^3 \frac{[5|2K_A|5]^3[5|13|5]^2}{x_{51|53}^2}\frac{-1}{s_{52}-m_B^2}\times \\ & \frac{2(p_1\cdot p_4 + z_{52}[5|43|5])(p_3\cdot p_4 - z_{52}[5|41|5]) + (m_A^2 + z_{52}[5|13|5])m_B^2}{2x_{52|13}}, \end{aligned} \tag{6.46}$$

with the spurious poles $x_{ij|kl}$ proportional to denominator factors evaluated at other residues

$$x_{ij|kl} = [5|P_{ij}K_A|5][5|P_{kl}K_A|5](z_{ij} - z_{kl}). \tag{6.47}$$

Through algebraic manipulations the spurious poles in the full expression can be cleared, and the amplitude can be symmetrized in K_A and K_B. The final expression is

$$\begin{aligned} \mathcal{A}_5^{(0)}\left(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^+\right) =& \frac{i\kappa^3}{8}\Bigg(\Bigg[\frac{-p_4\cdot p_2[5|13|5]^2}{s_{24}(s_{51}-m_A^2)(s_{53}-m_A^2)} + \frac{[5|K_AK_B|5]^2 - 8[5|13|5]^2}{16 s_{13}s_{24}} \\ &+ \frac{(m_A^2+m_B^2)[5|13|5]\big(2(s_{13}-s_{24})[5|13|5] + [5|K_B|5\rangle[5|K_AK_B|5]\big)}{8(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)} \\ &- \frac{K_A\cdot K_B(s_{24}-s_{13})^2[5|13|5]\big(4s_{24}[5|42|5] - [5|K_A|5\rangle[5|K_AK_B|5]\big)}{32 s_{13}s_{24}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)} \\ &- \frac{K_A\cdot K_B[5|42|5]\big([5|K_B|5\rangle[5|K_AK_B|5] - 4(s_{13}+s_{24})[5|13|5]\big)}{8s_{13}s_{24}(s_{52}-m_B^2)(s_{54}-m_B^2)} \\ &- \frac{K_A\cdot K_B[5|K_A|5\rangle[5|K_B|5\rangle\big([5|K_AK_B|5]^2 - 8[5|42|5]^2\big)}{64 s_{13}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)} \\ &+ \left(\mathrm{Tr}(\not{K}_A\not{K}_B\not{K}_A\not{K}_B) + 2[5|K_A|5\rangle^2 + 2[5|K_B|5\rangle^2 - 2s_{13}^2 - 2s_{24}^2\right) \\ &\times\Bigg(\frac{[5|K_A|5\rangle[5|K_B|5\rangle[5|13|5][5|42|5]}{64s_{13}s_{24}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)} \\ &\quad + \frac{[5|42|5](2(s_{13}-s_{24})[5|42|5] - [5|K_A|5\rangle[5|K_AK_B|5])}{64s_{13}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}\Bigg)\Bigg] + \big[(1,3,K_A)\leftrightarrow(2,4,K_B)\big]\Bigg), \end{aligned} \tag{6.48}$$

[5]Note that because the momenta are all incoming, the K_i are *not* momentum transfers.

and the negative helicity case is obtained simply by switching the square brackets for angle brackets. We find perfect agreement with the Feynman diagram calculation from section 6.1 when tested on rational kinematic points.

6.2.5 Six-point tree amplitude

The six-point tree amplitude can be computed using a standard BCFW shift $[5,6\rangle$,[6] where we consider

$$\mathcal{A}_6^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, \hat{5}^{\pm}, \hat{6}^{+}) \,, \tag{6.49}$$

which generates 10 factorization diagrams. All of these are of the general types of factorizations are shown in Fig. 6.8. We make use of the permutation invariance of the scalar particle by defining $\{\mathbf{I_1}, \mathbf{I_2}\} = \mathbf{P}\left(\{\mathbf{1^A}, \mathbf{3^A}\}\right)$ or $\{\mathbf{I_1}, \mathbf{I_2}\} = \mathbf{P}\left(\{\mathbf{2^B}, \mathbf{4^B}\}\right)$, with the complement set labelled as $\mathbf{J}_i$. There are four factorizations for each of the left and middle diagrams and two for the last, giving a total of ten residue contributions to the amplitude.

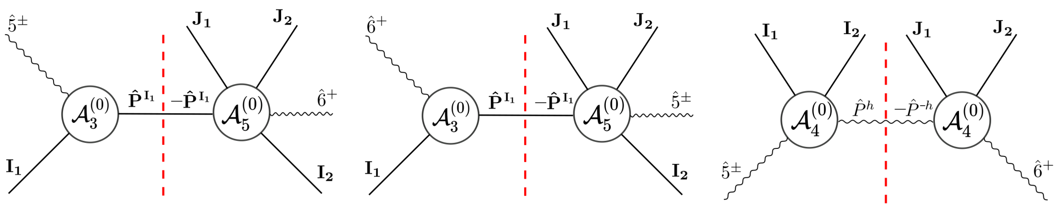


FIGURE 6.8: Schematic representation (up to crossing symmetry) of the three independent factorizations which are relevant for the construction of the six-point tree amplitude $\mathcal{A}_6^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^{\sigma_1}, 6^{\sigma_2})$ from the BCFW shift.

$$\mathcal{A}_6^{(0)}|_{5I_1} = \mathcal{A}_3^{(0)}\left(\mathbf{I_1}, \hat{\mathbf{P}}^{\mathbf{I_1}}, \hat{5}^{\pm}\right) \frac{i}{s_{5I_1} - m_I^2} \mathcal{A}_5^{(0)}(-\hat{\mathbf{P}}^{\mathbf{I_1}}, \mathbf{J}_1, \mathbf{I}_2, \mathbf{J_2}, \hat{6}^{+}) \tag{6.50}$$

$$\mathcal{A}_6^{(0)}|_{6I_1} = \mathcal{A}_3^{(0)}\left(\mathbf{I_1}, \hat{\mathbf{P}}^{\mathbf{I_1}}, \hat{6}^{+}\right) \frac{i}{s_{6I_1} - m_I^2} \mathcal{A}_5^{(0)}(-\hat{\mathbf{P}}^{\mathbf{I_1}}, \mathbf{J}_1, \mathbf{I}_2, \mathbf{J_2}, \hat{5}^{\pm}) \tag{6.51}$$

$$\mathcal{A}_6^{(0)}|_{I_1 I_2 5} = \sum_{\sigma=\pm} \mathcal{A}_4^{(0)}\left(\mathbf{I_1}, \mathbf{I_2}, \hat{5}^{\pm}, \hat{P}^{\sigma}\right) \frac{i}{s_{I_1 I_2 5}} \mathcal{A}_4^{(0)}\left(\mathbf{J_1}, \mathbf{J_2}, \hat{6}^{+}, -\hat{P}^{-\sigma}\right) \tag{6.52}$$

We confirm numerically the vanishing of the boundary (large-z) terms from the Feynman-diagram expression.[7] Moreover, we have verified that the reproduction of the amplitude, as the Feynman-diagram and on-shell calculations produce the same result on all (rational) numerical points tested.

[6]The three-line shift used in section 6.2.3 produces more factorisations making the form less efficient. Moreover the presence of a boundary term in the same-helicity case restricts its application to generic configurations.

[7]This occurs diagram by diagram for the mixed helicity case, but there are non-trivial cancellations in the case of the all-plus-helicity configuration.

6.3 The classical limit of scattering amplitudes with radiation: graviton interference is a quantum effect

In this section, we use the explicit calculation of the six-point tree amplitude $\mathcal{A}_6^{(0)}$ of the previous section to prove the coherence of the emitted semiclassical radiation field up to order $\mathcal{O}(G^4)$ for radiative observables. Moreover, assuming coherence to all orders as suggested by the arguments of section 5.4, we derive an infinite set of non-trivial relations between unitarity cuts in the classical limit. Those are relevant for the calculation of physical radiative observables, such as the waveform or the total linear and angular momentum emitted by the gravitons, because they suggest that only the 5-pt amplitude is required for the classical calculation and all the higher multiplicity amplitudes are not explicitly needed.

In order to take the classical limit, we follow the rules established in [166]. We express the massless momenta in terms of their wavenumbers and the momentum transfers of eq. (5.74),

$$k_i = \hbar \bar{k}_i \quad \text{for } i = 1, 2, 3, \dots ; \qquad q_j = \hbar \bar{q}_j, \quad w_j = \hbar \bar{w}_j \qquad \text{for } j = 1, 2; \tag{6.53}$$

and we use the parametrization of the massive momenta from eq. (5.76), which define the classical trajectory. They are therefore associated to classical velocities v_A and v_B,

$$p_j = \tilde{m}_j v_j, \qquad \tilde{m}_j^2 = m_j^2 - \hbar^2 \frac{\bar{q}^2}{4} \qquad \text{for } j = A, B\,. \tag{6.54}$$

Note that in section 6.2 we used notation which was more compact for the purposes of computing the amplitudes. We can translate to the notation introduced earlier in eq. (5.75) by noticing that

$$P_{13} = -w_1\,,\ P_{24} = -w_2\,. \tag{6.55}$$

Crucially, we also need to restore the powers of $\hbar$ in the coupling as

$$\kappa \to \frac{\kappa}{\sqrt{\hbar}}. \tag{6.56}$$

We use these equivalences to infer the $\hbar$ scaling of the amplitudes. We begin by extracting the leading classical scaling of the five-point and six-point amplitude, and we then discuss the consequences of coherence for classical radiative observables.

6.3.1 Classical limit of the five-point tree amplitude

We begin by computing the classical limit of the five-point tree amplitude, which was given previously in [95, 124] by an equivalent large mass expansion. An interesting alternative derivation can be made in supergravity theory by using the Kaluza-Klein compactification of amplitudes of massless particles in five dimensions, by taking advantage of a straightforward application of double copy [334].

The manifestly gauge invariant expression for $\mathcal{A}_5^{(0)}$ given in eq. (6.48) can easily be written in terms of the polarization tensor for the graviton through the identification

$$f^{\mu\nu} = p_5^{\mu}\varepsilon_5^{\nu} - p_5^{\nu}\varepsilon_5^{\mu} = [5|\gamma^\mu\gamma^\nu|5] \overset{\hbar\to 0}{\sim} \hbar, \tag{6.57}$$

and the following scalings also hold:

$$P_{13}, P_{24} \overset{\hbar\to 0}{\sim} \hbar, \qquad K_A^\mu\,,\ K_B^\nu \overset{\hbar\to 0}{\sim} \hbar^0, \qquad s_{13}\,,\ s_{24} \overset{\hbar\to 0}{\sim} \hbar^2,$$
$$s_{51}-m_A^2 = -2p_5\cdot p_1\,, \qquad s_{52}-m_B^2 = -2p_5\cdot p_2 \overset{\hbar\to 0}{\sim} \hbar. \tag{6.58}$$

Moreover, we can safely neglect the quantum shift in the masses $\tilde{m}_j \overset{\hbar\to 0}{=} m_j$. Using eq. (6.58), we can simply apply power counting to each of the terms in eq. (6.48). We deduce that, upon including the contribution from κ, the terms which contribute to leading behavior as $\hbar \to 0$ are

$$\begin{aligned}
&\mathcal{A}_5^{(0)}(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B,5^+) \overset{\hbar\to 0}{\sim}\\
&\quad \frac{i\kappa^3}{64}\Bigg(\Bigg[\frac{[5|K_AK_B|5]^2}{2s_{13}s_{24}} - \frac{K_A\cdot K_B[5|42|5][5|K_B|5\rangle[5|K_AK_B|5]}{s_{13}s_{24}(s_{52}-m_B^2)(s_{54}-m_B^2)}\\
&\quad - \frac{K_A\cdot K_B[5|K_A|5\rangle[5|K_B|5\rangle[5|K_AK_B|5]^2}{8s_{13}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}\\
&\quad + \mathrm{Tr}(\not{K}_A\not{K}_B\not{K}_A\not{K}_B)\Bigg(\frac{[5|K_A|5\rangle[5|K_B|5\rangle[5|13|5][5|42|5]}{8s_{13}s_{24}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}\\
&\quad - \frac{[5|42|5][5|K_A|5\rangle[5|K_AK_B|5]}{8s_{13}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}\Bigg)\Bigg] + \big[(1,3,K_A)\leftrightarrow(2,4,K_B)\big]\Bigg).
\end{aligned} \tag{6.59}$$

We can make the following replacements in order to match the notation in [95] at leading order in the classical expansion [8],

$$\begin{aligned}
&p_3 \overset{\hbar\to 0}{\sim} -m_Av_A, && p_4 \overset{\hbar\to 0}{\sim} -m_Bv_B,\\
&s_{13} \overset{\hbar\to 0}{\sim} -q_1^2, && s_{24} \overset{\hbar\to 0}{\sim} -q_2^2,\\
&[5|13|5] \overset{\hbar\to 0}{\sim} m_Af_{\mu\nu}v_A^\mu q_1^\nu, && [5|42|5] \overset{\hbar\to 0}{\sim} -m_Bf_{\mu\nu}v_B^\mu q_2^\nu,\\
&[5|K_A|5\rangle \overset{\hbar\to 0}{\sim} -4m_Ak\cdot v_A, && [5|K_B|5\rangle \overset{\hbar\to 0}{\sim} -4m_Bk\cdot v_B,\\
&(-s_{51}+m_A^2),(s_{53}-m_A^2) \overset{\hbar\to 0}{\sim} 2m_Ak\cdot v_A, && (-s_{52}+m_B^2),(s_{54}-m_B^2) \overset{\hbar\to 0}{\sim} 2m_Bk\cdot v_B,\\
&[5|K_AK_B|5] \overset{\hbar\to 0}{\sim} -4m_Am_Bf_{\mu\nu}v_A^\mu v_B^\nu, && K_A\cdot K_B \overset{\hbar\to 0}{\sim} 4m_Am_Bv_A\cdot v_B\,.
\end{aligned} \tag{6.60}$$

This implies that the leading order behaviour of the five-point tree amplitude is of order $\hbar^{-7/2}$. As we will see later, this will imply that the amplitude contributes to the total classical energy emitted in gravitational waves. In particular, we get

$$\begin{aligned}
&\mathcal{A}_5^{(0)}\left(m_Av_A,m_Bv_B,\hbar q_1-m_Av_A,\hbar q_2-m_Bv_B,\hbar k\right)\Big|_{\hbar^{-\frac{7}{2}}}\\
&= \frac{i\kappa^3}{4}\frac{m_A^2m_B^2f_{\mu\nu}f_{\rho\sigma}}{q_1^2q_2^2}\Bigg[v_A\cdot v_B\left(\frac{q_1^\mu v_A^\nu}{k\cdot v_A} - \frac{q_2^\mu v_B^\nu}{k\cdot v_B}\right)v_A^\rho v_B^\sigma + \frac{(q_1^2+q_2^2)v_A\cdot v_Bv_A^\mu v_B^\nu v_A^\rho v_B^\sigma}{2k\cdot v_Ak\cdot v_B}\\
&+ v_A^\mu v_B^\nu v_A^\rho v_B^\sigma + \mathrm{Tr}(\not{v}_A\not{v}_B\not{v}_A\not{v}_B)\Bigg(-\frac{v_A^\mu q_1^\nu v_B^\rho q_2^\sigma}{4k\cdot v_Ak\cdot v_B} - \frac{q_2^2v_A^\mu v_B^\nu v_B^\rho q_2^\sigma}{8k\cdot v_A(k\cdot v_B)^2} - \frac{q_1^2v_B^\mu v_A^\nu v_A^\rho q_1^\sigma}{8k\cdot v_B(k\cdot v_A)^2}\Bigg)\Bigg],
\end{aligned} \tag{6.61}$$

[8]Only for this case, we use an asymmetric parametrization of the external momenta in terms of the classical velocities just to show the agreement with the literature.

where $f_{\mu\nu}f_{\rho\sigma}$ is proportional to the linearized Riemann tensor and can be expressed in terms of the polarization tensor $\varepsilon_{\mu\nu}$

$$\begin{aligned}f_{\mu\nu}f_{\rho\sigma} &= 2R_{\mu\nu\rho\sigma} = k_\mu k_\rho \varepsilon_{\nu\sigma} + k_\nu k_\sigma \varepsilon_{\mu\rho} - k_\mu k_\sigma \varepsilon_{\nu\rho} - k_\nu k_\rho \varepsilon_{\mu\sigma} \\ &= \varepsilon^{\alpha\beta}\left(k_\mu k_\rho \eta_{\nu\alpha}\eta_{\sigma\beta} + k_\nu k_\sigma \eta_{\mu\alpha}\eta_{\rho\beta} - k_\mu k_\sigma \eta_{\nu\alpha}\eta_{\rho\beta} - k_\nu k_\rho \eta_{\mu\alpha}\eta_{\sigma\beta}\right) . \end{aligned} \tag{6.62}$$

Upon substituting the relation

$$\mathrm{Tr}(\not{v}_A \not{v}_B \not{v}_A \not{v}_B) = 8(v_A \cdot v_B)^2 - 4 \tag{6.63}$$

the amplitude can thus be expressed

$$\begin{aligned}&\mathcal{A}_5^{(0)}\left(m_A v_A, m_B v_B, \hbar q_1 - m_A v_A, \hbar q_2 - m_B v_B, \hbar k\right)\Big|_{\hbar^{-\frac{7}{2}}} \\ &= \frac{i\kappa^3}{4}\frac{m_A^2 m_B^2 \varepsilon_{\mu\nu}}{q_1^2 q_2^2}\Bigg[(k\cdot v_A)^2 v_B^\mu v_B^\nu + (k\cdot v_B)^2 v_A^\mu v_A^\nu - 2k\cdot v_A k\cdot v_B v_A^\mu v_B^\nu \\ &+ v_A\cdot v_B\left(\frac{(k\cdot v_A)^2 q_2^2 v_B^\mu v_B^\nu + (k\cdot v_B)^2 q_1^2 v_A^\mu v_A^\nu}{2k\cdot v_A\, k\cdot v_B} + k\cdot v_A\, q_2^\mu v_B^\nu + k\cdot v_B\, q_1^\mu v_A^\nu - (q_1^2+q_2^2) v_A^\mu v_B^\nu\right) \\ &+ \left(2(v_A\cdot v_B)^2 - 1\right)\left(\frac{\left(q_1^2 k\cdot v_B v_A^\mu - q_2^2 k\cdot v_A v_B^\mu\right)\left(q_1^\nu - q_2^\nu\right)}{2k\cdot v_B k\cdot v_A} - q_1^\mu q_2^\nu + \right. \\ &\qquad \left. + \frac{(q_1^2-q_2^2)^2\left((k\cdot v_A)^2 q_2^2 v_B^\mu v_B^\nu + (k\cdot v_B)^2 q_1^2 v_A^\mu v_A^\nu\right)}{4(k\cdot v_A)^2(k\cdot v_B)^2} - \frac{(q_1^2-q_2^2)^2 v_A^\mu v_B^\nu}{4k\cdot v_A k\cdot v_B}\right)\Bigg],\end{aligned} \tag{6.64}$$

which matches the result in [95] analytically.

6.3.2 Classical limit of the six-point tree amplitude

To compute the leading terms of the classical expansion of $\mathcal{A}_6^{(0)}$, we directly extract the $\hbar$ scaling of the BCFW residues in eq. (6.50) and eq. (6.52). In the following, we will use explicitly the rules extracted in eq. (6.53), eq. (6.56) and eq. (6.58). First we consider the terms which originate from the factorizations of the general type in eq. (6.50),

$$\mathcal{A}_6^{(0)}|_{5I_1} = \mathcal{A}_3^{(0)}\left(\mathbf{I_1}, \hat{\mathbf{P}}^{\mathbf{I_1}}, \hat{5}^{\pm}\right)\frac{i}{s_{5I_1} - m_I^2}\mathcal{A}_5^{(0)}(-\hat{\mathbf{P}}^{\mathbf{I_1}}, \mathbf{J}_1, \mathbf{I}_2, \mathbf{J}_2, \hat{6}^+). \tag{6.65}$$

For the scaling of the three-point amplitude $\mathcal{A}_3^{(0)}(\mathbf{I_1}, \hat{\mathbf{P}}_I, \hat{5}^{\pm})$, we first note that a shift in momenta does not modify the $\hbar$ scaling,

$$\hat{p}_5 = p_5 + z_{5I_1}|5\rangle[6| \to \hbar\hat{\bar{p}}_5, \tag{6.66}$$

which can be seen from the fact that z_{5I_1} takes the form

$$z_{5I_1}|5\rangle[6| = \frac{2p_5\cdot p_{I_1}}{[6|I_1|5\rangle}|5\rangle[6| \tag{6.67}$$

so that it scales in the same way as p_5. We can thus rearrange the amplitude to extract the scaling:

$$\begin{aligned}\mathcal{A}_3^{(0)}\left(\hat{\mathbf{I}}_1, \hat{\mathbf{P}}^{\mathbf{I_1}}, \hat{5}^+\right) &= i\kappa\frac{[\hat{5}|\hat{P}_I|\chi\rangle^2}{\langle 5\chi\rangle^2} \\ &= i\kappa\frac{[\hat{5}|\hat{P}_I\chi|\hat{5}]^2}{(2p_5\cdot p_\chi)^2} \\ &= i\kappa\frac{1}{(2p_5\cdot p_\chi)^2}\hat{f}_5^{\mu\nu}\hat{f}_5^{\rho\sigma}\hat{P}_\mu P_\rho(p_\chi)_\nu(p_\chi)_\sigma \\ &\overset{\hbar\to 0}{\sim} \hbar^{-\frac{1}{2}}\ ,\end{aligned} \tag{6.68}$$

where $\hat{f}_5 \equiv f$ as defined in eq. (6.57), but in terms of shifted momenta and polarization vectors. The shifted five-point amplitude inherits the $\hbar$ scaling of eq. (6.61),

$$\mathcal{A}_5^{(0)}(-\hat{\mathbf{P}}^{\mathbf{I_1}}, \mathbf{J}_1, \mathbf{I}_2, \mathbf{J}_2, \hat{6}^+) \overset{\hbar\to 0}{\sim} \hbar^{-\frac{7}{2}}. \tag{6.69}$$

Thus upon including the contribution from the pole, each term of the form in eq. (6.65) has the leading scaling behavior

$$\mathcal{A}_6^{(0)}|_{6I_1} \overset{\hbar\to 0}{\sim} \hbar^{-5}. \tag{6.70}$$

We now show how taking only the leading-classical term trivializes the kinematics. Using eq. (6.66) we have

$$\hat{P}^{I_1} = -p_{I_1} - \hat{p}_5 = -p_{I_1} + \mathcal{O}(\hbar)\ , \tag{6.71}$$

and we can make the statement

$$\mathcal{A}_5^{(0)}(-\hat{\mathbf{P}}^{\mathbf{I_1}}, \mathbf{J}_1, \mathbf{I}_2, \mathbf{J}_2, \hat{6}^+)\Big|_{\hbar^{-\frac{7}{2}}} = \mathcal{A}_5^{(0)}(\mathbf{I}_1, \mathbf{J}_1, \mathbf{I}_2, \mathbf{J}_2, \hat{6}^+)\Big|_{\hbar^{-\frac{7}{2}}}\ . \tag{6.72}$$

This is not the only simplification in the leading classical limit. We also observe that from $p_{I_1} = -p_{I_2} + \mathcal{O}(\hbar)$ we have

$$z_{I_1 5} = \frac{[5|I_1|5\rangle}{[6|I_1|5\rangle} = \frac{[5|I_2|5\rangle}{[6|I_2|5\rangle} + \mathcal{O}(\hbar). \tag{6.73}$$

Thus both the $\mathcal{A}_3^{(0)}$ and $\mathcal{A}_5^{(0)}$ factors in eq. (6.65) are invariant under the $I_1 \leftrightarrow I_2$ permutation. On the other hand the pole factor

$$\frac{1}{s_{I_1 5} - m_I^2} = \frac{1}{[5|I_1|5\rangle} = -\frac{1}{[5|I_2|5\rangle} \tag{6.74}$$

has the opposite sign under the $I_1 \leftrightarrow I_2$ switch, so these contributions cancel pairwise, giving

$$\mathcal{A}_6^{(0)}|_{5I_1} + \mathcal{A}_6^{(0)}|_{5I_2} \overset{\hbar\to 0}{\sim} \hbar^{-4}\ . \tag{6.75}$$

An identical argument for the terms of type in eq. (6.51) also gives

$$\mathcal{A}_6^{(0)}|_{6I_1} + \mathcal{A}_6^{(0)}|_{6I_2} \overset{\hbar\to 0}{\sim} \hbar^{-4}\ . \tag{6.76}$$

So the permutation invariance naturally leads to a drop in inverse-$\hbar$ scaling.

Finally, describing the scaling of terms of the type in eq. (6.52),

$$\mathcal{A}_6^{(0)}|_{I_1I_25} = \sum_{h=\sigma} \mathcal{A}_4^{(0)}\left(\mathbf{I_1}, \mathbf{I_2}, \hat{5}^{\pm}, \hat{P}^{\sigma}\right) \frac{i}{s_{I_1I_25}} \mathcal{A}_4^{(0)}\left(\mathbf{J_1}, \mathbf{J_2}, \hat{6}^{+}, -\hat{P}^{-\sigma}\right), \tag{6.77}$$

requires the scaling from the four-point single-flavor amplitude. From $\hbar$ counting we find

$$\mathcal{A}_4^{(0)}\left(\mathbf{I}_1, \mathbf{I}_2, \hat{P}^{\pm}, \hat{5}^{+}\right) \overset{\hbar\to 0}{\sim} \hbar^{-1}\,. \tag{6.78}$$

And as the massless poles contributes $\hbar^{-2}$, the massless factorizations manifestly scale as

$$\mathcal{A}_6^{(0)}|_{I_1I_25} \overset{\hbar\to 0}{\sim} \hbar^{-4}. \tag{6.79}$$

Thus we conclude that

$$\mathcal{A}_6^{(0)}(\mathbf{1}^A, \mathbf{2}^B, \mathbf{3}^A, \mathbf{4}^B, 5^{\pm}, 6^{+}) \overset{\hbar\to 0}{\sim} \hbar^{-4}. \tag{6.80}$$

We expect similar arguments to hold at higher points, which would imply that the general scaling of the $(n+4)$-point amplitude is

$$\mathcal{A}_{4+n}^{(0)} \overset{\hbar\to 0}{\sim} \hbar^{-3-\frac{n}{2}}\,. \tag{6.81}$$

6.3.3 Coherence of the final radiative state

Using the classical scaling discussed in eq.(6.53) and eq.(6.56), we can rewrite the graviton emission probability in our problem as

$$\begin{aligned}
P_n^{\lambda} = \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_n=\pm} \Bigg\langle\!\!\Bigg\langle \hbar^{4+2n} \int_{\bar\lambda} \prod_{i=1}^{n} d\Phi(\bar{k}_i) \int \frac{d^4\bar{q}}{(2\pi)^4} \delta\left(2p_A\cdot\bar{q}\right)\delta\left(2p_B\cdot\bar{q}\right)\Theta\left(p_A^0 + \hbar\frac{\bar{q}^0}{2}\right)\Theta\left(p_B^0 - \hbar\frac{\bar{q}^0}{2}\right) \\
\times \int d^4\bar{w}_1 d^4\bar{w}_2 e^{-ib\cdot\bar{q}} \delta^{(4)}\left(\bar{w}_1 + \bar{w}_2 + \sum_{i=1}^{n}\bar{k}_i\right) \Theta\left(p_A^0 + \hbar\bar{w}_1^0 - \hbar\frac{\bar{q}^0}{2}\right)\Theta\left(p_B^0 + \hbar\bar{w}_2^0 + \hbar\frac{\bar{q}^0}{2}\right) \\
\times\, \delta\left(2p_A\cdot\bar{w}_1 + \hbar\bar{w}_1^2 - \hbar\bar{q}\cdot\bar{w}_1\right)\delta\left(2p_B\cdot\bar{w}_2 + \hbar\bar{w}_2^2 + \hbar\bar{q}\cdot\bar{w}_2\right) \\
\times\, \mathcal{A}_{n+4}\left(p_A - \hbar\frac{\bar{q}}{2}, p_B + \hbar\frac{\bar{q}}{2} \to p_A + \hbar\bar{w}_1 - \hbar\frac{\bar{q}}{2}, p_B + \hbar\bar{w}_2 + \hbar\frac{\bar{q}}{2}, \hbar\bar{k}_1^{\sigma_1}, \dots, \hbar\bar{k}_n^{\sigma_n}\right) \\
\times\, \mathcal{A}_{n+4}^{*}\left(p_A + \hbar\frac{\bar{q}}{2}, p_B - \hbar\frac{\bar{q}}{2} \to p_A + \hbar\bar{w}_1 - \hbar\frac{\bar{q}}{2}, p_B + \hbar\bar{w}_2 + \hbar\frac{\bar{q}}{2}, \hbar\bar{k}_1^{\sigma_1}, \dots, \hbar\bar{k}_n^{\sigma_n}\right) \Bigg\rangle\!\!\Bigg\rangle.
\end{aligned} \tag{6.82}$$

The leading order contribution in the classically relevant region is

$$\begin{aligned}
P_n^{\lambda} = \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_n=\pm} \Bigg\langle\!\!\Bigg\langle \hbar^{4+2n} \int_{\bar\lambda} \prod_{i=1}^{n} d\Phi(\bar{k}_i) \int \frac{d^4\bar{q}}{(2\pi)^4} \delta\left(2p_A\cdot\bar{q}\right)\delta\left(2p_B\cdot\bar{q}\right)\Theta\left(p_A^0\right)\Theta\left(p_B^0\right) \\
\times \int d^4\bar{w}_1 d^4\bar{w}_2 e^{-ib\cdot\bar{q}} \delta^{(4)}\left(\bar{w}_1 + \bar{w}_2 + \sum_{i=1}^{n}\bar{k}_i\right) \delta\left(2p_A\cdot\bar{w}_1\right)\delta\left(2p_B\cdot\bar{w}_2\right) \\
\times\, \mathcal{A}_{n+4}\left(p_A - \hbar\frac{\bar{q}}{2}, p_B + \hbar\frac{\bar{q}}{2} \to p_A + \hbar\bar{w}_1 - \hbar\frac{\bar{q}}{2}, p_B + \hbar\bar{w}_2 + \hbar\frac{\bar{q}}{2}, \hbar\bar{k}_1^{\sigma_1}, \dots, \hbar\bar{k}_n^{\sigma_n}\right) \\
\times\, \mathcal{A}_{n+4}^{*}\left(p_A + \hbar\frac{\bar{q}}{2}, p_B - \hbar\frac{\bar{q}}{2} \to p_A + \hbar\bar{w}_1 - \hbar\frac{\bar{q}}{2}, p_B + \hbar\bar{w}_2 + \hbar\frac{\bar{q}}{2}, \hbar\bar{k}_1^{\sigma_1}, \dots, \hbar\bar{k}_n^{\sigma_n}\right) \Bigg\rangle\!\!\Bigg\rangle.
\end{aligned} \tag{6.83}$$

It is the scaling of the *energy* of the emitted radiation that determines if the amplitude contribution is classical or quantum, and in the following we take this as a guiding principle. The expectation value of the energy operator is given by the same unitarity cuts appearing in the mean of the graviton particle distribution, but weighted in the phase space integration by an energy factor $E_j := \hbar\omega_j$ for each of the emitted gravitons. The scaling in the classical limit has to be such that the total energy carried by the emitted gravitons, i.e. by the classical gravitational wave,

$$
\begin{aligned}
E^{\rm cl} = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\sigma_1,\ldots,\sigma_n=\pm}\Big\langle\!\!\Big\langle \hbar^{5+2n}\int\prod_{i=1}^{n}d\Phi(\bar{k}_i)\int\frac{d^4\bar{q}}{(2\pi)^4}\delta\left(2p_A\cdot\bar{q}\right)\delta\left(2p_B\cdot\bar{q}\right)\\
\times\int d^4\bar{w}_1 d^4\bar{w}_2 e^{-ib\cdot\bar{q}}\delta^{(4)}\left(\bar{w}_1+\bar{w}_2+\sum_{i=1}^{n}\bar{k}_i\right)\delta\left(2p_A\cdot\bar{w}_1\right)\delta\left(2p_B\cdot\bar{w}_2\right)\left(\sum_{j=1}^{n}\omega_j\right)\\
\times\mathcal{A}_{n+4}\left(p_A-\hbar\frac{\bar{q}}{2},p_B+\hbar\frac{\bar{q}}{2}\to p_A+\hbar\bar{w}_1-\hbar\frac{\bar{q}}{2},p_B+\hbar\bar{w}_2+\hbar\frac{\bar{q}}{2},\hbar\bar{k}_1^{\sigma_1},\ldots,\hbar\bar{k}_n^{\sigma_n}\right)\\
\times\mathcal{A}^{*}_{n+4}\left(p_A+\hbar\frac{\bar{q}}{2},p_B-\hbar\frac{\bar{q}}{2}\to p_A+\hbar\bar{w}_1-\hbar\frac{\bar{q}}{2},p_B+\hbar\bar{w}_2+\hbar\frac{\bar{q}}{2},\hbar\bar{k}_1^{\sigma_1},\ldots,\hbar\bar{k}_n^{\sigma_n}\right)\Big\rangle\!\!\Big\rangle,
\end{aligned}
\tag{6.84}
$$

is finite in the classical limit. While each separate probability of the emission of n gravitons in eq. (6.83) is infrared divergent when $\lambda\to 0$, in this paper we are interested only in the deviation from a Poissonian distribution in the $\hbar\to 0$ limit,

$$
\lim_{\hbar\to 0}\hbar\,\Delta_{\rm out}=\lim_{\hbar\to 0}\hbar\,(\Sigma^{\lambda}_{\rm out}-\mu^{\lambda}_{\rm out}).
\tag{6.85}
$$

As we have shown in section 5.4, this is an infrared-safe quantity. A naive power counting in $\hbar$ from Feynman diagrams for the five-point and six-point tree gives a series expansion starting with the following types of terms,

$$
\begin{aligned}
&\mathcal{A}_5^{(0)}\left(p_A-\hbar\frac{\bar{q}}{2},p_B+\hbar\frac{\bar{q}}{2},p_A+\hbar\bar{w}_1-\hbar\frac{\bar{q}}{2},p_B+\hbar\bar{w}_2+\hbar\frac{\bar{q}}{2},\hbar\bar{k}_1^{\sigma_1}\right)\\
&\qquad\qquad\qquad=\frac{C_1^{\mathcal{A}_5^{(0)}}}{\hbar^{\frac{9}{2}}}+\frac{C_2^{\mathcal{A}_5^{(0)}}}{\hbar^{\frac{7}{2}}}+\mathcal{O}\left(\frac{1}{\hbar^{\frac{7}{2}}}\right),\\
&\mathcal{A}_6^{(0)}\left(p_A-\hbar\frac{\bar{q}}{2},p_B+\hbar\frac{\bar{q}}{2},p_A+\hbar\bar{w}_1-\hbar\frac{\bar{q}}{2},p_B+\hbar\bar{w}_2+\hbar\frac{\bar{q}}{2},\hbar\bar{k}_1^{\sigma_1},\hbar\bar{k}_2^{\sigma_2}\right)\\
&\qquad\qquad\qquad=\frac{C_1^{\mathcal{A}_6^{(0)}}}{\hbar^{6}}+\frac{C_2^{\mathcal{A}_6^{(0)}}}{\hbar^{5}}+\frac{C_3^{\mathcal{A}_6^{(0)}}}{\hbar^{4}}+\mathcal{O}\left(\frac{1}{\hbar^{4}}\right),
\end{aligned}
\tag{6.86}
$$

but as we have seen in the preceding subsections, it turns out that some of the lower-order terms are zero,

$$
\begin{aligned}
&C_1^{\mathcal{A}_5^{(0)}}=0,\qquad C_2^{\mathcal{A}_5^{(0)}}\neq 0,\\
&C_1^{\mathcal{A}_6^{(0)}}=C_2^{\mathcal{A}_6^{(0)}}=0,\qquad C_3^{\mathcal{A}_6^{(0)}}\neq 0.
\end{aligned}
\tag{6.87}
$$

The cancellation of the leading term in the $\hbar$ expansion was shown already in [95] for $\mathcal{A}_5^{(0)}$, but the new result here is the double cancellation of two leading terms in the $\hbar$

expansion for $\mathcal{A}_6$.[9] This has physical consequences, as we have seen: we find that

$$\lim_{\hbar\to 0} \hbar P_1^{(0,0)} \sim \hbar^0, \qquad \lim_{\hbar\to 0} \hbar\, P_2^{(0,0)} = 0, \tag{6.88}$$

where for simplicity we have kept the powers of $\hbar$ coming from the coupling in eq. (6.56) implicit inside the probabilities.[10] This will be assumed for all the rest of our arguments in this section.

Therefore, while the 5-point tree-level amplitude gives a classical contribution to classical radiative observables, the 6-point tree-level amplitude gives a "quantum" contribution

$$\lim_{\hbar\to 0} \hbar\, \Delta_{\text{out}}\Big|_{\mathcal{O}(G^4)} = 0, \tag{6.89}$$

which proves that we can describe the final semiclassical radiation state as a coherent state at least up to order $\mathcal{O}(G^4)$ for classical radiative observables.

6.3.4 Classical relations for unitarity cuts from all-order coherence

Assuming coherence to all orders in perturbation theory implies a set of (integral) relations between loop and tree amplitudes with emission of gravitons.

For example, we expect that unitarity cuts involving tree-level amplitudes with two or more gravitons emitted, and their conjugates, would give vanishing contributions in the classical limit. The reason is that having a coherent state as an exact final semiclassical state for the radiation would imply that all the gravitons emitted are uncorrelated. Indeed, our conjectural classical scaling for tree-level amplitudes in eq. (6.81),

$$\mathcal{A}^{(0)}_{n+4}(\phi_A\phi_B \to \phi_A\phi_B h_1 h_2 .....h_n) \sim \hbar^{-3-\frac{n}{2}}, \tag{6.90}$$

would imply that

$$\lim_{\hbar\to 0} \hbar P_n^{(0,0)} = 0 \qquad \text{for } n \geq 2. \tag{6.91}$$

This follows directly from our main result in eq. (5.104). While the expansion of $\Gamma^{(n),\lambda}_{\text{out}}$ starts at order G^{2+n}, the lowest order contribution to $(\mu^{\lambda}_{\text{out}})^n$ is of order G^{2n+n}: clearly then for $n \geq 2$ the eq. (6.91) must hold, as a simple consequence of coherence.

In order to make definite statements about the probabilities at higher orders, we need to combine them at a given loop order, so let us define

$$P_n^{(L)} := \sum_{j=0}^{L} P_n^{(j,L-j)}. \tag{6.92}$$

[9] A similar result has been obtained in scalar QED in [6].

[10] Alternatively, we should have written

$$\lim_{\hbar\to 0} \hbar \left(\frac{G^3}{\hbar^3}\right) P_1^{(0,0)} \sim \hbar^0, \qquad \lim_{\hbar\to 0} \hbar \left(\frac{G^4}{\hbar^4}\right) P_2^{(0,0)} = 0.$$

We have decided to avoid this cumbersome notation here.

Once we feed eq. (6.92) and eq. (6.91) into the constraints given by considering the factorial moments

$$\lim_{\hbar\to 0} \hbar\, \Delta^{(j)}_{\text{out}} = 0 \qquad \text{for } j \geq 2, \tag{6.93}$$

we can conclude that

$$\lim_{\hbar\to 0} \hbar P^{(L)}_n = \lim_{\hbar\to 0} \hbar \left(\sum_{j=0}^{L} P^{(j,L-j)}_n \right) = 0, \qquad \text{for } n \geq L+2. \tag{6.94}$$

This is the loop-level generalization of eq. (6.91), which is essentially saying that coherence implies that classically we can only have, at a given loop order $L_1 + L_2$, contributions from product of amplitudes with $n < L_1 + L_2 + 2$ external gravitons.

We would like to make further progress in understanding exactly which amplitudes are relevant in the classical limit, and in particular this requires to go beyond eq. (6.91) and eq. (6.94). If we consider the expansion of $\Delta^{(m)}_{\text{out}}$ with $m = 2, 3, 4$ that we found in eq. (5.96), eq. (5.105) and eq. (5.106),

$$\begin{aligned}
\Delta^{(2)}_{\text{out}} &= 2G^4 P^{(0,0)}_2 + 6G^5 P^{(0,0)}_3 + 12G^6 P^{(0,0)}_4 + 20G^7 P^{(0,0)}_5 \\
&\quad + G^5(2P^{(1,0)}_2 + 2P^{(0,1)}_2) + G^6(2P^{(2,0)}_2 + 2P^{(0,2)}_2 + 6P^{(1,0)}_3 + 6P^{(0,1)}_3) \\
&\quad + G^7(2P^{(3,0)}_2 + 2P^{(0,3)}_2 + 6P^{(2,0)}_3 + 6P^{(0,2)}_3 + 6P^{(1,1)}_3 + 12P^{(1,0)}_4 + 12P^{(0,1)}_4 - 4P^{(0,0)}_1 P^{(0,0)}_2) \\
&\quad + \Big[G^6(2P^{(1,1)}_2 - (P^{(0,0)}_1)^2) + G^7(2P^{(1,2)}_2 + 2P^{(2,1)}_2 - 2P^{(0,1)}_1 P^{(0,0)}_1 - 2P^{(1,0)}_1 P^{(0,0)}_1)\Big], \\
\Delta^{(3)}_{\text{out}} &= 6G^5 P^{(0,0)}_3 + 24G^6 P^{(0,0)}_4 + 60G^7 P^{(0,0)}_5 \\
&\quad + G^6(6P^{(1,0)}_3 + 6P^{(0,1)}_3) + G^7(6P^{(0,2)}_3 + 6P^{(2,0)}_3 + 6P^{(1,1)}_3 + 24P^{(1,0)}_4 + 24P^{(0,1)}_4), \\
\Delta^{(4)}_{\text{out}} &= 24G^6 P^{(0,0)}_4 + 120G^7 P^{(0,0)}_5 \\
&\quad + G^7(24P^{(1,0)}_4 + 24P^{(0,1)}_4)\,,
\end{aligned} \tag{6.95}$$

we get, after imposing all the constraints in eq. (6.91) and eq. (6.94) in the expansion in the coupling,

$$\begin{aligned}
\lim_{\hbar\to 0} \hbar\Delta^{(2)}_{\text{out}} &= \lim_{\hbar\to 0} \hbar \left(G^6(2P^{(2,0)}_2 + 2P^{(0,2)}_2) \right) \\
&\quad + \lim_{\hbar\to 0} \hbar \left(G^7(2P^{(3,0)}_2 + 2P^{(0,3)}_2 + 6P^{(2,0)}_3 + 6P^{(0,2)}_3 + 6P^{(1,1)}_3 \right) \\
&\quad + \lim_{\hbar\to 0} \hbar \Big[G^6(2P^{(1,1)}_2 - (P^{(0,0)}_1)^2) + G^7(2P^{(1,2)}_2 + 2P^{(2,1)}_2 - 2P^{(0,1)}_1 P^{(0,0)}_1 - 2P^{(1,0)}_1 P^{(0,0)}_1)\Big], \\
\lim_{\hbar\to 0} \hbar\Delta^{(3)}_{\text{out}} &= \lim_{\hbar\to 0} \hbar \left(G^7(6P^{(0,2)}_3 + 6P^{(2,0)}_3 + 6P^{(1,1)}_3) \right), \\
\lim_{\hbar\to 0} \hbar\Delta^{(4)}_{\text{out}} &= 0.
\end{aligned} \tag{6.96}$$

Assuming coherence, we can now impose

$$\lim_{\hbar\to 0} \hbar\Delta^{(3)}_{\text{out}} = \lim_{\hbar\to 0} \hbar \left(G^7(6P^{(0,2)}_3 + 6P^{(2,0)}_3 + 6P^{(1,1)}_3) \right) = 0, \tag{6.97}$$

which implies for $\Delta^{(2)}_{\rm out}$

$$\begin{aligned}\lim_{\hbar\to 0}\hbar\Delta^{(2)}_{\rm out} &= \lim_{\hbar\to 0}\hbar\left(G^6(2P_2^{(2,0)}+2P_2^{(0,2)})+G^7(2P_2^{(3,0)}+2P_2^{(0,3)})\right)\\ &+\lim_{\hbar\to 0}\hbar\Big[G^6(2P_2^{(1,1)}-(P_1^{(0,0)})^2)+G^7(2P_2^{(1,2)}+2P_2^{(2,1)}-2P_1^{(0,1)}P_1^{(0,0)}-2P_1^{(1,0)}P_1^{(0,0)})\Big].\end{aligned} \tag{6.98}$$

We see now that the contributions in the first line manifestly involve the six-point tree amplitude and six-point loop amplitudes. We expect, based also on the uncertainty principle [6], that these contributions must be irrelevant in the classical limit because the six-point tree amplitude does not contribute to the classical field. But we cannot prove this directly from the coherence property, so we therefore assume that this is the case. As a consequence, we conjecture that

$$\lim_{\hbar\to 0}\hbar(P_2^{(n,0)}+P_2^{(0,n)}) \qquad \text{for } n\geq 2. \tag{6.99}$$

which is equivalent to saying that the leading classical term in the expansion of the n-loop six-point amplitudes with $n \geq 2$ will not conspire with the quantum $\hbar$ scaling of the six-point tree amplitude to give a classical contribution. It would be nice to have a direct check of eq. (6.99) and its higher order generalizations. A first consequence of eq. (6.97) and eq. (6.99) is

$$\lim_{\hbar\to 0}\hbar P_3^{(1,1)} = 0, \tag{6.100}$$

which is equivalent to the statement that the seven-point one-loop amplitude is classically suppressed. More generally, from the equations eq. (6.98) and eq. (6.99) a very interesting set of relations follow directly,

$$\begin{aligned}\hbar P_2^{(1,1)} &\overset{\hbar\to 0}{=} \frac{1}{2}\hbar(P_1^{(0,0)})^2\\ \hbar(P_2^{(1,2)}+P_2^{(2,1)}) &\overset{\hbar\to 0}{=} \hbar(P_1^{(0,1)}P_1^{(0,0)}+P_1^{(1,0)}P_1^{(0,0)}).\end{aligned} \tag{6.101}$$

Those relations have the common feature that they relate particular combinations of unitarity cuts involving more than one graviton emitted at higher loops to other unitarity cuts involving the 5-point amplitude at a lower loop level. We have represented the simplest of these relations, involving the one-loop amplitude with two gravitons emitted and the tree-level amplitude $\mathcal{A}_5^{(0)}(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B,5^{\sigma_1})$ in Fig. 6.9.

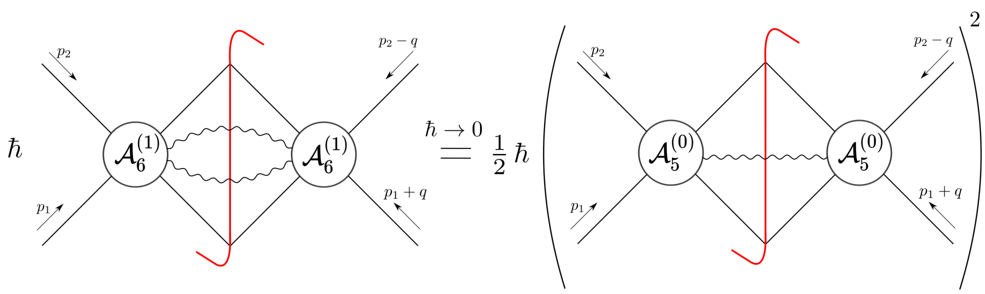


FIGURE 6.9: If the final graviton particle distribution is Poissonian, this implies non-trivial classical relations between cut contributions of classical amplitudes with more than one graviton emitted to the cuts of the 5-point amplitude $\mathcal{A}_5(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B,5^{\sigma_1})$.

The outcome of this section is that we have strong evidence that the fundamental data to describe the final semiclassical state are encoded in the 5-point amplitude at all orders in the coupling constant, providing that eq. (6.99) and its higher order generalizations hold. All the higher-multiplicity amplitudes are either suppressed in the classical regime, or related to the 5-point amplitude by a classical relation. This suggests, purely from the S-matrix perspective, that we can describe the radiation in the two-body problem entirely with a coherent state where the 5-point amplitude $\mathcal{A}_5(\phi_A\phi_B \to \phi_A\phi_B h_1)$ plays an essential role, as suggested in [6].

Chapter 7

Eikonal with radiation and spin

We would like to understand the exact structure of the final semiclassical state, including classical radiation. Many recent insights, coming both from a pure worldline description [124, 275, 276, 282] and from a different parametrization of the kinematics in the classical limit [122, 138, 139, 335], suggest that we should expect an (eikonal) exponentiation at all orders in the impact parameter space. This is clear from the path integral point of view, since the saddle-point estimate gives an exponential with the classical action evaluated on the classical trajectory, up to possible boundary terms. The situation is less clear when we allow particle production. We got some insights on the problem in chapter 5, where we showed that it is natural to describe the outgoing classical radiation with a single coherent state. Motivated by that, in this chapter we provide a new evidence in favor of a representation of the classical S-matrix for the two-body problem in terms of an eikonal phase and a coherent state of gravitons. Finally, we show how to extend the eikonal description to classically spinning particles.

7.1 Generalising the eikonal

We have now seen that the uncertainty, or the variance, in the measurement of a scattering observable can be computed in terms of amplitudes and, moreover, that the classical absence of uncertainty leads to an infinite set of relationships among fragments of amplitudes expanded in powers of momentum transfer, which is a Laurent series in $\hbar$. In a purely conservative limit, these relationships can be understood in terms of eikonal exponentiation. Our goal now is to review the eikonal formula, emphasising its connection to final state dynamics. We will then build on this eikonal state to incorporate radiative dynamics as a kind of coherent state so that the variance is naturally small.

7.1.1 Eikonal final state

Eikonal methods have long been used to extract classical physics from quantum mechanics. Recent years have seen a renewed surge of interest in this approach, especially in the context of gravitational scattering [110, 113, 115, 116, 118, 119, 124, 132, 136, 138, 139, 180, 263, 282, 301, 336–344], though this has roots in earlier work [345–347]. Originally born out of the study of high energy/Regge scattering [348–355] where the Feynman diagrammatics dramatically simplify, eikonal physics now have much wider application. The simplification in this regime allows diagrams, expressed in impact parameter space rather than momentum space, to be summed exactly to an exponential form. This exponential depends on the $2 \to 2$ scattering amplitude, and contains information about classical quantities such as the deflection angle. There are rich connections to soft/IR physics and Wilson lines [275–277, 280, 281, 356] which lead to a formal proof of the exponentiation quite generally [275]. Nowadays the exponentiation is taken as a starting point and applied to various scattering regimes.

In this section our goal is to explain the link between eikonal methods and the KMOC approach. Firstly it is worth noting that the small $\hbar$ expansion in the KMOC formalism is essentially the same as the soft expansion in the eikonal literature; the $\hbar$ scaling counts the order of softness. The key connection is to compute the final state using the methods of KMOC instead of computing observables directly. We will see that this final, outgoing, state is controlled by the usual eikonal function. In this section we restrict to a purely conservative scattering scenario: then eikonal exponentiation is exact. We take two incoming particles and (since the scattering is conservative) assume that the outgoing state is also an element of the two-particle Hilbert space.

We begin with the standard definition of the eikonal as the transverse Fourier transform of the four-point amplitude

$$e^{i\chi(x;\,s)/\hbar}(1+i\Delta(x;\,s))-1=i\int \hat{\mathrm{d}}^4q\,\delta(2p_1\cdot q)\delta(2p_2\cdot q)e^{-iq\cdot x/\hbar}\mathcal{A}_4(s,q^2)\,, \tag{7.1}$$

where $\chi(x;\,s)$ is the eikonal function and $\Delta(x;\,s)$ is the so-called quantum remainder which takes into account contributions that do not exponentiate (see for example [116]). This remainder is important for computing the eikonal function — but it will play no role in this thesis, so we will omit it. Meanwhile the two Dirac delta functions appearing in eq. (7.1) ensure that we integrate only over the components of q transverse to the momenta. This is often just written instead as $\hat{d}^{D-2}q_\perp$ (times a Jacobian factor). Indeed, the parameter x should be thought of as an element of the $D-2$ dimensional spatial slice perpendicular to p_1 and p_2: this is most evident on the right-hand-side of the eq. (7.1), where the Dirac delta functions project away any components of q in the (timelike) p_1 and p_2 directions. Consequently no components of x in the space spanned by p_1 and p_2 enter the dot product $q\cdot x$.

The eikonal can be written as an expansion in powers of the generic coupling g

$$\chi(x;\,s)=\sum_{n=0}^{\infty}\chi_n(x;\,s)\quad,\quad\chi_n(x;\,s)\sim g^{2n}\,, \tag{7.2}$$

and we will, as others have (see for example [110, 116, 138, 139, 340, 357]), assume that this holds to all orders. The structure has been formally proven at leading order, (see for example [275, 301]), however an all orders proof has not been given (to the best of our knowledge).

Starting with the in state in eq. (2.9) from chapter 2, we obtain the final state by acting with the S matrix. Writing $S=1+iT$ and inserting a complete set of states, we have

$$\begin{aligned}S|\psi_{\text{in}}\rangle=&|\psi_{\text{in}}\rangle+i\int \mathrm{d}\Phi(p_1',p_2',p_1,p_2)\,\psi_b(p_1,p_2)|p_1'p_2'\rangle\langle p_1'p_2'|T|p_1p_2\rangle\\=&|\psi_{\text{in}}\rangle+i\int \mathrm{d}\Phi(p_1',p_2',p_1,p_2)\,\psi_b(p_1,p_2)\mathcal{A}_4(s,q^2)\delta^{(4)}(p_1+p_2-p_1'-p_2')|p_1'p_2'\rangle\,.\end{aligned} \tag{7.3}$$

Notice that we made explicit use of our assumption of conservative scattering by restricting the complete set of states to the two-particle Hilbert space.

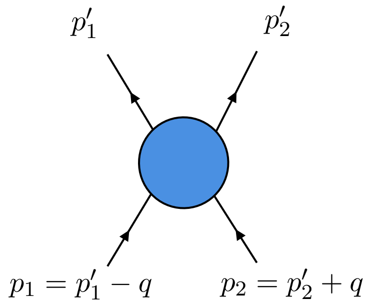


FIGURE 7.1: Momentum labelling at four points

With the momentum labelling in Fig. 7.1 we can convert the p_1 and p_2 phase space integrals to integrals over q. Doing so, we may write

$$S|\psi_{\text{in}}\rangle = |\psi_{\text{in}}\rangle \\ + i\int \mathrm{d}\Phi(p_1', p_2')\, \hat{\mathrm{d}}^4 q\, \delta(2p_1'\cdot q - q^2)\delta(2p_2'\cdot q + q^2)\, \psi_b(p_1' - q, p_2' + q)\mathcal{A}_4(s, q^2)|p_1' p_2'\rangle. \tag{7.4}$$

At this point, the q integral is tantalising similar to the q integral in the eikonal formula in eq. (7.1). However there is a key difference in the nature of the delta functions: those in eq. (7.4) involve q^2 terms which are absent in eq. (7.1). This issue appeared recently in refs. [122, 335]: there the authors proceeded using a "HEFT" phase, which is analogous to yet distinct from the eikonal phase. We will instead continue with the eikonal phase.

It may be worth remarking that the q^2 terms in these delta functions are suppressed in specific examples. One such example is the leading order impulse [166]. Nevertheless the impulse at NLO does indeed involve the full delta functions [166].

To incorporate the full delta functions, we follow the route described in [139]. We introduce new momentum variables $\tilde{p}$ as

$$\tilde{p}_1 = p_1' - \frac{q}{2} \qquad \tilde{p}_2 = p_2' + \frac{q}{2}. \tag{7.5}$$

Now, rather than using the eikonal eq. (7.1) directly we can take advantage of its inverse Fourier transform in the following form[1]

$$i\delta(2\tilde{p}_1\cdot q)\delta(2\tilde{p}_2\cdot q)\mathcal{A}_4(s, q^2) = \frac{1}{\hbar^4}\int \mathrm{d}^4 x\, e^{iq\cdot x/\hbar}\left\{e^{i\chi(x_\perp;\, s)/\hbar} - 1\right\}, \tag{7.6}$$

where we have written $x_\perp$ as one of the arguments of the eikonal function $\chi(x_\perp;\, s)$ to emphasise that $\chi(x_\perp;\, s)$ only depends on components of x which are orthogonal to the space spanned by $\tilde{p}_1$ and $\tilde{p}_2$. Indeed, integrating over the two components of x which *are* in the space spanned by $\tilde{p}_1$ and $\tilde{p}_2$, one recovers the two Dirac delta functions on the left-hand-side of eq. (7.6). In this way, we find that the final state is

$$S|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi(p_1', p_2')\, |p_1' p_2'\rangle \frac{1}{\hbar^4}\int \mathrm{d}^4 q\, \mathrm{d}^4 x\, e^{iq\cdot x/\hbar}\, e^{i\chi(x_\perp;\, s)/\hbar}\, \psi_b(p_1' - q, p_2' + q)\,. \tag{7.7}$$

It is worth emphasising once again that $x_\perp$ is perpendicular to $\tilde{p}_i$, rather than to p_i, so that

$$\tilde{p}_1\cdot x_\perp = 0 = \tilde{p}_2\cdot x_\perp. \tag{7.8}$$

[1]As noted above, we dropped the quantum remainder Δ which plays no role in our analysis.

In particular, $x_\perp$ depends on q.

7.1.2 The impulse from the eikonal

In this subsection, we will recover one beautiful result from the literature on the eikonal function: the scattering angle can be extracted from the eikonal function using a stationary phase argument. As our interest is not so much in this conservative case but rather in its radiative generalisation (which we discuss below), we wish to emphasise that it is, in fact, possible to extract the *full* final momentum from the eikonal using stationary phase. Later, in section 7.1.4, we will use the same ideas to extract the final momentum in the case of radiation — with radiation, of course, knowledge of the direction of the final momentum is insufficient to recover the full momentum.

The impulse is the observable

$$\begin{aligned}\Delta p_1^\mu &\equiv \langle\psi_{\text{in}}|S^\dagger \mathbb{P}_1^\mu S|\psi_{\text{in}}\rangle - \langle\psi_{\text{in}}|\mathbb{P}_1^\mu|\psi_{\text{in}}\rangle \\ &= \langle\psi_{\text{in}}|S^\dagger[\mathbb{P}_1^\mu, S]|\psi_{\text{in}}\rangle\,.\end{aligned} \tag{7.9}$$

It is convenient to focus on

$$[\mathbb{P}_1^\mu, S]|\psi_{\text{in}}\rangle\,. \tag{7.10}$$

As we shall see, in essence the operator $[\mathbb{P}_1^\mu, S]$ pulls out a factor of the momentum transfer multiplying $S|\psi_{\text{in}}\rangle$. We will evaluate $[\mathbb{P}_1^\mu, S]|\psi_{\text{in}}\rangle$ by stationary phase; it is then trivial to determine $\langle\psi_{\text{in}}|S^\dagger$ in the same way.

We begin our expression for the final-state wavepacket, eq. (7.7), quickly finding

$$\begin{aligned}[\mathbb{P}_1^\mu, S]|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi(p_1', p_2')\, e^{ib\cdot p_1'/\hbar}\, |p_1' p_2'\rangle \frac{1}{\hbar^4} \int \mathrm{d}^4 q\, \mathrm{d}^4 x\, e^{iq\cdot x/\hbar} e^{-ib\cdot q/\hbar} \\ \times\, e^{i\chi(x_\perp;\, s)/\hbar}\, \psi(p_1' - q, p_2' + q)\, q^\mu\,.\end{aligned} \tag{7.11}$$

To obtain formulae for the impulse, we apply the stationary phase approximation to the x and q integrals. (Our approach is very similar to that of Ciafaloni and Colferai [358] who previously discussed wavepacket dynamics, the eikonal, and stationary phase.)

The stationary phase condition for x is[2]

$$q_\mu = -\frac{\partial}{\partial x^\mu}\chi(x_\perp, s)\,. \tag{7.12}$$

One thing to note immediately is that this q^μ is *not* of order $\hbar$: it is a classical momentum, of order g^2. Further, it is useful to note that χ is actually a function of $x_\perp^2$ (since the four-point function is a function of s and q^2.) Therefore we may write the momentum transfer as

$$q^\mu = -2\chi'(x_\perp^2, s)x_\perp^\mu\,, \tag{7.13}$$

where $\chi'(x_\perp^2, s)$ is the derivative of χ with respect to its first argument. Since $\tilde{p}_1 \cdot x_\perp = 0 = \tilde{p}_2 \cdot x_\perp$, it now follows that $\tilde{p}_1 \cdot q = 0 = \tilde{p}_2 \cdot q$: thus the on-shell delta functions in

[2] One can find this expression also by promoting the eikonal phase to an operator $\hat{\chi}(x_\perp, s)$: the impulse on particle i in the transverse plane $\hat{Q}^\mu_{\perp i}$ is then be related to the standard commutator $[\hat{Q}^\mu_{\perp i}, e^{i\hat{\chi}(\hat{x}_\perp;\, s)/\hbar}] = e^{i\hat{\chi}(\hat{x}_\perp;\, s)/\hbar}\partial^\mu_{\perp i}\hat{\chi}(\hat{x}_\perp, s)$ and the result follows from unitarity.

eq. (7.6) have reappeared, now as "equations of motion" following from the stationary phase conditions[3].

The second stationary phase condition, associated with q, is

$$x^\mu - b^\mu + \frac{\partial}{\partial q_\mu}\chi(x_\perp^2, s) = 0\,. \tag{7.14}$$

The q derivative is non-vanishing because $x_\perp$ depends on q. It is often useful to introduce a particular notation for the variables q and x when they satisfy the stationary phase conditions: we will denote these by q_* and x_*. Armed with this notation, we may use eq. (7.12) we may write eq. (7.14) as

$$x^\mu = b^\mu + q_{*\nu}\frac{\partial}{\partial q_\mu}x_\perp^\nu\,. \tag{7.15}$$

Performing the derivative is straightforward, but requires some notation which we relegate to appendix E. The result may be expressed in the form

$$x_{*\perp}^\mu = b^\mu - \tilde{N}_q(p_1^\mu - p_2^\mu) - \tilde{N}_{0q}(p_1^\mu + p_2^\mu)\,, \tag{7.16}$$

where $\tilde{N}_q$ and $\tilde{N}_{0q}$ can be interpreted as Lagrange multipliers ensuring that $x_\perp \cdot \tilde{p}_1 = 0 = x_\perp \cdot \tilde{p}_2$.

Our result for $x_{*\perp}$ takes a familiar form in the CM frame. Then $x_{*\perp}^0 = 0$ and, using, $\boldsymbol{p_2} = -\boldsymbol{p_1}$, we have

$$\begin{aligned} \boldsymbol{x}_{*\perp} &= \boldsymbol{b} - 2N_q\boldsymbol{p} \\ \Rightarrow \boldsymbol{b}\cdot\boldsymbol{x}_{*\perp} &= \boldsymbol{b}^2\,. \end{aligned} \tag{7.17}$$

Denoting the scattering angle by Ψ, we may write this result[4] as

$$|\boldsymbol{b}| = |\boldsymbol{x}_{*\perp}|\cos(\Psi/2)\,, \tag{7.18}$$

where Ψ is the scattering angle.

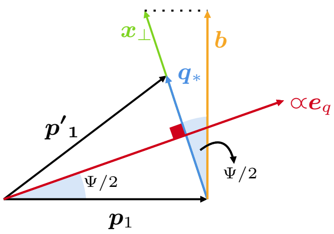


FIGURE 7.2: Geometry of eikonal scattering.

In this way, we have performed the integrals over q and x. The result has been to evaluate the factor q^μ as a derivative of the eikonal function. To obtain the full expectation, we simply evaluate $\langle\psi_{\text{in}}|S^\dagger$ using stationary phase in precisely the same way: the only difference (apart from the obvious Hermitian conjugation) is the absence

[3] Since some of the x integrations may be performed exactly to yield delta functions, we are slightly abusing terminology by referring to all of the conditions on x and q as "stationary phase conditions". Some of the conditions arise approximately by stationary phase, and some are exact conditions due to the delta functions. Nevertheless it is most convenient to treat all the conditions as one set.

[4] In this context, the quantity $x_\perp$ is sometimes referred to as the "eikonal" impact parameter.

of the q^μ factor. We then exploit unitarity to conclude that[5]

$$\Delta p_1^\mu = q^\mu = -\partial^\mu \chi(x_\perp, s)\,, \tag{7.19}$$

where all quantities are defined on using the solution of the stationary phase conditions.

Notice that we have determined the *complete* impulse four-vector, not just the scattering angle. The distinction between these quantities is obviously unimportant at the level of conservative dynamics, but it is important when radiation occurs. The key aspect of our argument which leads to the full impulse rather than the scattering angle is the presence of a perpendicular projector. To see how this works, let us discuss an explicit example: the impulse at next-to-leading order in gravitational fast scattering.

Focusing on the scattering between two massive bodies in general relativity, the eikonal phase at next-to-leading order in G is

$$\chi = -2Gm_Am_B\left(\frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}}\log|x_\perp| - \frac{3\pi}{8}\frac{(5\gamma^2-1)}{\sqrt{\gamma^2-1}}\frac{G\,(m_A+m_B)}{|x_\perp|}\right)\,, \tag{7.20}$$

where we have defined $\gamma = v_A \cdot v_B$ as the scalar product between the four-velocities of the particles. Using eq. (7.19) and straightforward differentiation, we obtain the following expression in terms of $x_\perp^\mu$,

$$\Delta p_1^\mu = \frac{2Gm_Am_Bx_\perp^\mu}{|x_\perp^2|}\left(\frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}} + \frac{3\pi}{8}\frac{(5\gamma^2-1)}{\sqrt{\gamma^2-1}}\frac{G\,(m_A+m_B)}{|x_\perp|}\right)\,, \tag{7.21}$$

where the four-velocities in γ can now be identified with the incoming four-velocities of the particles due to the integrals over the wavepackets that took place to arrive at eq. (7.19). At this point, it is important to remember that $x_\perp^\mu$ is not quite b^μ. It is trivial to show that $x_\perp^\mu$ coincides with b^μ at leading order in the gravitational coupling, but this is no longer the case at next-to-leading order; then instead

$$x_\perp^\mu = b^\mu - \frac{G(2\gamma^2-1)}{(\gamma^2-1)^{3/2}}\left(v_A^\mu(m_B+\gamma m_A) - v_B^\mu(m_A+\gamma m_B)\right)\,, \tag{7.22}$$

where

$$|x_\perp^2| = |b^2|\,. \tag{7.23}$$

We can now express the impulse in terms of the impact parameter b^μ. At the order we are interested in we find,

$$\begin{aligned}\Delta p_1^\mu = {}& \frac{2Gm_Am_Bb^\mu}{|b^2|}\left(\frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}} + \frac{3\pi}{8}\frac{(5\gamma^2-1)}{\sqrt{\gamma^2-1}}\frac{G\,(m_A+m_B)}{|b|}\right)\\ &-\frac{G^2m_Am_B\left(2\gamma^2-1\right)^2\left((\gamma m_A+m_B)\,v_A^\mu-(\gamma m_B+m_A)\,v_B^\mu\right)}{\left(\gamma^2-1\right)^2|b^2|}\,,\end{aligned} \tag{7.24}$$

in perfect agreement with the literature [136].

From the perspective of this work, the key achievement of the eikonal resummation is that negligible variance becomes automatic in the stationary phase argument.

[5] We discuss the details of unitarity more explicitly in appendix F, performing all the remaining integrals.

Indeed since the stationary phase condition in eq. (7.12) sets the momentum transfer to a specific, classical, value it is clear that the expectation value of any polynomial in the momentum operator will evaluate to the classical expectation.

7.1.3 Extension with coherent radiation

The exponentiated eikonal final state of eq. (7.7) beautifully describes semiclassical conservative dynamics, leading to a transparent method for extracting the impulse (or scattering angle) from amplitudes in a manner which automatically enforces minimal uncertainty. We have also seen that coherent states naturally enforce minimal uncertainty for radiation. Now let us put these two ideas together to form a proposal for an eikonal-type final state in the fully dynamical, radiative, case.

It is very natural to consider a modification of the eikonal formula which includes radiation, and indeed this idea has received attention [263] in the literature. Given that our motivation is to extend the eikonal while maintaining its minimal uncertainty property, an obvious way to proceed is to include an additional factor in the eikonal formula which has the structure of a coherent state like in eq. (5.50). If this radiative part of the state has large occupation number, expectations of products of field-strength operators will naturally factorise into products of expectations of the operators.

We will simply propose one possibility for the structure of this final state, depending on a coherent waveshape parameter α^σ (of helicity σ) in addition to an eikonal function χ. We believe there is strong evidence in favour of the basic structure of our proposal, and in particular in the idea that two objects χ and α^σ suffice to define it; however, it seems possible to implement the idea in somewhat different ways. We will discuss the basic virtues of our proposal in the remainder of this section, leaving it to future work to determine further details. Since we are primarily interested in classical effects, we will continue to neglect the quantum remainder Δ in this discussion[6].

To describe our proposal, we begin with the eikonal final state in eq. (7.7). With an eye towards a situation where momentum is lost to radiation, we need a description in which the sum of the momenta of the two final particles differs from the initial momenta. A first step, then, is to Fourier transform the wavepacket to position space:

$$\begin{aligned} S|\psi_{\text{in}}\rangle|_{\text{conservative}} = \int \mathrm{d}\Phi(p_1', p_2') \int \hat{\mathrm{d}}^4\bar{q}\, \mathrm{d}^4x\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}_b(x_1, x_2)\, e^{i(p_1'\cdot x_1 + p_2'\cdot x_2)/\hbar} \\ \times e^{i[q\cdot(x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar}\, |p_1' p_2'\rangle\,. \end{aligned} \tag{7.25}$$

Our proposal is now very straightforward: we simply incorporate a coherent state by assuming that

$$\begin{aligned} S|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi(p_1', p_2') \int \hat{\mathrm{d}}^4\bar{q}\, \mathrm{d}^4x\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}_b(x_1, x_2)\, e^{i(p_1'\cdot x_1 + p_2'\cdot x_2)/\hbar} \\ \times\; e^{i[q\cdot(x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} \exp\left[\sum_{\sigma=\pm1} \int \mathrm{d}\Phi(k)\alpha^\sigma(k, x_1, x_2) a_\sigma^\dagger(k)\right] |p_1' p_2'\rangle\,. \end{aligned} \tag{7.26}$$

This is a minimal proposal: more generally, one could imagine that the coherent waveshape parameter α^σ depends on other variables, for example x or q which appear in the eikonal dynamics. We will nevertheless restrict throughout this work to our minimal proposal. However, it is important that the state is not merely an outer

[6] Indeed radiative quantum effects will require Δ to be upgraded to an operator.

product of a conservative eikonal state with a radiative factor. Some entanglement is necessary so that the radiation can backreact on the motion. In the case of the present proposal, the integrals over the variables x_1, x_2 and x perform this role. We have already presented a partial derivation of this proposal in section 5.1, showing that the leading low-frequency classical radiation does indeed exponentiate as anticipated. There is a connection between our proposal here and recent work [343] on the exponential structure of the S matrix.

Following the discussion in section 5.2, we know that the waveshape should be proportional to $\hbar^{-3/2}$ so we may also write the state as

$$S|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi(p'_1, p'_2) \int \hat{\mathrm{d}}^4\bar{q}\, \mathrm{d}^4x\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}_b(x_1, x_2)\, e^{i(p'_1\cdot x_1 + p'_2\cdot x_2)/\hbar} \\ \times\; e^{i[q\cdot(x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} \exp\left[\frac{1}{\hbar^{3/2}} \sum_{\sigma=\pm 1} \int \mathrm{d}\Phi(k) \bar{\alpha}^\sigma(k, x_1, x_2) a^\dagger_\sigma(k)\right] |p'_1 p'_2\rangle\,. \tag{7.27}$$

In this expression, the classical waveshape $\bar{\alpha}^\sigma$ is *independent* of $\hbar$, just as the eikonal function χ is independent of $\hbar$.

In order to determine α^σ, we follow the same steps as in section 7.1.1; we act on the incoming state with the S matrix, and then expand in terms of integrals of amplitudes. To isolate the waveshape, we consider the overlap of our proposed final state with the bra $\langle p'_1\, p'_2\, k^\sigma|$:

$$\langle p'_1\, p'_2\, k^\sigma|S|\psi_{\text{in}}\rangle = \int \hat{\mathrm{d}}^4\bar{q}\, \mathrm{d}^4x\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}_b(x_1, x_2)\, e^{i(p'_1\cdot x_1 + p'_2\cdot x_2)/\hbar} \\ \times\; e^{i[q\cdot(x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar}\, \alpha^\sigma(k, x_1, x_2)\,. \tag{7.28}$$

Next we expand the S matrix as

$$\begin{aligned} \langle p'_1\, p'_2\, k^\sigma|S|\psi_{\text{in}}\rangle &= \int \mathrm{d}\Phi(p_1, p_2)\, \psi_b(p_1, p_2)\, \langle p'_1\, p'_2\, k^\sigma|S|p_1 p_2\rangle \\ &= \int \mathrm{d}\Phi(p_1, p_2) \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}_b(x_1, x_2)\, e^{i(p_1\cdot x_1 + p_2\cdot x_2)/\hbar} \\ &\qquad \times\; i\mathcal{A}_5(p_1, p_2 \to p'_1, p'_2, k^\sigma)\, \delta^4(p_1 + p_2 - p'_1 - p'_2 - k)\,. \end{aligned} \tag{7.29}$$

We note that the five-point amplitude appearing here could in principle include disconnected components beginning at order g. This order g disconnected term would involve exactly zero-energy photons, and does not contribute to observables such as the radiated momentum or the asymptotic Newman-Penrose scalar. We have already briefly discussed this type of contributions in section 4.6, and for the purpose of this section we will omit this term.

To continue, it is useful to perform a change of variable in the phase space measures, taking $q_1 \equiv p_1 - p'_1$ and $q_2 \equiv p_2 - p'_2$ as variables of integration. Neglecting Heaviside theta functions (which will always be unity in the domain of validity of our calculation) we find

$$\langle p'_1\, p'_2\, k^\sigma|S|\psi_{\text{in}}\rangle = \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}_b(x_1, x_2)\, e^{i(p'_1\cdot x_1 + p'_2\cdot x_2)/\hbar} \\ \times \int \hat{\mathrm{d}}^4q_1\, \hat{\mathrm{d}}^4q_2\, \delta(2p'_1\cdot q_1 + q_1^2)\delta(2p'_2\cdot q_2 + q_2^2)\, e^{i(q_1\cdot x_1 + q_2\cdot x_2)/\hbar} \\ \times\; i\mathcal{A}_5(p'_1 + q_1, p'_2 + q_2 \to p'_1, p'_2, k^\sigma)\, \delta^4(q_1 + q_2 - k)\,. \tag{7.30}$$

Requiring eq. (7.28) and eq. (7.30) to be equal for any (appropriately classical) initial wavepacket $\tilde{\psi}_b(x_1, x_2)$ we deduce that

$$\alpha^\sigma(k, x_1, x_2) = i \int \hat{\mathrm{d}}^4 q_1 \, \hat{\mathrm{d}}^4 q_2 \, \delta(2p_1' \cdot q_1 + q_1^2)\delta(2p_2' \cdot q_2 + q_2^2)\delta^4(q_1 + q_2 - k) e^{i(\bar{q}_1 \cdot x_1 + \bar{q}_2 \cdot x_2)} \times \mathcal{A}_5(p_1' + q_1, p_2' + q_2 \to p_1', p_2', k^\sigma) \left[\int \hat{\mathrm{d}}^4 q \, \mathrm{d}^4 x \, e^{iq\cdot x} e^{iq\cdot(x_2 - x_1)/\hbar} e^{i\chi(x_\perp; s)/\hbar} \right]^{-1} . \tag{7.31}$$

It is easy to use the eikonal eq. (7.1) to show that equivalently we may write the waveshape as

$$\alpha^\sigma(k, x_1, x_2) = i \int \hat{\mathrm{d}}^4 q_1 \, \hat{\mathrm{d}}^4 q_2 \, \delta(2p_1' \cdot q_1 + q_1^2)\delta(2p_2' \cdot q_2 + q_2^2)\delta^4(q_1 + q_2 - k) e^{i(\bar{q}_1 \cdot x_1 + \bar{q}_2 \cdot x_2)} \times \mathcal{A}_5(p_1' + q_1, p_2' + q_2 \to p_1', p_2', k^\sigma) \left[1 + \int \hat{\mathrm{d}}^4 q \, \delta(2\tilde{p}_1 \cdot q)\delta(2\tilde{p}_2 \cdot q) \, e^{iq\cdot(x_2 - x_1)/\hbar} i\mathcal{A}_4(s, q^2) \right]^{-1} . \tag{7.32}$$

This last expression makes the physical meaning transparent: the waveshape is obtained by removing iterated contributions of four-point amplitudes from the five-point amplitude.

To see this in more detail, it is instructive to expand the α^σ order-by-order in perturbation theory. We again consider a generic coupling g and expand the waveshape as

$$\alpha^\sigma = \alpha_0^\sigma + \alpha_1^\sigma + \cdots . \tag{7.33}$$

The leading order term, α_0^σ, follows immediately from eq. (7.32):

$$\alpha_0^\sigma(k, x_1, x_2) = i \int \hat{\mathrm{d}}^4 q_1 \, \hat{\mathrm{d}}^4 q_2 \, \delta(2p_1' \cdot q_1 + q_1^2)\delta(2p_2' \cdot q_2 + q_2^2)\delta^4(q_1 + q_2 - k) \times e^{i(q_1\cdot(x_1 + b) + q_2 \cdot x_2)/\hbar} \mathcal{A}_5^{(0)}(p_1' + q_1, p_2' + q_2 \to p_1', p_2', k^\sigma) . \tag{7.34}$$

It is determined by the tree-level five-point amplitude $\mathcal{A}_5^{(0)}$, so it is of order g^3 in gauge theory and gravity. The fact that the leading-order classical radiation field is intimately related to five-point amplitudes was already discussed in [4, 95, 166, 171]. The basic structure of this leading-order waveshape is strikingly reminiscent of a coherent state which describes the *static* Coulomb/Schwarzschild background [106] on analytic continuation from Minkowski signature to $(2, 2)$ signature $(+, +, -, -)$.

More precisely, α_0^σ is really determined by $\mathcal{A}_5^{(0)}$. This follows by counting powers of $\hbar$. Indeed extracting dominant powers of $\hbar$ using

$$\begin{aligned} \mathcal{A}_5^{(0)}(i \to f) &= \hbar^{-7/2} \left(\mathcal{A}_5^{(0)}(i \to f) + \hbar \mathcal{A}_5^{(0)}(i \to f) + \cdots \right) , \\ \mathcal{A}_5^{(1)}(i \to f) &= \hbar^{-9/2} \left(\mathcal{A}_5^{(1)}(i \to f) + \hbar \mathcal{A}_5^{(1)}(i \to f) + \cdots \right) , \end{aligned} \tag{7.35}$$

we find

$$\alpha_0^\sigma(k, x_1, x_2) = \frac{i}{\hbar^{3/2}} \int \mathrm{d}^4 \bar{q}_1 \, \mathrm{d}^4 \bar{q}_2 \, \delta(2p_1' \cdot \bar{q}_1)\delta(2p_2' \cdot \bar{q}_2)\delta^4(\bar{q}_1 + \bar{q}_2 - \bar{k}) \times e^{i(\bar{q}_1 \cdot x_1 + \bar{q}_2 \cdot x_2)} \mathcal{A}_5^{(0)}(p_1' + q_1, p_2' + q_2 \to p_1', p_2', k^\sigma) . \tag{7.36}$$

Note that the factor $\hbar^{-3/2}$ arises as expected on general grounds. The conclusion is that the leading-in-$\hbar$ part of the five-point tree amplitude determines the radiation. The amplitude itself contains higher order terms in $\hbar$; rather than arising from the radiative factor in our proposal in eq. (7.27), these terms arise from a generalised quantum remainder.

The next-to-leading order correction to the waveshape following from eq. (7.32) is

$$\begin{aligned}\alpha_1^\sigma(k,x_1,x_2) = i\int \hat{\mathrm{d}}^4 q_1\,\hat{\mathrm{d}}^4 q_2\,\delta(2p_1'\cdot q_1+q_1^2)\delta(2p_2'\cdot q_2+q_2^2)\delta^4(q_1+q_2-k)\\ \times e^{i(q_1\cdot x_1+q_2\cdot x_2)/\hbar}\mathcal{A}_5^{(1)}(p_1'+q_1,p_2'+q_2\to p_1',p_2',k^\sigma)\\ -\alpha_0^\sigma(k,x_1,x_2)\int \hat{\mathrm{d}}^4 q\,\delta(2\tilde{p}_1\cdot q)\delta(2\tilde{p}_2\cdot q)\,e^{iq\cdot(x_2-x_1)/\hbar}i\mathcal{A}_4^{(0)}(s,q^2)\,.\end{aligned} \tag{7.37}$$

This correction involves the five-point one-loop amplitude, after subtracting an iteration term. To understand the role of the subtraction, it is instructive to extract the leading-in-$\hbar$ part of α_1^σ:

$$\begin{aligned}\alpha_1^\sigma(k,x_1,x_2) = \frac{i}{\hbar^{5/2}}\int \mathrm{d}^4\bar{q}_1\,\mathrm{d}^4\bar{q}_2\,\delta(2p_1'\cdot\bar{q}_1)\delta(2p_2'\cdot\bar{q}_2)\delta^4(\bar{q}_1+\bar{q}_2-\bar{k})\\ \times e^{i(\bar{q}_1\cdot x_1+\bar{q}_2\cdot x_2)}\mathcal{A}_5^{(1)}(p_1'+q_1,p_2'+q_2\to p_1',p_2',k^\sigma)\\ -\frac{1}{\hbar^{5/2}}\bar{\alpha}_0^\sigma(k,x_1,x_2)\int \mathrm{d}^4\bar{q}\,\delta(2p_1'\cdot\bar{q})\delta(2p_2'\cdot\bar{q})\,e^{i\bar{q}\cdot(x_2-x_1)}i\mathcal{A}_{4,0}^{(0)}(s,q^2)\\ +\mathcal{O}(\hbar^{-3/2})\,.\end{aligned} \tag{7.38}$$

At this stage it seems that there is an unwanted order $\hbar^{-5/2}$ term in the NLO waveshape! Consistency with our proposal therefore demands

$$\begin{aligned}\int \mathrm{d}^4\bar{q}_1\,\mathrm{d}^4\bar{q}_2\,\delta(2p_1'\cdot\bar{q}_1)\delta(2p_2'\cdot\bar{q}_2)\delta^4(\bar{q}_1+\bar{q}_2-\bar{k})e^{i(\bar{q}_1\cdot x_1+\bar{q}_2\cdot x_2)}\mathcal{A}_{5,0}^{(1)}(p_1'+q_1,p_2'+q_2\to p_1',p_2',k^\sigma)\\ = -\bar{\alpha}_0^\sigma(k,x_1,x_2)\int \mathrm{d}^4\bar{q}\,\delta(2p_1'\cdot\bar{q})\delta(2p_2'\cdot\bar{q})\,e^{i\bar{q}\cdot(x_2-x_1)}\mathcal{A}_{4,0}^{(0)}(s,q^2)\,.\end{aligned} \tag{7.39}$$

Since $\bar{\alpha}_0^\sigma$ is determined by $\mathcal{A}_{5,0}^{(0)}$, this requirement relates $\mathcal{A}_{5,0}^{(1)}$ to $\mathcal{A}_{5,0}^{(0)}$ and $\mathcal{A}_{4,0}^{(0)}$. The requirement is nothing but a Fourier transform of the zero-variance relation in eq. (4.66) which we encountered in section 4.2.4. So we see that the zero-variance relations retain their importance in the context of this eikonal/coherent resummation: their validity admits the possibility of exponentiation.

In the same vein, it is interesting to project our proposal onto a two-photon final state:

$$\begin{aligned}\langle p_1'p_2'k_1^{\sigma_1}k_2^{\sigma_2}|S|\psi_{\text{in}}\rangle = \int \hat{\mathrm{d}}^4\bar{q}\,\mathrm{d}^4x\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\tilde{\psi}_b(x_1,x_2)\,e^{i(p_1'\cdot x_1+p_2'\cdot x_2)/\hbar}\\ \times e^{i[q\cdot(x-x_1+x_2)+\chi(x_\perp;s)]/\hbar}\,\alpha^{\sigma_1}(k_1,x_1,x_2)\alpha^{\sigma_2}(k_2,x_1,x_2)\,.\end{aligned} \tag{7.40}$$

Since the waveshape is at least of order g^3, it follows that this overlap begins at order g^6. However by expanding the S matrix out directly, we encounter a six-point amplitude. The conclusion is that our proposal does not populate the (order g^4) tree-level six-point amplitude. Of course this is as it should be: we saw that the six-point tree is suppressed in the classical region. Similarly the seven-point tree and one-loop amplitudes are suppressed, etc.

As a final remark, note that we did not introduce any normalisation factor in our proposal. This may be surprising, but the origin of the difference is simply that unitarity must already guarantee the normalisation of the final state. As is by now well understood, at two loops the eikonal function χ ceases to be real in the radiative case. Instead the imaginary part of χ is related to $\sum_{\sigma=\pm1} |\alpha^\sigma|^2$; this supplies the necessary normalisation.

7.1.4 Radiation reaction

Once there is radiation, there must also be radiation reaction: the particle's motion must change in the radiative case relative to the conservative case to account for the loss of momentum to radiation. In this section we will see that the waveshape indeed contributes to the impulse of a particle in the manner required.

We begin by acting on our conjectural final state, eq. (7.26), with the momentum operator of the field corresponding to particle 1:

$$\begin{aligned}\mathbb{P}_1^\mu S|\psi_{\text{in}}\rangle = &\int \mathrm{d}\Phi(p_1', p_2') \int \hat{\mathrm{d}}^4\bar{q}\, \mathrm{d}^4x\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \tilde{\psi}_b(x_1, x_2)\, e^{i(p_1'\cdot x_1 + p_2'\cdot x_2)/\hbar} \\ &\times\, e^{i[q\cdot(x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} \exp\left[\sum_{\sigma=\pm1} \int \mathrm{d}\Phi(k)\alpha^\sigma(k, x_1, x_2) a_\sigma^\dagger(k)\right] p_1'^\mu |p_1' p_2'\rangle\,.\end{aligned} \tag{7.41}$$

The operator simply inserts a factor $p_1'^\mu$. We proceed by rewriting this factor in terms of a derivative $-i\hbar\partial/\partial x_{1\mu}$ acting on the exponential factor in the first line of eq. (7.41), and then integrating by parts. Neglecting the boundary term, the result is

$$\begin{aligned}\mathbb{P}_1^\mu S|\psi_{\text{in}}\rangle = &\int \mathrm{d}\Phi(p_1', p_2') \int \hat{\mathrm{d}}^4\bar{q}\, \mathrm{d}^4x\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\;\; e^{i(p_1'\cdot x_1 + p_2'\cdot x_2)/\hbar} e^{i(q\cdot x + \chi(x_\perp;\, s))/\hbar} \\ &\times\, i\hbar\partial_1^\mu \left(\tilde{\psi}_b(x_1, x_2) e^{iq\cdot(x_2 - x_1)/\hbar} \exp\left[\sum_{\sigma=\pm1} \int \mathrm{d}\Phi(k)\, \alpha^\sigma(k, x_1, x_2) a_\sigma^\dagger(k)\right]\right) |p_1' p_2'\rangle\,.\end{aligned} \tag{7.42}$$

Expanding out the derivative, we encounter three terms. In the first, the derivative acts on the spatial wavefunction: as usual in quantum mechanics, this term will evaluate (in the expectation value of the final momentum) to the contribution of the initial momentum. The second term arises when the derivative operator acts on $e^{iq\cdot(x_2 - x_1)/\hbar}$, which inserts a factor of q^μ. This term is familiar from eq. (7.11) in section 7.1.2, and contributes to the impulse as a (suitably projected) derivative of the eikonal function. Only the final term is new: it involves the waveshape, and must then be the origin of radiation reaction in our approach.

Since we have discussed the conservative impulse in detail in section 7.1.2, we focus on the final (new) term here. In this term, the derivative brings down a factor of

$$\sum_{\sigma=\pm1} \int \mathrm{d}\Phi(k)\, \partial_1^\mu \alpha^\sigma(k, x_1, x_2) a_\sigma^\dagger(k). \tag{7.43}$$

Now to extract the momentum observable we multiply by $\langle\psi_{\text{in}}|S^\dagger$. Since this will introduce yet more integrals, it is helpful to define a modified KMOC style 'classical average angle brackets' $\left\langle\!\!\left\langle \dots \right\rangle\!\!\right\rangle$, defined by

$$\begin{aligned}\left\langle\!\!\left\langle \dots \right\rangle\!\!\right\rangle = &\int \mathrm{d}\Phi(p_1', p_2') \int \hat{\mathrm{d}}^4\bar{q}\, \mathrm{d}^4x\, \mathrm{d}^4\bar{Q}\, \mathrm{d}^4y\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \tilde{\psi}_b^*(y_1, y_2)\tilde{\psi}_b(x_1, x_2)\\ &\times e^{i(p_1'\cdot(x_1-y_1)+p_2'\cdot(x_2-y_2))/\hbar} e^{i(q\cdot(x_2-x_1)-Q\cdot(y_2-y_1))/\hbar} e^{i(q\cdot x+\chi(x_\perp;s)-Q\cdot y-\chi^*(y_\perp;s))/\hbar}\\ &\times \exp\left[-\frac{1}{2}\sum_{\sigma=\pm1}\int \mathrm{d}\Phi(k)\, \left|\alpha^\sigma(k, x_1, x_2) - (\alpha^\sigma)^*(k, y_1, y_2)\right|^2\right](\dots)\,. \end{aligned} \tag{7.44}$$

The $a^\dagger(k)$ will act on this left state will produce a delta function and α^*. After using the delta function it gives just a factor α^*. Putting this together, and using the angle brackets shorthand we get

$$\langle \mathbb{P}_1^\mu \rangle_{\text{reaction}} = \left\langle\!\!\left\langle i \sum_{\sigma=\pm1}\int \mathrm{d}\Phi(k)\, \alpha^{(\sigma),*}(k, x_1, x_2)\partial_1^\mu \alpha^\sigma(k, x_1, x_2) \right\rangle\!\!\right\rangle. \tag{7.45}$$

Notice that the manipulations leading to eq. (7.45) were exact (under the assumption of eq. (7.26)). However eq. (7.45) involves a number of integrals which would need to be performed to arrive at a concrete expression for the impulse. In section 7.1.2, we performed these integrals by stationary phase. In the present (radiative) case a similar approach would be possible, but the stationary phase conditions are significantly more complicated. For example, demanding the phase of the q integral to be stationary leads to a condition involving the variables x_2 and x_1; further demanding that the phases of these x_i integrals should be stationary leads to an equation involving the integral of a quadratic function of the waveshape. In this way the stationary phase conditions involve an intricate interplay of the eikonal and the waveshape. Of course this is as it should be: the complexity of radiation reaction must be captured by the final state.

As a simpler sanity check of our machinery, we evaluate the radiation reaction contribution to the impulse at lowest non-trivial perturbative order. In [166] the leading order radiation reaction term was written as

$$I^\mu_{rad} = e^6 \left\langle\!\!\left\langle \int \mathrm{d}\Phi(\bar{k}) \prod_{i=1,2} \hat{\mathrm{d}}^4\bar{q}_i \hat{\mathrm{d}}^4\bar{q}_i'\, \bar{q}_1^\mu\, \mathcal{Y}(\bar{q}_1, \bar{q}_2, \bar{k})\mathcal{Y}^*(\bar{q}_1', \bar{q}_2', \bar{k}) \right\rangle\!\!\right\rangle, \tag{7.46}$$

with

$$\mathcal{Y}(\bar{q}_1, \bar{q}_2, \bar{k}) = \delta(p_1\cdot\bar{q}_1)\delta(p_2\cdot\bar{q}_2)\hat{\delta}^4(\bar{q}_1+\bar{q}_2-k)e^{ib\cdot\bar{q}_1}\mathcal{A}^{(0)}_{5,0}(\bar{q}_1, \bar{k}^\sigma). \tag{7.47}$$

It is straightforward to verify that this is equivalent to the second term in eq. (7.45). In fact, recalling the leading order waveshape formula in eq. (7.36), the match between the two expressions of eq. (7.46) and eq. (7.45) is immediate, once our definition of average over wave packets in eq. (7.44) is taken into account.

We note in passing that the expectation $\langle\psi_{\text{in}}|S^\dagger \mathbb{F}_{\mu\nu} S|\psi_{\text{in}}\rangle$ (or $\langle\psi_{\text{in}}|S^\dagger \mathbb{R}_{\mu\nu\rho\sigma} S|\psi_{\text{in}}\rangle$) can be determined in a similar way. The annihilation operators in the field strength operator immediately bring down a single power of the waveshape. At leading non-trivial order in g, it is then straightforward to see that the field strength is determined by the five-point tree amplitude (specifically the leading fragment in $\hbar$) consistent with reference [4]. Similarly, the momentum radiated into messengers can be computed as the expectation

$$\langle\psi_{\rm in}|S^\dagger \sum_{\sigma=\pm1}\int \mathrm{d}\Phi(k)\, k^\mu\, a^\dagger_\sigma(k) a_\sigma(k) S|\psi_{\rm in}\rangle\,. \tag{7.48}$$

In this case, the creation and annihilation operators bring down the waveshape times its conjugate. At leading perturbative order, the momentum radiated is the square of the five-point tree, as observed in [166]. The result is also consistent with classical field theory: in that context, radiation is described the the energy-momentum tensor, which is quadratic in the field strength. Finally, we note that conservation of momentum holds as discussed in [166].

7.2 Classical spin dynamics from the eikonal

So far, we have only discussed the classical dynamics of spinless charges in electrodynamics or point masses in gravity. However, the application of scattering amplitudes to classical systems involving colour and spin is also an important topic [82, 121, 128, 246, 247, 252, 295, 342, 359–364], motivated in large part by the dynamics of inspiraling black holes with angular momentum. In this section, we expand this discussion to include minimum uncertainty spin states. For simplicity, however, we revert to the purely conservative case throughout this section, leaving radiation of spin to future work.

7.2.1 Final state including spin

In section 7.1, we derived a formula for the final state involving an eikonal operator acting on the two-particle state. In general, states are labelled by an assortment of quantum numbers which can also evolve during a scattering event, and so in this section we will generalise the eikonal S-matrix operator to include Lie-algebra valued quantum numbers, focussing on spin for clarity.

We begin with a generalisation of eq. (2.19) to include spin, resulting in a state of the form

$$\begin{aligned}|\psi_{\rm in}\rangle &\equiv \int \mathrm{d}\Phi(p_1,p_2)\psi_b(p_1,p_2)\xi_1^{a_1}\xi_2^{a_2}a^\dagger(p_1)_{a_1}a^\dagger(p_2)_{a_2}|0\rangle\\ &\equiv \int \mathrm{d}\Phi(p_1,p_2)\psi_b(p_1,p_2)\xi_1^{a_1}\xi_2^{a_2}|p_1,a_1;p_2,a_2\rangle\\ &\equiv \int \mathrm{d}\Phi(p_1,p_2)\psi_b(p_1,p_2)|p_1,\xi_1;p_2,\xi_2\rangle\,,\end{aligned} \tag{7.49}$$

where a_1 and a_2 are the spins (and the indices are summed over). When spin is included, the expectation value of the T matrix is given by

$$\langle p_1',a_1';p_2',a_2'|T|p_1,a_1;p_2,a_2\rangle = \mathcal{A}_4(p_1,p_2\to p_1',p_2')^{a_1'a_2'}_{a_1a_2}\delta^{(4)}(p_1+p_2-p_1'-p_2')\,. \tag{7.50}$$

Notice that the scattering amplitudes are now matrices in spin space. In view of this matrix structure, we follow [114, 295, 365] and assume that eikonal exponentiation takes a matrix form[7]

$$\exp\left(i\chi(x_\perp,s)/\hbar\right)^{a_1',a_2'}_{a_1,a_2} = \delta^{a_1'}_{a_1}\delta^{a_2'}_{a_2} + i\int \hat{\mathrm{d}}^4 q\,\delta(2p_1\cdot q)\delta(2p_2\cdot q)\,e^{-iq\cdot x/\hbar}\,\mathcal{A}_4(s,q^2)^{a_1',a_2'}_{a_1,a_2}\,. \tag{7.51}$$

[7]We continue to ignore the quantum remainder in this discussion.

Acting with the S-matrix on the initial state in eq. (7.49) and following steps similar those in section 7.1.1, we find the out state to be

$$\begin{aligned} S|\psi_{\text{in}}\rangle = |\psi_{\text{in}}\rangle + \int \mathrm{d}\Phi(p_1', p_2') \hat{\mathrm{d}}^4 q\, \psi_b\left(p_1' - q, p_2' + q\right) |p_1', a_1'; p_2', a_2'\rangle \\ \times \hat{\delta}\left(2p_1' \cdot q - q^2\right) \hat{\delta}\left(2p_2' \cdot q + q^2\right) i\mathcal{A}_4(s, q^2)_{a_1 a_2}^{a_1' a_2'}\, \xi_1^{a_1} \xi_2^{a_2}\,, \end{aligned} \tag{7.52}$$

which clearly recovers the spinless case of eq. (7.4) if we take the spin group to be trivial. Again following section 7.1.1, we invert the Fourier transform in eq. (7.51) to find

$$i\delta(2\tilde{p}_1 \cdot q)\delta(2\tilde{p}_2 \cdot q)\mathcal{A}_4(s, q^2)_{a_1, a_2}^{a_1' a_2'} = \frac{1}{\hbar^4} \int \mathrm{d}^4 x\, e^{iq\cdot x/\hbar} \left\{ \exp\left(i\chi(x_\perp, s)/\hbar \right)_{a_1, a_2}^{a_1', a_2'} - \delta_{a_1}^{a_1'} \delta_{a_2}^{a_2'} \right\}. \tag{7.53}$$

As a result, we can express the final state in terms of the eikonal operator as

$$\begin{aligned} S|\psi_{\text{in}}\rangle = \frac{1}{\hbar^4} \int \mathrm{d}\Phi(p_1', p_2')\, |p_1', a_1'; p_2', a_2'\rangle \int \mathrm{d}^4 q \mathrm{d}^4 x\, \psi_b(p_1' - q, p_2' + q) e^{iq\cdot x/\hbar} \\ \times \exp\left(i\chi(x_\perp, s)/\hbar \right)_{a_1, a_2}^{a_1', a_2'} \xi_1^{a_1} \xi_2^{a_2}\,. \end{aligned} \tag{7.54}$$

While we have constructed this with spin in mind, it is applicable to any operator with quantum numbers a_i, and so we will now consider the expectation value of a generic Lie-algebra valued operator $\mathcal{O}$. The change in an observable due to a scattering event is defined as

$$\Delta\mathcal{O} = \langle\Psi|S^\dagger \hat{\mathcal{O}} S|\Psi\rangle - \langle\Psi|\hat{\mathcal{O}}|\Psi\rangle = \langle\Psi|S^\dagger [\hat{\mathcal{O}}, S]|\Psi\rangle. \tag{7.55}$$

For concreteness, we restrict ourselves to an important class of operators with sensible classical limits, namely the momentum operator $\mathbb{P}^\mu$, the Pauli-Lubanski operator $\mathbb{W}^\mu$ defined as

$$\mathbb{W}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\delta} \mathbb{P}_\nu \mathbb{S}_{\rho\delta}\,, \tag{7.56}$$

and (in the Yang-Mills case) the colour charge operator $\mathbb{C}^a$. Such operators act on one particle states, are linear in the momentum and obey momentum conservation, meaning they can be written

$$\langle p, a|\mathcal{O}|k, b\rangle = \delta_\Phi(p - k) \mathcal{O}^a{}_b(p). \tag{7.57}$$

Examining an operator that acts only on the spin space of particle one, we find

$$\begin{aligned} \mathcal{O}_1 S|\psi_{\text{in}}\rangle = \frac{1}{\hbar^4} \int \mathrm{d}\Phi(p_1', p_2')\, |p_1', b_1; p_2', a_2'\rangle \int \mathrm{d}^4 q\, \mathrm{d}^4 x\, \psi_b(p_1' - q, p_2' + q) e^{iq\cdot x/\hbar} \\ \times \mathcal{O}_1(p_1')^{b_1}{}_{a_1'} \exp\left(i\chi(x_\perp, s)/\hbar \right)_{a_1, a_2}^{a_1', a_2'} \xi_1^{a_1} \xi_2^{a_2}\,, \end{aligned} \tag{7.58}$$

where, in order to evaluate the operator, we have inserted a complete set of spin states

$$1 = \sum_{b_1, b_2} \int \mathrm{d}\Phi(k_1, k_2)\, |k_1, b_1; k_2, b_2\rangle\langle k_1, b_1; k_2, b_2|. \tag{7.59}$$

Similarly, acting with $S\mathcal{O}_1|\psi_{\text{in}}\rangle$ gives

$$S\mathcal{O}_1|\psi_{\text{in}}\rangle = \frac{1}{\hbar^4}\int \mathrm{d}\Phi(p_1', p_2')\,|p_1', b_1; p_2', a_2'\rangle \int \mathrm{d}^4q\,\mathrm{d}^4x\,\psi_b(p_1'-q, p_2'+q)e^{iq\cdot x/\hbar} \\ \times \exp\Big(i\chi(x_\perp, s)/\hbar\Big)^{b_1,a_2'}_{a_1',a_2}\mathcal{O}_1(p_1'-q)^{a_1'}{}_{a_1}\,\xi_1^{a_1}\xi_2^{a_2}\,. \tag{7.60}$$

Notice that we encounter the operator $\mathcal{O}_1(p_1'-q)$ at a shifted value of the momentum. Since the class of operators we consider is at most linear in the momentum, we proceed by writing

$$\mathcal{O}_1(p_1'-q) = \mathcal{O}_1(p_1') - \mathcal{O}_1(q)\,. \tag{7.61}$$

Notice that our notation $\mathcal{O}_1(q)$ indicates that momentum factors in the operator are evaluated at momentum q. Using this notation, we can eventually write

$$[\mathcal{O}_1(p_1'), S]|\psi_{\text{in}}\rangle = \frac{1}{\hbar^4}\int \mathrm{d}\Phi(p_1', p_2')\,|p_1', b_1; p_2', a_2'\rangle \int \mathrm{d}^4q\,\mathrm{d}^4x\,\psi_b(p_1'-q, p_2'+q)e^{iq\cdot x/\hbar} \\ \times\left\{\left[\mathcal{O}_1(p_1'), \exp\left(i\chi(x_\perp, s)/\hbar\right)\right]^{b_1,a_2'}_{a_1,a_2} + \exp\Big(i\chi(x_\perp, s)/\hbar\Big)^{b_1,a_2'}_{a_1',a_2}\mathcal{O}_1(q)^{a_1'}{}_{a_1}\right\}\xi_1^{a_1}\xi_2^{a_2}\,. \tag{7.62}$$

Two matrix structures have appeared under the integral: one is a commutator, while the second is a more simple matrix product. This structure is a generalisation of the structure discussed in [360], where the second term was called the "direct" term in contrast to the "commutator" term.

We will shortly perform the q and x integrals in eq. (7.62) using stationary phase. In preparation for doing so, it is convenient to rewrite the equation such that the matrix $\exp(i\chi(x_\perp, s)/\hbar)$ stands to the left of any other matrices. Then in the evaluation of the overlap $\langle\psi_{\text{in}}|S^\dagger[\mathcal{O}_1(p_1'), S]|\psi\rangle$, the matrix and (on the solution of the stationary phase conditions) its inverse will simplify. We only have to move the eikonal operator though the commutator term in eq. (7.62); we can do so using the Baker-Hausdorff lemma in the form [366]

$$[\mathcal{O}, e^{i\chi/\hbar}] = e^{i\chi/\hbar}\left(-\frac{i}{\hbar}[\mathcal{O},\chi] + \frac{1}{2\hbar^2}[\chi,[\mathcal{O},\chi]] + \cdots + \frac{-i^n}{n!\hbar^n}[\chi,[\chi,[\chi,...[\mathcal{O},\chi]]]...]\right)\,, \tag{7.63}$$

where the last term contains n nested commutators involving χ.

So far, we have not yet taken advantage of any simplifications available for large, classical spins. Classical spin representations must be large, in the sense that the spin quantum number S times $\hbar$ is a classical angular momentum $S\hbar$. It can be useful to think of this as the limit $S\to\infty$, $\hbar\to 0$ with $S\hbar$ fixed. Large spin representations have the property that

$$\langle S_{\mu\nu}S_{\rho\sigma}\rangle = \langle S_{\mu\nu}\rangle\langle S_{\rho\sigma}\rangle + \mathcal{O}(\hbar)\,, \tag{7.64}$$

where $S_{\mu\nu}$ is a spin Lorentz generator and $\langle S_{\mu\nu}\rangle$ is its expectation value on the spin state, as shown in section 2.3. The key point here is that the small correction term is of order $S\hbar^2$ compared to the explicit term on the right-hand side, which is of order $S^2\hbar^2$ (see appendix A of [270] for a recent review.) We would therefore like to take advantage of this kind of simplification in the context of eq. (7.62).

In fact, we can easily take advantage of this simplification provided we first use the Baker-Hausdorff lemma (7.63) to move the eikonal operator to the left. The reason is that this exposes the infinite series to the right-hand-side of eq. (7.63) which is a series in inverse powers of $\hbar$. These inverse powers are compensated by powers in commutators of operators; for example in the case of spin we have

$$[\mathbb{W}_\mu, \mathbb{W}_\nu] = -i\hbar\epsilon_{\mu\nu\rho\sigma}\mathbb{W}^\rho\mathbb{P}^\sigma \,. \tag{7.65}$$

Had we not simplified the matrix structure and first tried to replace operators by expectation values we would make an error because the small correction in eq. (7.64), which is intimately related to the commutator in eq. (7.65), would be omitted.

Having taken advantage of the Baker-Hausdorff lemma, then, we may replace all operators by commutators. In doing so, we must retain the infinite set of non-vanishing commutators. This is easily done: we simply introduce the Poisson bracket notation defined by

$$\{\langle\mathbb{W}_\mu\rangle, \langle\mathbb{W}_\nu\rangle\} = -\epsilon_{\mu\nu\rho\sigma}\langle\mathbb{W}^\rho\rangle\langle\mathbb{P}^\sigma\rangle \,. \tag{7.66}$$

Notice that the Poisson brackets are directly inherited from the underlying algebraic structure of the quantum field theory. We have scaled out appropriate factors of $i\hbar$.

We therefore pass from eq. (7.62) to

$$\begin{aligned}[\mathcal{O}_1(p_1'), S]|\psi_\text{in}\rangle = \frac{1}{\hbar^4}\int \mathrm{d}\Phi(p_1', p_2')\int \mathrm{d}^4q\, \mathrm{d}^4x\, \psi_b(p_1'-q, p_2'+q)\, e^{iq\cdot x/\hbar} \\ \times\left(\left\{\mathcal{O}_1(p_1'), e^{i\chi(x_\perp, s;\langle w\rangle)/\hbar}\right\}_\text{B.H.} + \mathcal{O}_1(q)e^{i\chi(x_\perp, s;\langle w\rangle)/\hbar}\right)|p_1', \xi_1; p_2', \xi_2\rangle,\end{aligned} \tag{7.67}$$

where all operators have been replaced with scalar functions, at the expense of introducing Poisson brackets. The notation is

$$\left\{\mathcal{O}_1(p_1'), e^{i\chi(x_\perp, s;\langle w\rangle)/\hbar}\right\}_\text{B.H.} \equiv e^{i\chi/\hbar}\left(-\{\mathcal{O},\chi\} - \frac{1}{2}\{\chi,\{\mathcal{O},\chi\}\} + \cdots\right). \tag{7.68}$$

We emphasise that $\{\cdot,\cdot\}_\text{B.H.}$ is *not* a Poisson bracket: it is simply convenient notation for the commutator structure in eq. (7.63).

In this form, the eikonal has also become a scalar function $\chi(x_\perp, s;\langle w\rangle)$ which now depends on the expectation value of operators, eg Pauli-Lubanski $\langle w\rangle$ (and/or in the Yang-Mills case, the colour). We can therefore perform the x and q integrals by stationary phase, following precisely the methods of section 7.1.2. The stationary phase conditions are

$$q_\mu = -\frac{\partial}{\partial x^\mu}\chi(x_\perp, s) \,. \tag{7.69}$$

We now note that χ is now potentially a linear function of $x_\perp^\mu$ and so q_*^μ need not point only along $x_\perp^\mu$, and will in general point along some other direction in the plane perpendicular to $\tilde{p}_1$ and $\tilde{p}_2$. The final result for the out state is then

$$\begin{aligned}[\mathcal{O}_1(p_1'), S]|\psi_\text{in}\rangle = \int \mathrm{d}\Phi(p_1', p_2')\, \psi_b(p_1'-q_*, p_2'+q_*)e^{iq_*\cdot(x_*)/\hbar} \\ \times\left(\left\{\mathcal{O}_1(p_1'), e^{i\chi(x_{\perp *}, s)/\hbar}\right\}_\text{B.H.} + \mathcal{O}_1(q_*)e^{i\chi(x_{\perp *}, s)/\hbar}\right)|p_1', \xi_1; p_2', \xi_2\rangle.\end{aligned} \tag{7.70}$$

We can evaluate $\langle\psi_\text{in}|S^\dagger$ on the stationary phase too, which results in

$$\langle\psi_\text{in}|S^\dagger = \int \mathrm{d}\Phi(p_1, p_2)\psi_b^\dagger(p_1-q_*, p_2+q_*)e^{-iq_*\cdot(x_*)/\hbar}e^{-i\chi^\dagger(x_{\perp *}, s)}\langle p_1, \xi_1; p_2, \xi_2|, \tag{7.71}$$

such that

$$\begin{aligned}\Delta\mathcal{O}_1 &= \langle\psi_{\rm in}|S^\dagger[\mathcal{O}_1, S]|\psi_{\rm in}\rangle \\ &= \int \mathrm{d}\Phi(p_1,p_2)|\psi(p_1-q_*,p_2+q_*)|^2\left(e^{-i\chi^\dagger(x_{\perp *},s)/\hbar}\left\{\mathcal{O}_1(p_1), e^{i\chi(x_{\perp *},s)/\hbar}\right\}_{\rm B.H.} + \mathcal{O}_1(q_*)\right) \\ &= \left\langle\!\!\left\langle e^{-i\chi^\dagger(x_{\perp *},s)/\hbar}\left\{\mathcal{O}_1(p_1), e^{i\chi(x_{\perp *},s)/\hbar}\right\}_{\rm B.H.} + \mathcal{O}_1(q_*)\right\rangle\!\!\right\rangle .\end{aligned} \tag{7.72}$$

The Baker-Hausdorff lemma in the form of eq. (7.68) immediately tells us how to expand the Poisson bracket, leading to a neat expression for the change in $\mathcal{O}_1$

$$\Delta\mathcal{O}_1 = \mathcal{O}_1(q_*) - \{\mathcal{O}_1(p_1),\chi\} - \frac{1}{2}\{\chi,\{\mathcal{O}_1(p_1),\chi\}\} + \frac{1}{6}\{\chi,\{\chi,\{\mathcal{O}_1(p_1),\chi\}\}\} + \cdots . \tag{7.73}$$

where we recall that χ depends on $x^\mu_{\perp *}$ along with spin, colour etc, and we have dropped the brackets since we are in the fully classical regime. It is useful to note that for operators without momentum dependence, the term involving q_* vanishes, since it was induced by a shift in momentum and the linearity of the operator.

To check this, we can consider the momentum operator of particle one, which acts trivially in spin-space meaning we can choose

$$\mathcal{O}_1 = \mathbb{P}_1^\mu, \qquad \rightarrow \qquad {\mathcal{O}_1}^a{}_b(p) = p_1^\mu \delta^a{}_b. \tag{7.74}$$

Since the identity commutes with everything in spin-space, only the $\mathcal{O}(q_*)$ term contributes, giving

$$\Delta p_1^\mu = \left\langle\!\!\left\langle q_*^\mu \right\rangle\!\!\right\rangle = -\left\langle\!\!\left\langle \partial^\mu\chi(x_\perp,s)\right\rangle\!\!\right\rangle, \tag{7.75}$$

which matches the expression given in eq. (7.19). Note, however, that in general the $x^\mu_\perp$ dependence of χ will be different in the spinning case, which will lead to additional spin contributions to the impulse.

7.2.2 Classical spin from the Eikonal

The angular impulse for particle one is given by

$$\Delta s_1^\mu = \frac{1}{m_A}\langle\Psi|S^\dagger[\mathbb{W}_1^\mu, S]|\Psi\rangle \tag{7.76}$$

$$= \left\langle\!\!\left\langle e^{-i\chi^\dagger(x_{\perp *},s_i)/\hbar}\left\{s_1^\mu(p_1), e^{i\chi(x_{\perp *},s_i)/\hbar}\right\}_{\rm B.H.} + s_1^\mu(q_*)\right\rangle\!\!\right\rangle, \tag{7.77}$$

where the expection of the Pauli-Lubanski pseudovector is

$$\langle p, a_i|\mathbb{W}^\mu|p', a_j\rangle \equiv m s^\mu_{ij}(p)\hat{\delta}(p-p'). \tag{7.78}$$

The spin is more complicated to consider than colour, since the Pauli-Lubanski operator is a product of both the linear and angular momentum operators. This means that the q_*-dependent piece contributes to the spin and we find that the expansion is therefore given by

$$\Delta s_1^\mu = s_1^\mu(q_*) - \{s_1^\mu,\chi\} - \frac{1}{2}\{\chi,\{s_1^\mu,\chi\}\} + \cdots \tag{7.79}$$

The first term is the spin vector of particle one evaluated at q_*, and so we can write it as

$$s_1^\mu(q_*) = \frac{1}{2m_A}\epsilon^{\mu\nu\rho\sigma}q_{*\nu}S_{\rho\sigma}(p_1), \tag{7.80}$$

where the spin tensor is defined as $S_{\rho\sigma}(p_1) = \frac{1}{m_A}\epsilon_{\rho\sigma\mu\nu}p_1^\mu s_1^\nu(p_1)$. We can input this definition to find that

$$\begin{aligned} s_1^\mu(q_*) &= \frac{1}{2m_A^2}\epsilon^{\mu\nu\rho\sigma}q_{*\nu}\epsilon_{\rho\sigma\lambda\tau}p_1^\lambda s_1^\tau(p_1) = \frac{1}{m_A^2}\left((q_*\cdot p_1)s_1^\mu(p_1) - (s_1\cdot q_*)p_1^\mu\right) \\ &= \frac{1}{m_A^2}\Bigg((\Delta p_1\cdot p_1)s_1^\mu(p_1) - (s_1\cdot\Delta p_1)p_1^\mu\Bigg). \end{aligned} \tag{7.81}$$

We can compute the angular impulse in GR at the spin-1/2×spin-0 order by considering the lowest order eikonal phase

$$\chi = -\frac{2Gm_Am_B}{\sqrt{\gamma^2-1}}\left[(2\gamma^2-1)\log\left|\frac{x_\perp}{L}\right| - \frac{2\gamma}{m_A}\frac{\epsilon^{\mu\nu\rho\sigma}v_{A\mu}x_{\perp\nu}v_{B\rho}s_{1\sigma}}{|x_\perp|^2}\right]. \tag{7.82}$$

The angular impulse is given by

$$\begin{aligned} \Delta s_1^\mu &= s_1^\mu(q_*) - \{s_1^\mu, s_1^\nu\}\frac{\partial\chi}{\partial s_1^\nu} \\ &= -\frac{1}{m_A^2}p_1^\mu(s_1(p_1)\cdot\Delta p_1) - \frac{1}{m_A}\epsilon^{\mu\nu\rho\sigma}\frac{\partial\chi}{\partial s_1^\nu}p_{1\rho}s_{1\sigma}, \end{aligned} \tag{7.83}$$

where we have used $\{s_1^\mu, s_1^\nu\} = \frac{1}{m_A}\epsilon^{\mu\nu\rho\sigma}p_{1\rho}s_{1\sigma}$. The derivative with respect to s_1^μ is given by

$$\frac{\partial\chi}{\partial s_{1\alpha}} = \frac{4Gm_B\gamma}{\sqrt{\gamma^2-1}}\frac{\epsilon^{\alpha\mu\nu\rho}v_{A\mu}x_{\perp\nu}v_{B\rho}}{|x_\perp|^2}, \tag{7.84}$$

such that the angular impulse is finally

$$\Delta s_1^\mu = -\frac{1}{m_A^2}(s_1\cdot\Delta p_1)p_1^\mu - \frac{4Gm_B\gamma}{\sqrt{\gamma^2-1}}\frac{\epsilon^{\mu\nu\rho\delta}s_{1\nu}v_{A\rho}\epsilon_{\sigma\alpha\beta\gamma}v_A^\alpha v_B^\beta x_\perp^\gamma}{|x_\perp|^2}, \tag{7.85}$$

which matches the existing literature [360] and where we have dropped the higher order $\Delta p_1\cdot p_1$ term.

Chapter 8

Conclusion

We are only at the beginning of a new exciting adventure. The inspiral regime of the coalescence of compact objects in the sky can be conveniently described in the post-Minkowskian expansion by scattering amplitudes in the classical limit, which has offered a new perspective on the calculation of observables in general relativity. The Kosower-Maybee-O'Connell (KMOC) formalism [166] is a way to derive classical observables in the two-body scattering problem directly from amplitudes, which follows from the classical on-shell reduction of the in-in formalism, i.e. from the LSZ reduction with appropriate wavefunctions localizing the massive fields on their point-particle trajectory as $\hbar \to 0$.

In chapter 2 we have extended this formalism to include waves, which are made of an infinite superposition of massless particles, in terms of coherent states. The waveshape $\alpha(k)$ of the coherent state determines the structure of the wave, and it is interesting to analyze it in different physical situations. For example, in section 4.1 we studied the scattering of a very sharply-localized wave off a massive particle by using the geometric optics approximation, which is relevant for the light deflection around a massive compact body. But we can also derive the waveshape from the two-body scattering of a pair of massive point particles: at large distances and in the classical limit, this will be given by the five-point amplitude with one external messenger as discussed in section 4.3. Interestingly, the waveform offers a first example of a localized observable: a detector located at spatial infinity will register the deviation from an empty space, which is exactly what happens for the LIGO-VIRGO interferometers. But this is not the only localized observable we can detect: there are analogues of the global momentum and angular momentum emitted in gravitational waves, which we call collectively gravitational event shapes. These can be studied in the on-shell approach by considering a system of light-ray operators for linearized gravity near null infinity, as done in chapter 3. They offered quite interesting observables: gravitational energy event shapes are linked directly to the amplitude of the waveform and we have shown explicitly that they are infrared-finite, see section 4.5 and section 4.4 respectively. Moreover, zero-energy graviton contributions to the displacement memory are linked to the choice of the BMS frame at the amplitude level, as discussed in 4.6.

Since we know that gravitational waves are composed of many gravitons, it might be surprising to hear that only the five-point amplitude is relevant in the classical limit. What happens is that all the gravitons contribute coherently to the wave, as we have proved in section 5 and section 6. This can be done from different perspectives, and it is useful to do so in order to understand the physics behind it. First of all, we expect on general grounds that the unitary evolution of a pure state gives a pure state, and coherent state are the only classical pure states for radiation in quantum mechanics. It is well-known how to derive exactly the final coherent state from the soft dynamics, as we also discussed in section 5.1 within the worldline formalism and in section 5.5 by using asymptotic symmetries. The challenge is extend this to the full classical dynamics in the two-body problem. A detailed analysis of the particle distribution shows that we can study the deviation from a Poisson statistics for the emitted gravitons, as we did in section 5.4. But we can also take advantage of the uncertainty principle for the radiative field: since we expect to have only one classical field, all expectation values of product of field operators must factorize. Both approaches gave the same answer: at leading order, the six-point tree level amplitude must be classically suppressed. We have checked this in section 6.1, both by using Feynman diagrams from a convenient field redefinition of the Einstein-Hilbert action and also from a BCFW massive recursion relation, confirming thus that coherence holds at least at the lowest order in the coupling.

Exponentiation is very natural in classical physics as it follows directly from the uncertainty principle, not only in the Fock space of gravitons but also in spin space for the external massive spinning particles. Mathematically, this translates in the "zero-variance" property in the Glauber-Sudarshan representation which has an exponential as a solution as we showed in section 5.2 and section 5.3. All this evidence for coherence in the final state calls for an extension of the standard eikonal formulation with a coherent state of gravitons and with spin coherent states for the massive spinning external particles. We have carried out this program successfully in chapter 7, where we have also explored the infinity of amplitude relations which follows as a consequence of this representation.

Finally, we conclude with some important open questions. First, we have primarily focused on the scattering case but it is known that classical equations of motion can be used also for the bound case by changing boundary conditions [66, 162–165]. It would be very exciting if both χ and α can be analytically continued in a similar way: this would lead to a direct, quantum-first connection between amplitudes and binary black hole physics. Second, it would be nice to have a full derivation of the conservative eikonal phase from physical principles. While in the traditional formulation a quantum remainder is present which prevents the full exponentiation of the conservative part, this might not be the case if we use a more symmetric parametrization of the external momenta for the classical trajectory [122, 139]. Last but not least, it is known that the classical description breaks down at sufficiently high energies because of quantum radiation reaction effects, which ultimately make the emitted gravitons interfere with each other [135, 136, 263, 337]. This is actually important to have a consistent resummation of radiation reaction effects, and perhaps a simpler setup where analytic calculations are possible at very high orders – like working in a fixed background – can give us some useful lessons in this direction [192, 367–372].

Appendix A

Review of Schwinger proper time method

In this appendix, we present a short review of the Schwinger proper time method, which allows propagators in quantum field theory can be rewritten in terms of path integrals in ordinary quantum mechanics. We set $\hbar = 1$ only for this appendix, but we have restored for the application of this formalism in section 5.1. Consider first the Feynman propagator for a free scalar field of mass m:

$$D_F(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2} e^{ik\cdot(x-y)}, \tag{A.1}$$

which satisfies the position space equation

$$i(\hat{H} - i\varepsilon) D_F(x-y) = \delta^{(d)}(x-y), \quad \hat{H} = \Box + m^2, \tag{A.2}$$

where we have introduced the Klein-Gordon operator $\hat{H}$, which appears in the quadratic terms in the scalar field Lagrangian. Eq. (A.2) corresponds to the well-known fact that the propagator is associated with the inverse of the operator $\hat{H}$, and we may formally write

$$-i(\hat{H} - i\varepsilon)^{-1} = \int_0^\infty dT e^{-iT(\hat{H} - i\varepsilon)}, \tag{A.3}$$

where the $i\varepsilon$ prescription guarantees convergence of the integral, and T is conventionally called a Schwinger parameter. The integral in eq. (A.3) contains the operator

$$\hat{U}(T) = e^{-i\hat{H}T}, \tag{A.4}$$

and if we interpret T as a time variable. this has the known form of the evolution operator in quantum mechanics where $\hat{H}$ is the Hamiltonian, given by

$$\hat{H} = -\hat{p}^2 + m^2. \tag{A.5}$$

We can introduce a Hilbert space of states on which this Hamiltonian acts. Complete sets are provided by the position or momentum states $\{|x\rangle\}$ or $\{|p\rangle\}$ (eigenstates of $\hat{x}$ and $\hat{p}$ respectively). Consider the evolution operator sandwiched between a state of given initial position and final momentum. For a small time separation δT, one finds

$$\langle p| e^{-i\hat{H}\delta T} |x\rangle = e^{-iH\delta T + \mathcal{O}(\delta T^2)} \langle p|x\rangle, \tag{A.6}$$

where H denotes the replacement of the position and momentum operators in $\hat{H}$ with their corresponding eigenvalues. Note that, strictly speaking, we must ensure that the Hamiltonian is first written in Weyl-ordered form, such that all momentum operators appear to the left of all position ones. This is not an issue for the Hamiltonian of eq. (A.5), but will be relevant when a gauge field is included. The above simple result applies only for small time separations. For a large time separation T, one may divide the time interval into N steps

$$\delta T = \frac{T}{N}, \tag{A.7}$$

where the limit $N \to \infty$ will eventually be taken. Then we may introduce a complete set of intermediate position and momentum states at each time-step, so that the matrix element of the evolution operator between a final momentum state $|p_f\rangle$ and initial position state $|x_i\rangle$ becomes

$$\begin{aligned}\langle p_f|e^{-i\hat{H}T}|x_i\rangle = &\int \mathrm{d}x_1 \ldots \mathrm{d}x_N \int \mathrm{d}p_0 \ldots \mathrm{d}p_{N-1} \langle p_f|e^{-i\hat{H}\delta t}|x_N\rangle \\ &\times \langle x_N|p_{N-1}\rangle\langle p_{N-1}|e^{-i\hat{H}\delta t}|x_{N-1}\rangle\langle x_{N-1}|p_{N-2}\rangle\langle p_{N-2}|e^{-i\hat{H}\delta t}|x_{N-2}\rangle \ldots \\ &\times \langle x_1|p_0\rangle\langle p_0|e^{-i\hat{H}\delta t}|x_i\rangle.\end{aligned} \tag{A.8}$$

Introducing the notation $x_0 \equiv x_i, p_N \equiv p_f$ for the fixed boundary conditions, we can write the previous equation more succinctly as

$$\begin{aligned}\langle p_f|e^{-i\hat{H}T}|x_i\rangle = &\int \mathrm{d}x_1 \ldots \int \mathrm{d}x_N \int \mathrm{d}p_0 \ldots \mathrm{d}p_{N-1} \exp\left[-i\sum_{k=0}^{N} H\left(p_k, x_k\right)\delta T\right] \\ &\times \prod_{k=0}^{N}\langle p_k|x_k\rangle \prod_{k=0}^{N-1}\langle x_{k+1}|p_k\rangle \\ = &\int \mathrm{d}x_1 \ldots \int \mathrm{d}x_N \int \mathrm{d}p_0 \ldots \mathrm{d}p_{N-1} \exp\left[-i\sum_{k=0}^{N} H\left(p_k, x_k\right)\delta T\right] \\ &\times \left[\prod_{k=0}^{N-1} \exp\left(i\frac{p_k \cdot (x_{k+1} - x_k)}{\delta T}\delta T\right)\right] e^{-ip_f \cdot x_N},\end{aligned} \tag{A.9}$$

where in the second line we have used that the inner product of position and momentum states gives

$$\langle x|p\rangle = e^{-ip\cdot x}. \tag{A.10}$$

We have written eq. (A.9) in a form such that its continuum limit $N \to \infty$ – eq. (5.5) – may be straightforwardly recognised.

This has the form of a double path integral in position and momentum, subject to the boundary conditions we imposed above. With further manipulations, eq. (5.5) can be related to the usual expression for the evolution operator sandwiched between two position states:

$$\langle x_f|e^{-i\hat{H}T}|x_i\rangle = \int_{x(0)=x_i}^{x(T)=x_f} \mathcal{D}x \exp\left[i\int_0^T \mathrm{d}t L(x,\dot{x})\right], \tag{A.11}$$

where $L(x, \dot{x})$ is the Lagrangian[1]. We have here considered a free particle, but the extension to a particle in a background gauge field is straightforward: one simply replaces the Hamiltonian of eq. (A.5) with the corresponding quadratic operator when a gauge field is present, as in eq. (5.6).

[1]To do this, one can add an arbitrary potential to the Hamiltonian, and complete the Gaussian path integral over p.

Appendix B

Shear-inclusive ANEC and Raychaudhuri equation

The connection of the ANEC operator and its expectation value with causality properties of the underlying quantum field theory is well-known [373]. A congruence of null (complete, achronal) geodesics obeys Raychaudhuri equation

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} = -\frac{1}{2}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - 8\pi G T^{\text{matter}}_{\mu\nu} k^\mu k^\nu , \tag{B.1}$$

where λ is the affine parameter, $\Theta = \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}\lambda}$[1] is the expansion, $\sigma_{\mu\nu}$ is the shear and k^μ is a null vector of unit affine length which in our case we will take the light-sheet generator $k^\mu = \delta^\mu_-$. The right hand side of eq. (B.1) immediately reminds us the enlarged definition of the matter ANEC with the gravitational contribution once we integrate along the null direction of the generator, albeit here eq. (B.1) is purely classical in its nature. If we consider a generic congruence of null geodesics with starts at $\mathcal{I}^-$ and ends on $\mathcal{I}^+$, one can ask how the expansion parameter changes along the congruence. Classically, the ANEC condition $T^{\text{matter}}_{\mu\nu} k^\mu k^\nu \geq 0$ implies the classical focussing theorem, that is light-rays never "anti-focus" as long as matter has positive energy. Quantum mechanically, quantum fields can developed negative energy so that we generically need a quantum version of the ANEC [374] and of the focussing theorem [375]. In general, there are problems in including quantized gravity perturbations in this story. Nevertheless, assuming we have matter minimally coupled with gravity, for a stationary null surface one can quantize $h_{\mu\nu}$ on the null surface in light-front quantization [204] and at the first order in the perturbation one can define a shear-inclusive ANEC operator of the form

$$\mathcal{E}_{\text{shear-inclusive}} = \int \mathrm{d}\lambda \left[\frac{1}{8\pi G}\sigma_{\mu\nu}\sigma^{\mu\nu} + T^{\text{matter}}_{\mu\nu} k^\mu k^\nu\right], \tag{B.2}$$

whose expectation value is positive definite $\langle \mathcal{E}^{\text{shear-inclusive}} \rangle \geq 0$ as it can be proved from the linearized Raychaudhuri equation [376]. The connection to our previous definition of the shear-inclusive ANEC in the Bondi gauge consists in considering the asymptotically flat region for the geodesic congruence. Indeed in such region one can define a boundary shear as the limit of the rescaled shear of the congruence on future null infinity,

$$\lim_{r\to+\infty} r^2 \left[\frac{1}{8\pi G}\sigma_{\mu\nu}\sigma^{\mu\nu} + T^{\text{matter}}_{\mu\nu} k^\mu k^\nu\right] \sim \frac{1}{32\pi G} N_{\zeta\zeta} N^{\zeta\zeta} + \lim_{r\to\infty} r^2 T^{\text{matter}}_{vv} \tag{B.3}$$

as proved in [377, 378] in order to study asymptotic entropy bounds.

[1] A is the cross sectional area here.

Appendix C

Light-ray operators for a light-sheet in the bulk

In the general, one would like to generalize the light-ray operator definition to allow them to be localized on general light-sheets in the bulk of the spacetime. If the theory is scale invariant (i.e. for all CFT for example) then the two definitions will coincide but in general they will be different.[1] To simplify our calculations, we can define

$$\tilde{x}_- := \frac{x^0 + x^3}{2} \qquad \tilde{x}_+ := \frac{x^0 - x^3}{2}, \tag{C.1}$$

so that light-sheets at fixed $\tilde{x}_- = \text{const}$ will be parametrized by the coordinate $\tilde{x}_+$. It is worth noticing that as $\tilde{x}_- \to +\infty$ we will have $\tilde{x}_\pm \to x_\pm$, so that there is no ambiguity with the previous notation in appendix B. For the light-sheet defined by $\tilde{x}_- = c$ with $c \in \mathbb{R}$, we define the light-ray family as

$$\begin{aligned}
\mathcal{E}(\tilde{x}_\perp) &:= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, T_{--}(\tilde{x}_+ = c, \tilde{x}_-, \tilde{x}_\perp), \\
\mathcal{K}(\tilde{x}_\perp) &:= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, (\tilde{x}_+) \, T_{--}(\tilde{x}_+ = c, \tilde{x}_-, \tilde{x}_\perp), \\
\mathcal{N}_A(\tilde{x}_\perp) &:= (\partial_A \tilde{x}^\mu) \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ T_{-\mu}(\tilde{x}_+ = c, \tilde{x}_-, \tilde{x}_\perp).
\end{aligned} \tag{C.2}$$

C.0.1 Spin 0

For a general light-sheet $c \in \mathbb{R}$ one needs to compute

$$\begin{aligned}
T_{--}(x) &= \frac{1}{4}\left(T_{00}(x) + T_{33}(x) - 2T_{03}(x)\right) = \frac{1}{4}\left(\partial_0\phi(x) - \partial_3\phi(x)\right)^2 = \partial_-\phi(x)\partial_-\phi(x), \\
T_{1-}(x) &= \frac{1}{2}\left(T_{10}(x) - T_{13}(x)\right) = \frac{1}{2}\partial_1\phi(x)\left(\partial_0\phi(x) - \partial_3\phi(x)\right) = \partial_1\phi(x)\partial_-\phi(x), \\
T_{2-}(x) &= \frac{1}{2}\left(T_{20}(x) - T_{23}(x)\right) = \frac{1}{2}\partial_2\phi(x)\left(\partial_0\phi(x) - \partial_3\phi(x)\right) = \partial_2\phi(x)\partial_-\phi(x).
\end{aligned} \tag{C.3}$$

[1] See the explicit Hofman-Maldacena conformal transformation [149].

The light-ray operators are then defined as

$$\begin{aligned}
\mathcal{E}_{\text{scalar}}(\tilde{x}_\perp) &:= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \partial_- \phi(\tilde{x}) \partial_- \phi(\tilde{x}) \Big|_{\tilde{x}_-=c}, \\
\mathcal{N}_{A,\text{scalar}}(\tilde{x}_\perp) &:= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \partial_A \phi(\tilde{x}) \partial_- \phi(\tilde{x}) \Big|_{\tilde{x}_-=c}, \\
\mathcal{K}_{\text{scalar}}(\tilde{x}_\perp) &:= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \tilde{x}_+ \, \partial_- \phi(\tilde{x}) \partial_- \phi(\tilde{x}) \Big|_{\tilde{x}_-=c}.
\end{aligned} \tag{C.4}$$

It is quite remarkable that interactions drop out completely. This generically happens for scalar theories without derivative couplings and it implies some universality of the scalar light-ray algebra.

C.0.2 Spin 1

We have

$$\begin{aligned}
T_{--} &= \frac{1}{2}\eta^{ij} \left[(F^a_{0i} - F^a_{3i})(F^b_{j0} - F^b_{j3}) \right] \mathrm{Tr}(T^a T^b) = (\partial_- A^a_1)(\partial_- A^a_1) + (\partial_- A^a_2)(\partial_- A^a_2), \\
T_{1-} &= F^a_{1\beta}\eta^{\alpha\beta}(F^b_{\alpha 0} - F^b_{\alpha 3})\mathrm{Tr}(T^a T^b) = (2\partial_{[1}A^a_{2]} + g f^{acd} A^c_1 A^d_2)(\partial_- A^a_2) + 2(\partial_- A^a_+)(\partial_- A^a_1), \\
T_{2-} &= F^a_{2\beta}\eta^{\alpha\beta}(F^b_{\alpha 0} - F^b_{\alpha 3})\mathrm{Tr}(T^a T^b) = (2\partial_{[2}A^a_{1]} + g f^{acd} A^c_2 A^d_1)(\partial_- A^a_1) + 2(\partial_- A^a_+)(\partial_- A^a_2)
\end{aligned} \tag{C.5}$$

where in the last line have imposed the light-cone gauge

$$A_0 = A_3 \Leftrightarrow A_- = 0. \tag{C.6}$$

We see that while $T_{--}(\tilde{x})$ is interaction independent in this gauge, $T_{A-}(\tilde{x})$ is not. Nevertheless, we are interested in light-ray operators where we integrate over the light-cone time $\tilde{x}_+$ and therefore we can take advantage of the equations of motion

$$\mathcal{D}_\mu F^{\mu,a}_{-} = \partial_- \left[\partial_1 A^a_1 + \partial_2 A^a_2 - 2\partial_- A^a_+ \right] + g f^{abc}(A^b_1 \partial_- A^c_1 + A^b_2 \partial_- A^c_2) = 0, \tag{C.7}$$

provided that we take into account possible boundary terms at $\tilde{x}_+ \to \pm\infty$. To fix completely the gauge for the boundary contributions, we can enforce the radiation gauge where together with the light-cone condition $A_- = 0$ we have

$$\nabla^\mu A^a_\mu(x)\Big|_{\mathcal{I}^-} = \lim_{\tilde{x}_+ \to -\infty} \left[2\partial_- A^a_+ - \partial_1 A^a_1 - \partial_2 A^a_2 \right] = 0, \tag{C.8}$$

thanks to which

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, T_{1-}(\tilde{x}) &= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \left[(\partial_1 A^a_2 - \partial_2 A^a_1)(\partial_- A^a_2) + (\partial_1 A^a_1 + \partial_2 A^a_2)(\partial_- A^a_1) \right] \\
&\quad + \frac{1}{2} A^a_1 \left[2\partial_- A^a_+ - \partial_1 A^a_1 - \partial_2 A^a_2 \right]_{\tilde{x}_+ \to +\infty}, \\
\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, T_{2-}(\tilde{x}) &= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \left[(\partial_2 A^a_1 - \partial_1 A^a_2)(\partial_- A^a_1) + (\partial_1 A^a_1 + \partial_2 A^a_2)(\partial_- A^a_2) \right] \\
&\quad + \frac{1}{2} A^a_2 \left[2\partial_- A^a_+ - \partial_1 A^a_1 - \partial_2 A^a_2 \right]_{\tilde{x}_+ \to +\infty}.
\end{aligned} \tag{C.9}$$

The light-ray operators are then defined as

$$
\begin{aligned}
\mathcal{E}_{\text{gluon}}(\tilde{x}_\perp) &:= \sum_{i=1}^{2} \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \partial_- A_i^a(\tilde{x}) \partial_- A_i^a(\tilde{x}) \Big|_{\tilde{x}_-=c} , \\
\mathcal{K}_{\text{gluon}}(\tilde{x}_\perp) &:= \sum_{i=1}^{2} \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \tilde{x}_+ \, \partial_- A_i^a(\tilde{x}) \partial_- A_i^a(\tilde{x}) \Big|_{\tilde{x}_-=c} , \\
\mathcal{N}_{1,\text{gluon}}(\tilde{x}_\perp) &:= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \left[(\partial_1 A_2^a - \partial_2 A_1^a)(\partial_- A_2^a) + (\partial_1 A_1^a + \partial_2 A_2^a)(\partial_- A_1^a) \right] \Big|_{\tilde{x}_-=c} \\
&\quad + \frac{1}{2} A_1^a \left[2\partial_- A_+^a - \partial_1 A_1^a - \partial_2 A_2^a \right]_{\tilde{x}_+ \to +\infty} , \\
\mathcal{N}_{2,\text{gluon}}(\tilde{x}_\perp) &:= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}_+ \, \left[(\partial_2 A_1^a - \partial_1 A_2^a)(\partial_- A_1^a) + (\partial_1 A_1^a + \partial_2 A_2^a)(\partial_- A_2^a) \right] \Big|_{\tilde{x}_-=c} \\
&\quad + \frac{1}{2} A_2^a \left[2\partial_- A_+^a - \partial_1 A_1^a - \partial_2 A_2^a \right]_{\tilde{x}_+ \to +\infty} .
\end{aligned} \tag{C.10}
$$

If we compare to the self-interacting scalar case, we see that in YM theory interactions do not affect the definition of the operators $\mathcal{E}_{\text{gluon}}(\tilde{x}_\perp)$ and $\mathcal{K}_{\text{gluon}}(\tilde{x}_\perp)$ but they appear in $\mathcal{N}_{A,\text{gluon}}(\tilde{x}_\perp)$ through the boundary contribution.

Appendix D

Poissonian distributions and coherent states

The graviton coherent states introduced in the main text can be expanded in the Fock space basis of a definite number of gravitons,

$$|\alpha^\sigma\rangle = \exp\left(-\frac{1}{2}\int \mathrm{d}\Phi(k)|\alpha^\sigma(k)|^2\right)\sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{i=1}^{n}\left[d\Phi(k_i)\alpha^\sigma(k_i)\right]|k_1^\sigma...k_n^\sigma\rangle, \tag{D.1}$$

and a direct calculation of the probability of detecting n gravitons with helicity σ' gives

$$P_n^{\sigma'} := \delta_{\sigma\sigma'}\exp\left(-\int \mathrm{d}\Phi(k)|\alpha^\sigma(k)|^2\right)\frac{1}{n!}\left(\int \mathrm{d}\Phi(k)|\alpha^\sigma(k)|^2\right)^n, \tag{D.2}$$

which corresponds exactly to Poissonian statistics. A straightforward calculation of the mean and the variance in a coherent state gives

$$\mu_{\alpha^\sigma} = \Sigma_{\alpha^\sigma} = \int \mathrm{d}\Phi(k)|\alpha^\sigma(k)|^2. \tag{D.3}$$

Poissonian statistics are equivalent to having a coherent state, as can be seen by computing $P_n^{\sigma'}$ for a generic probability distribution in the Glauber-Sudarshan representation,

$$\mathrm{Tr}\left(P_n^{\sigma'}\right)_{\rho_{\text{radiation,GS}}} = \sum_{\sigma=\pm}\int \mathcal{D}^2\alpha^\sigma\,\mathcal{P}^\sigma(\alpha)\,P_n^\sigma, \tag{D.4}$$

which requires $\mathcal{P}^\sigma(\alpha) = \delta^2(\alpha^\sigma - \alpha_\star^\sigma)$ to match the Poissonian distribution.

In classical physics, however, we can have more general statistics for the classical radiation field. In particular, the variance of the distribution can be greater than the mean,[1]

$$\mu_\rho < \Sigma_\rho, \tag{D.5}$$

which defines the so-called super-Poissonian statistics. This applies, for example, to thermal classical distributions. In our case, as discussed in the main text, the fact that we are working with pure states that are evolved with a unitary map suggests that all the classical states will have to obey the minimum uncertainty principle [166].

[1]We expect the opposite inequality, i.e. $\mu_\rho > \Sigma_\rho$, to be relevant for purely quantum particle statistics. Indeed by using the Jensen inequality it is possible to prove that any mixture of coherent states will produce only super-Poissonian distributions.

Appendix E

Projection in the plane of scattering in the eikonal approach

The relation between the eikonal impact parameter $x_\perp^\mu$ and b^μ can be stated clearly by first introducing some notation. Let us define the following four-vectors in momentum space

$$\begin{cases} e_0^\mu \equiv N_0(\tilde{p}_1^\mu + \tilde{p}_2^\mu) \\ e_q^\mu \equiv N_q(\tilde{p}_1^\mu - \tilde{p}_2^\mu) - N_{0q}(\tilde{p}_1^\mu + \tilde{p}_2^\mu) \ , \end{cases} \tag{E.1}$$

where the normalization factors N_0, N_q and N_{0q} are fixed by requiring $e_0^2 = 1$, $e_q^2 = -1$ and $e_0 \cdot e_q = 0$. By definition of $x_\perp^\mu$, the following identities hold:

$$e_0 \cdot x_\perp = 0 \quad , \quad e_q \cdot x_\perp = 0 \ . \tag{E.2}$$

As a consequence, we can write the projection of x^μ on the plane orthogonal to $\tilde{p}_1^\mu$ and $\tilde{p}_2^\mu$ as

$$x_\perp^\mu = x^\mu - (x \cdot e_0) e_0^\mu + (x \cdot e_q) e_q^\mu \ , \tag{E.3}$$

where the different signs in the the last two terms are a consequence of e_0^μ being time-like while e_q^μ space-like. Using eq. (E.3) we can easily compute some of the derivatives involved in the evaluation of the stationary phase on x^μ such as

$$\frac{\partial x_\perp^2}{\partial x_\mu} = 2x_{\perp,\nu}(\eta^{\mu\nu} - e_0^\mu e_0^\nu + e_q^\mu e_q^\nu) = 2x_\perp^\mu \ . \tag{E.4}$$

Another example where the use of eq. (E.3) is useful is when we apply the stationary phase for the integral over q^μ. In this case, the stationary condition for q^μ can be expressed as

$$x^\mu = b^\mu + q_{\nu,*} \frac{\partial x_\perp^\nu}{\partial q_\mu}\bigg|_{q=q_*} \ , \tag{E.5}$$

where q_*^μ satisfies the stationary phase condition on x^μ given by $q_*^\mu = -2\chi'(x_\perp)x_\perp^\mu$. One of the advantages in the definition in eq. (E.1) of e_0^μ is that it is q^μ independent so that the previous stationary condition can be expressed as

$$x^\mu = b^\mu + q_{\nu,*} \frac{\partial}{\partial q_\mu}\left[(x \cdot e_q) e_q^\nu\right]\bigg|_{q=q_*} \ . \tag{E.6}$$

Since q_*^μ is parallel to $x_\perp^\mu$, we know that $q_* \cdot e_q \propto x_\perp \cdot e_q = 0$, and so eq. (E.6) simplifies to

$$x^\mu = b^\mu + q_{\nu,*}(x \cdot e_q) \frac{\partial e_q^\nu}{\partial q_\mu}\bigg|_{q=q_*} \ . \tag{E.7}$$

The remaining derivative can be easily performed using the definition of e_q^μ. The result is

$$q_\nu \frac{\partial e_q^\nu}{\partial q_\mu}\bigg|_{q=q_*} = -(N_q\, q^\mu)\bigg|_{q=q_*} . \tag{E.8}$$

We can then write the eikonal impact parameter as

$$x_\perp^\mu = b^\mu - (x\cdot e_0)e_0^\mu - \big[(x\cdot e_q)N_q q^\mu\big]\big|_{q=q_*} + \big[(x\cdot e_q)e_q^\mu\big]\big|_{q=q_*} . \tag{E.9}$$

The scalar products $x\cdot e_0$ and $x\cdot e_q$, which can be viewed as Lagrange multipliers for the phase which we are minimizing, are fixed by requiring the eikonal impact parameter to be orthogonal to e_0^μ and e_q^μ. A straightforward calculation gives

$$x_\perp^\mu = b^\mu - \big[(e_q\cdot b)(N_q q^\mu - e_q^\mu)\big]\big|_{q=q_*} . \tag{E.10}$$

Expressing e_q^μ in terms of p_1^μ, p_2^μ and q^μ we obtain

$$x_\perp^\mu = b^\mu - \Big[(e_q\cdot b)\big[N_{0q}(p_1^\mu + p_2^\mu) - N_q(p_1^\mu - p_2^\mu)\big]\Big]\Big|_{q=q_*} , \tag{E.11}$$

which agrees — when evaluated in the center of mass frame — with the expression for the eikonal impact parameter in eq. (7.17), where $\tilde{N}_q = -(e_q\cdot b)N_q$ and $\tilde{N}_{0q} = (e_q\cdot b)N_{0q}$.

Appendix F

Unitarity of the final state in the eikonal approach

After stationary phase in q^μ and x^μ, the final state in the conservative sector is

$$S|\psi\rangle = \int \mathrm{d}\Phi(p_1',p_2')\,|p_1'p_2'\rangle\, e^{iq_*(s)\cdot x_*(s)/\hbar} e^{i\chi\left(x_{*,\perp}(s);s\right)/\hbar}\psi_b\left(p_1'-q_*(s),p_2'+q_*(s)\right) \ , \tag{F.1}$$

where $q_*(s)$ and $x_*(s)$ are solutions of the stationary phase condition depending on the Mandelstam variable $s=(p_1'+p_2')^2$. To check unitarity of the final state we can proceed by computing

$$\langle\psi|S^\dagger S|\psi\rangle = \int \mathrm{d}\Phi\left(p_1,p_2\right)|\psi\left(p_1-q_*(s),p_2+q_*(s)\right)|^2 \ , \tag{F.2}$$

where we have dropped the subscript b from the wavepacket, and the prime indices on the momenta. We then introduce the following change of variable

$$\begin{cases} P_1^\mu = p_1^\mu - q_*^\mu(s) \\ P_2^\mu = p_2^\mu + q_*^\mu(s) \ . \end{cases} \tag{F.3}$$

Due to the dependence of $q_*(s)$ on s, we obtain the Jacobian determinant of eq. (F.3) as well as a dependence in the conserving momentum Dirac delta on $q_*(s)$

$$\begin{aligned}\langle\psi|S^\dagger S|\psi\rangle = \int \hat{\mathrm{d}}^4P_1\hat{\mathrm{d}}^4P_2\,\hat{\delta}^{(+)}(P_1^2-m_A^2+2P_1\cdot q_*+q_*^2)\,\hat{\delta}^{(+)}(P_2^2-m_B^2-2P_2\cdot q_*+q_*^2)\\ \times|\det J(P_1,P_2)|\,|\psi\left(P_1,P_2\right)|^2 \ .\end{aligned} \tag{F.4}$$

At first sight, these two additional contributions seem to spoil the unitarity of the final state. However, a closer look reveals that this is not the case. First of all, the Dirac delta functions are trivial as a consequence of $e_0\cdot q_* = e_q\cdot q_* = 0$:

$$2P_1\cdot q_* + q_*^2 = 0 \quad , \quad 2P_2\cdot q_* - q_*^2 = 0 \ . \tag{F.5}$$

As for the Jacobian determinant, this can be evaluated in (p_1,p_2) variables as

$$\begin{aligned}|\det J(P_1,P_2)|(p_1,p_2) =& 1+\mathrm{Tr}\left(\frac{\partial q_*^\mu(s)}{\partial p_2^\nu}-\frac{\partial q_*^\mu(s)}{\partial p_1^\nu}\right)\\ &= 1+\mathrm{Tr}\left(\frac{\partial q_*^\mu(s)}{\partial s}\frac{\partial s}{\partial p_2^\nu}-\frac{\partial q_*^\mu(s)}{\partial s}\frac{\partial s}{\partial p_1^\nu}\right) \ .\end{aligned} \tag{F.6}$$

Since the derivative of the Mandelstam variable s is symmetric with respect to p_1 and p_2, we can conclude that eq. (F.6) is indeed equal to one. Thus,

$$\langle\psi|S^\dagger S|\psi\rangle = \int \mathrm{d}\Phi(P_1, P_2)|\psi\,(P_1, P_2)\,|^2 = 1 \tag{F.7}$$

as expected.

Appendix G

Explicit six-point tree amplitude calculation in scalar QED

We now compute the leading contribution in $\hbar$ of the six-point tree amplitude. We will discuss the computation explicitly in electromagnetism, and explain only the mechanism for cancellation of apparent excess powers of $\hbar$ in gravity.

Suppose particle 1 has charge Q_1 while particle 2 has charge Q_2. Then there are three gauge-invariant six-point tree partial amplitudes:

$$\mathcal{A}_6^{(0)}(p_1+q_1, p_2+q_2 \to p_1, p_2, k) = Q_1^3 Q_2\, A_{(3,1)} + Q_1^2 Q_2^2\, A_{(2,2)} + Q_1 Q_2^3\, A_{(1,3)}\,. \tag{G.1}$$

The "charge-ordered" partial amplitudes $A_{(3,1)}$, $A_{(2,2)}$, and $A_{(1,3)}$ are analogues of color-ordered amplitudes in gauge theory, which motivates our choice of notation.

Evidently there can be no cancellation of powers of $\hbar$ between these partial amplitudes because of the different powers of the charges. Thus the problem reduces to computing the leading-in-$\hbar$ terms in these amplitudes. There are two partial amplitudes to consider, since $A_{(1,3)}$ can be obtained from $A_{(3,1)}$ by trivially swapping the labels 1 and 2.

We have performed two separate computations of these partial amplitudes. Firstly, we made use of standard automated tools to directly compute the amplitude in full detail. Specifically, we used FeynArt [379] to create a model for scalar QED, extracting the Feynman rules directly from the Lagrangian. FeynCalc [380, 381] can automatically generate all the topologies relevant for the calculation of $\mathcal{A}_6^{(0)}$. There are 42 diagrams to be computed, and FeynCalc provided direct automatic expressions for each of these. We processed our expressions further in Mathematica with the tensor package xAct [317], which help to extract the classical limit. The final result for $\mathcal{A}_{6,0}^{(0)}$ is of order $\hbar^{-4}$.

To gain further insight we performed a separate computation of the partial amplitudes $A_{(3,1)}$ and $A_{(2,2)}$ in a convenient gauge which greatly reduced the labour necessary to see that two powers of $\hbar$ cancel. The gauge we chose (referring to the momentum routing in equation (G.1)) is

$$p_1 \cdot \varepsilon(k_i) = 0 \text{ for } i = 1, 2, \tag{G.2}$$

where p_1 is the momentum of particle 1 while k_i is the outgoing momentum of photon i.

The effect of this choice is twofold; it removes many diagrams from the calculation and those that remain get an $\hbar$ enhancement from each emission vertex. For example, consider $A_{(3,1)}$: in this case, the emitted photon is radiated from particle 1. With our momentum labelling convention the emission vertices will produce factors of the form $(2p_1+\hbar\bar{Q})\cdot\varepsilon(k_i)$, where $\bar{Q}$ is some combination of wavenumbers. The first part vanishes

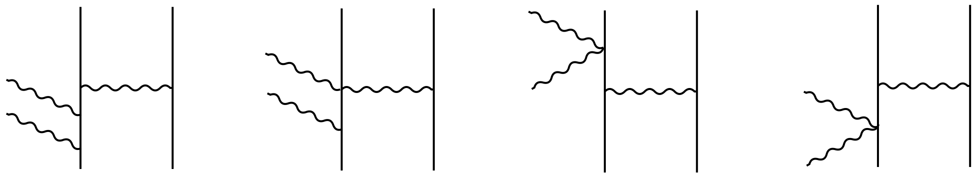


FIGURE G.1: Diagrams in $Q_1^3Q_2$ sector.

leaving only the $\hbar$ enhanced $\varepsilon(k_i)\cdot\bar{Q}$ term. Any diagram with a photon emitted from the outgoing line of particle 1 vanishes, as it is proportional to $(2p_1+\hbar\bar{k}_i)\cdot\varepsilon(k_i)=0$. In what follows we will write $\varepsilon(k_i)=\varepsilon_i$. We will also suppress the $i\epsilon$ factors in the propagators[1]; they all implicitly come with $+i\epsilon$. Finally we will refer to each diagram contributing to the amplitude by, for example, $D(3,1)$ so that

$$iA_{(3,1)} = iD_{(3,1)cubic} + iD_{(3,1)quartic}. \tag{G.3}$$

Some of the sub-amplitudes are given by a single diagram, whereas others are made up of multiple diagrams.

Our gauge choice is most powerful in the case of $A_{(3,1)}$, so we discuss this case in most detail. The Feynman diagrams that constitute this amplitude can be split into 3 classes: the first involves single photon emissions coming from cubic vertices, the second has precisely one photon emitted into the final state from a quartic vertex, while the third class has two photons emitted from the same quartic vertex. These classes are shown in Fig. G.1, after removing diagrams which vanish by gauge choice. The first diagram is an example of the first class, the second an example of the second class and the last two diagrams are in the third class.

We choose a particular ordering of k_1 and k_2 for the calculation, and include the permuted case by swapping $k_1 \leftrightarrow k_2$. For the first class there is a single diagram to compute after gauge fixing, and it is trivial to write down the leading term in the amplitude and see that it has the desired scaling. This diagram is

$$\begin{aligned}
D_{(3,1)}|_{cubic} =& \frac{1}{\hbar^4\bar{q}_2^2}\left[\frac{2(\varepsilon_2\cdot\bar{q}_1)(\varepsilon_1\cdot(2\bar{q}_1-\bar{k}_2))(2p_2+\hbar\bar{q}_2)\cdot(2p_1-\hbar\bar{q}_2)}{(-2p_1\cdot\bar{q}_2+\hbar\bar{q}_2^2)(2p_1\cdot(\bar{q}_1-\bar{k}_1)+\hbar(\bar{q}_1-\bar{k}_1)^2)}\right]\\
=& \frac{1}{\hbar^4\bar{q}_2^2}\left[\frac{4p_1\cdot p_2(\varepsilon_2\cdot\bar{q}_1)(\varepsilon_1\cdot(2\bar{q}_1-\bar{k}_2))}{2(-p_1\cdot\bar{q}_2)(p_1\cdot(\bar{q}_1-\bar{k}_1)}+\mathcal{O}(\hbar)\right]\\
=& \frac{2p_1\cdot p_2}{\hbar^4\bar{q}_2^2}\left[\frac{(\varepsilon_2\cdot\bar{q}_1)(\varepsilon_1\cdot(\bar{q}_1-\bar{q}_2))}{(p_1\cdot(\bar{k}_1+\bar{k}_2))(p_1\cdot\bar{k}_1)}\right]+\mathcal{O}(\hbar^{-3}).
\end{aligned} \tag{G.4}$$

We used momentum conservation $q_1+q_2=k_1+k_2$ to write our expressions in terms of just the k_i or the q_i. The choice here is most natural for obtaining a similarly simple expression for the amplitude with k_1 and k_2 swapped.

The second class is actually tractable without fixing a gauge, as there is only a single cancellation to show. However with our gauge fixing this class becomes trivial, and there is only a single diagram to compute. This is

$$D_{(3,1)}|_{cubic/quartic} = \frac{4(p_2\cdot\varepsilon_2)(\bar{q}_1\cdot\varepsilon_1)}{\hbar^4\bar{q}_2^2 p_1\cdot\bar{k}_1}. \tag{G.5}$$

[1]The $i\epsilon$ factors are often important — and indeed play an important role in appendix G — but in this computation they are spectators.

The final class is unaffected by our choice of gauge as it is proportional to $\varepsilon_1 \cdot \varepsilon_2$. After gauge fixing this is naively $\hbar^{-5}$, so we must find a single cancellation. The mechanism of the cancellation is identical to the case of the five-point tree amplitude. This is done in [95, 166], but we shall review it here for completeness. The key step is to make use of the on-shell conditions

$$(p_1 + q_1)^2 = m_A^2, \qquad (p_2 + q_2)^2 = m_B^2, \tag{G.6}$$

which allows us, after the $\hbar$ rescaling, to replace $2p_i \cdot \bar{q}_i \to -\hbar \bar{q}_i^2$ in the propagators. Lastly, we Taylor expand. There are two diagrams to compute which are,

$$\begin{aligned} D_{(3,1)}|_{quartic,1} =& -\frac{2\varepsilon_1 \cdot \varepsilon_2}{\hbar^5 \bar{q}_2^2}\left[\frac{(2p_1 + \hbar(2\bar{q}_1 + \bar{q}_2)) \cdot (2p_2 + \hbar\bar{q}_2)}{2p_1 \cdot (\bar{k}_1 + \bar{k}_2) + \hbar(\bar{k}_1 + \bar{k}_2)^2}\right] \\ =& -\frac{2\varepsilon_1 \cdot \varepsilon_2}{\hbar^5 \bar{q}_2^2}\left[\frac{4p_1 \cdot p_2}{2p_1 \cdot (\bar{k}_1 + \bar{k}_2)}\right. \\ & \left. + \hbar\left(\frac{4p_2 \cdot \bar{q}_1 + 2p_1 \cdot \bar{q}_2}{2p_1 \cdot (\bar{k}_1 + \bar{k}_2)} - \frac{4p_1 \cdot p_2 (\bar{k}_1 + \bar{k}_2)^2}{(2p_1 \cdot (\bar{k}_1 + \bar{k}_2))^2}\right)\right] \end{aligned} \tag{G.7}$$

and

$$\begin{aligned} D_{(3,1)}|_{quartic,2} =& -\frac{2\varepsilon_1 \cdot \varepsilon_2}{\hbar^5 \bar{q}_2^2}\left[\frac{4p_1 \cdot p_2}{-2p_1 \cdot (\bar{k}_1 + \bar{k}_2)}\right. \\ & \left. + \hbar\left(\frac{4p_1 \cdot \bar{q}_2}{-2p_1 \cdot (\bar{k}_1 + \bar{k}_2)} + \frac{4p_1 \cdot p_2((\bar{k}_1 + \bar{k}_2)^2 - \bar{q}_1^2)}{(2p_1 \cdot (\bar{k}_1 + \bar{k}_2))^2}\right)\right]. \end{aligned} \tag{G.8}$$

Notice the most singular terms are equal up to a sign, and so cancel. Combining the remaining terms we obtain

$$D_{(3,1)}|_{quartic} = -\frac{\varepsilon_1 \cdot \varepsilon_2}{\hbar^4 \bar{q}_2^2}\left[\frac{4(p_1 \cdot p_2)(\bar{q}_2 \cdot (\bar{k}_1 + \bar{k}_2))}{(p_1 \cdot (\bar{k}_1 + \bar{k}_2))^2} + \frac{4p_2 \cdot \bar{q}_1 + 2p_1 \cdot \bar{q}_2}{p_1 \cdot (\bar{k}_1 + \bar{k}_2)}\right]. \tag{G.9}$$

These can be combined as $A_{(3,1)} = D_{(3,1)cubic} + D_{(3,1)quartic}$ yielding,

$$\begin{aligned} A_{(3,1)} =& \frac{1}{\hbar^4 \bar{q}_2^2}\left[\frac{4p_1 \cdot p_2 (\varepsilon_2 \cdot \bar{q}_1)(\varepsilon_1 \cdot (\bar{q}_1 - \bar{q}_2))}{2(p_1 \cdot (\bar{k}_1 + \bar{k}_2))(p_1 \cdot \bar{k}_1)} + \frac{4p_1 \cdot p_2 (p_2 \cdot \varepsilon_2)(\bar{q}_1 \cdot \varepsilon_1)}{p_1 \cdot \bar{k}_1}\right. \\ & \left. - (\varepsilon_1 \cdot \varepsilon_2)\left(\frac{4(p_1 \cdot p_2)(\bar{q}_2 \cdot (\bar{k}_1 + \bar{k}_2))}{(p_1 \cdot (\bar{k}_1 + \bar{k}_2))^2} + \frac{4p_2 \cdot \bar{q}_1 + 2p_1 \cdot \bar{q}_2}{p_1 \cdot (\bar{k}_1 + \bar{k}_2)}\right)\right] \\ & + (k_1 \leftrightarrow k_2). \end{aligned} \tag{G.10}$$

The story is very similar for $A_{(2,2)}$. We can split into the same 3 classes, use gauge fixing to get rid of one factor of $\hbar$ and then massage using the on-shell constraints to show the final cancellation. Here we just quote the result and give details in the appendix. The result is

$$
\begin{aligned}
A_{(2,2)} =& \frac{4}{\hbar^4(\bar{q}_2 - \bar{k}_2)^2} \Bigg[4\varepsilon_1 \cdot \varepsilon_2 + \frac{(\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot \bar{q}_2)}{p_2 \cdot \bar{k}_2} - \frac{(\varepsilon_2 \cdot p_2)(\varepsilon_1 \cdot (\bar{q}_2 + \bar{q}_1))}{2 p_2 \cdot \bar{k}_2} \\
& \frac{p_1 \cdot p_2 (\varepsilon_1 \cdot \bar{q}_1)(\varepsilon_2 \cdot \bar{q}_2)}{(p_2 \cdot \bar{k}_2)(p_1 \cdot \bar{k}_1)} - \frac{(p_1 \cdot \bar{k}_2)(\varepsilon_1 \cdot \bar{q}_1)(\varepsilon_2 \cdot p_2)}{(p_2 \cdot \bar{k}_2)(p_1 \cdot \bar{k}_1)} \\
& - \frac{(\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot p_2)(\bar{q}_2 \cdot \bar{k}_2)}{(p_2 \cdot \bar{k}_2)^2} - \frac{p_1 \cdot p_2 (\varepsilon_1 \cdot \bar{q}_1)(\varepsilon_2 \cdot p_2) \bar{q}_2 \cdot \bar{k}_2}{(p_2 \cdot \bar{k}_2)^2 (p_1 \cdot \bar{k}_1)} \Bigg] \\
& + (k_1 \leftrightarrow k_2).
\end{aligned}
\tag{G.11}
$$

Finally $A_{(1,3)}$ can be obtained by swapping the labels $1 \leftrightarrow 2$ in the expression for $A_{(3,1)}$ (written in the gauge where $p_2 \cdot \varepsilon_i = 0$. Of course the partial amplitudes themselves are gauge-invariant.) In all cases, the $\hbar$ scaling is as required from negligible variance.

Appendix H

One loop factorisation of the five-point amplitude in scalar QED

We now turn to verifying eq. (4.66). For simplicity we focus on the case of scalar QED, though the general nature of our arguments indicates that the result should also hold in gravity and in Yang-Mills theory. In order to keep the computational labour to the minimum necessary, we take advantage of lessons we learned in the context of the six-point tree amplitude in appendix H. First, we note that the scalar QED five-point amplitudes can be reduced to gauge-invariant partial amplitudes analogous to colour-ordered amplitudes in Yang-Mills theory. In particular we write

$$\begin{aligned}
\mathcal{A}^{(0)}_{4,0}(p_1,p_2\to p_1+w_1,p_2+w_2) &= Q_1Q_2A_{(1,1)}\,,\\
\mathcal{A}^{(0)}_{5,0}(p_1,p_2\to p_1+q_1,p_2+q_2,k) &= Q_1Q_2^2A_{(1,2)}+Q_1^2Q_2A_{(2,1)}\,,\\
\mathcal{A}^{(1)}_{5,0}(p_1,p_2\to p_1+q_1,p_2+q_2,k) &= Q_1^2Q_2^3A_{(2,3)}+Q_1^3Q_2^2A_{(3,2)}\,.
\end{aligned} \tag{H.1}$$

In view of the symmetry between $A_{2,3}$ and $A_{3,2}$ we may compute just one choice: we choose to focus on the charge sector $Q_1^2Q_2^3$.

Second, we find it useful to choose an explicit gauge, namely

$$\varepsilon_\sigma(\bar{k})\cdot p_2 = 0\,. \tag{H.2}$$

This choice drastically reduces the relevant number of terms in the $\hbar$ expansion. It is trivial to determine the tree partial amplitudes in this gauge, which are

$$A_{(1,1)} = e^2\frac{4p_1\cdot p_2}{\bar{w}_1^2}\,, \tag{H.3}$$

and

$$A_{(1,2)} = \frac{4e^3}{\bar{q}_1^2}\left(p_1\cdot\varepsilon_\sigma(k)+\frac{p_1\cdot p_2\,\varepsilon_\sigma(k)\cdot\bar{q}_1}{p_2\cdot\bar{k}}\right)\,. \tag{H.4}$$

The anatomy of eq. (4.66) is that the leading-in-$\hbar$ fragment of the one-loop five-point amplitude $A_{(2,3)}$ will organise itself into a product of $A_{(1,1)}$ times the four terms that constitute $A_{(1,2)}$. Our strategy will be to isolate these terms one by one. Let us start gathering the relevant diagrams of $A_{(2,3)}$. At one loop, and in the classical limit, we need to consider the transfer expansion of the following five-point diagrams with a photon emitted in the final state:

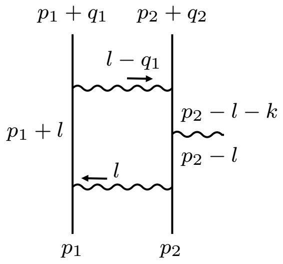
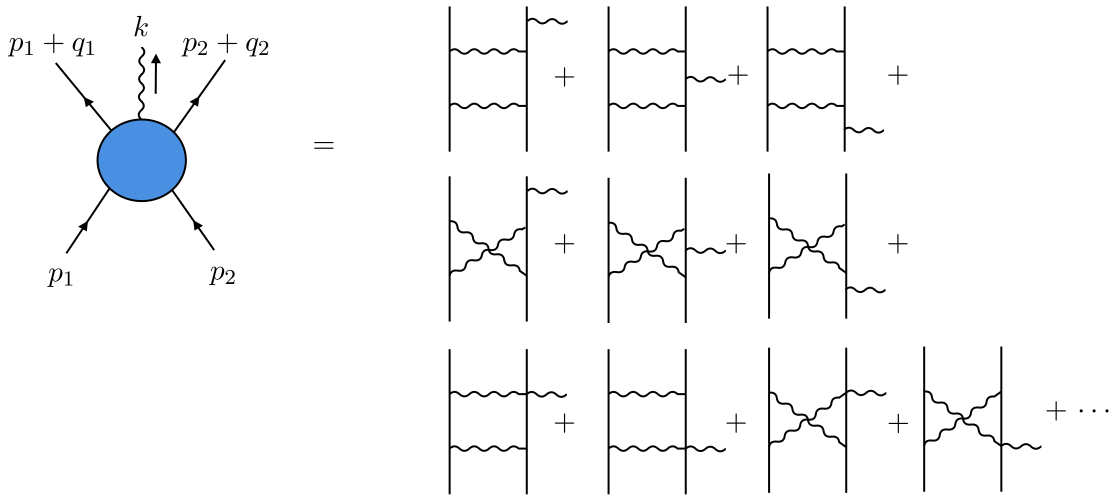


FIGURE H.1: Momentum routing for the pentagon contribution to the five-point one-loop amplitude.



The ellipsis indicate purely quantum diagrams which are not relevant for us.

As in appendix H we tidy our expressions up by making use of the on-shell conditions. Using the momentum labelling in the figure above these read

$$p_1 \cdot \bar{q}_1 = p_2 \cdot \bar{q}_2 = \mathcal{O}(\hbar). \tag{H.5}$$

Keeping this in mind we start to compute the diagrams. It is helpful to compute the first six diagrams in the figure above, which involve only three-point vertices, separately from the rest of the diagrams (which involved contact four-point vertices). We therefore write

$$A_{(2,3)} = A^{\text{3pt}}_{(2,3)} + A^{\text{4pt}}_{(2,3)}\,. \tag{H.6}$$

We focus first on the six diagrams which constitute $A^{\text{3pt}}_{(2,3)}$. For clarity, let us begin by describing the contribution from the pentagon diagram, in our gauge of eq. (H.2) in detail. We choose the momentum routing shown in Fig. H.1. On the support of the momentum-conserving delta function $\hat{\delta}^4(q_1 + q_2 + k)$, its contribution to $A_{(2,3)}$ is

$$ie^5 \int \frac{\hat{\mathrm{d}}^4 \bar{l}}{\bar{l}^2(\bar{l} - \bar{q}_1)^2} \frac{(4p_1 \cdot p_2)^2(-2\,\varepsilon_\sigma(k) \cdot \bar{l})}{(2p_1 \cdot \bar{l})(-2p_2 \cdot \bar{l})(-2p_2 \cdot (\bar{k} + \bar{l}))}. \tag{H.7}$$

The sum of the cubic diagrams yields

$$
\begin{aligned}
A^{\text{3pt}}_{(2,3)} &= ie^5 \int \frac{\hat{\mathrm{d}}^4\bar{l}}{\bar{l}^2(\bar{l}-\bar{q}_1)^2}(4p_1\cdot p_2)^2 \\
&\times \left(\frac{-2\varepsilon_\sigma(k)\cdot\bar{l}}{(2p_1\cdot\bar{l})(-2p_2\cdot\bar{l})(-2p_2\cdot(\bar{k}+\bar{l}))} - \frac{2\varepsilon_\sigma(k)\cdot\bar{q}_1}{(2p_1\cdot\bar{l})(-2p_2\cdot\bar{l})(2p_2\cdot\bar{k})}\right. \\
&+ \left.\frac{2\varepsilon_\sigma(k)\cdot(\bar{l}-\bar{q}_1)}{(2p_1\cdot\bar{l})(2p_2\cdot(\bar{k}+\bar{l}))(2p_2\cdot\bar{l})} - \frac{2\varepsilon_\sigma(k)\cdot\bar{q}_1}{(2p_1\cdot\bar{l})(2p_2\cdot(\bar{k}+\bar{l}))(2p_2\cdot\bar{k})}\right).
\end{aligned}
\tag{H.8}
$$

Note that the signs in the linearised propagators are important! We use them to indicate the hidden $i\epsilon$'s,

$$
\frac{1}{\pm p\cdot\bar{l}} \equiv \frac{1}{\pm p\cdot\bar{l}+i\epsilon} \neq \pm\frac{1}{p\cdot\bar{l}} = \frac{1}{\pm p\cdot\bar{l}\pm i\epsilon}. \tag{H.9}
$$

which allows us to make use of the following identity

$$
-i\hat{\delta}(p\cdot\bar{l}) = \frac{1}{p\cdot\bar{l}} + \frac{1}{-p\cdot\bar{l}}. \tag{H.10}
$$

In order to make use of this identity we apply a change of variables $l \to \bar{q}_1 - l$ in the last two terms in eq. (H.8). This, along with the on-shell conditions, allows pairs of terms to take an almost identical form — denominators differ only by a sign in the $p_1\cdot l$ term which is precisely what is needed to apply eq. (H.10). The amplitude then reduces to

$$
A^{\text{3pt}}_{(2,3)} = 4e^5(p_1\cdot p_2)^2 \int \frac{\hat{\mathrm{d}}^4\bar{l}}{\bar{l}^2(\bar{l}-\bar{q}_1)^2}\frac{\hat{\delta}(p_1\cdot\bar{l})}{-p_2\cdot\bar{l}}\left(-\frac{\varepsilon_\sigma(k)\cdot\bar{q}_1}{p_2\cdot\bar{k}} - \frac{\varepsilon_\sigma(k)\cdot\bar{l}}{-p_2\cdot(\bar{k}+\bar{l})}\right). \tag{H.11}
$$

It is possible to expose a second delta function in this expression by writing $-\varepsilon_\sigma(k)\cdot\bar{q}_1 = \varepsilon_\sigma(k)\cdot(\bar{l}-\bar{q}_1) - \varepsilon_\sigma(k)\cdot\bar{l}$. The two terms involving $\varepsilon_\sigma(k)\cdot\bar{l}$ under the integral sign can be simplified by a partial fraction, yielding

$$
\begin{aligned}
A^{\text{3pt}}_{(2,3)} &= 4e^5(p_1\cdot p_2)^2 \int \frac{\hat{\mathrm{d}}^4\bar{l}}{\bar{l}^2(\bar{l}-\bar{q}_1)^2}\frac{\hat{\delta}(p_1\cdot\bar{l})}{p_2\cdot\bar{k}}\left(\frac{\varepsilon_\sigma(k)\cdot(\bar{l}-\bar{q}_1)}{-p_2\cdot\bar{l}} - \frac{\varepsilon_\sigma(k)\cdot\bar{l}}{-p_2\cdot(\bar{l}-\bar{q}_1)}\right) \\
&= 4ie^5(p_1\cdot p_2)^2 \int \frac{\hat{\mathrm{d}}^4\bar{l}}{\bar{l}^2(\bar{l}-\bar{q}_1)^2}\hat{\delta}(p_1\cdot\bar{l})\hat{\delta}(p_2\cdot(\bar{l}-\bar{q}_1))\frac{\varepsilon_\sigma(k)\cdot\bar{l}}{p_2\cdot\bar{k}}.
\end{aligned}
\tag{H.12}
$$

To obtain the second of these equalities, we redefined the variable of integration to $\bar{l}' = -(\bar{l}-\bar{q}_1)$ in the first term, and dropped the prime.

Next, we address the remaining diagrams contributing to $A_{(2,3)}$ which now involve four-point vertices. After a straightforward computation, we find

$$
A^{\text{4pt}}_{(2,3)} = 4ie^5 p_1\cdot p_2\,\varepsilon_\sigma(k)\cdot p_1 \int \frac{\hat{\mathrm{d}}^4\bar{l}}{\bar{l}^2(\bar{l}-\bar{q}_1)^2}\hat{\delta}(p_1\cdot\bar{l})\hat{\delta}(p_2\cdot(\bar{l}-\bar{q}_1)). \tag{H.13}
$$

At this stage we can see the structure of the required factorisation — we have exposed the delta functions present in eq. (4.66).

Combining the contact terms of eq. (H.13) with the rest of the diagrams, eq. (H.12), we find that the total expression for the amplitude fragment is

$$
A_{(2,3)} = 4ie^5\,p_1\cdot p_2 \int \frac{\hat{\mathrm{d}}^4\bar{l}}{\bar{l}^2(\bar{l}-\bar{q}_1)^2}\hat{\delta}(p_1\cdot\bar{l})\hat{\delta}(p_2\cdot(\bar{l}-\bar{q}_1))\left(\varepsilon_\sigma(k)\cdot p_1 + p_1\cdot p_2\frac{\varepsilon_\sigma(k)\cdot\bar{l}}{p_2\cdot\bar{k}}\right). \tag{H.14}
$$

The final step is to compare this result with the prediction for $A_{(2,3)}$ from eq. (4.66). Using the amplitudes of eq. (H.1), it is easy to see that the prediction is

$$
\begin{aligned}
A_{(2,3)} = \int & \hat{\mathrm{d}}^4\bar{w}_1\hat{\mathrm{d}}^4\bar{w}_2\,\hat{\delta}(2p_1\cdot\bar{w}_1)\hat{\delta}(2p_2\cdot\bar{w}_2)\hat{\delta}^4(\bar{q}_1+\bar{q}_2-\bar{w}_1-\bar{w}_2) \\
& \times\frac{4e^3}{\bar{w}_1^2}\left(p_1\cdot\varepsilon^\sigma+\frac{p_1\cdot p_2\,\varepsilon^\sigma(k)\cdot\bar{w}_1}{p_2\cdot\bar{k}}\right)\frac{4e^2}{(\bar{q}_1-\bar{w}_1)^2}p_1\cdot p_2\,.
\end{aligned}
\tag{H.15}
$$

Upon performing the integral over $\bar{w}_2$ using the four-fold delta function, relabelling $\bar{w}_1=\bar{l}$ and recognising that $\bar{k}=-\bar{q}_1-\bar{q}_2$, we immediately recover eq. (H.14).

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